Singularities of normal affine 3-folds containing $\mathbb{A}^1$-cylinderlike open subsets

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Contents

1. Introduction
2. The main results
3. Reduction via $\sharp$-Minimal Model Program
4. Proof of Theorem 2.1
5. Proof of Theorem 2.2 and Corollary 2.1

Abstract

The normal affine varieties with $\mathbb{A}^1$-cylinderlike open subsets play important roles in affine algebraic geometry and of interest in itself. In [Miy81], Miyanishi investigated how to construct normal affine surfaces Y with $\mathbb{A}^1$-cylinderlike open subsets, explicitly. According to this explicit construction, he proved that such surfaces Y have at most cyclic quotient singularities. In this article, we consider the 3-dimensional generalization of this Miyanishi's result. Namely, we shall investigate how to construct normal affine 3-folds X containing $\mathbb{A}^1$-cylinderlike open subsets and the possibility of singularities on such X under a certain geometric assumption concerning the compactifications.

Keywords: an $\mathbb{A}^1$-cylinderlike open subset, log Kodaira dimension, singularities, $\sharp$-minimal model program

1. Introduction

Throughout the present article we work over the field of complex numbers $\mathbb{C}$. In affine algebraic geometry the varieties with $\mathbb{A}^1$-*cylinderlike open subsets* (cf. Definition 3.1) play the important roles. We often call such varieties as *affine ruled* varieties and use the these two terminologies "containing an $\mathbb{A}^1$-cylinderlike open subset" and "affine ruled" interchangeably in the subsequent. It is obvious that normal one-dimensional affine ruled affine variety is the affine

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line $\mathbb{A}^1$ only. In 1981, Miyanishi investigated the construction and singularities on a normal affine surface with an $\mathbb{A}^1$-cylinderlike open subset (cf. [Miy81]). As a consequence, he obtained the following result:

Theorem 1.1 (cf. [Miy81])**.** *Let Y be a normal affine surface with an $\mathbb{A}^1$-cylinderlike open subset. Then Y has at most cyclic quotient singularities.*

In his proof, the theory of singular fibers of $\mathbb{P}^1$-fibrations on smooth surfaces (cf. [Giz70, Miy78]) and the affineness played the crucial roles. Roughly speaking, his argument to prove Theorem 1.1 is summarized as follows: Let $\mu : \widehat{Y} \to Y$ be the minimal resolution. Then $\widehat{Y}$ inherits an $\mathbb{A}^1$-fibration from that on Y and can be embedded into a smooth projective surface W with a structure of a $\mathbb{P}^1$-fibration, say π, so that π contains an $\mathbb{A}^1$-fibration on $\widehat{Y}$. Then Miyanishi investigated the possibility of configurations of the singular fibers of π the proper images of which pass through singularities on Y in consideration of the results in [Giz70, Miy78]. Theorem 1.1 above has several significant applications for not only the theory on affine surfaces but also that on higher-dimensional affine varieties. For instance, the algebro-geometric characterization of the affine 3-space $\mathbb{A}^3$ in [Miy88] makes use of it effectively.

Now our main interest lies in the 3-dimensional generalization of the above mentioned Miyanishi's result (cf. Theorem 1.1). Namely:

Problem **1.2.** Let X be a normal affine 3-fold with only isolated, $\mathbb{Q}$-factorial singularities such that X contains an $\mathbb{A}^1$-cylinderlike open subset. What kinds of singularities can appear on X ?

In order to tackle Problem 1.2, it seems to be enough to argue along the similar line as performed in Miyanishi's proof in [Miy81]. Namely, let $\mu : \widehat{X} \to X$ be the resolution of singularities of X, and we embed $\widehat{X}$ into a smooth projective 3-fold $\widehat{T}$ so that $\widehat{T}$ has a structure of a $\mathbb{P}^1$-fibration, say π, which contains an $\mathbb{A}^1$-fibration on $\widehat{X}$ induced from that on X. In this setting, you may try to determine the types of singularities on X by observing configurations of fibers of π the images of which contain the singularities on X. But this attempt is not suitable for our purpose. Indeed, π is not flat, and the singular fibers of non-flat $\mathbb{P}^1$-fibration offer little information to determine the singularities.

In this present article, we make a certain assumption with respect to the compactification of X and consider Problem 1.2. Our assumption is geometric in nature: We embed X into a normal projective 3-fold T so that T is smooth along the boundary $T - X$ and $T - X$ is an S.N.C. divisor on T. Then we assume the following:

ASSUMPTION (†): The complete linear system $\mathcal{H} := |D|$ on T is nef and contains a smooth member, say S, where D is the reduced boundary divisor with respect to $X \hookrightarrow T$.

Under this assumption (†), we shall establish the following two:

(A) Explicit construction of X,

(B) Determination of the possibility of singularities on X.

We shall state the scheme of this paper.

In Section 2, we shall state the main results concerning (A) and (B) (cf. Theorems 2.1, 2.2 and Corollary 2.1), and some remarks about them. The detailed proof for them are performed in the subsequent Sections 3, 4 and 5.

In Section 3, we prove a part of Theorem 2.1. More precisely to say, we shall construct birational maps ϕ', ϕ'', ψ' and μ' in the diagram in Theorem 2.1 (see Section 2, Theorem 2.1 for the notation). For this purpose, first of all, we shall perform the $\sharp$-Minimal Model Program associated to $(\widehat{T}, \widehat{\mathcal{H}})$ as in [Me02]. In the original context of $\sharp$-Minimal Model Program, it works to control the Minimal Model Program starting from $\widehat{T}$ in a right direction with respect to the linear system $\widehat{\mathcal{H}}$. But, in addition to this, we now need to observe how the inside quasi-projective variety $\widehat{X}$ (and also X) changes carefully via $\sharp$-Minimal Model Program.

In Section 4, we shall complete the proof of Theorem 2.1 by constructing birational morphisms ψ'' and $\mu^\sharp$ in the diagram in Theorem 2.1. We note that the $\mathbb{Q}$-factoriality on X is needed at this stage.

In Section 5, we shall prove Theorem 2.2. This result concerning the structure of Mori fiber space $\pi : \widehat{T}^\sharp \to W$ is essentially obtained in [C-F93, Me02]. But, note that since $(\widehat{T}, \widehat{\mathcal{H}})$ may not be *a pair with a big uniruled system* (see [Me02, Definition 5.1]), we can not make use of the result in [C-F93, Me02] directly to our present situation. Furthermore, we have to observe the birational morphism $\mu^\sharp : \widehat{T}^\sharp \to T^\sharp$. As seen in Theorem 2.2, $\mu^\sharp$ is, in fact, an isomorphism except for the case where $\pi : \widehat{T}^\sharp \to W$ is of $C2$-type. After proving Theorem 2.2, we finally prove Corollary 2.1.

We shall employ the following convention of notation in this paper:

Convention of Notation.

- $D|_W$: a scheme-theoretic restriction of a Cartier divisor D on V to a subvariety $W \subset V$;
- $D_1 \sim D_2$: D_1 is linearly equivalent to D_2;
- $D_1 \equiv D_2$: D_1 is numerically equivalent to D_2;
- $\mathbb{Q}^3$: a smooth quadric hypersurface in $\mathbb{P}^4$;
- $V \cong W$: V is isomorphic to W;
- $V \approx W$: V is birationally equivalent to W;
- $\varrho(V)$: the Picard number of a variety V;
- $\mathbb{P}(a_0, \cdots, a_n)$: a weighted projective space of weights $\mathrm{wt}(x_0, \cdots, x_n) = (a_1, \cdots, a_n)$, i.e., $\mathbb{P}(a_0, \cdots, a_n) = \mathrm{Proj}\mathbb{C}[x_0, \cdots, x_n]$ with $\mathrm{wt}(x_i) = a_i$;
- $T_{\alpha_1, \cdots \alpha_r} \subset \mathbb{P}(a_0, \cdots, a_n)$: a weighted complete intersection of degree $\alpha_1, \cdots, \alpha_r$ in $\mathbb{P}(a_0, \cdots, a_n)$;
- $\frac{1}{r}(a, b, c)$: a germ of the quotient singularity of analytic type $o \in \mathbb{C}^3 : (x, y, z)/\mathbb{Z}_r(a, b, c)$, where the cyclic group $\mathbb{Z}_r =< \xi >$ acts on $\mathbb{C}^3$ via $\xi \cdot (x, y, z) = (\xi^a x, \xi^b y, \xi^c z)$. ($\xi \in \mathbb{C}^*$ is a primitive r-th root of unity.)

An *E2-type divisorial contraction* means a birational morphism between two algebraic 3-folds so that an exceptional divisor E of it satisfies $(E, E|_E) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$ (cf. [Mori82]).

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2. The main results

In this section, we shall state the results concerning (A) and (B) in Section 1. The first result is stated as follows:

Theorem 2.1. *Let X be a normal affine 3-fold with only isolated, $\mathbb{Q}$-factorial singularities such that it contains an $\mathbb{A}^1$-cylinderlike open subset. Assume that X is embedded into a normal projective 3-fold T satisfying* (†). *Let $\mu : \widehat{T} \to T$ be the resolution of singularities of T, let $\widehat{\mathcal{H}} := \mu^*(\mathcal{H})$ and let $\widehat{D} := \mu^*(D)$, $\widehat{S} := \mu^*(S) \in \widehat{\mathcal{H}}$. Then there exists a following commutative diagram satisfying the conditions (1)∼(4) below:*

$$\begin{array}{ccccc} \widehat{T} & \overset{\phi'}{\cdots\cdots\to} & \widehat{T}' & \xrightarrow{\phi''} & \widehat{T}^\sharp \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu'} & & \downarrow{\scriptstyle\mu^\sharp} \\ T & \overset{\psi'}{\cdots\cdots\to} & T' & \xrightarrow{\psi''} & T^\sharp, \end{array}$$

(1) *$\phi := \phi'' \circ \phi' : \widehat{T} \cdots \to \widehat{T}^\sharp$ is a process of the $\sharp$-Minimal Model Program associated to $(\widehat{T}, \widehat{\mathcal{H}})$ (cf. Section 3 and [Me02]) and $\widehat{T}^\sharp$ has a structure of Mori fiber space, say $\pi : \widehat{T}^\sharp \to W$. $\phi'' : \widehat{T}' \to \widehat{T}^\sharp$ is a composite of E2-type divisorial contractions unless $\phi'' = id$;*

Let $\widehat{D}'$ (resp. $\widehat{D}^\sharp$) be the proper transform of $\widehat{D}$ on $\widehat{T}'$ (resp. on $\widehat{T}^\sharp$). Let D' (resp. $D^\sharp$) be the proper transform of D on T' (resp. on $T^\sharp$). Put $\widehat{X}' := \widehat{T}' - \mathrm{Supp}\,(\widehat{D}')$, $\widehat{X}^\sharp := \widehat{T}^\sharp - \mathrm{Supp}\,(\widehat{D}^\sharp)$, $X' := T' - \mathrm{Supp}\,(D')$ and $X^\sharp := T^\sharp - \mathrm{Supp}\,(D^\sharp)$. Then:

(2) *$X \cong X'$;*

(3) *$\widehat{X}^\sharp$ (resp. $X^\sharp$) is an open subset of $\widehat{X}'$ (resp. of $X' \cong X$) such that $\widehat{X}' - \widehat{X}^\sharp \cong X' - X^\sharp$ is a union of several affine planes unless $\widehat{X}' = \widehat{X}^\sharp$ and $X' = X^\sharp$. More precisely, $\widehat{X}'$ (resp. X') is obtained from $\widehat{X}^\sharp$ (resp. $X^\sharp$) via the composite of suitable half-point attachments (cf. Definition 3.2) unless $\widehat{X}' = \widehat{X}^\sharp$ and $X' = X^\sharp$;*

(4) $\mathrm{Sing}\,(X) = \mathrm{Sing}\,(X^\sharp)$.

