A new proof of the Abhyankar-Moh-Suzuki Theorem

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Abstract

This preprint contains a new proof of the Abhyankar-Moh-Suzuki theorem, which Abhyankar calls occasionally the high school lemma, in characteristic zero case. Preprint **can** be read without knowing what is locally nilpotent derivations.

Introduction.

In the preprint MPIM2004-92 I showed that a locally nilpotent derivation (lnd for short) of an affine domain is equivalent to the restriction of a Jacobian type derivation of a polynomial ring. As an example where such a representation of lnd can be useful a new proof of the Abhyankar-Moh-Suzuki (AMS) Theorem in the zero characteristic case is given below.

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In this case the AMS Theorem which was independently proved by Abhyankar-Moh and Suzuki (see [AM] and [Su]) and later reproved by many authors (see [Es] for the references) states the following (field $\mathbb{C}$ of complex numbers can be replaced by any field of characteristic zero).

AMS Theorem. Let f and g be polynomials in $\mathbb{C}[z]$ such that $\mathbb{C}[f, g] = \mathbb{C}[z]$. If the degree of f is n and the degree of g is m then the greatest common divisor of m and n is equal to the minimum of m and n.

The only lnd which will be used here to prove the Theorem is the ordinary derivative $\partial = \frac{d}{dz}$ of $\mathbb{C}[z]$. It is locally nilpotent in the following sense: $\partial^i(h) = 0$ for any $h \in \mathbb{C}[z]$ if i is sufficiently large (larger then $\deg(h)$) but ∂^i is not zero for any i.

Suppose now that f, $g \in \mathbb{C}[z]$ and let S be a subalgebra of $\mathbb{C}[z]$ which is spanned by these polynomials. Then $S = \mathbb{C}[z]$ if and only if $S \neq \mathbb{C}$ and $\partial(S) \subset S$. One direction of this equivalence is obvious. To check the other direction take any non-constant polynomial $h \in S$ and apply ∂ to it sufficiently many times to get a linear polynomial.

In order to use this observation we should find a suitable presentation of ∂ through f and g. It can be done as follows.

Any two polynomials in one variable are algebraically dependent; that is, there is an irreducible polynomial in two variables which is satisfied by these two polynomials. So let P be an irreducible polynomial which is satisfied by f and g: $P(f, g) = 0$. Of course the factor algebra $\mathbb{C}[x, y]/(P) \cong S$ since the kernel of the mapping $\pi : \mathbb{C}[x, y] \to \mathbb{C}[z]$ defined by $\pi(x) = f(z)$, $\pi(y) = g(z)$ is the principal ideal (P).

Recall that any linear homomorphism ϕ of an algebra A which satisfies the Leibnitz law $\phi(ab) = \phi(a)b + a\phi(b)$ is called a derivation of A. One can use P to produce a derivation on $\mathbb{C}[x, y]$: define $\mathcal{D}_P(Q) = \mathrm{J}(P, Q)$ where $\mathrm{J}(P, Q)$ is the Jacobian of P and Q, i. e. the determinant $P_x Q_y - P_y Q_x$ of the corresponding Jacobi matrix.

The derivation $\mathcal{D}_P$ induces a derivation on $\mathbb{C}[x, y]/(P)$ by $\partial_P(\pi(Q)) = \pi(\mathcal{D}_P(Q)$. Since $\mathbb{C}[x, y]/(P) \cong S$ we can think about ∂_P as a derivation of S.

A more general statement which is proved in [LML] implies that if $S = \mathbb{C}[z]$ then $\partial_P = c\frac{d}{dz}$ where $c \in \mathbb{C} \setminus 0$. To make preprint more self-contained this particular fact is proved in the appendix.

If $\partial_P = c\frac{d}{dz}$ then $\pi(\mathcal{D}_P(x)) = \partial_P(\pi(x)) = cf'(z)$. Since $\mathcal{D}_P(x) = -\frac{\partial P}{\partial y}$ we have $\deg(\frac{\partial P}{\partial y}(f(z), g(z)) = \deg(f(z)) - 1$ and actually we will use only this relation to prove that $(\deg(f), \deg(g)) = \min(\deg(f), \deg(g))$. (It is also easy to show that this relation implies that $\pi(\mathcal{D}_P) = c\frac{d}{dz}$.)

In order to be able to connect these two numerical relations we should know more about P.

Irreducible dependence of two polynomials.

Let us assume in this section that f and g are any two polynomials from $K[z]$ where K is a field of any characteristic. It is well-known that they are algebraically dependent and there are many ways to show it. Unfortunately existence per se is not sufficient for our purposes.

