FUNCTIONAL ENVELOPE OF A DYNAMICAL SYSTEM

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Abstract. If (X, f) is a dynamical system given by a compact metric space X and a continuous map $f : X \to X$ then by the functional envelope of (X, f) we mean the dynamical system $(S(X), F_f)$ whose phase space $S(X)$ is the space of all continuous selfmaps of X and the map $F_f : S(X) \to S(X)$ is defined by $F_f(\varphi) = f \circ \varphi$ for any $\varphi \in S(X)$. The functional envelope of a system always contains a copy of the original system.

Our motivation for the study of dynamics in functional envelopes comes from the semigroup theory, from the theory of functional difference equations and from dynamical systems theory. The paper mainly deals with the connection between the properties of a system and the properties of its functional envelope. A special attention is paid to orbit closures, ω-limit sets, (non)existence of dense orbits and topological entropy.

1. Introduction

Our motivation to study the dynamics in what we will call the functional envelope of a dynamical system comes from the semigroup theory, the theory of functional difference equations, and the proper theory of dynamical systems. Before we describe the motivation, one remark to a notation: Given spaces X and Y, we will denote the set of all continuous maps $X \to Y$ by $C(X, Y)$. Nevertheless, if $X = Y$ then, instead of $C(X, X)$, we prefer the shorter notation $S(X)$ which is common in the semigroup theory — $S(X)$ is a semigroup under composition; 'S' stands for 'semigroup' (the notation $C(X)$ would be confusing because this usually denotes the set of continuous real or complex functions defined on X).

Semigroup theory. Let X be a compact metrizable space and $S(X)$ be the set of all *continuous* maps from X into X. Obviously, $S(X)$ is a topological semigroup under composition and the compact-open topology. Since X is compact then, as is well known, the compact-open topology coincides with the topology of uniform convergence derived from a metric ϱ compatible with the topology on X. Moreover, the latter topology is independent of the choice of the metric ϱ. Another topology on $S(X)$ equivalent with the two already mentioned is the topology given by the Hausdorff metric (see [30]). In fact, the maps in $S(X)$ have closed graphs and so we can consider a distance between elements of $S(X)$ defined as the Hausdorff distance (derived from, say, maximum metric in $X \times X$) of their graphs. We will work

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with the uniform metric in $S(X)$ which is usually easier to handle, but also with the Hausdorff metric because this one has a big advantage — it can in a natural way be extended from continuous functions to multifunctions (set valued maps) with closed graphs. This fact will be substantially used in our study of topological entropy in Section 6.

For general information about the theory of semigroups of continuous selfmaps, one may consult [26], [27], [28], [13]. Here we only wish to touch on a problem closely related to what we will study in Section 5. $S(X)$ may be very small, as in the case of rigid continua. Recall that a *rigid continuum* is a nendegenerate continuum (i.e., a nondegenerate compact connected Hausdorff topological space) X such that $S(X)$ consists only of the constant maps and the identity. Rigid continua exist even in the class of metric continua. On the other hand, if X is, say, the interval $I = [0,1]$, then $S(I)$ is very large. When one thinks about the diversity of the functions in $S(I^N)$, $N \in \mathbb{N}$, it is surprising that the so called density index of $S(I^N)$ is two. Recall that if S is a topological semigroup, the *density index* of S, denoted $D(S)$, is the smallest positive integer n, if such exists, with the property that S contains a dense subsemigroup with n generators. If no such integer exists, then set $D(S) = \infty$. Thus, in this notation, $D(S(I^N)) = 2$. The story of this result started with J. Schreier and S. Ulam [32] and W. Sierpiński [35] and finished with H. Cook and W. Ingram [5] and S. Subbiah [37]. From the last two papers it follows that $D(S(X)) = 2$ not only for I^N but also for the Cantor discontinuum and, among noncompact spaces, the countably infinite discrete space, the space of rationals and the space of irrationals. Any time X is separable and metrizable, $S(X)$ will be separable. However, it can happen that $D(S(X))$ is infinite. This is in particular true for any Euclidean N-sphere: $D(S(\mathbb{S}^N)) = \infty$, see [28]. If the density index of $S(X)$ is two, which is the case of special interest, we have two generators f and φ for a dense subsemigroup of $S(X)$. One can of course ask whether in such a case there is a dense *set* in $S(X)$ containing only elements of very simple form, say $f^n \circ \varphi$, $n = 0, 1, 2, \dots$. This set is the *orbit* of φ in $(S(X), F_f)$, where $F_f : S(X) \to S(X)$ is defined by $F_f(\psi) = f \circ \psi$ (first apply ψ) for any $\psi \in S(X)$. Here $(S(X), F_f)$ can be viewed as an unary topological algebra or as a dynamical system.