As stated in Theorem 2.1, the original affine 3-fold X is obtained from $X^\sharp = T^\sharp - \mathrm{Supp}\,(D^\sharp)$ via the composite of suitable half-point attachments (cf. Definition 3.2). Therefore, we have only to investigate the construction of $X^\sharp$ and singularities on $X^\sharp$. For this purpose, we shall look into the structure of Mori fiber space $\pi : \widehat{T}^\sharp \to W$ and the morphism $\mu^\sharp : \widehat{T}^\sharp \to T^\sharp$ (cf. Theorem 2.1). Namely, the following result, which is essentially obtained in [C-F93] and [Me02], follows:

Theorem 2.2 (cf. [C-F93, Me02]). *Let the notation and the assumption be the same as above. Let $\widehat{\mathcal{H}}^\sharp := \phi_*(\widehat{\mathcal{H}})$ be the proper transform by $\phi := \phi'' \circ \phi' : \widehat{T} \cdots \to \widehat{T}^\sharp$ of $\widehat{\mathcal{H}}$. Then the Mori fiber space $\pi : \widehat{T}^\sharp \to W$, the boundary divisor $\widehat{D}^\sharp \in \widehat{\mathcal{H}}^\sharp$ and the morphism $\mu^\sharp : \widehat{T}^\sharp \to T^\sharp$ are described as one of the following:*

– C_2**-type:** *$\pi : \widehat{T}^\sharp \cong \mathbb{P}(\mathcal{E}) \to W$ is a $\mathbb{P}^1$-bundle structure over a smooth projective surface W, where $\mathcal{E} := \pi_*\mathcal{O}_{\widehat{T}^\sharp}(\widehat{D}^\sharp)$ is a rank 2 vector bundle on W, and $\widehat{D}^\sharp \sim \mathcal{O}(1)$. Unless $\mu^\sharp$ is an isomorphism (this is satisfied if X is not smooth), then $W \cong \mathbb{P}^2$, $\widehat{T}^\sharp \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a))$ $(a \geq 2)$, $\widehat{D}^\sharp$ is a section corresponding to the surjection $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a) \to \mathcal{O}_{\mathbb{P}^2}(a)$ and $\mu^\sharp$ is the contraction of the negative section $E^\sharp$ of π with $(E^\sharp, E^\sharp|_{E^\sharp}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-a))$;*

– D_2'**-type:** *$\pi : \widehat{T}^\sharp \to W$ is a quadric bundle over a smooth curve with at most cA_1 singularities of analytic type $o \in (xy + z^2 + t^k = 0) \subset \mathbb{C}^4 : (x, y, z, t)$ for $k \geq 2$, and $\widehat{D}^\sharp|_F \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)|$, where F is a general fiber of π. $\mu^\sharp$ is an isomorphism, hence $X^\sharp \cong \widehat{X}^\sharp = \widehat{T}^\sharp - \mathrm{Supp}\,(\widehat{D}^\sharp)$;*

– D_3**-type:** *$\pi : \widehat{T}^\sharp \cong \mathbb{P}(\mathcal{E}) \to W$ is a $\mathbb{P}^2$-bundle structure over a smooth curve W, where $\mathcal{E} := \pi_*\mathcal{O}_{\widehat{T}^\sharp}(\widehat{D}^\sharp)$ is a rank 3 vector bundle over W. $\mu^\sharp$ is an isomorphism, hence $X^\sharp \cong \widehat{X}^\sharp = \widehat{T}^\sharp - \mathrm{Supp}\,(\widehat{D}^\sharp)$;*

– D_3'**-type:** *$\pi : \widehat{T}^\sharp \to W$ is a $\mathbb{P}^2$-fibration over a smooth curve W with a general fiber $(F, \widehat{D}^\sharp|_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and with at most finitely many singular fibers $G \cong \mathbb{S}_4$, where $\mathbb{S}_4 \subset \mathbb{P}^5$ is a cone over the quartic rational curve $\subset \mathbb{P}^4$ and the vertex of G sits in a hyper-quotient singularity of analytic type $o \in (xy + z^2 + t^k = 0) \subset \mathbb{C}^4 : (x, y, z, t)/\mathbb{Z}_2(1, 1, 1, 0)$ for $k \geq 1$. $\mu^\sharp$ is an isomorphism, hence $X^\sharp \cong \widehat{X}^\sharp = \widehat{T}^\sharp - \mathrm{Supp}\,(\widehat{D}^\sharp)$;*

– $\mathbb{Q}$**-Fano:** *$\widehat{T}^\sharp$ is a $\mathbb{Q}$-Fano 3-fold with the Picard number $\varrho(\widehat{T}^\sharp) = 1$. $\mu^\sharp$ is an isomorphism, hence $X^\sharp \cong \widehat{X}^\sharp = \widehat{T}^\sharp - \mathrm{Supp}\,(\widehat{D}^\sharp)$. More precisely, the classification of the pair $(\widehat{T}^\sharp, \widehat{\mathcal{H}}^\sharp)$ up to deformation is given as one of the following:*

(i) $(\mathbb{P}(1, 1, 2, 3), \mathcal{O}(6))$;
(ii) $(T_6 \subset \mathbb{P}(1, 1, 2, 3, a), \{x_4 = 0\} \cap T_6)$ *with* $a \in \{3, 4, 5\}$;
(iii) $(T_6 \subset \mathbb{P}(1, 1, 2, 2, 3), \{x_3 = 0\} \cap T_6)$;
(iv) $(T_6 \subset \mathbb{P}(1, 1, 1, 2, 3), \{x_0 = 0\} \cap T_6)$;
(v) $(\mathbb{P}(1, 1, 1, 2), \mathcal{O}(b))$ *with* $b \in \{2, 4\}$;

(vi) $(T_4 \subset \mathbb{P}(1,1,1,1,2), \{x_0 = 0\} \cap T_4)$*;*
(vii) $(T_4 \subset \mathbb{P}(1,1,1,2,a), \{x_4 = 0\} \cap T_4)$ *with* $a \in \{2,3\}$*;*
(viii) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(b))$ *with* $b \in \{1,2,3\}$*,* $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}}(b))$ *with* $b \in \{1,2\}$*;*
(ix) $(T_3 \subset \mathbb{P}(1,1,1,1,2), \{x_4 = 0\} \cap T_3)$;
(x) $(T_3 \subset \mathbb{P}^4, \mathcal{O}(1))$*;*
(xi) $(T_{2,2} \subset \mathbb{P}^5, \mathcal{O}(1))$*;*
(xii) $(V_5, \mathcal{O}(1))$*, where* $V_5 \hookrightarrow \mathbb{P}^6$ *is a linear section of the Grassmann variety* $\mathrm{Gr}(2,5) \hookrightarrow \mathbb{P}^9$ *parametrizing lines in* $\mathbb{P}^4$;

Remark 1. Let us consider, for instance, cubic hypersurface $\subset \mathbb{P}^4$, (x) in Theorem 2.2. Of course, a general cubic hypersurface is smooth. On the other hand, we consider the following cubic hypersurface:

$$T_3 := (x_0x_1x_2 + x_3^3 + x_4^3 = 0) \subset \mathbb{P}^4.$$

Then T_3 is a $\mathbb{Q}$-Fano 3-fold with $\varrho = 1$ having three singularities of analytic type $o \in (xy + z^3 + t^3 = 0)$. By the similar argument, it follows that the possibility of singularities on $\mathbb{Q}$-Fano 3-folds with $\varrho(\widehat{T}^\sharp) = 1$, (ii), (iii), (iv), (vi), (vii), (ix), (x), (xi) and (xii) in Theorem 2.2 is not unique in general. But, when $\widehat{T}^\sharp$ in question is *quasi-smooth*, we can determine the singularities on $\widehat{T}^\sharp$ concretely. Now, we recall the definition of quasi-smoothness. Let $p : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}(a_0, \cdots, a_n)$ be the quotient map induced by the $\mathbb{C}^*$-action on $\mathbb{A}^{n+1} - \{0\}$ defined by:

$$\lambda \cdot (x_0, \cdots, x_n) = (\lambda^{a_0}x_0, \cdots, \lambda^{a_n}x_n), \quad \lambda \in \mathbb{C}^*.$$

A weighted complete intersection $T := T_{\alpha_1,\cdots,\alpha_r} \subset \mathbb{P}(a_0, \cdots, a_n) =: \mathbb{P}$ is said to be *quasi-smooth* if the inverse image $C_T^* := p^{-1}(T) \subset \mathbb{A}^{n+1} - \{0\}$ is smooth (cf. [Fl00, 6.3 Definition]). There is a criterion in order to see whether or not a general weighted complete intersection $T_{\alpha_1,\cdots,\alpha_r} \subset \mathbb{P}(a_0, \cdots, a_n)$ is quasi-smooth in terms of weights $a_0, \cdots, a_n$ and degrees $\alpha_1, \cdots, \alpha_r$ (cf. [Fl00, 8.1 Theorem]). Here we must clarify the meaning of a "general" weighted complete intersection $T_{\alpha_1,\cdots,\alpha_r} \subset \mathbb{P}(a_0, \cdots, a_n)$. The weighted hypersurfaces T_{α_i} in $\mathbb{P}(a_0, \cdots, a_n)$ are parametrized by the projective space:

$$\mathbb{P}\Big(H^0(\mathbb{P}(a_0, \cdots, a_n), \mathcal{O}_{\mathbb{P}}(\alpha_i))\Big) = \mathbb{P}(\mathbb{V}_i),$$

where $\mathbb{V}_i$ is the basis of $H^0(\mathbb{P}(a_0, \cdots, a_n), \mathcal{O}_{\mathbb{P}}(\alpha_i))$ as a $\mathbb{C}$-vector space:

$$\mathbb{V}_i := \{x_0^{d_0} \cdots x_n^{d_n} \ ; \ \sum_{j=0}^{n} a_jd_j = \alpha_i \ \}.$$

The correspondence is given by:

$$\sum_{x_0^{d_0} \cdots x_n^{d_n} \in \mathbb{V}_i} f_{d_0,\cdots,d_n} x_0^{d_0} \cdots x_n^{d_n} \mapsto (\cdots : f_{d_0,\cdots,d_n} : \cdots) \in \mathbb{P}\big(\mathbb{V}_i\big).$$

On the other hand, the weighted complete intersections $T_{\alpha_1,\cdots,\alpha_r} = T_{\alpha_1} \cap \cdots \cap T_{\alpha_r}$ are parametrized by:

$$\prod_{i=1}^{r} \mathbb{P}\Big(H^0(\mathbb{P}(a_0,\cdots,a_n), \mathcal{O}_{\mathbb{P}}(\alpha_i))\Big) = \prod_{i=1}^{r} \mathbb{P}(\mathbb{V}_i).$$

Let U_i denote the complement of all coordinate hyperplanes in $\mathbb{P}(\mathbb{V}_i)$ for $1 \leq i \leq r$ and we put $U := \prod_{i=1}^{r} U_i$. When a point on $\prod_{i=1}^{r} \mathbb{P}(\mathbb{V}_i)$ is contained in U, the corresponding weighted complete intersection $T_{\alpha_1,\cdots,\alpha_r}$ in $\mathbb{P}$ is said to be general.