Let f be a polynomial in $K[z]$ with $\deg_z(f) = n$. Let $E = K(z)$ and $F = K(f(z))$ be the fields of rational functions in z and $f(z)$ correspondingly. Since $F \subset E$ we can consider E as a vector space over F. Denote by

$[E:F]$ the dimension of this vector space.

The next two Lemmas may be skipped by a reader who knows that there exists an irreducible polynomial dependence between f and g which is given by a polynomial $P(x,y)$ which is monic in y.

Lemma 1. $[E:F]=n$ and $\{1,z,\dots,z^{n-1}\}$ is a basis of E over F.

Proof. It is clear that $\{1,z,\dots,z^{n-1}\}$ are linearly independent over F because the degrees of $\alpha_i z^i$ where $\alpha_i\in F$ and $0\le i<n$ are different for different i's. (As usual $\deg_z(h)$ for $h\in K(z)$ is the difference between the degrees of numerator and denominator of h.) It is also clear that $K[z]=\bigoplus_{i=0}^{n-1} z^iK[f(z)]$ since $\bigoplus_{i=0}^{n-1} z^iK[f(z)]$ contains monomials z^k for any non-negative k. So $K[z]\subset\bigoplus_{i=0}^{n-1} z^iF$ and for $p\in K[z]$ the elements $1,p,\dots,p^n$ satisfy a linear relation $\sum_{i=0}^{n}\alpha_ip^i=0$ where $\alpha_i\in F$ and some of them are not equal to zero. By cancelling an appropriate power of p from this relation we may assume that $\alpha_0\neq 0$. So $p^{-1}=-\sum_{i=1}^{n}\alpha_0^{-1}\alpha_ip^{i-1}$ and $p^{-1}q=-\sum_{i=1}^{n}\alpha_0^{-1}\alpha_ip^{i-1}q\in\bigoplus_{i=0}^{n-1} z^iF$ for any $q\in K[z]$.

Let $g\in K[z]$. By the previous Lemma there exists a non-trivial relation $\sum_{i=0}^{n}\alpha_ig^i=0$. So there exists a non-zero element $P(x,y)\in A=K(x)[y]$ for which $P(f(z),g(z))=0$. We will assume that $k=\deg_y(P)$ is minimal possible. Then P is an irreducible element of A and if $Q(f,g)=0$ for some $Q\in A$ then Q is divisible by P by the Euclidean algorithm. We also assume that

P is a monic polynomial in y.

Lemma 2. $P \in K[x, y]$.

Proof. Since $P = y^k + \sum_{i=0}^{k-1} p_i(x) y^i$ where $p_i \in K(x)$ we can multiply P by the least common denominator $D(x) \in K[x]$ of p_i and obtain a polynomial $DP \in K[x, y]$ which is irreducible in $K[x, y]$. In order to prove that $D = 1$ it is sufficient to find an element $Q \in K[x, y]$ such that $Q(f, g) = 0$ and Q is monic in y. Indeed, by a Gauss lemma Q must be divisible by DP in $K[x, y]$ which is possible only if $D = 1$.

Let us define $Q_m \in K[x, y]$ for all natural numbers m so that $Q_m = y^m + R_m$ where $\deg_y(R_m) < m$ and $\deg_z(Q_m(f, g))$ is the minimal possible. Let $d_m = \deg_z(Q_m(f, g))$ if $Q_m(f, g) \neq 0$. If $a > b$ and $d_a \equiv d_b \pmod{n}$ then $d_a < d_b$ because otherwise we can find $j \in \mathbb{Z}^+$ and $c \in K$ so that $\deg_z(Q_a(f, g) - cf^j Q_b(f, g)) < \deg_z(Q_a(f, g))$. Therefore we can have only a finite number of d_a which means that $Q_a(f, g) = 0$ for a sufficiently large a.

Let us describe now a procedure which will produce P (over a field of any characteristic).

First an informal description. Raise g in the smallest possible power d so that by subtracting some power of f (with an appropriate coefficient) the degree can be decreased. If the result has a degree which can be decreased by subtracting a monomial in f and g, do it and continue until the degree of the result cannot be decreased. Since different monomials in f and g can have the same degree, use only monomials with power of g less than d. Then the choice of a monomial with given degree is unique. If the result h is zero it gives the dependence we are looking for. If not, raise h to the smallest possible power so that the degree can be decreased by subtracting some monomial of f and g and on further steps use for reduction purposes the monomials in f, g, and h with appropriately restricted powers of g and h, and so on. After several steps like that an algebraic dependence will be obtained. It is easy to implement this procedure and it works nicely on examples. On the other hand why should it stop? Also it is not clear which monomials to use in the reduction: say, should we use monomials with negative powers of f?