Once we came to the problem of the existence/nonexistence of dense orbits in the dynamical system $(S(X), F_f)$ it is natural to study the dynamics in this system as a whole and not to restrict ourselves only to the problem directly related to the problem of the density index of the semigroup $S(X)$.

Functional difference equations. Consider a functional difference equation of the form

$$x(t+1) = f(x(t)), \qquad t \geq 0,$$

where f is a function from a compact real interval I into itself. Solutions of this equation are functions $x : [0, \infty) \to I$. Every solution is uniquely determined by its values on the initial interval $[0, 1)$. In fact, for any function

$\varphi : [0, 1) \to I$, the initial condition

$$x(t) = \varphi(t), \qquad \text{for } t \in [0, 1)$$

defines a unique solution x_φ which can be described by using the iterations of f:

$$x_\varphi(t + i) = f^i(\varphi(t)) \qquad \text{for } t \in [0, 1),\ i = 0, 1, 2, \dots .$$

We will assume that f is continuous. Then, the solution x_φ is continuous if and only if φ is continuous and satisfies the condition $\lim_{t\to 1^-} \varphi(t) = f(\varphi(0))$. Thus, we can describe all continuous solutions by

$$x_\varphi(t + i) = f^i(\varphi(t)) \qquad \text{for } t \in [0, 1],\ i = 0, 1, 2, \dots$$

where $\varphi : [0, 1] \to I$ (defined on the *closed* interval [0,1]) is continuous and $\varphi(0) = \varphi(1)$. As it was observed in the works by A. N. Sharkovsky et al. (see, e.g., [33], [34]), in a natural way we have obtained a dynamical system given by the space of all continuous maps $\varphi : [0, 1] \to I$ with the property $\varphi(0) = \varphi(1)$ and a continuous selfmap F_f of this space defined by $F_f(\varphi) = f \circ \varphi$. Obviously, the investigation of the asymptotic behavior of the solution given by the initial condition φ is equivalent with the investigation of the asymptotic behavior of the trajectory $\varphi, f \circ \varphi, f^2 \circ \varphi, \dots$ of the mentioned dynamical system. Of course, if one is going to study the dynamics in such systems, it seems too restrictive to consider only maps $\varphi : [0, 1] \to I$ with the property $\varphi(0) = \varphi(1)$. Instead, it would be natural to take any two compact metric spaces $(X; d_1)$ and $(Y; d_2)$ together with the set $C(X, Y)$ of all continuous maps from X to Y (endowed with a suitable metric), then to fix f, a continuous map $Y \to Y$, and to study the dynamics of the map F_f given on $C(X, Y)$ by $F_f(\varphi) = f \circ \varphi$. On the other hand, it turns out that in such a general formulation there is a weaker relation between the properties of f and F_f than in the case when $X = Y$ and so it seems reasonable to start the investigation with $X = Y$, especially if this case corresponds to the previous motivation.

Dynamical systems. Apparently the first consideration of dynamics on a space of functions was that of the Bebutov system. The space is the bounded uniformly continuous functions from the real to the complex numbers, provided with the compact open topology and made into a dynamical system by translation of functions (see [31] or [2] and for recent related results [24] and [23]).

The Ellis enveloping semigroup was defined later, and has proved to be very useful in the study of dynamical systems. Let X be a compact Hausdorff space and f be a continuous selfmap of X. The Ellis semigroup $E(f)$ is defined as the closure of the set of all iterates of f in the space X^X of all (not necessarily continuous) maps $X \to X$ with the product topology. As a general reference, see [7] or [2]. One can find interesting recent results in [10].

The main impulse for our study of functional envelopes were works by A. N. Sharkovsky, Yu. L. Maĭstrenko and E. Yu. Romanenko who, without assigning any name to them, studied the systems $(C([0,1],[a,b]),F_f)$ in the connection with their applications to difference equations as already indicated above. Our aim is to develop the theory of systems $(S(X),F_f)$ for compact dynamical systems (X,f). Of course sometimes we will need to restrict ourselves to more specific spaces X.