By making use of the method in [Fl00, Section 10], we can determine singularities on quasi-smooth weighted complete intersections. In fact, we can see that general 3-folds (in the above mentioned sense) in the family (ii), (iii), (iv), (vi), (vii), (ix), (x), (xi) and (xii) in Theorem 2.2 are quasi-smooth, and, consequently, ascertain the following concerning singularities on general (weighted) complete intersections (ii), (iii), (iv), (vi), (vii), (ix), (x), (xi) and (xii):

Type	singularities on quasi-smooth $\widehat{T}^\sharp$
(ii)	$2 \times \frac{1}{3}(1,1,-1)$ if $a = 3$; $\frac{1}{2}(1,1,-1) + \frac{1}{4}(1,1,-1)$ if $a = 4$; $\frac{1}{5}(1,2,-2)$ if $a = 5$
(iii)	$3 \times \frac{1}{2}(1,1,-1)$
(iv)	smooth
(vi)	smooth
(vii)	$2 \times \frac{1}{2}(1,1,-1)$ if $a = 2$; $\frac{1}{3}(1,1,-1)$ if $a = 3$
(ix)	$\frac{1}{2}(1,1,-1)$
(x)	smooth
(xi)	smooth
(xii)	smooth

Table 1.

By making use of Theorem 2.1 and the argument to prove Theorem 2.2, we obtain the following result concerning (A) and (B):

Corollary 2.1. *Let X be a normal affine 3-fold with only isolated, $\mathbb{Q}$-factorial singularities such that it contains an $\mathbb{A}^1$-cylinderlike open subset. Assume the condition (†) is satisfied. Let $X^\sharp$, $\widehat{X}^\sharp$ and Mori fiber space $\pi : \widehat{T}^\sharp \to W$ be the same as in Theorems* 2.1 *and* 2.2*:*

(1) *X is constructed from $X^\sharp$ via the composite of half-point attachments (cf. Definition* 3.2*) unless $X \cong X^\sharp$. If π is of type D2, D3, D3′ or $\mathbb{Q}$-Fano 3-fold, then $X^\sharp \cong \widehat{X}^\sharp$. Moreover, if π is of type D3, then $X \cong X^\sharp \cong \widehat{X}^\sharp$.*

(2) *If π is either of type D2, D3 or D3′ then X is smooth.*

(3) *If* $\widehat{T}^\sharp$ *is a* $\mathbb{Q}$*-Fano 3-fold with* $\varrho(\widehat{T}^\sharp) = 1$ *then* X *has at most terminal singularities which are lying on* $\widehat{T}^\sharp$*, (i)∼ (xii) in Theorem* 2.2*. Moreover, in the case where* $\widehat{T}^\sharp$ *is general (cf. Remark* 1*),* Sing (X) *is described as follows:*

$$\mathrm{Sing}(X) = \alpha\times\frac{1}{2}(1,1,-1)+\beta\times\frac{1}{3}(1,1,-1)+\gamma\times\frac{1}{4}(1,1,-1)+\delta\times\frac{1}{5}(1,2,-2),$$

where $\alpha,\beta,\gamma,\delta \in \mathbb{Z}_{\geq 0}$ *satisfy one of the following:*
- $\alpha \leq 3$*,* $\beta = \gamma = \delta = 0$: (iii), (v), (vii) $a = 2$*,* (ix)*;*
- $\alpha = 0$*,* $\beta \leq 3$*,* $\gamma = \delta = 0$: (ii) $a = 3$*,* (vii) $a = 3$*;*
- $\alpha,\beta \leq 1$*,* $\gamma = \delta = 0$: (i)*;*
- $\alpha,\gamma \leq 1$*,* $\beta = \delta = 0$: (ii) $a = 4$*;*
- $\alpha = \beta = \gamma = 0$*,* $\delta \leq 1$: (ii) $a = 5$*.*
- $\alpha = \beta = \gamma = \delta = 0$*, i.e.,* X *is smooth*: (iv), (vi), (viii), (x), (xi), (xii).

(4) *If* π *is of* $C2$*-type and* X *is not smooth, then* $X^\sharp = \mu^\sharp(\widehat{X}^\sharp)$*, where* $\widehat{X}^\sharp$ *is the complement in* $\widehat{T}^\sharp = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a))$ *of the section* $\widehat{D}^\sharp$ *corresponding to* $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a) \to \mathcal{O}_{\mathbb{P}^2}(a)$ *and* $\mu^\sharp$ *is the contraction of the negative section* $E^\sharp$ *with* $(E^\sharp, E^\sharp|_{E^\sharp}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-a))$ *with* $a \geq 2$*. In particular, we have* Sing $(X) = p := \mu^\sharp(E^\sharp)$ *and* $(p \in X) {\approx} o \in \frac{1}{a}(1,1,1)$.

Remark 2. Although Corollary 2.1 indicates the *possibility* of singularities on X in question, we do not know whether or not an affine 3-fold X constructed from $X^\sharp$ in Theorem 2.2 via the composite of suitable half-point attachments contains an $\mathbb{A}^1$-cylinderlike open subset actually except for some special cases.

Problem **2.3.** Let X be a normal affine 3-fold constructed from $X^\sharp = T^\sharp - \mathrm{Supp}\,(D^\sharp)$ stated in Theorem 2.2 and Corollary 2.1 via the composite of half-point attachments. Does X contain an $\mathbb{A}^1$-cylinderlike open subset ?

Nevertheless the possibility of singularities stated in Corollary 2.1, we shall say the following conjecture:

Conjecture: Let X be a normal affine 3-fold with only isolated, $\mathbb{Q}$-factorial singularities so that X contains an $\mathbb{A}^1$-cylinderlike open subset. Then does X have at most cyclic quotient singularities ?

Note that any 3-fold cyclic quotient singularity can be realized as a singular point on a normal affine 3-fold X with an $\mathbb{A}^1$-cylinderlike open subset. Indeed, let us consider a cyclic quotient singularity $o \in \mathbb{C}^3 : (x,y,z)/\mathbb{Z}_r(1,a,b) =: X$ with $1 \leq a \leq b < r$ and $\gcd(a,r) = \gcd(b,r) = 1$, where the action of the finite cyclic group $\mathbb{Z}_r =< \xi >$ is given by $\xi \cdot (x,y,z) = (\xi x, \xi^a y, \xi^b z)$ (ξ is a primitive r-th root of unity). Then there exists an $\mathbb{A}^1$-fibration:

$$p : X \longrightarrow Y := {\mathbb{C}^2}_{(x,y)}/\mathbb{Z}_r(1,a) \quad (\overline{x},\overline{y},\overline{z}) \mapsto (\overline{x},\overline{y}),$$

over a normal affine surface Y with one cyclic quotinet singularity $o \in Y$. X contains an $\mathbb{A}^1$-cylinderlike open subset $p^{-1}(Y - \{o\})$.

Note that the recent work due to Flenner and Zaidenberg [F-Z03, Theorem 0.3] asserts that if a scheme Z with an isolated Cohen-Macaulay singularity $p \in Z$ such that a Zariski open subset of Z can be covered by rational curves which do not pass through p, then $p \in Z$ is a rational singularity. In my opinion, the geometric condition that "X contains an $\mathbb{A}^1$-cylinderlike open subset" is very restrictive and, so, we expect that singularities on such X form smaller class of singularities (e.g. cyclic quotient singularities) than that of rational singularities as in 2-dimensional case (cf. Theorem 1.1).

In the remaining Sections 3, 4 and 5, we shall give the proof of Theorems 2.1, 2.2 and Corollary 2.1.

3. Reduction via $\sharp$-Minimal Model Program

First of all, we shall start with the definition:

Definition 3.1. An $\mathbb{A}^1$-*cylinderlike open subset* U of a quasi-projective variety Y is an open subset of the form $U \cong U_0 \times \mathbb{A}^1$, where U_0 is a smooth quasi-projective variety of $\dim U_0 = \dim Y - 1$.

Let X be a normal affine 3-fold with an $\mathbb{A}^1$-cylinderlike open subset such that satisfies **ASSUMPTION** ($\dagger$), and let the notations $(T, \mathcal{H})$, $(\widehat{T}, \widehat{\mathcal{H}})$, $D, S \in \mathcal{H}$ and $\widehat{D}, \widehat{S} \in \widehat{\mathcal{H}}$ be the same there. In this section we shall construct the birational maps ϕ', ϕ'', ψ' and the birational morphism μ' in the diagram of Theorem 2.1. Note that since X contains an $\mathbb{A}^1$-cylinderlike open subset, $\widehat{T}$ is uniruled. We reduce the pair $(\widehat{T}, \widehat{\mathcal{H}})$ to a suitable pair $(\widehat{T}^\sharp, \widehat{\mathcal{H}}^\sharp)$ via a suitably chosen Minimal Model Program $\phi : \widehat{T} \cdots \to \widehat{T}^\sharp$ associated to $\widehat{\mathcal{H}}$ (so called $\sharp$-*Minimal Model Program* in the terminology of [Me02]), where $\widehat{T}^\sharp$ has a structure of Mori fiber space, say $\pi : \widehat{T}^\sharp \to W$. As performed in [Me02], after several divisorial contractions and flips we can finally reach a Mori fiber space $\pi : \widehat{T}^\sharp \to W$. But we, furthermore, have to investigate how the inside varieties $\widehat{X}$ and X change via this process of $\sharp$-Minimal Model Program for our purpose to determine the possibility of singularities on X. From this point of view, the main obstacles lie in the following:

OBSTACLE.