Here is an example. Take $f = z^4$, $g = z^6 - z$. We have to start with $g^2 - f^3 = -2z^7 + z^2$ and $h = -2z^7 + z^2$. Next $h^2 - 4f^2g = z^4$ and $h^2 - 4f^2g - f = 0$. So $(g^2 - f^3)^2 - 4f^2g - f = 0$. Let us assume now that the characteristic of the ground field is 2. In this case $g^2 - f^3 = z^2$ and we can proceed with reduction of degree but we should use the monomial $f^{-1}g$. So here $h = g^2 - f^3 - f^{-1}g = z^{-3}$ and $h^2 - f^{-3}g - f^{-2}h = 0$ is a dependence in which miraculously all negative powers disappear: $g^4 - f^6 - f^{-2}g^2 - f^{-3}g - f^{-2}g^2 - f - f^{-3}g = g^4 - f^6 - f$.

So it seems reasonable to include monomials with negative powers of f in the process.

FORMAL DESCRIPTION.

Below deg denotes the z-degree of a rational function from $K(z)$.

First step.

Put $g_0 = g$. Let $\deg(g_0) = m_0$ and $\deg(f) = n$. Find the greatest common divisor $d_0 = (n, m_0)$ of n and m_0. Find the smallest positive integers a_0, b_0 so that $a_0 m_0 = b_0 n$. Then $\deg(g_0^{a_0}) = \deg(f^{b_0})$. Find $k_0 \in K$ for which $m_{0,1} = \deg(g_0^{a_0} - k_0 f^{b_0}) < \deg(g_0^{a_0})$. If $m_{0,1}$ is divisible by d_0 find a monomial $f^i g_0^{j_0}$ with $0 \leq j_0 < a_0$ and $\deg(f^i g_0^{j_0}) = m_{0,1}$, find $k_1 \in K$ for which $m_{0,2} = \deg(g_0^{a_0} - k_0 f^{b_0} - k_1 f^i g_0^{j_0}) < m_{0,1}$ and so on.

If after a finite number of reductions zero is obtained, we have a dependence.

If after a finite number of reductions $m_{0,i}$ which is not divisible by d_0 is obtained, denote the corresponding expression by g_1 and make the next step.

If the procedure does not stop we failed.

Generic step.

Assume that after s steps we obtained $g_0, \dots, g_s$ where $g_s \neq 0$. Denote $\deg(g_i)$ by m_i and $g.c.d.(n, m_0, \dots, m_i)$ by d_i. The numbers d_i are positive while m_i can be negative. Put $d_{-1} = n$ and $a_i = \frac{d_{i-1}}{d_i}$ for $0 \leq i \leq s$. (Clearly $a_s m_s$ is divisible by d_{s-1} and a_s is the smallest integer with this property.) Call a monomial $\mathbf{m} = f^i g_0^{j_0} \dots g_s^{j_s}$ with $0 \leq j_k < a_k$ *s-standard.*

Find an s-standard monomial $\mathbf{m}_{s,0}$ with $\deg(\mathbf{m}_{s,0}) = a_s m_s$ and $k_0 \in K$ for which $m_{s,1} = \deg(g_s^{a_s} - k_0 \mathbf{m}_{s,0}) < a_s m_s$.

If $m_{s,1}$ is divisible by d_s find an s-standard monomial $\mathbf{m}_{s,1}$ with $\deg(\mathbf{m}_{s,1}) =$

$m_{s,1}$ and $k_1 \in K$ for which $m_{s,2} = \deg(g_s^{a_s} - k_0\mathbf{m}_{s,0} - k_1\mathbf{m}_{s,1}) < m_{s,1}$ and so on.

If after a finite number of reductions zero is obtained, we have a dependence.

If after a finite number of reductions $m_{s,i}$ which is not divisible by d_s is obtained, denote the corresponding expression by g_{s+1} and make the next step.

If the procedure does not stop we failed.

First we prove that failure is not an option.

Lemma 3. After a finite number of reductions, either zero or $m_{s,i}$ which is not divisible by d_s will be obtained.

Proof. In this proof we consider s-standard monomials which do not contain f. The degrees of different s-standard monomials are different $\bmod n$. Indeed, if $\sum\limits_{k=0}^{s} j_k m_k \equiv \sum\limits_{k=0}^{s} i_k m_k \pmod{n}$ then $j_s m_s \equiv i_s m_s \pmod{d_{s-1}}$ and $j_s = i_s$ because $0 \leq j_s < a_s$ and $0 \leq i_s < a_s$ and $|j_s - i_s| m_s$ is not divisible by d_{s-1} if $0 < |j_s - i_s| < a_s$. So $j_s = i_s$ and we can omit them from the sums and proceed to prove that $j_{s-1} = i_{s-1}$, etc..