The present paper is closely related also to the recent notion of a quasi-factor. Let us recall the definition. Let Z be a compact metric space, 2^Z be the compact space of all closed subsets of Z with the Hausdorff topology. For a dynamical system (Z,T), where $T \in S(Z)$, a *factor map* is a continuous surjective map $\pi : Z \to Z'$ such that $S \circ \pi = \pi \circ T$ for a continuous map $S \in S(Z')$. For any $z \in Z'$ the set $\pi^{-1}(z)$ ranges over points of 2^Z and T acts on these points just like S does on Z'. In general a *quasi-factor* of (Z,T) is any subsystem of $(2^Z,T)$. Though there is a relation between dynamical properties of a system and its factor, this cannot in general be said on a system and some of its quasi-factors (see [4], [12], [8], [3], [9]).

Given a system (X,f), denote $R_X := \{\text{range}(\varphi) : \varphi \in S(X)\}$. Then (R_X,f) is one of the quasi-factors of (X,f). In fact, f sends the range of a map φ to the range of the map $f\circ\varphi$. The map $\varphi \mapsto \text{range}(\varphi)$ defines a homomorphism of $(S(X),F_f)$ onto (R_X,f), i.e., into (the compact space) $(2^X,f)$. Of course the dynamics of $(S(X),F_f)$ is much more interesting than that of (R_X,f). (For instance, if f and φ are surjective, range(φ) is a fixed point in (R_X,f).) Still, in Section 4 we will see that there is a connection between properties of $(S(X),F_f)$ and (R_X,f).

Let (X,g) and (Y,f) be dynamical systems, where $(X;d_1)$ and $(Y;d_2)$ are compact metric spaces, $g \in S(X)$, $f \in S(Y)$. Consider the product system $(X\times Y, g\times f)$. Of course, the map $g\times f$ may be applied also to any subset of $X\times Y$, in particular to graphs of continuous maps $X \to Y$. Denote by $\text{gr}(\varphi)$ the graph of a map $\varphi \in C(X,Y)$, i.e., $\text{gr}(\varphi) := \{(x,y) \in X\times Y : y = \varphi(x)\}$. The set $\text{Gr}(X,Y) := \{\text{gr}(\varphi) : \varphi \in C(X,Y)\}$ of the graphs of all continuous maps from X to Y is a subset of $2^{X\times Y}$. When g is a homeomorphism then one of the quasi-factors of the dynamical system $(X\times Y, g\times f)$ is the dynamical system $(\text{Gr}(X,Y), g\times f)$. In particular, if g is the identity map on X and we will make no distinction between maps and their graphs then the system we get is $(C(X,Y),F_f)$ where $F_f(\varphi) = f\circ\varphi$ and if $X = Y$ then we get $(S(X),F_f)$.

Taking into account all the said above, we come to the following

Definition 1.1. If (X,f) is a dynamical system given by a compact metric space X and a continuous map $f : X \to X$ then the *functional envelope* of (X,f) is the dynamical system $(S(X),F_f)$ whose phase space is the space of all continuous selfmaps of X with the compact-open topology (considered as a metric space with the uniform metric or with the Hausdorff metric applied to the graphs of maps) and the map $F_f : S(X) \to S(X)$ is defined by $F_f(\varphi) = f\circ\varphi$ for any $\varphi \in S(X)$.

Note that the functional envelope of a system always contains an isometric copy of the original system (see Proposition 2.4) — from this the name 'functional envelope' comes.

In the present paper we fix the following

> **Topic for the research:** Study the relations between the dynamical properties of a dynamical system (X, f) given by a compact metric space X and a continuous map $f : X \to X$ and its functional envelope $(S(X), F_f)$.

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2. First observations. Standing hypotheses.

If A is a set in a metric space, by $B_\varepsilon[A]$ or $\overline{B}_\varepsilon[A]$ we will denote the union of all open or closed, respectively, balls of radius $\varepsilon > 0$ whose centers run over A. If $A = \{a\}$, we write only a instead of $\{a\}$ in these notations.

Recall that the *Hausdorff distance* d_H between two sets A_1 and A_2 in a metric space X is given by

$$d_H(A_1, A_2) = \inf\{\varepsilon > 0 : \ \overline{B}_\varepsilon[A_1] \supseteq A_2 \text{ and } \overline{B}_\varepsilon[A_2] \supseteq A_1\} \ .$$

This is a metric on the family of all bounded, nonempty closed subsets of X. Note that if X is compact we can apply this metric to (the graphs of) continuous maps from X to X.