(1) When a divisorial contraction occurs in the process of $\sharp$-Minimal Model Program associated to $(\widehat{T}, \widehat{\mathcal{H}})$, some exceptional divisor may not be contained in the proper transform of the boundary $\widehat{D}$. Hence $\widehat{X}$ (at the same time X) may be shrinked. In particular, some of singular points on X may be shrinked via the process.

(2) When a flip occurs in the process of $\sharp$-Minimal Model Program, some flipping or flipped curves may not be contained in the proper transform of the boundary $\widehat{D}$. This may result in new singularities inside varieties.

This is why it is usually not suitable to apply 3-dimensional Minimal Model Program for the study on affine 3-folds. But, under our assumption (†), we can control the process of the $\sharp$-Minimal Model Program in a right direction, namely, so that it does not bring too bad influence on $\widehat{X}$ and X although $\widehat{X}$ and X may be shrinked.

From now on, we shall recall and rewrite the notion of $\sharp$-Minimal Model Program (cf. [Me02, Theorem 3.2, Proposition 3.6]) in our context. Put $\rho := \rho(\widehat{T}, \widehat{\mathcal{H}}) := \sup\{\lambda \in \mathbb{Q}\ ;\ \widehat{\mathcal{H}} + \lambda K_{\widehat{T}} \text{ is effective}\}$ which we call *threshold of the pair* $(\widehat{T}, \widehat{\mathcal{H}})$. Moreover, put $t^0 := \sup\{\lambda \in \mathbb{Q}\ ;\ \widehat{\mathcal{H}} + \lambda K_{\widehat{T}} \text{ is nef}\}$ which we call *nef value of the pair* $(\widehat{T}, \widehat{\mathcal{H}})$. First of all, we prove the following:

Lemma 3.1. $0 \leq t^0 \leq \rho < 1$.

Proof. Since the cone of nef divisors is the closure of the ample cone by virtue of Kleiman's criterion of ampleness (cf. [Kle66]), the nef cone is contained in the closure of the cone of effective divisors. Hence $t^0 \leq \rho$. Note that $\widehat{\mathcal{H}}$ is nef. So, we have $t^0 \geq 0$. Note that an $\mathbb{A}^1$-cylinderlike open subset $U \cong U_0 \times \mathbb{A}^1$ is disjoint from the singular points Sing (X). Hence $\widehat{X}$ contains canonically an $\mathbb{A}^1$-cylinderlike open subset $\mu^{-1}(U) \cong U$. In particular, $\overline{\kappa}(\widehat{X}) = -\infty$ (see [Ii77] for the definition and the relevant results on log Kodaira dimension $\overline{\kappa}$). Since $\widehat{X}$ is embedded into a smooth projective 3-fold $\widehat{T}$ so that $\widehat{T} - \widehat{X} = \widehat{D}$ is an S.N.C. divisor, we have $\overline{\kappa}(\widehat{X}) = \kappa(\widehat{T}; K_{\widehat{T}} + \widehat{D}) = -\infty$. Therefore, $\widehat{\mathcal{H}} + K_{\widehat{T}} \sim \widehat{D} + K_{\widehat{T}}$ can not be effective, i.e., $\rho < 1$. □

The next proposition and its proof yield how to proceed in order to reach a suitable Mori fiber space starting with the pair $(\widehat{T}, \widehat{\mathcal{H}})$.

Proposition 3.1 (cf. [Me02]). *There exists a pair $(\widehat{T}^\sharp, \widehat{\mathcal{H}}^\sharp)$ satisfying the following three conditions:*

(i) *there is a birational map $\phi : \widehat{T} \cdots \to \widehat{T}^\sharp$ with $\widehat{\mathcal{H}}^\sharp = \phi_*(\widehat{\mathcal{H}})$;*

(ii) *$\widehat{T}^\sharp$ has a structure of Mori fiber space, say $\pi : \widehat{T}^\sharp \to W$;*

(iii) *$\widehat{\mathcal{H}}^\sharp + \rho K_{\widehat{T}^\sharp} \equiv_\pi 0$, i.e., $\widehat{\mathcal{H}}^\sharp + \rho K_{\widehat{T}^\sharp}$ is numerically π-trivial.*

Proof. Although our proof is parallel to that performed in [Me02], we shall prove in our context for the purpose to establish Theorem 2.1 in the subsequent. We shall proceed in an inductive way. There exists an extremal ray $R^0 = \mathbb{R}_+[C^0]$ in the hyperplane $\overline{\mathrm{NE}}\,(\widehat{T}) \cap \{\widehat{\mathcal{H}} + t^0 K_{\widehat{T}}\}^\perp$ (cf. [K-M-M87]). Let $\phi^0 : \widehat{T} \cdots \to \widehat{T}^1$ be a rational map associated with R^0 which is either a Mori fiber space, a divisorial contraction or a flip. (In fact, since $\widehat{T}$ is a smooth 3-fold, the case ϕ^0 is a flip does not occur. cf. [CKM88].) Put $\widehat{\mathcal{H}}^1 := \phi^0_*(\widehat{\mathcal{H}})$ and let $\widehat{S}^1$, $\widehat{D}^1 \in \widehat{\mathcal{H}}^1$ be the proper transforms of $\widehat{D}$, $\widehat{S}$ on $\widehat{T}^1$, respectively. Now suppose that we have already constructed the i-th step, namely,

– a birational map $\phi^{i-1} : \widehat{T}^{i-1} \cdots \to \widehat{T}^i$ which is associated with an extremal ray $R^{i-1} = \mathbb{R}_+[C^{i-1}]$ in $\overline{\mathrm{NE}}\,(\widehat{T}^{i-1}) \cap \{\widehat{\mathcal{H}}^{i-1} + t^{i-1} K_{\widehat{T}^{i-1}}\}^\perp$;

– a nef linear system $\widehat{\mathcal{H}}^i := \phi^{i-1}_*(\widehat{\mathcal{H}}^{i-1})$ on $\widehat{T}^i$;

– the proper transform $\widehat{D}^i := \phi^{i-1}_*(\widehat{D}^{i-1})$ and a smooth member $\widehat{S}^i := \phi^{i-1}_*(\widehat{S}^{i-1})$ of $\widehat{\mathcal{H}}^i$.

Put $t^i := \sup\{\lambda \in \mathbb{Q}\ ;\ \widehat{\mathcal{H}}^i + \lambda K_{\widehat{T}^i}$ is nef$\}$. Since $\widehat{\mathcal{H}}^i$ is nef and $\widehat{T}^i$ is a uniruled varieties, it is obvious that $t^i \geq 0$. On the other hand, it follows that $0 \leq t^{i-1} \leq t^i \leq \rho < 1$ (cf. Lemma 3.1 and [Me02, Claim 3.4]). There exist an extremal ray $R^i = \mathbb{R}_+[C^i]$ in $\overline{\mathrm{NE}}(\widehat{T}^i) \cap \{\widehat{\mathcal{H}}^i + t^i K_{\widehat{T}^i}\}^{\perp}$. Let $\phi^i : \widehat{T}^i \cdots \to \widehat{T}^{i+1}$ be a rational map associated to this extremal ray R^i. If ϕ^i yields a Mori fiber space, then we put $\pi := \phi^i$, $\widehat{T}^\sharp := \widehat{T}^i$ and $\widehat{\mathcal{H}}^\sharp := \widehat{\mathcal{H}}^i$ to obtain the assertion. If ϕ^i is a birational map (either a divisorial contraction or a flip), we put $\widehat{\mathcal{H}}^{i+1} := \phi^i_*(\widehat{\mathcal{H}}^i)$. We denote by $\widehat{D}^{i+1}$ (resp. $\widehat{S}^{i+1}$) the proper transform on $\widehat{T}^{i+1}$ of $\widehat{D}^i$ (resp. $\widehat{S}^i$). Then we claim the following:

Claim . $\widehat{\mathcal{H}}^{i+1}$ *is a nef Cartier linear system.* $\widehat{S}^{i+1}$ *is a smooth member of* $\widehat{\mathcal{H}}^{i+1}$.

Proof of Claim. It is not difficult to see that $\widehat{\mathcal{H}}^{i+1}$ is nef. We shall prove that $\widehat{S}^{i+1}$ is smooth. Note that since $\widehat{S}^i$ is a smooth Cartier divisor, it is disjoint from the singular locus of $\widehat{T}^i$. We shall consider according as the type of ϕ^i, separately.

flip case: If ϕ^i is a flip, then $(-K_{\widehat{T}^i} \cdot C^i) < 1$ (cf. [CKM88]). Since $0 \leq t^i < 1$, we have $(\widehat{S}^i \cdot C^i) = (\widehat{\mathcal{H}}^i \cdot C^i) = t^i(-K_{\widehat{T}^i} \cdot C^i) < 1$. By noting that $\widehat{S}^i$ is nef and Cartier, we have $(\widehat{S}^i \cdot C^i) = 0$. This means that $\widehat{S}^i$ is disjoint from the flipping curves. (Note that the case where $\widehat{S}^i$ contains flipping curves is impossible because $\widehat{S}^i$ is disjoint from the singular locus of $\widehat{T}^i$ as remarked above.) Therefore, $\widehat{S}^{i+1}$ is isomorphic to $\widehat{S}^i$ and, in particular, is smooth. (Moreover, we have $t^i = 0$ and $\widehat{\mathcal{H}}^i \cdot C^i = 0$.)

In the case where ϕ^i is a divisorial contraction, we denote by $\widehat{E}^i \subset \widehat{T}^i$ the exceptional divisor of ϕ^i.

$(2:1)$-type: When $\widehat{E}^i$ is contracted onto a curve, it follows that $(-K_{\widehat{T}^i} \cdot C^i) = 1$, where C^i represent a general fiber of $\phi^i|_{\widehat{E}^i} : \widehat{E}^i \to \phi^i(\widehat{E}^i)$. Then by the similar argument as above we have $(\widehat{S}^i \cdot C^i) = 0$. This means that $\widehat{S}^i$ is disjoint from $\widehat{E}^i$ or $\widehat{S}^i \cap \widehat{E}^i$ consists of several mutually disjoint fibers of $\phi^i|_{\widehat{E}^i}$ which are (-1)-curves on $\widehat{S}^i$. Hence $\widehat{S}^{i+1}$ is also smooth. (Moreover, we have $t^i = 0$ and $\widehat{\mathcal{H}}^i \cdot C^i = 0$.)