Consider now the field $E = K(z)$ as a vector space over its subfield $F = K(f(z))$. The s-standard monomials $\mathbf{m_j}$ are linearly independent over F since their degrees are different $\bmod n$. Let us denote by V_s the subspace generated by all s-standard monomials over F and call the corresponding basis of V_s standard.

We have to look at two cases: $g_s^{a_s} \in V_s$ and $g_s^{a_s} \notin V_s$.

If $g_s^{a_s} \in V_s$ then $\Delta g_s^{a_s} = \sum \delta_{\mathbf{j}} \mathbf{m}_{\mathbf{j}}$ for some non-zero polynomial $\Delta(f)$ and polynomials $\delta_{\mathbf{j}}(f)$. Of course we can assume that these polynomials do not have a common divisor and so are relatively prime.

Let us show that $\Delta g_s^{a_s} - \sum \delta_{\mathbf{j}} \mathbf{m}_{\mathbf{j}} = 0$ is the irreducible polynomial relation we are looking for. The elements $g_i \in L = K[f, f^{-1}, g]$. Therefore all $\mathbf{m}_{\mathbf{j}} \in L$. So $\Delta g_s^{a_s} - \sum \delta_{\mathbf{j}} \mathbf{m}_{\mathbf{j}} = P(f, f^{-1}, g)$. It is sufficient to verify that P is an irreducible element of L.

First of all P is not identically zero. Indeed, it is easy to check by induction that $\deg_g(g_i) = a_0 \dots a_{i-1}$. The base $\deg_g(g_0) = 1$ is clear since $g_0 = g$. Assume that $\deg_g(g_k) = a_0 \dots a_{k-1}$ for $k < i+1$. Now, $g_{i+1} = g_i^{a_i} - r_i(f, g_0, \dots, g_i)$. Since $\deg_{g_k}(r_i) < a_k$ it follows from the assumption that $\deg_g(r_i) \leq \sum_{k=0}^{i} (a_k - 1) \deg_g(g_k) = a_0 \dots a_i - 1 = a_i \deg_g(g_i) - 1$, hence $\deg_g(g_{i+1}) = a_i \deg_g(g_i)$. So $\deg_g(P) = a_0 \dots a_s = a_s \deg_g(g_s)$ and it is a monic polynomial in g.

Let us check that P is irreducible. Assume for a moment that any non-zero polynomial $Q(g) \in K(f)[g]$ with $\deg_g(Q) < a_s \deg_g(g_s) = \deg_g(P)$ is an element of V_s. Any non-zero $Q(g)$ is not equal to zero as a rational function of z since the z-degrees of the elements of the standard basis of V_s are different $\bmod n$. So P is irreducible since otherwise $P = Q_1 Q_2$ where $\deg_g(Q_i) < \deg_g(P)$.

It remains to show that for each $d < a_s \deg_g(g_s)$ there is an s-standard monomial $\mathbf{m}$ with $\deg_g(\mathbf{m}) = d$. There is exactly $a_0 \dots a_s = a_s \deg_g(g_s)$ s-standard monomials. If $\sum_{k=0}^{s} j_k \deg_g(g_k) = \sum_{k=0}^{s} i_k \deg_g(g_k)$ then $j_0 \equiv i_0 \pmod{a_0}$ and $j_0 = i_0$ because $0 \leq j_0 < a_0$ and $0 \leq i_0 < a_0$ and we can proceed to prove that $j_1 = i_1$, etc.. So a standard monomial $\mathbf{m}$ is completely determines by $\deg_g(\mathbf{m})$ and different monomials have different g-degree. Therefore we have a standard monomial with g-degree equal to d for any $d < \deg_g(P)$.

Since the irreducible polynomial should be monic in g, we have $\Delta = 1$ and so $g_s^{a_s} - \sum \delta_{\mathbf{j}} \mathbf{m}_{\mathbf{j}} = 0$ after a finite number of reductions.

Now let $g_s^{a_s} \notin V_s$. As we know E is n-dimensional over F and $\{1, z, \ldots, z^{n-1}\}$ is a basis of E over F (Lemma 1). Extend the standard basis of V_s by the appropriate powers of z to the basis of E over F. Write $g_s^{a_s} = \sum \delta_{\mathbf{j}} \mathbf{m}_{\mathbf{j}} + \sum \epsilon_k z^k$ for some rational functions $\delta_{\mathbf{j}}(f)$, $\epsilon_k(f)$ from F. The second sum is not zero and any k in it is not divisible by d_s. Hence $d = \deg(\sum \epsilon_k z^k)$ cannot be divisible by d_s and is not equal to z-degree of any s-standard monomial.