Given a compact metric space (X, d) and the set (semigroup) $S(X)$ of all continuous maps from X to X, we will consider the following metrics on this set:

- the metric d_U of uniform convergence: $d_U(\varphi, \psi) = \sup_{x \in X} d(\varphi(x), \psi(x))$. The corresponding space will be denoted by $S_U(X)$. Note that $d_U(\varphi, \psi)$ is well defined for any bounded selfmaps of X (in fact, for any selfmaps of X, since X is bounded).
- the Hausdorff metric d_H (derived from the metric $d_{\max}((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\}$ in $X \times X$) applied to the graphs of maps (we identify a map and its graph, so we will write $d_H(\varphi, \psi)$). The corresponding space will be denoted by $S_H(X)$.

For any $\varphi, \psi \in S(X)$ we have $d_H(\varphi, \psi) \le d_U(\varphi, \psi)$. If X is a compact metric space then the topologies given by the uniform metric and Hausdorff metric are equivalent in $S(X)$. In fact, a bit stronger result holds (see [30]) and for completeness we give here its proof:

Proposition 2.1 ([30])**.** *Let (Z, ϱ) be a compact metric space, (Z^*, ϱ^*) be any bounded metric space and $C(Z, Z^*)$ be the set of all continuous maps*

$Z \to Z^*$. Then the topologies given by the uniform metric $d_U(\varphi, \psi) = \sup_{z\in Z} \varrho^*(\varphi(z), \psi(z))$ and by the Hausdorff metric d_H (derived from the metric $d_{\max}((z_1, z_1^*), (z_2, z_2^*)) = \max\{\varrho(z_1, z_2), \varrho^*(z_1^*, z_2^*)\}$ in $Z \times Z^*$) applied to the graphs of maps are equivalent on $C(Z, Z^*)$.

Proof. Let $\varepsilon > 0$ and let for $\varphi \in C(Z, Z^*)$, $U_U(\varphi, \varepsilon) = \{\psi \in C(Z, Z^*) : d_U(\varphi, \psi) < \varepsilon\}$ and $U_H(\varphi, \varepsilon) = \{\psi \in C(Z, Z^*) : d_H(\varphi, \psi) < \varepsilon\}$. Clearly $U_U(\varphi, \varepsilon) \subseteq U_H(\varphi, \varepsilon)$. We now show that there exists a $\delta > 0$ such that $U_H(\varphi, \delta) \subseteq U_U(\varphi, \varepsilon)$. Since $\varphi \in C(Z, Z^*)$ and Z is compact, there exists a $\delta > 0$ ($\delta < \varepsilon/2$) such that for all $u, v \in Z$, $\varrho(u, v) < \delta$ implies $\varrho^*(\varphi(u), \varphi(v)) < \varepsilon/2$. Now let $\psi \in U_H(\varphi, \delta)$. Fix any $a \in Z$. Then there exists a $b \in Z$ such that $\varrho(a, b) < \delta$ and $\varrho^*(\psi(a), \varphi(b)) < \delta$. Thus $\varrho^*(\varphi(a), \psi(a)) \leq \varrho^*(\varphi(a), \varphi(b)) + \varrho^*(\varphi(b), \psi(a)) < \varepsilon/2 + \delta$. This shows that $\psi \in U_U(\varphi, \varepsilon)$. □

In particular, a subset of $S(X)$ is compact in $S_U(X)$ if and only if it is compact in $S_H(X)$. Nevertheless, in general the two metrics are not uniformly equivalent (two maps which are close each to other in the Hausdorff metric, may have a large distance in the uniform metric). Of course, on compact subsets of $S(X)$ they are uniformly equivalent. Finally, recall that the space $S_U(X)$ is complete but in general not totally bounded and the space $S_H(X)$ is totally bounded but in general not complete and so in general none of them is compact.

Proposition 2.2. *If X is a compact metric space and $f : X \to X$ is continuous, then the map $F : \varphi \mapsto f \circ \varphi$ is uniformly continuous both as a map $S_U(X) \to S_U(X)$ and as a map $S_H(X) \to S_H(X)$.*

Proof. Since f is uniformly continuous, the claim is obvious for the uniform metric. Only the uniform continuity of f and the definition of Hausdorff metric are needed to show that F_f is continuous as a map $S_H(X) \to S_H(X)$. □

By a *dynamical system* we mean a pair (X, f) where X is a (usually compact metrizable) topological space and f is a continuous map $X \to X$. Given a point $x \in X$, its *trajectory* is the sequence $x, f(x), f^2(x), \dots$ and its *orbit* is the set $\{x, f(x), f^2(x), \dots\}$.