$(2:0)$-type: We consider the case where $\widehat{E}^i$ is contracted to a point. Of course, we may assume that $\widehat{S}^i$ meets $\widehat{E}^i$. Then we can actually see that $\widehat{E}^i \cong \mathbb{P}^2$ with a normal bundle $\widehat{E}^i|_{\widehat{E}^i} \cong \mathcal{O}_{\mathbb{P}^2}(-1)$ (cf. [Me02, Claim 3.7]). Then $\widehat{S}^i \cdot C^i$ is a positive integer. On the other hand, as $\widehat{S}^i \cdot C^i = t^i(-K_{\widehat{T}^i} \cdot C_i) = 2t^i$ with $0 \leq t^i < 1$ we can see $t^i = 1/2$ and $\widehat{S}^i \cdot C^i = 1$, where C^i is a line on $\widehat{E}^i \cong \mathbb{P}^2$. Therefore, $\widehat{S}^i \cap \widehat{E}^i$ is a (-1)-curve on $\widehat{S}^i$ and $\widehat{S}^{i+1}$ is smooth. □

Thus we can construct the $(i+1)$-th step. By an inductive argument (and the fact that there is no infinite sequence of flips, see e.g. [FA]), after the composite of divisorial contractions and flips, we finally obtain a Mori fiber space $\pi : \widehat{T}^r \to W$ and a nef Cartier linear system $\widehat{\mathcal{H}}^r$ on $\widehat{T}^r$, where π is obtained from the contraction of an extremal ray $R^r = \mathbb{R}_+[C^r]$ in $\overline{\mathrm{NE}}\,(\widehat{T}^r) \cap \{\widehat{\mathcal{H}}^r + t^r K_{\widehat{T}^r}\}^{\perp}$ with $0 \leq t^r \leq \rho < 1$. In fact, it follows that $t^r = \rho$ (cf. [Me02, Claim 3.4]). Put $(\widehat{T}^\sharp, \widehat{\mathcal{H}}^\sharp) := (\widehat{T}^r, \widehat{\mathcal{H}}^r)$ to complete the proof. $\square$

Remark 3. Since $(\widehat{T}, \widehat{\mathcal{H}})$ is not necessarily *a pair with a big uniruled system* (see [Me02, Definition 5.1]), we can not deduce that $\rho > 0$ at the present time. But, as seen in Lemma 5.1, we can show $\rho > 0$ easily.

Although we have obtained the Mori fiber space $\pi : \widehat{T}^\sharp \to W$, we, furthermore, need to observe how the inside varieties X and $\widehat{X}$ change in this process of $\sharp$-Minimal Model Program performed in the proof of Proposition 3.1. The next proposition asserts that $\sharp$-Minimal Model Program does not lead to so bad influence on X and $\widehat{X}$.

Proposition 3.2. *Let $\phi : \widehat{T} \overset{\phi^0}{\cdots\to} \widehat{T}^1 \cdots\cdots \to \widehat{T}^{r-1} \overset{\phi^{r-1}}{\cdots\to} \widehat{T}^r =: \widehat{T}^\sharp$ be the process of $\sharp$-Minimal Model Program performed in the proof of Proposition* 3.1*. Among the nef values $t^i \geq 0$ of the pair $(\widehat{T}^i, \widehat{\mathcal{H}}^i)$ for $0 \leq i < r$ (see the proof of Proposition* 3.1*), let $t^i = 0$ for $0 \leq i < s$ and $t^i > 0$ for $i \geq s$ (if there exist at all). Then:*

(1) *there exists a following commutative diagram:*

$$\begin{array}{ccccccccccccc}
\widehat{T} & \overset{\phi^0}{\cdots\to} & \widehat{T}^1 & \overset{\phi^1}{\cdots\to}\cdots\cdots\to & \widehat{T}^{s-1} & \overset{\phi^{s-1}}{\cdots\to} & \widehat{T}^s & \overset{\phi^s}{\to}\cdots\to & \widehat{T}^{r-1} & \overset{\phi^{r-1}}{\to} & \widehat{T}^\sharp \\
{\scriptstyle\mu}\downarrow & & {\scriptstyle\mu^1}\downarrow & & {\scriptstyle\mu^{s-1}}\downarrow & & {\scriptstyle\mu^s}\downarrow & & & & \\
T & \overset{\psi^0}{\cdots\to} & T^1 & \overset{\psi^1}{\cdots\to}\cdots\cdots\to & T^{s-1} & \overset{\psi^{s-1}}{\cdots\to} & T^s & & & &
\end{array}$$

so that each square

$$\begin{array}{ccc}
\widehat{T}^i & \overset{\phi^i}{\cdots\cdots\to} & \widehat{T}^{i+1} \\
{\scriptstyle\mu^i}\downarrow & & \downarrow{\scriptstyle\mu^{i+1}} \\
T^i & \overset{\psi^i}{\cdots\cdots\to} & T^{i+1}
\end{array}$$

satisfies the following according as the type of the birational map $\phi^i : \widehat{T}^i \cdots\to \widehat{T}^{i+1}$ (flip or divisorial contraction) for $0 \leq i < s$;

–In the case where ϕ^i is a flip, let $C_1^i, \cdots, C_t^i \subset \widehat{T}^i$ exhaust all the flipping curves and let $C_1^{i+}, \cdots, C_t^{i+} \subset \widehat{T}^{i+1}$ denote the corresponding flipped curves. Then exactly one of the following two cases occurs concerning the location of these curves:

(a) $C_1^i \cup \cdots \cup C_t^i \subset \mathrm{Supp}\,(\widehat{D}^i)$ *and* $C_1^{i+} \cup \cdots \cup C_t^{i+} \subset \mathrm{Supp}\,(\widehat{D}^{i+1})$*; or*

(b) $C_1^i \cup \cdots \cup C_t^i \subset \mathrm{Exc}(\mu^i)$ *and* $C_1^{i+} \cup \cdots \cup C_t^{i+} \subset \mathrm{Exc}(\mu^{i+1})$.

–In the case where ϕ^i is divisorial, an exceptional divisor $\widehat{E}^i$ of ϕ^i is contained in $\mathrm{Supp}(\widehat{D}^i)$ *or* $\mathrm{Exc}(\mu^i)$. *If $\widehat{E}^i \subset \mathrm{Supp}(\widehat{D}^i)$, then ψ^i is the contraction of $E^i := \mu^i(\widehat{E}^i) \subset \mathrm{Supp}\,(D^i)$ and μ^{i+1} is the naturally induced morphism. Meanwhile, if $\widehat{E}^i \subset \mathrm{Exc}(\mu^i)$ then $T^{i+1} = T^i$ and μ^{i+1} is the induced morphism.*

(2) *For the remainings $\phi^i : \widehat{T}^i \to \widehat{T}^{i+1}$ $(s \le i < r)$ if any, we have: $t^i = 1/2$ and ϕ^i is a divisorial contraction of $E2$-type, i.e., an exceptional divisor $\widehat{E}^i := \mathrm{Exc}(\phi^i)$ satisfies $(\widehat{E}^i, \widehat{E}^i|_{\widehat{E}^i}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$. Furthermore, $\widehat{D}^i \cdot C^i = 1$, where C^i is the class of a line on $\widehat{E}^i \cong \mathbb{P}^2$.*

(3) *Put $D^i := \mu^i_*(\widehat{D}^i)$ and $X^i := T^i - \mathrm{Supp}\,(D^i)$ for $0 \le i \le s$. Put $\widehat{X}^i := \widehat{T}^i - \mathrm{Supp}\,(\widehat{D}^i)$ for $0 \le i \le r$. Then $X \cong X^i$ for $0 \le \forall i \le s$. When $s \le i < r$ and $\widehat{E}^i$ is not contained in* $\mathrm{Supp}\,(\widehat{D}^i)$, *$\widehat{X}^i$ is obtained from $\widehat{X}^{i+1}$ via a half-point attachment (cf. Definition* 3.2*) and $\widehat{X}^i - \widehat{X}^{i+1} \cong \mathbb{A}^2$.*

Proof of Proposition 3.2. We have already obtained a birational map $\phi : \widehat{T} \cdots \to \widehat{T}^\sharp$ in Proposition 3.1, so we have only to verify the following two:

(i) the construction of birational maps $\psi^{i-1} : T^{i-1} \cdots \to T^i$ and birational morphisms $\mu^i : \widehat{T}^i \to T^i$ for $1 \le i \le s$ which fit into a commutative diagram in the assertion;

(ii) the detailed description of $\widehat{T}^s \cdots \to \widehat{T}^\sharp$.

Our proof consists of several steps and proceed in an inductive way.

Step 1. At first we shall prove (1) and the first part of (3). So, let $t^i = 0$. Note that then $t^0 = \cdots t^1 = \cdots = t^i = 0$. Assume that we have already obtained the i-th step:

$$\begin{array}{ccc} \widehat{T}^{i-1} & \overset{\phi^{i-1}}{\cdots\cdots\to} & \widehat{T}^i \\ {\scriptstyle \mu^{i-1}}\downarrow & & \downarrow{\scriptstyle \mu^i} \\ T^{i-1} & \overset{\psi^{i-1}}{\cdots\cdots\to} & T^i \end{array}$$

satisfying the conditions in the statement and we shall construct the $(i+1)$-th step. Since $t^i = 0$ we have $\widehat{D}^i \cdot C^i = \widehat{\mathcal{H}}^i \cdot C^i = 0$, where C^i is a generating curve of the extremal ray R^i. Then any irreducible curve $C \in R^i$ is contained in either $\mathrm{Supp}\,(\widehat{D}^i)$ or $\mathrm{Exc}(\mu^i)$. Indeed, assume to the contrary that $C \not\subset \mathrm{Supp}\,(\widehat{D}^i)$, $\not\subset \mathrm{Exc}(\mu^i)$. Then $\mu^i(C)$ is a complete curve contained in $T^i - \mathrm{Supp}\,(D^i) = X^i$. Note that $X^i \cong X$ by an inductive hypothesis. Since X is affine, this is absurd.

Step 2. First of all, we consider:

$\boxed{\textbf{Case: } \phi^i \text{ is a flip}}$

Let $C_1^i, \cdots, C_t^i \subset \widehat{T}^i$ exhaust all the flipping curves and let $C_1^{i+}, \cdots, C_t^{i+} \subset \widehat{T}^{i+1}$ be the corresponding flipped curves. Then:

Lemma 3.2.
(1) *$C_1^i \cup \cdots \cup C_t^i$ is contained in either* $\mathrm{Supp}\,(\widehat{D}^i)$ *or* $\mathrm{Exc}(\mu^i)$*;*

(2) *If $C_1^i \cup \cdots \cup C_t^i \subset \mathrm{Supp}\,(\widehat{D}^i)$ then $C_1^{i+} \cup \cdots \cup C_t^{i+} \subset \mathrm{Supp}\,(\widehat{D}^{i+1})$. In this case, there exists a commutative diagram:*

$$\begin{array}{ccc} \widehat{T}^i & \overset{\phi^i}{\cdots\cdots\to} & \widehat{T}^{i+1} \\ {\scriptstyle \mu^i}\big\downarrow & & \big\downarrow{\scriptstyle \mu^{i+1}} \\ T^i & \overset{\psi^i}{\cdots\cdots\to} & T^{i+1}, \end{array}$$

where $\psi^i : T^i \cdots \to T^{i+1}$ *is a flip of the curves* $\mu^i(C_1^i), \cdots, \mu^i(C_t^i) \subset \mathrm{Supp}\,(D^i)$ *with flipped curves* $\mu^{i+1}(C_1^{i+}), \cdots, \mu^{i+1}(C_t^{i+}) \subset \mathrm{Supp}(D^{i+1})$ *and* $\mu^{i+1} : \widehat{T}^{i+1} \to T^{i+1}$ *is the naturally induced morphism;*

(3) *If* $C_1^i \cup \cdots \cup C_t^i \subset \mathrm{Exc}(\mu^i)$ *then there exists a following commutative diagram:*

$$\begin{array}{ccc} \widehat{T}^i & \overset{\phi^i}{\cdots\cdots\to} & \widehat{T}^{i+1} \\ {\scriptstyle \mu^i}\big\downarrow & & \big\downarrow{\scriptstyle \mu^{i+1}} \\ T^i & \overset{\psi^i}{=} & T^{i+1}, \end{array}$$

where $\mu^{i+1} : \widehat{T}^{i+1} \to T^{i+1}$ *is the naturally indeced morphism and all the flipped curves* $C_1^{i+}, \cdots, C_t^{i+}$ *are contained in* $\mathrm{Exc}(\mu^{i+1})$*;*

(4) $X^{i+1} \cong X^i$.