A rational function $\frac{p(f)}{\Delta(f)}$ can be written in the form $\frac{p(f)}{\Delta(f)} = \sum_{k=-N}^{M} c_i f^i - r_N$ where $c_i \in K$ and $\deg_f(r_N) < -N$. Therefore we can write $g_s^{a_s} - \sum c_{\mathbf{j},i} f^i \mathbf{m}_{\mathbf{j}} = \sum \epsilon_k z^k - R_N$ where $\deg(R_N)$ is smaller than d. We can also assume that $\deg(c_{\mathbf{j},\mathbf{i}} f^i \mathbf{m}_{\mathbf{j}}) > d$ by moving all "small" summands into the right side. Then $g_s^{a_s} - \sum c_{\mathbf{j},i} f^i \mathbf{m}_{\mathbf{j}} = g_{s+1}$.

Remark. A standard monomial $\mathbf{m} = f^i g_0^{j_0} \ldots g_s^{j_s}$ is completely determined by i and $\deg_g(\mathbf{m})$.

Lemma 4. After a finite number of steps we obtain zero.

Proof. From the definition of V_i it follows that $\dim(V_i) < \dim(V_{i+1})$ if $g_i^{a_i} \notin V_i$. Since $\dim(V_i) \leq n$, after a finite number of steps $g_i^{a_i} \in V_i$ which leads to a relation.

So the algorithm works and we even know that on the last step all negative degrees of f will disappear. What is interesting though is that in our experiment we did not have negative powers of f in the intermediate steps when characteristic was zero. Let us prove that this is always the case.

Lemma 5. If characteristic of K is zero then all g_i are polynomials in f and g.

Proof. Order the monomials $f^i g^j$ of $L = K[f, f^{-1}, g]$ lexicographically by $\deg_g, \deg_f$. Call a monomial *negative* if its f-degree is negative, otherwise call it *positive*. For an element $h \in L$ introduce a function *gap* as follows. If $h \in K[f, g]$ then $\operatorname{gap}(h) = \infty$. Otherwise find the largest monomial $\overline{h}$ of h and the largest negative monomial $\widetilde{h}$ of h. Then $\operatorname{gap}(h) = \overline{h} \div \widetilde{h}$. Define ∞ to be larger than any monomial.

We will use the following properties of gap which are easy to check:

(a) $\operatorname{gap}(h_1 h_2) \geq \min(\operatorname{gap}(h_1), \operatorname{gap}(h_2))$;

(b) $\operatorname{gap}(h^d) = \operatorname{gap}(h)$ if h is monic in g and the characteristic is zero;

(c) $\operatorname{gap}(fh) \geq \operatorname{gap}(h)$.

Call $h \in L \setminus K[f, g]$ a Laurent polynomial. Of course h is a Laurent polynomial if and only if $\operatorname{gap}(h) < \infty$. We will show that $\operatorname{gap}(g_{j+1}) \leq \operatorname{gap}(g_j)$. Since we know that the last g_s which gives an irreducible dependence of $f(z)$ and $g(z)$ is a polynomial, this will imply that $\operatorname{gap}(g_j) = \infty$ for all j and hence the Lemma.

Let us use induction. The base of induction $\operatorname{gap}(g_1) \leq \operatorname{gap}(g_0)$ is obvious since $\text{gap}(g_0) = \infty$. Assume that $\text{gap}(g_{j+1}) \leq \text{gap}(g_j)$ if $j < k$. If $g_k \in K[f,g]$ then $\text{gap}(g_{k+1}) \leq \text{gap}(g_k)$. So let g_k be a Laurent polynomial. Since $g_{k+1} = g_k^{a_k} - r_k$ it is sufficient to check that the largest negative monomial of r_k cannot cancel the largest negative monomial of $g_k^{a_k}$.

As above, call a k-standard monomial *negative* if its f-degree is negative and *positive* otherwise. Let $\mathbf{m} = f^i g_0^{j_0} \ldots g_k^{j_k}$ be a k-standard monomial. From the properties of gap mentioned above it follows that $\text{gap}(g_0^{j_0} \ldots g_k^{j_k}) \geq \text{gap}(g_k)$. Indeed $\text{gap}(g_i^{j_i}) = \text{gap}(g_i) \geq \text{gap}(g_k)$ since g_i is monic in g and $\text{gap}(h_1 h_2) \geq \min(\text{gap}(h_1), \text{gap}(h_2))$. Also if $i > 0$ then $\text{gap}(f^i h) \geq \text{gap}(h)$, so $\text{gap}(\mathbf{m}) \geq \text{gap}(g_k)$ for a positive k-standard monomial $\mathbf{m}$. If $i < 0$ then $\text{gap}(\mathbf{m}) = 1$ since $g_0^{j_0} \ldots g_k^{j_k}$ is monic in g and the largest monomial of $\mathbf{m}$ is negative.