¿From now on we will keep the following

> **Standing hypotheses and notation:** (X, d) is a compact metric space, $S_U(X)$ resp. $S_H(X)$ is the space of all continuous maps from X to X endowed with the metric d_U of uniform convergence resp. the Hausdorff metric d_H (applied to the graphs of maps). If the properties under consideration do not depend on which of the two metrics is used in $S(X)$, we will sometimes keep the notation $S(X)$ instead of $S_U(X)$ or $S_H(X)$. So, $S(X)$ will mean either of the two metric spaces rather than a topological space with compact-open topology.

The symbol f denotes a fixed continuous (hence uniformly continuous) map $X \to X$ and the symbol F (or, more precisely, F_f) denotes the (uniformly continuous) map from $S(X)$ to itself defined by $F(\varphi) = f \circ \varphi$. Thus, $(X; f)$ as well as $(S_U(X); F)$ and $(S_H(X); F)$ are dynamical systems.

Two dynamical systems (Y_1, g_1) and (Y_2, g_2) are called *topologically conjugate* if there is a homeomorphism $h : Y_1 \to Y_2$ with $h \circ g_1 = g_2 \circ h$. The systems $(S_U(X), F)$ and $(S_H(X), F)$ are obviously topologically conjugate (the conjugating homeomorphism being the identity). In fact, more is true:

Proposition 2.3. *Let X_i be a compact metric space and $f_i : X_i \to X_i$ be continuous, i=1,2. Assume that the systems (X_1, f_1) and (X_2, f_2) are topologically conjugate. Then the four systems $(S_U(X_1), F_{f_1})$, $(S_H(X_1), F_{f_1})$ $(S_U(X_2), F_{f_2})$ and $(S_H(X_2), F_{f_2})$ are all topologically conjugate.*

Proof. This is routine to check. The uniform continuity of the conjugating homeomorphism $X_1 \to X_2$ and its inverse are used. □

Given a point $a \in X$, the (constant) map in $S(X)$ sending all points from X to the point a will be denoted by con_a. In the present paper we mainly concentrate on the problem what is the relation between the dynamical properties of f and those of F. One thing is obvious: the dynamics of the system $(S(X), F)$ cannot be simpler than that of the system (X, f), because the system $(X; f)$ is present in systems $(S_U(X), F)$ and $(S_H(X), F)$ as their compact subsystem:

Proposition 2.4. *The system $(S(X), F)$ contains (properly, if $\operatorname{card} X \geq 2$) an isomorphic copy of (X, f).*

Proof. The map $\iota : X \to S(X)$ sending a point $a \in X$ to the constant map con_a, is an isometry (regardless of whether the uniform or the Hausdorff metric is used in $S(X)$) and also a topological conjugacy, i.e., $\iota \circ f = F \circ \iota$. Finally, if $\operatorname{card} X \geq 2$ then $S(X)$ contains also a nonconstant map (the identity). □

Let P be a property a dynamical system may or may not have. We say that P is *hereditary down* if all subsystems of a system having this property also have this property. We say that P is *hereditary up* if a system has this property whenever there is such a subsystem of the system which has this property. In view of the said above, at least *all properties which are hereditary down carry over from $(S_U(X), F)$ and $(S_H(X), F)$ to (X, f)* and at least *all properties which are hereditary up carry over from (X, f) to $(S_U(X), F)$ and $(S_H(X), F)$.*

The paper is organized as follows. In Section 3 we show that some (but not all) properties which are usually synonyms of simplicity of a system are carried over from a system to its functional envelope. In Section 4 we study orbit closures in the functional envelope of a system. We also obtain some results on minimal sets. Several properties of maps will be shown to be range properties, i.e., it depends only on the range of a map whether the map has or has not this property. The main result of Section 5 says that, under weak assumptions, the functional envelope of a system does not contain any dense orbit. Finally, in Section 6 we address the problem of the relation between topological entropy of a system and that of its functional envelope.

3. Properties related to the simplicity of a system

The *surjectivity* of a system (X, f) means that $f(X) = X$. The assumption of surjectivity of f sometimes is very natural (convenient) and can always be obtained, given compactness, by replacing X by its largest strongly f invariant subset if necessary. The *injectivity* of a system $(X; f)$ means that the map f is injective. The identity map $X \to X$ will be denoted by id_X and the restriction of a map g to a subset A of its domain by $g_{|A}$.

If $X = I := [0, 1]$ and $f \in C(I)$ is defined by $f(0) = 0$, $f(1/4) = f(3/4) = 1/2$ and $f(1) = 1$ and f is linear in between, then it is easy to see that there is no map $\varphi \in C(I)$ with $f \circ \varphi = id_I$. Hence, the surjectivity of f does not imply the surjectivity of F. The converse is true.