Proof. (1) Assume that C_a^i is contained in Supp $(\widehat{D}^i)$ and C_b^i is contained in $\mathrm{Exc}(\mu^i)$. Since C_a^i, $C_b^i \in R^i$, we can write $C_b^i \equiv \alpha C_a^i$ with $\alpha > 0$. Let A be a very ample Cartier divisor on T^i. Then we have:

$$0 = ({\mu^i}^*(A) \cdot C_b^i) = \alpha({\mu^i}^*(A) \cdot C_a^i) = \alpha(A \cdot {\mu^i}_*(C_a^i)) > 0,$$

which is a contradiction. Hence $C_1^i \cup \cdots \cup C_t^i \subset \mathrm{Supp}\,(\widehat{D}^i)$ or $C_1^i \cup \cdots \cup C_t^i \subset \mathrm{Exc}(\mu^i)$.

(2) Let $C_1^i \cup \cdots \cup C_t^i$ be contained in Supp $(\widehat{D}^i)$. Then we obtain the commutative diagram in the assertion (2), and we need to observe the location of the flipped curves $C_1^{i+}, \cdots, C_t^{i+}$. Assume to the contrary that several flipped curves, say $C_1^{i+}, \cdots, C_a^{i+}$ $(1 \le a \le t)$, are not contained in Supp $(\widehat{D}^{i+1})$ and the remainings $C_{a+1}^{i+}, \cdots, C_t^{i+}$ (if any) are contained in Supp $(\widehat{D}^{i+1})$. Then $\widehat{X}^i \cong \widehat{T}^{i+1} - (\mathrm{Supp}\,(\widehat{D}^{i+1}) \cup C_1^{i+} \cup \cdots \cup C_a^{i+})$. Hence the affine 3-fold X^i is embedded into T^{i+1} in such a way that $T^{i+1} - X^i = \mathrm{Supp}\,(D^{i+1}) \cup \mu^{i+1}(C_1^{i+}) \cup \cdots \cup \mu^{i+1}(C_a^{i+})$, which is not of pure codimension one. This is absurd.

(3) Let $C_1^i \cup \cdots \cup C_t^i$ be contained in $\mathrm{Exc}(\mu^i)$. Then $\phi^i : \widehat{T}^i \cdots \to \widehat{T}^{i+1}$ is a part of the Minimal Model Program over T^i. Therefore, there exists a birational morphism μ^{i+1} which fits into a commutative diagram stated in (3).

(4) By the arguments in (1), (2) and (3), we can easily see (4). □

Next we consider:

$\boxed{\textbf{Case: } \phi^i \text{ is divisorial}}$

Since an exceptional divisor $\widehat{E}^i$ of ϕ^i is contained in either Supp $(\widehat{D}^i)$ or $\mathrm{Exc}(\mu^i)$, it is not difficult to obtain the corresponding assertion of Proposition

3.2 in this case. For example, if $\widehat{E}^i \subset \mathrm{Exc}(\mu^i)$ then $\phi^i : \widehat{T}^i \to \widehat{T}^{i+1}$ is a part of the process of MMP$/T^i$. Therefore, there exists a morphism $\mu^{i+1} : \widehat{T}^{i+1} \to T^i =: T^{i+1}$. We shall omit the detail.

Step 3. The remainings we have to prove are the assertion (2) and the latter assertion in (3) of Proposition 3.2. So let $t^i > 0$. Then $t^i = 1/2$ and ϕ^i is a divisorial contraction of $E2$-type with an exceptional divisor $\widehat{E}^i$ with a normal bundle $(\widehat{E}^i, \widehat{E}^i|_{\widehat{E}^i}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$ (see the proof of Proposition 3.1). Hence $\widehat{D}^i \cdot C^i = 1$, where C^i is a line on $\widehat{E}^i \cong \mathbb{P}^2$. Thus, if $\widehat{E}^i$ is not contained in the boundary $\widehat{D}^i$, then $\widehat{D}^i$ intersects $\widehat{E}^i$ in such a way that $\widehat{D}^i|_{\widehat{E}^i} \in |\mathcal{O}_{\mathbb{P}^2}(1)|$. Hence $\widehat{X}^i$ is obtained from $\widehat{X}^{i+1}$ by a single half-point attachment (cf. Definition 3.2) and so $\widehat{X}^i - \widehat{X}^{i+1} \cong \mathbb{A}^2$. Thus we complete the proof of Proposition 3.2. □

In the assertion of Proposition 3.2, we use the notion of half-point attachment which is a straightforward 3-dimensional version of the original definition given in [Miy01, 4.4.1. Definition.].

Definition 3.2. Let Z be a normal quasi-projective 3-fold and let $Z \hookrightarrow V$ be a compactification into a normal projective 3-fold V with the boundary $V - T = \mathrm{Supp}\,(B)$. Let p be a point on $\mathrm{Supp}\,(B)$ where V is smooth. Let $f : V' \to V$ be the blow-up at p with an exceptional divisor $E \cong \mathbb{P}^2$ and we denote by $B' := f_*^{-1}B$ the proper transform of B. Then $Z' := V' - \mathrm{Supp}\,(B')$ is said to be a *half-point attachment* to Z. It is clear, by the definition, that Z is an open subset of Z' so that $Z' - Z \cong \mathbb{A}^2$.

4. Proof of Theorem 2.1

In this section we shall complete the proof of Theorem 2.1. More precisely speaking, we shall construct the birational morphisms $\psi'' : T' \to T^\sharp$ and $\mu^\sharp : \widehat{T}^\sharp \to T^\sharp$ in the diagram of Theorem 2.1, where we put $T' := T^s$ with the notation of Section 3, Proposition 3.2. At first, we have:

Lemma 4.1. *Let the notations be the same as in Proposition* 3.2*. The exceptional set* $\mathrm{Exc}(\mu^s)$ *of* μ^s *is of pure codimension one and is disjoint from the boundary* $\mathrm{Supp}\,(\widehat{D}^s)$.

Proof. Note that X has $\mathbb{Q}$-factorial singularities by our setting. Since the process $T \cdots \to T^s$ constructed in Proposition 3.2 has no influence on X and is a part of Minimal Model Program, T^s also has $\mathbb{Q}$-factorial singularities. Hence, since both $\widehat{T}^s$ and T^s have $\mathbb{Q}$-factorial singularities, the exceptional set of $\mu^s : \widehat{T}^s \to T^s$ is of pure codimension one (cf. the explanation in the proof of [Ma01, Theorem 4.1.3]). By the construction of $\widehat{T} \cdots \to \widehat{T}^s$, we can see $\mathrm{Exc}(\mu^s) \cap \mathrm{Supp}\,(\widehat{D}^s) = \emptyset$. □

For the construction of $\psi'' : T^s \to T^\sharp$ and $\mu^\sharp : \widehat{T}^\sharp \to T^\sharp$, we proceed in an inductive way. Namely, we prove the following:

Lemma 4.2. *Let the notations be as in Section 3. There exist birational morphisms* $\psi^i : T^i \to T^{i+1}$ $(s \le i < r)$ *and* $\mu^i : \widehat{T}^i \to T^i$ $(s \le i \le r)$ *so that they satisfy the following:*

(1) *there exists a commutative diagram:*

$$\begin{array}{ccc} \widehat{T}^i & \overset{\phi^i}{\to} & \widehat{T}^{i+1} \\ {\scriptstyle \mu^i}\downarrow & & \downarrow{\scriptstyle \mu^{i+1}} \\ T^i & \overset{\psi^i}{\to} & T^{i+1}, \end{array}$$

so that:

$(a)_i$ ϕ^i *is of E2-type with* $\widehat{E}^i := \mathrm{Exc}(\phi^i)$ *such that* $(\widehat{E}^i, \widehat{E}^i|_{\widehat{E}^i}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$ *and* $\widehat{E}^i \cap \mathrm{Exc}(\mu^i) = \emptyset$*;*

$(b)_i$ ψ^i *is of E2-type with* $E^i := \mu^i(\widehat{E}^i) = \mathrm{Exc}(\psi^i)$ *such that* $(E^i, E^i|_{E^i}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$*;*

$(c)_{i+1}$ $\mathrm{Supp}\,(\widehat{D}^{i+1}) \cap \mathrm{Exc}(\mu^{i+1}) = \emptyset$*.*

(2) *Put* $\widehat{X}^i := \widehat{T}^i - \mathrm{Supp}\,(\widehat{D}^i)$ *and* $X^i := T^i - \mathrm{Supp}\,(D^i)$*. If the exceptional divisor* $\widehat{E}^i$ *of* ϕ^i *is contained in* $\mathrm{Supp}\,(\widehat{D}^i)$*, then* $\widehat{X}^i \cong \widehat{X}^{i+1}$ *and* $X^i \cong X^{i+1}$*. Meanwhile, if* $\widehat{E}^i$ *is not contained in* $\mathrm{Supp}\,(\widehat{D}^i)$*, then* $\widehat{X}^i$ *(resp.* X^i*) is obtained from* $\widehat{X}^{i+1}$ *(resp. from* X^{i+1}*) by a single half-point attachment (cf. Definition 3.2), so* $\widehat{X}^i - \widehat{X}^{i+1} \cong X^i - X^{i+1} \cong \mathbb{A}^2$*.*

(3) *Put* $T^\sharp := T^r$*,* $D^\sharp := D^r$ *and* $X^\sharp := X^r = T^\sharp - \mathrm{Supp}\,(D^\sharp)$*. Then* X *is obtained from* $X^\sharp$ *via the composite of suitable half-point attachments. In particular,* $\mathrm{Sing}\,(X) = \mathrm{Sing}\,(X^\sharp)$*.*