Recall that r_k is defined as a linear combination of k-standard monomials. Let $\mathbf{m}$ be a positive monomial of r_k. Then since $\deg_g(\mathbf{m}) < \deg_g(g_k^{a_k})$ and $\text{gap}(\mathbf{m}) \geq \text{gap}(g_k)$ we see that even if $\mathbf{m}$ as an element of L is a Laurent polynomial, the negative monomials of $\mathbf{m}$ are smaller than the largest negative monomial of $g_k^{a_k}$. So if e. g. r_k does not contain negative k-standard monomials then $\text{gap}(g_{k+1}) = \text{gap}(g_k)$.

We consider now two cases to make reading less unpleasant. In what follows j-standard monomials are ordered lexicographically by their g-degree and f-degree.

(a) $\text{gap}(g_k) < \text{gap}(g_{k-1})$. Since $g_k = g_{k-1}^{a_{k-1}} - r_{k-1}$ and $\text{gap}(g_{k-1}^{a_{k-1}}) = \text{gap}(g_{k-1}) > \text{gap}(g_k)$ we can conclude that the largest negative monomial of r_{k-1} is larger than negative monomials of $g_{k-1}^{a_{k-1}}$. Since all summands of r_{k-1} have different g-degree this monomial is $\overline{\nu_{k-1}}$ for the largest negative k-

standard monomial ν_{k-1} of r_{k-1}. So $\operatorname{gap}(g_k) = g^{b_k} \div \overline{\nu_{k-1}}$ where $b_k = \deg_g(g_k)$.

Next, $g_{k+1} = (g_{k-1}^{a_{k-1}} - r_{k-1})^{a_k} - r_k = g_{k-1}^{a_{k-1}a_k} - R_k - r_k$. Since $\deg_g(R_k) < \deg_g(g_{k+1})$ we know that $R_k \in V_k$ (see the proof of Lemma 3). Present R_k through the standard basis as a sum of k-standard monomials. The largest negative k-standard monomial in R_k is $\nu_{k-1}g_k^{a_k-1}$. Indeed $\operatorname{gap}(g_{k-1}^{a_{k-1}a_k} - R_k) = \operatorname{gap}(g_k^{a_k}) = \operatorname{gap}(g_k) < \operatorname{gap}(g_{k-1})$; therefore the largest negative monomial of $g_{k-1}^{a_{k-1}a_k}$ is smaller than the largest negative monomial of R_k. Hence $g^{b_k} \div \overline{\nu_{k-1}} = \operatorname{gap}(g_k) = \overline{g_{k-1}^{a_{k-1}a_k}} \div \overline{\mu}$ where μ is the largest negative k-standard monomial of R_k. Since $\overline{g_{k-1}^{a_{k-1}a_k}} = g^{b_{k+1}}$ we have $\overline{\mu} = g^{b_{k+1}-b_k}\overline{\nu_{k-1}} = \overline{g_k^{a_k-1}}\,\overline{\nu_{k-1}}$. Since μ is determined by $\overline{\mu}$ (see Remark to Lemma 3) $\mu = \nu_{k-1}g_k^{a_k-1}$. Let us compute its z-degree: $\deg(\nu_{k-1}g_k^{a_k-1}) = \deg(\nu_{k-1}) + (a_k-1)m_k > a_k m_k$ because $\deg(\nu_{k-1}) > m_k$ since ν_{k-1} is a $k-1$-standard monomial of r_{k-1}. But $\deg(r_k) = a_k m_k$ and all k-standard monomials in r_k have z-degree not exceeding $a_k m_k$. So $\nu_{k-1}g_k^{a_k-1}$ is not a summand of r_k and cannot be cancelled.

(b) $\operatorname{gap}(g_k) = \operatorname{gap}(g_{k-1})$. Since $\operatorname{gap}(g_0) = \infty$ and $\operatorname{gap}(g_k) < \infty$ we can find such a p that $\operatorname{gap}(g_k) = \operatorname{gap}(g_{k-1}) = \ldots = \operatorname{gap}(g_p) < \operatorname{gap}(g_{p-1})$. Just as above, $g_{k+1} = g_{p-1}^{a_{p-1}\ldots a_k} - R_k - r_k$ where $R_k \in V_k$. Since $\operatorname{gap}(g_{p-1}^{a_{p-1}\ldots a_k}) = \operatorname{gap}(g_{p-1}) > \operatorname{gap}(g_{p-1}^{a_{p-1}\ldots a_k} - R_k) = \operatorname{gap}(g_k) = \operatorname{gap}(g_p)$ we can conclude that the maximal negative k-standard monomial in the standard representation of R_k is $\nu_{p-1}g_p^{a_p-1}\ldots g_k^{a_k-1}$, where ν_{p-1} is the largest negative $p-1$-standard monomial in r_{p-1}. But $\deg(\nu_{p-1}g_p^{a_p-1}\ldots g_{k-1}^{a_k-1}) = \deg(\nu_{p-1}) + (a_p-1)m_p + \ldots + (a_k-1)m_k > a_k m_k = \deg r_k$ since $\deg(\nu_{p-1}) > m_p$ and $a_j m_j > m_{j+1}$. So again this monomial cannot be cancelled by a monomial from r_k.