Proposition 3.1. *F is injective if and only if f is injective. Further, the following are equivalent:*

(a) *F is surjective,*
(b) *there is a map $\varphi \in S(X)$ with $F(\varphi) = \mathrm{id}_X$,*
(c) *there is a set $L \subseteq X$ such that $f_{|L}$ is a homeomorphism of L onto X.*

Proof. The claim on injectivity as well as (a)⇒(b) are trivial. To prove (b)⇒(c) assume that for some $\varphi \in S(X)$ we have

$$f(\varphi(x)) = x \qquad \text{for all } x \in X\ . \tag{3.1}$$

This implies that φ is injective and, since φ is continuous and the metric space X is compact, the set $L := \varphi(X)$ is a homeomorphic copy of X. By (3.1), $f_{|L} \circ \varphi = \mathrm{id}_X$ and so $f_{|L} : L \to X$ is a homeomorphism. To prove (c)⇒(a), take any $\psi \in S(X)$ and put $\varphi := (f_{|L})^{-1} \circ \psi$. Then $\varphi \in S(X)$ and $f \circ \varphi = \psi$, i.e., $F(\varphi) = \psi$. □

Corollary 3.2. *The following are equivalent:*

(a) *f is a homeomorphism,*
(b) *f is a uniform homeomorphism,*
(c) *F is a homeomorphism (both in $S_U(X)$ and $S_H(X)$),*
(d) *F is a uniform homeomorphism (both in $S_U(X)$ and $S_H(X)$).*

Proof. Since X is compact, we have (a)⇒(b). Now assume (b). In view of Proposition 3.1, F is a bijection and by Lemma 2.2, F is uniformly continuous. To get (d), we need to show that F^{-1} is uniformly continuous. Since obviously $F^{-1}(\psi) = f^{-1} \circ \psi$ for any ψ, we get the result from Lemma 2.2. The implication (d)⇒(c) is trivial and (c)⇒(a) follows from Proposition 3.1. □

Recall some definitions. A system (X, f) is *equicontinuous* if $\{f^n : n = 0, 1, 2, \dots\}$ is an equicontinuous family of maps, i.e., if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$ then $d(f^n(x), f^n(y)) < \varepsilon$ for $n = 0, 1, 2, \dots$. A system (X, f) is called *uniformly rigid* if f^{n_i} converges uniformly to id_X for some $n_i \to \infty$.

A pair x, y in (X, f) is called *asymptotic* if $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$. If all pairs are asymptotic then the system is called asymptotic. A pair (x, y) is called *proximal* if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and *distal* if it is not proximal. A system is called proximal if all pairs are proximal and it is called distal if all pairs with $x \neq y$ are distal.

The isometry, equicontinuity, uniform rigidity, distality, asymptoticity and proximality are all hereditary down and so they are always carried over from F to f (regardless of whether we have the uniform or the Hausdorff metric on $S(X)$). For the direction from f to F we have the following proposition and two (counter)examples.

Proposition 3.3. (a) *If f is an isometry on X then F is an isometry both on $S_U(X)$ and $S_H(X)$.*
(b) *If f is equicontinuous on X then F is equicontinuous both on $S_U(X)$ and $S_H(X)$.*
(c) *If f is uniformly rigid on X then F is uniformly rigid both on $S_U(X)$ and $S_H(X)$.*
(d) *If f is distal on X then F is distal on $S_U(X)$.*

Proof. The first three parts are obvious. In (d), if (φ, ψ) were a proximal pair for F, then for every $a \in X$, $(\varphi(a), \psi(a))$ would be a proximal pair for f. □

Example 3.4. *There is a surjective asymptotic system (X, f) such that the corresponding systems $(S_U(X), F)$ and $(S_H(X), F)$ contain distal pairs (and so are not even proximal.*

The set $X = \{0\} \cup \{2^{-n} : n = 0, 1, 2, \dots\}$ is a compact metric space with the Euclidean metric. Let f be a continuous surjective selfmap of X defined by $f(0) = f(1) = 0$ and $f(2^{-n}) = 2^{-n+1}$ for $n = 1, 2, 3, \dots$ (so, X is a homoclinic orbit of f). Since every point in the system (X, f) is eventually mapped to the fixed point 0, the system is asymptotic. Nevertheless, the maps con_0 and id_X form a distal pair for F. In fact, $F^k(\mathrm{con}_0)(x) = f^k \circ \mathrm{con}_0(x) = 0$ for all $x \in X$, while for every $k = 0, 1, 2, \dots$ we have $F^k(\mathrm{id}_X)(2^{-k}) = f^k \circ \mathrm{id}_X(2^{-k}) = f^k(2^{-k}) = 1$. Thus, for all k, $d_U(F^k(\mathrm{con}_0), F^k(\mathrm{id}_X)) = d_H(F^k(\mathrm{con}_0), F^k(\mathrm{id}_X)) = 1$. □

If two metrics are uniformly equivalent then a system is distal with respect to one of them if and only if it is distal with respect to the other. Nevertheless, the distality is not a topological property – on noncompact topological spaces it depends on the metric. Therefore it is not surprising that (d) does not work for $S_H(X)$.