Proof. (1) Assume that we have already constructed the $(i-1)$-th step:

$$\begin{array}{ccc} \widehat{T}^{i-1} & \overset{\phi^{i-1}}{\to} & \widehat{T}^{i} \\ {\scriptstyle \mu^{i-1}}\downarrow & & \downarrow{\scriptstyle \mu^{i}} \\ T^{i-1} & \overset{\psi^i}{\to} & T^{i}, \end{array}$$

satisfying $(a)_{i-1}$, $(b)_{i-1}$ and $(c)_i$, and we shall construct the i-th step satisfying $(a)_i$, $(b)_i$ and $(c)_{i+1}$ for $i \geq s$. As seen in Section 3 (cf. Proposition 3.1), the contraction of the extremal ray $R^i = \mathbb{R}_+[C^i]$ yields an $E2$-type divisorial contraction $\phi^i : \widehat{T}^i \to \widehat{T}^{i+1}$ with an exceptional divisor $\widehat{E}^i \cong \mathbb{P}^2$ such that $\widehat{E}^i|_{\widehat{E}^i} \cong \mathcal{O}_{\mathbb{P}^2}(-1)$ and $\widehat{D}^i \cdot C^i = 1$, where C^i is a line on $\widehat{E}^i \cong \mathbb{P}^2$. If $\widehat{E}^i$ is contained in $\mathrm{Supp}\,(\widehat{D}^i)$, then we can easily construct the desired i-th step by noting that $\mathrm{Supp}(\widehat{D}^i)$ is disjoint from $\mathrm{Exc}(\mu^i)$ (the property $(c)_i$). On the other hand, if $\widehat{E}^i$ is not contained in $\mathrm{Supp}(\widehat{D}^i)$, then $\widehat{E}^i \cap \mathrm{Exc}(\mu^i) = \emptyset$. In fact, assume to the contrary that $\widehat{E}^i \cap \mathrm{Exc}(\mu^i) \neq \emptyset$. Note that since $\widehat{T}^s$ and T^s have $\mathbb{Q}$-factorial singularities, the varieties $\widehat{T}^i$ and T^i have also $\mathbb{Q}$-factorial singularities. Hence the exceptional set $\mathrm{Exc}(\mu^i)$ is purely of codimension one. Thus, $\widehat{E}^i \cap \mathrm{Exc}(\mu^i) \neq \emptyset$ implies that we can take an irreducible curve $C \subseteq \widehat{E}^i \cap \mathrm{Exc}(\mu^i)$. But, since $\widehat{E}^i \cong \mathbb{P}^2$ and $\widehat{D}^i$ meets $\widehat{E}^i$ this implies that $\mathrm{Supp}\,(\widehat{D}^i) \cap \mathrm{Exc}(\mu^i) \neq \emptyset$, which is a contradiction to $(c)_i$. Therefore, $\widehat{E}^i \cap \mathrm{Exc}(\mu^i) = \emptyset$ and we can construct the birational morphisms $\psi^i : T^i \to T^{i+1}$, $\mu^{i+1} : \widehat{T}^{i+1} \to T^{i+1}$ and the desired square at the i-th step.

(2) Note that $\widehat{X}^i - \widehat{X}^{i+1} \cong X^i - X^{i+1}$ because $\widehat{E}^i$ is disjoint from $\mathrm{Exc}(\mu^i)$ as seen in the proof of (1). If $\widehat{E}^i$ is contained in Supp $(\widehat{D}^i)$, then the assertion is easy to see. If $\widehat{E}^i$ is not contained in Supp $(\widehat{D}^i)$, then $\widehat{D}^i$ meets $\widehat{E}^i$ in such a way that $\widehat{D}^i|_{\widehat{E}^i} \in |\mathcal{O}_{\mathbb{P}^2}(1)|$ because of $\widehat{D}^i \cdot C^i = 1$, where C^i is a line on $\widehat{E}^i \cong \mathbb{P}^2$. Therefore, in this case, we have $\widehat{X}^{i+1} \cong \widehat{T}^i - (\mathrm{Supp}\,(\widehat{D}^i) \cup \widehat{E}^i)$. Similarly, we have $X^{i+1} \cong T^i - (\mathrm{Supp}(D^i) \cup E^i)$. Hence $\widehat{X}^i$ (resp. X^i) is obtained from $\widehat{X}^{i+1}$ (resp. X^{i+1}) via a single half-point attachment. The assertion (3) is easily seen by the repetition of (2). $\square$

By combining Propositions 3.1 and 3.2 and Lemma 4.2 we complete the proof of Theorem 2.1.

5. Proof of Theorem 2.2 and Corollary 2.1

As seen in the previous sections, the original normal affine 3-fold X in question is constructed from $X^\sharp = T^\sharp - \mathrm{Supp}\,(D^\sharp)$ via the composite of half-point attachments (cf. Definition 3.2). Hence, the investigation of the Mori fiber space $\pi : \widehat{T}^\sharp \to W$ and the birational morphism $\mu^\sharp : \widehat{T}^\sharp \to T^\sharp$ are necessary for the analysis the structure on X. In this final section, we shall classify the pair $(\widehat{T}^\sharp, \widehat{D}^\sharp)$ (hence, classify $\widehat{X}^\sharp = \widehat{T}^\sharp - \mathrm{Supp}(\widehat{D}^\sharp)$) and investigate $\mu^\sharp$ according as Mori fiber space $\pi : \widehat{T}^\sharp \to W$ to obtain Theorem 2.2. This result is essentially contained in [C-F93, Me02]. Hereafter we shall determine the possibility of singularities on X. First of all, we have:

Lemma 5.1. *The threshold $\rho := \rho(\widehat{T}, \widehat{\mathcal{H}}) < 1$ of the pair $(\widehat{T}, \widehat{\mathcal{H}})$ (cf. Section 3) is greater than zero.*

Proof. Note that $\widehat{\mathcal{H}}^\sharp + \rho K_{\widehat{T}^\sharp} \equiv_\pi 0$. If $\rho = 0$ then $\widehat{\mathcal{H}}^\sharp \cdot C^\sharp = \widehat{D}^\sharp \cdot C^\sharp = 0$, where $C^\sharp$ is any extremal rational curve yielding π. This implies that $C^\sharp \subset \mathrm{Supp}(\widehat{D}^\sharp)$. Indeed, otherwise, $\widehat{D}^\sharp \cdot C^\sharp = 0$ means that $C^\sharp$ is disjoint from Supp $(\widehat{D}^\sharp)$, so is contained in $\widehat{X}^\sharp = \widehat{T}^\sharp - \mathrm{Supp}\,(\widehat{D}^\sharp)$. Note that π is a Mori fiber space, so $C^\sharp \not\subset \mathrm{Exc}(\mu^\sharp)$ for a general extremal rational curve. But then the image $\mu^\sharp(C^\sharp)$ is a complete curve contained in $X^\sharp$ which is an open affine subset of X. This is absurd. Hence $C^\sharp \subset \mathrm{Supp}\,(\widehat{D}^\sharp)$ for a general curve $C^\sharp$ contracted by π. This is obviously a contradiction. $\square$

Since $0 < \rho < 1$, the same argument as in the proof of [Me02, Theorem 5.3] says that the possibility of the pair $(\widehat{T}^\sharp, \widehat{D}^\sharp)$ of $\mathbb{Q}$-factorial terminal 3-fold Mori fiber space $\widehat{T}^\sharp$ and the proper transform $\widehat{D}^\sharp$ of $\widehat{D}$ is described as in Theorem 2.2. We shall prove the following where we make use of the terminologies in Theorem 2.2:

Lemma 5.2. *Assume that $\pi : \widehat{T}^\sharp \to W$ is either of type $D'2$, $D3$, $D3'$ or $\mathbb{Q}$-Fano. Then:*

(1) *$\mu^\sharp$ is an isomorphism.*

(2) *$X^\sharp$ and X are smooth.*

Proof. (1) Assume to the contrary that $\mu^\sharp$ is not isomorphism. Then $E^\sharp := \mathrm{Exc}(\mu^\sharp)$ is a divisor which is disjoint from $\widehat{D}^\sharp$. This is obviously impossible when π is of type $D'2$, $D3$, $D3'$ or $\mathbb{Q}$-Fano.

(2) If π is of $D3$-type, then $\widehat{T}^\sharp$ is smooth. Hence $X^\sharp$ is smooth. Therefore, X is also smooth. From now on, we shall consider the case where π is of $D3'$-type. If $\widehat{T}^\sharp$ is smooth (i.e., all the fibers of π are isomorphic to $\mathbb{P}^2$), then the assertion is clear to see. Assume that $\widehat{T}^\sharp$ is not smooth, so there exists a singular fiber $G \cong \mathbb{S}_4 \subset \mathbb{P}^5$ such that the vertex $v \in G$ sits in the hyper-quotient singularity of anlytic type $o \in (xy + z^2 + t^k = 0) \subset \mathbb{C}^4 \ : (x, y, z, t)/\mathbb{Z}_2(1, 1, 1, 0)$ for $k \geq 1$. This singularity $v \in \widehat{T}^\sharp$ can be resolved via the following diagram ($\natural$) which is a composite of elementary links of Type II in the sense of the Sarkisov Program (cf. [Me02], [Cor95, Cor00]):

($\natural$)

$$\begin{tikzcd}
& U^{(0)} \arrow[dl, "\mu_0"'] \arrow[dr, "\nu_0"] & & \cdots\cdots\cdots \arrow[dl] \arrow[dr] & & U^{(k-1)} \arrow[dl, "\mu_{k-1}"'] \arrow[dr, "\nu_{k-1}"] & \\
\widehat{T}^\sharp =: T^{(0)} \arrow[d, "\pi"] & & T^{(1)} \arrow[d, "\pi^{(1)}"] & & T^{(k-1)} \arrow[d, "\pi^{(k-1)}"] & & T^{(k)} \arrow[d, "\pi^{(k)}"] \\
W & = & W & = \cdots = & W & = & W,
\end{tikzcd}$$

where each elementary link $T^{(i)}/W \xleftarrow{\mu_i} U^{(i)} \xrightarrow{\nu_i} T^{(i+1)}/W$ is constructed, inductively, in the following fashion:

– Let $G^{(i)}$ be the fiber of $\pi^{(i)}$ over the point $w := \pi(G)$, which is isomorphic to $G^{(i)} \cong \mathbb{S}_4 \subset \mathbb{P}^5$ such that the vertex $v_i \in G^{(i)}$ sits in the hyper-quotient singularity of analytic type $o \in (xy + z^2 + t^{k-i} = 0) \subset \mathbb{C}^4 : (x, y, z, t)/\mathbb{Z}_2(1, 1, 1, 0)$ for $0 \leq i < k$.

– Let $\mu_i : U^{(i)} \to T^{(i)}$ be the blow-up at the maximal ideal $\mathfrak{m}_{v_i}$ of v_i with an exceptional divisor $E^{(i)}$. It follows that $E^{(i)} \cong \mathbb{S}_4$ if $0 \leq i < k - 1$ and $E^{(i)} \cong \mathbb{P}^2$ if $i = k - 1$.