Remark. An explanation for the different behavior in the case of finite characteristic is in the different behavior of the function gap. Property (b) is not true if d is divisible by characteristic. Say, in our example the largest negative monomial in h^2 is $f^{-2}g^2$ and its z-degree is too small. When we make it 1-standard by substituting $g^2 = h - f^3 - f^{-1}g$ we obtain $f^{-2}(h - f^3 - f^{-1}g) = f - f^{-2}h - f^{-3}g$ and negative monomials in this expression are the terms in the corresponding r.

Lemma 5 gives us the information sufficient for a proof of AMS Theorem.

Let $A = \mathbb{C}[x, y]$ and let π be a projection of A into $\mathbb{C}[z]$ which is given by $\pi(x) = f(z)$, $\pi(y) = g(z)$. For an $a \in A$ denote $\deg_z(\pi(a))$ by $\deg(a)$. As usual put $\deg(0) = -\infty$.

Let us introduce a *defect* function for elements of A: $\mathrm{def}(a) = \deg(\mathrm{J}(a, x)) - \deg(a)$. If both $\pi(\mathrm{J}(a, x)) = 0$ and $\pi(a) = 0$ then $\mathrm{def}(a)$ is not defined. Here are some obvious properties of this function.

a) $\mathrm{def}(a^k) = \mathrm{def}(a)$;

b) $\mathrm{def}(ab) \le \max(\mathrm{def}(a), \mathrm{def}(b))$;

c) $\mathrm{def}(ab) = \max(\mathrm{def}(a), \mathrm{def}(b))$ if $\mathrm{def}(a) \ne \mathrm{def}(b)$;

d) if $\deg(a) > \deg(b)$ then $\mathrm{def}(a - b) \le \max(\mathrm{def}(a), \mathrm{def}(b) - (\deg(a) - \deg(b)))$.

We proved in Lemma 5 that g_i are polynomials in f and g. Put $G_i = g_i(x, y)$. Recall that $g_{i+1} = g_i^{a_i} - r_i$ where r_i is a linear combination of the

i-standard monomials. So $G_{i+1} = G_i^{a_i} - R_i$ where R_i can be presented as a linear combination of the i-standard monomials of $G_0, \dots, G_i$.

Lemma 6. $\operatorname{def}(R_i) < \operatorname{def}(G_i) < \operatorname{def}(G_{i+1})$.

Proof. First of all $\operatorname{def}(x^i y^j) = -m_0$ if $j \neq 0$ and $\operatorname{def}(x^i) = -\infty$. Hence $\operatorname{def}(G_0) = -m_0$ since $G_0 = y$.

Now, $G_1 = G_0^{a_0} - R_0$ where $R_0 = cx^{b_0} + \sum c_{ij} x^i y^j$ and $b_0 \deg(x) = a_0 \deg(G_0) > \deg(\sum c_{ij} x^i y^j)$. Therefore $\operatorname{def}(R_0) = \deg(\sum j c_{ij} x^i y^{j-1}) - b_0 n < (b_0 n - m_0) - b_0 n = -m_0 = \operatorname{def}(G_0)$ since $in + jm_0 < b_0 n$. Next, $\operatorname{def}(G_1) = (a_0 - 1)m_0 - m_1 > -m_0$ since $a_0 m_0 - m_0 > \deg(j c_{ij} x^i y^{j-1})$ and by definition $m_1 < a_0 m_0$. So $\operatorname{def}(R_0) < \operatorname{def}(G_0) < \operatorname{def}(G_1)$.