Example 3.5. *There is a distal system (X, f) such that $(S_H(X), F)$ contains asymptotic pairs.*

To show this let (X, f) be the unit disk rotated at different rates around the center. Precisely, define X in the complex notation and f in polar coordinates as follows:

$$X = \{z \in \mathbb{C} : \ |z| \leq 1\},$$
$$f(r, \vartheta) = (r, \vartheta + r) \ .$$

Clearly, this is a distal system (notice that the homeomorphism f is not equicontinuous). We are going to prove that any two rotations of X are asymptotic in $(S_H(X), F)$.

Consider $\alpha \in \mathbb{R}$ and the rotation

$$\mathrm{rot}_{\alpha}(r, \vartheta) = (r, \vartheta + \alpha) \ .$$

Call a *cell* any subset of X of the form

$$C = \{(r, \vartheta) : \ r_1 \leq r \leq r_2, \ \vartheta_1 \leq \vartheta \leq \vartheta_2\}$$

which has nonempty interior in X. We will say that the annulus $\{(r, \vartheta) : r_1 \leq r \leq r_2\}$ is given by the cell C.

If we apply the rotation rot_α to a cell we again get a cell but the f-image of a cell is not a cell in general. Nevertheless, given a cell $C = \{(r, \vartheta) : \ r_1 \leq r \leq r_2, \ \vartheta_1 \leq \vartheta \leq \vartheta_2\}$ there is always n_0 (depending on C) such that for every $n \geq n_0$ the set $f^n(C)$, a subset of the annulus given by C, has the full ϑ-projection, i.e., $f^n(C)$ intersects every radius $\vartheta =$ const (to see this, realize that $f^i(C)$ is connected for every i and that the rate of the 'rotation' f is an increasing function of r, so the points on the circle with larger radius r_2 'overrun' those on the circle with smaller radius r_1).

Now we are ready to show that, given $\alpha, \beta \in \mathbb{R}$, the rotations $\mathrm{rot}_\alpha, \mathrm{rot}_\beta$ are asymptotic in $(S_H(X), F)$. Fix $\varepsilon > 0$. It is sufficient to show that for all sufficiently large n's, the Hausdorff distance of the graphs of $f^n \circ \mathrm{rot}_\alpha$ and $f^n \circ \mathrm{rot}_\beta$ is smaller than ε.

To this end, partition the disk X by finitely many circles $r =$ const and finitely many radii $\vartheta =$ const into cells with diameters smaller than ε. Denote this family of cells by $\mathcal{C}$. Since $\mathcal{C}$ is finite and we have only two rotations, from what was said above we get that there is $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$ and for every cell $C \in \mathcal{C}$, both $f^n \circ \mathrm{rot}_\alpha(C)$ and $f^n \circ \mathrm{rot}_\beta(C)$, subsets of the annulus given by the cell C, have full ϑ-projections. This implies the following: If we take any $n \geq N_0$ and any point $(A, f^n \circ \mathrm{rot}_\alpha(A))$ from the graph of the map $f^n \circ \mathrm{rot}_\alpha$ then, if we denote by C a cell from $\mathcal{C}$ containing the point A, there is a point $B \in C$ such that the point $(B, f^n \circ \mathrm{rot}_\beta(B))$ from the graph of the map $f^n \circ \mathrm{rot}_\beta$ has the property that the ϑ-projection of $f^n \circ \mathrm{rot}_\beta(B)$ is the same as the ϑ-projection of $f^n \circ \mathrm{rot}_\alpha(A)$. Since the distance of the points A and B is at most the diameter of the cell C (which is smaller than ε) and the distance of the points $f^n \circ \mathrm{rot}_\alpha(A)$ and $f^n \circ \mathrm{rot}_\beta(B)$ (both lying in the annulus given by C and having the same