– Let $\nu_i : U^{(i)} \to T^{(i+1)}$ be the contraction of the proper transform $G_{U^{(i)}} \cong \mathbb{F}_4$ of $G^{(i)}$ along the ruling on it. Thus we obtain the $(i+1)$-th step $\pi^{(i+1)} : T^{(i+1)} \to W$.

For other singular points on $\widehat{T}^\sharp$ (if there exist at all), they can be resolved by the same fashion. Thus we have the birational map $\Phi : \widehat{T}^\sharp/W \cdots \to \overline{T}/W$ which is a composite of elementary links of Type II (in the Sarkisov Category) centered at singularities, where $\overline{T}/W$ is a $\mathbb{P}^2$-bundle structure over W. Let $\overline{D} := \Phi_*(\widehat{D}^\sharp)$ be the proper transform of $\widehat{D}^\sharp$. It is easy to see that $\overline{D}|_{\overline{F}} \in |\mathcal{O}_{\mathbb{P}^2}(2)|$, where $\overline{F} \cong \mathbb{P}^2$ is a fiber of $\overline{T}/W$. We can see that $\widehat{D}^\sharp$ contains all the singular points of $\widehat{T}^\sharp$ by investigating the inverse procedure $\Phi^{-1} : \overline{T}/W \cdots \to \widehat{T}^\sharp/W$. Therefore, the complement $\widehat{X}^\sharp$ is smooth. Since $\mu^\sharp$ is an isomorphism, $X^\sharp$ and hence X also are smooth. By the similar argument, we can show that X in the case where π is of type $D2'$ is smooth. We shall omit the detail. $\square$

Lemma 5.3.
(1) *If $\pi : \widehat{T}^\sharp \to W$ is of type $D'2$ (resp. $D'3$ or $\mathbb{Q}$-Fano), then the singularities* Sing (X) *on X are described as in (i) (resp. (ii) or (iii)) in Corollary* 2.1.
(2) *If π is of type $D3$, then $X \cong \widehat{X}^\sharp$ and smooth.*

Proof. (1) By Lemma 5.2, $X^\sharp \cong \widehat{X}^\sharp = \widehat{T}^\sharp - \mathrm{Supp}\,(\widehat{D}^\sharp)$. Since Sing $(X) =$ Sing $(X^\sharp)$ (cf. Theorem 2.1), we can easily see the assertion in consideration of Theorem 2.2.

(2) Let us assume that π is of $D3$-type, i.e., a $\mathbb{P}^2$-bundle structure over a smooth curve W. We claim the following:

Claim. $s = r$, i.e., $t^0 = t^1 = \cdots = t^{r-1} = 0$.

Proof of Claim. Note that, by the construction of $\sharp$-Minimal Model Program associated to the pair $(\widehat{T}, \widehat{\mathcal{H}})$ (see Section 3), inequalities $0 \le t^0 \le t^1 \le \cdots \le t^{r-1} \le t^r = \rho < 1$ hold. Since π is of type $D3$ we have $(\widehat{D}^\sharp \cdot C^\sharp) = 1$, i.e., $\rho = 1/3$, where $C^\sharp$ is a line on a fiber $F \cong \mathbb{P}^2$ of π. If $s < r$, i.e., $t^i > 0$ for some $0 \le i \le r-1$, then $\phi^i : \widehat{T}^i \to \widehat{T}^{i+1}$ is an $E2$-type divisorial contraction (see the arguments in the proof of Propositions 3.1, 3.2). In particular, $t^i = 1/2$. But then $1/2 = t^i > 1/3 = \rho$, a contradiction. $\square$

The above mentioned claim means $\phi = \phi'$ in the notation of Theorem 2.1, in particular, $X \cong X^\sharp$. Since $\mu^\sharp$ is an isomorphism (cf. Lemma 5.2), we have $X^\sharp \cong \widehat{X}^\sharp$. Thus $X \cong X^\sharp \cong \widehat{X}^\sharp$. Since $\widehat{T}^\sharp$ is smooth, so is $\widehat{X}^\sharp$ obviously. $\square$

The remaining is the case where $\pi : \widehat{T}^\sharp \to W$ is of $C2$-type.

Lemma 5.4. *Assume that $\pi : \widehat{T}^\sharp \to W$ is of $C2$-type. If X is not smooth, then we have $\widehat{T}^\sharp \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a))$ $(a \ge 2)$. Furthermore, $\widehat{D}^\sharp$ (resp. $E^\sharp := \mathrm{Exc}(\mu^\sharp)$) is a section of π corresponding to $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a) \to \mathcal{O}_{\mathbb{P}^2}(a)$ (resp. $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a) \to \mathcal{O}_{\mathbb{P}^2}$). In particular, $E^\sharp$ is a negative section of π with $E^\sharp|_{E^\sharp} \cong \mathcal{O}_{\mathbb{P}^2}(-a)$ and* Sing $(X) = p := \mu^\sharp(E^\sharp)$ *such that $(p \in X) \approx o \in \frac{1}{a}(1,1,1)$.*

Proof. Our proof consists of three steps.

Step 1. Let us assume that X is not smooth. Note that $\mu^\sharp$ is not an isomorphism. Indeed, if $\mu^\sharp$ is an isomorphism, then $X^\sharp$ is smooth and so is X (cf. Theorem 2.1 (4)). This is a contradiction. Hence $E^\sharp := \mathrm{Exc}(\mu^\sharp)$ is a divisor and disjoint from $\widehat{D}^\sharp$ (see the proof for Lemma 4.2). By noting that $\widehat{D}^\sharp \cdot C^\sharp = 1$, $\widehat{D}^\sharp$ is a meromorphic section of π, where $C^\sharp$ is the class of a fiber of π. In fact, $\widehat{D}^\sharp$ is a (holomorphic) section of π. Indeed, if $\widehat{D}^\sharp$ contains a fiber of π then it meets $E^\sharp$, a contradiction. $E^\sharp$ is also a (holomorphic) scetion of π.

Step 2. Now we claim the following:

Claim. The base surface W (and hence $E^\sharp$ also) is a ruled surface.

Proof of Claim. Let $p : X \to Z$ denote the $\mathbb{A}^1$-fibration induced naturally from an $\mathbb{A}^1$-cylinderlike open subset of X. At first, we consider the case where $X^\sharp$ has still an $\mathbb{A}^1$-fibration induced from p. We denote by $p^\sharp := p|_{X^\sharp}$ the restricted $\mathbb{A}^1$-fibration. (Note that $X^\sharp$ is an open affine subset of X.) Each of the closures in $T^\sharp$ of general fibers of $p^\sharp$ meets $D^\sharp := \mu^\sharp(\widehat{D}^\sharp)$ in a single point. If these closures do not define a $\mathbb{P}^1$-fibration on $T^\sharp$ then, after resolving the indeterminacy, we obtain a birational morphism $\theta : V \to T^\sharp$ and a $\mathbb{P}^1$-fibration $f : V \to W_0$, which is an extension of $p^\sharp$, over a ruled surface W_0. Therefore $\widehat{T}^\sharp \approx V \approx W_0 \times \mathbb{P}^1$ On the other hand, we have $\widehat{T}^\sharp \approx W \times \mathbb{P}^1$ associated to the conic bundle structure $\pi : \widehat{T}^\sharp \to W$. Therefore $W \times \mathbb{P}^1 \approx W_0 \times \mathbb{P}^1$, so we have $W \approx W_0$ (cf. [Ka87]). Thus W is a ruled surface. On the other hand, if the closures in $T^\sharp$ of general fibers of $p^\sharp$ define a $\mathbb{P}^1$-fibration g on $T^\sharp$, g has $D^\sharp$ as a section. Therefore, $\widehat{T}^\sharp$ has two essentially distinct $\mathbb{P}^1$-fibrations π and $\widehat{g}$, where $\widehat{g}$ is naturally inherited from g. This is possible only if the base surface W is ruled. Next, we shall consider the case where $p : X \to Z$ can not induce an $\mathbb{A}^1$-fibration on $X^\sharp$. Then at least one of components of the complement $X - X^\sharp$ meets general fibers of p. Note that $X - X^\sharp$ is a union of several affine planes (cf. Theorem 2.1 (3)). Hence the base surface Z of p is dominated by $\mathbb{A}^2$ and is rational. Therefore X is a rational 3-fold. Since $E^\sharp$ is a section of a $\mathbb{P}^1$-bundle π on a rational 3-fold $\widehat{T}^\sharp$, we have $E^\sharp \times \mathbb{P}^1 \approx \mathbb{P}^3$. Therefore, $E^\sharp$ is rational (cf. [Ka87]). □

Step 3. Since $E^\sharp$ is a ruled surface, we can write $K_{\widehat{T}^\sharp} = (\mu^\sharp)^*(K_{T^\sharp}) + eE^\sharp$ with $e > -1$. Since $K + (1-\varepsilon)E^\sharp$ is not nef over $T^\sharp$ and klt for a sufficiently small $0 < \varepsilon << 1$, there exists an extremal ray, say Q, in $\overline{\mathrm{NE}}(\widehat{T}^\sharp/T^\sharp)_{K+(1-\varepsilon)E^\sharp<0}$ (cf. [K-M-M87]). Since $\widehat{T}^\sharp$ and $T^\sharp$ are $\mathbb{Q}$-factorial and $\mathrm{Exc}(\mu^\sharp) = E^\sharp$, we have $N_1(\widehat{T}^\sharp/T^\sharp) \cong \mathbb{R}$ and $\overline{\mathrm{NE}}(\widehat{T}^\sharp/T^\sharp) = Q$. Hence the contraction of Q coincides with $\mu^\sharp$. In particular, all the curves on $E^\sharp$ are numerically proportional. But since $E^\sharp$ is a section of π we have $\varrho(E^\sharp) = 1$. Hence $E^\sharp \cong \mathbb{P}^2$ and π is a $\mathbb{P}^1$-bundle structure over $W \cong \mathbb{P}^2$ admitting a section $\widehat{D}^\sharp$. Hence we can write $\widehat{T}^\sharp \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a))$ for some $a \geq 1$ and $E^\sharp$ is a negative section. Since the contraction of $E^\sharp$ have to result in the singular point, we have $a \geq 2$. The remaining assertions are easy to see. Thus we complete the proof. □

By summarizing the arguments, we obtain Theorem 2.2. Finally, we shall prove Corollary 2.1. The assertions except for the latter part in (2) are already proved in the previous argument. So, let us consider the case where $\widehat{T}^\sharp$ is a $\mathbb{Q}$-Fano 3-fold with $\varrho = 1$ such that $\widehat{T}^\sharp$ is general in the sense of Remark 1. Hence $\widehat{T}^\sharp$ in question is quasi-smooth (cf. Remark 1) and, so, the singularities on $\widehat{T}^\sharp$ are described in Table 1. Then we can easily see the assertion. (If $\widehat{T}^\sharp$ is (i), (v) or (viii), then we need not to consider generality condition, and, in fact, we can easily observe the singularities on it.)

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