Let us apply induction. As we know $G_{i+1} = G_i^{a_i} - R_i$ where R_i is a linear combination of i-standard monomials. Let $\nu = x^j G_0^{j_0} G_1^{j_1} \dots G_i^{j_i}$ be an i-standard monomial from R_i. By c) $\operatorname{def}(\nu) = \operatorname{def}(G_k)$ where k is the largest number for which G_k is contained in ν since by assumption $\operatorname{def}(G_a) < \operatorname{def}(G_b)$ if $a < b < i + 1$. So $\operatorname{def}(\nu) \leq \operatorname{def}(G_i)$ for any ν from R_i. The i-standard monomial μ of R_i with the largest z-degree does not contain G_i since otherwise $\widetilde{a}_i m_i$ is divisible by d_{i-1} for some $\widetilde{a}_i < a_i$. So $\operatorname{def}(\mu) \leq \operatorname{def}(G_{i-1})$.

Since all i-standard monomials have different z-degrees (see the proof of Lemma 3), by d) $\operatorname{def}(R_i) = \operatorname{def}(\mu + (R_i - \mu)) \leq \max(\operatorname{def}(G_{i-1}), \operatorname{def}(G_i) - 1) < \operatorname{def}(G_i)$. Therefore $\deg(\mathrm{J}(R_i, x)) = \operatorname{def}(R_i) + \deg(R_i) < \operatorname{def}(G_i) + \deg(R_i) = \operatorname{def}(G_i^{a_i}) + \deg(G_i^{a_i}) = \deg(\mathrm{J}(G_i^{a_i}, x))$ because $\deg(R_i) = \deg(G_i^{a_i})$. Hence $\deg(\mathrm{J}(G_i^{a_i} - R_i, x)) - \deg(G_i^{a_i}) = \deg(\mathrm{J}(G_i^{a_i}, x)) - \deg(G_i^{a_i}) = \operatorname{def}(G_i^{a_i}) = \operatorname{def}(G_i) < \operatorname{def}(G_{i+1}) = \deg(\mathrm{J}(G_i^{a_i} - R_i, x) - \deg(G_{i+1})$ since $\deg(G_{i+1}) <$

$\deg(G_i^{a_i})$.

Lemma 7. $\deg(\mathrm{J}(G_{i+1}, x)) = \sum_{k=0}^{i} (a_k - 1)m_k$.

Proof. Indeed, $G_{i+1} = G_i^{a_i} - R_i$. So $\mathrm{J}(G_{i+1}, x) = \mathrm{J}(G_i^{a_i}, x) - \mathrm{J}(R_i, x)$. In the proof of Lemma 6 we checked that $\deg(\mathrm{J}(R_i, x)) < \deg(\mathrm{J}(G_i^{a_i}, x))$ and so $\deg(\mathrm{J}(G_{i+1}, x)) = \deg(\mathrm{J}(G_i^{a_i}, x)) = (a_i - 1)m_i + \deg(\mathrm{J}(G_i, x))$. Since $\deg(\mathrm{J}(G_0, x)) = \deg(1) = 0$ the Lemma is proved.

Finally, the numerical relation of the AMS Theorem.

Lemma 8. $\deg(\mathrm{J}(G_s, x)) = \deg(x) - 1$ only if $\min(\deg(f), \deg(g)) = (\deg(f), \deg(g))$.

Proof. By Lemma 7 we have $\deg(\mathrm{J}(G_s, x)) = \sum_{k=0}^{s-1} (a_k - 1)m_k$. So $n - 1 \geq (a_0 - 1)m_0$ since all $m_k > 0$. Now, $a_0 d_0 = n$ for $d_0 = (n, m_0)$. So $d_0(n-1) \geq d_0(a_0 - 1)m_0 = (n - d_0)m_0$ and $0 \geq (n - d_0)(m_0 - d_0) + d_0 - d_0^2$. It is possible only if $(n - d_0)(m_0 - d_0) = 0$ since otherwise $n - d_0 \geq d_0$ and $m_0 - d_0 \geq d_0$.

Appendix.

Let us check that $\partial_P = c\frac{d}{dz}$ if $\mathbb{C}[x, y]/(P) \cong \mathbb{C}[z]$. Take any Q for which $\pi(Q) = z$. Then $\pi(x - f(Q)) = 0$ and $\pi(y - g(Q)) = 0$ since $\pi(x) = f(z)$ and $\pi(y) = g(z)$. So $x - f(Q) = x_1 P$, $y - g(Q) = y_1 P$ and $1 =$

$\mathrm{J}(x,y)=\mathrm{J}(f(Q)+x_1P,g(Q)+y_1P)\equiv \mathrm{J}(P,Q)(x_1g'(Q)-f'(Q)y_1) \pmod{P}$. Therefore $\pi(\mathrm{J}(P,Q))=c\in\mathbb{C}^*$ and $\partial_P(z)=\partial_P(\pi(Q))=c$. Since a derivation of $\mathbb{C}[z]$ is completely determined by is value on z we should have $\partial_P=c\frac{d}{dz}$.

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