ϑ-projection) is again smaller than ε, we have shown that for every point from the graph of $f^n \circ \mathrm{rot}_\alpha$ there is a point from the graph of $f^n \circ \mathrm{rot}_\beta$ which is closer to it than ε. Since in the above argument we could replace the roles of α and β, the Hausdorff distance of the (graphs of) maps $f^n \circ \mathrm{rot}_\alpha$ and $f^n \circ \mathrm{rot}_\beta$ is smaller than ε. □

Nevertheless, *on the interval* (c) works also for the Hausdorff metric:

Proposition 3.6. *Let $X = I$ be a real compact interval. If f is distal on I then F is distal on $S_H(I)$.*

Proof. If an interval map f is distal, then $f^2 = \mathrm{id}_I$ and so the trajectory of φ under F is a periodic sequence $\varphi, f \circ \varphi, \varphi, f \circ \varphi, \dots$. Since f is injective, this implies that F is distal on $S_H(I)$. □

Contrary to the general case, on the interval the proximality and the asymptoticity are carried over from f to F. In fact, for the interval maps we have even the equivalence of the following conditions:

Proposition 3.7. *Let $X = I$ be a real compact interval. Then the following are equivalent:*

(1) *f is asymptotic,*
(2) *f is proximal,*
(3) *F is asymptotic on $S_U(I)$,*
(4) *F is proximal on $S_U(I)$,*
(5) *F is asymptotic on $S_H(I)$,*
(6) *F is proximal on $S_H(I)$.*

Proof. $I^* := \bigcap_{n=0}^\infty f^n(I)$ is a singleton or a nondegenerate compact interval. Moreover, $f(I^*) = I^*$ and so if f is proximal, then I^* is necessarily a singleton. For if it is a compact interval then f, being surjective on I^* has at least two periodic points, a contradiction with proximality. Thus if f is proximal, then I^* is a singleton and then f is asymptotic. Moreover, if I^* is a singleton, one can easily see that $\operatorname{diam} f^n(I) \to 0$ when n tends to infinity and hence F is asymptotic. The other implications needed for the completion of the proof are trivial (use also that $d_H(\varphi, \psi) \leq d_U(\varphi, \psi)$ and that the considered properties are hereditary down). □

4. Orbit closures, ω-limit sets and range properties

If (Y, g) is a dynamical system (with Y not necessarily compact) then a subset M of Y is called a *minimal set* if M is closed, nonempty and invariant (i.e., $g(M) \subseteq M$), and if M has no proper subsets with these three properties. Note that a nonempty subset M of Y is minimal if and only if it is the orbit closure of each of its points (i.e., the orbit of every point in M is a dense subset of M). If the whole space Y is a minimal subset of Y then the system (Y, g) is called *minimal*. So, minimality of a system means the density of all orbits. Note that the density of an orbit does not depend on whether we consider the uniform or the Hausdorff metric in $S(X)$.

If $\operatorname{card} X \geq 2$ then the system $(S(X), F)$ cannot have *all* orbits dense, i.e., it cannot be minimal. In fact, the F-orbit of a constant map is a subset of the family of constant maps and so it cannot approach the map id_X. (If X is a singleton we do have minimality.) To describe minimal sets in a functional envelope is much more difficult. Of course, the map F has, as its minimal sets, at least the copies of the minimal sets of f. Further, realize that a minimal set is either finite (a periodic orbit) or uncountable. Again, due to the equivalence of the two considered metrics in $S(X)$, it is sufficient to prove this in $S_U(X)$. So, let M be a minimal set in $S_U(X)$ which is not a periodic orbit, and let $\varphi \in M$. Then $M = \overline{\mathrm{orb}_F(\varphi)}$. For every k, the closure of the set $\{F^k(\varphi), F^{k+1}(\varphi), \dots\}$ is closed, nonempty and invariant subset of M and so φ belongs to it. Hence φ is recurrent. It follows immediately that all $F^n(\varphi)$ are recurrent, and, since every $\psi \in M$ is in the orbit closure of φ, then M has no isolated points. Thus M is a Polish space (since it is a closed subset of the Polish space $S_U(X)$) with no isolated points, and hence uncountable. In particular, an infinite orbit cannot be a minimal set. [1]

Since a minimal set is the orbit closure of each of its points, it is obvious that the study of orbit closures is not only interested in itself but also could help to understand minimal sets in functional envelopes. This section is therefore devoted to orbit closures. We also obtain some results on minimal sets.

Definition 4.1. Let P be a property a map from $S(X)$ may or may not have. It is said to be a *range property* if

$$\operatorname{range}\theta = \operatorname{range}\varphi \Longrightarrow (\varphi \text{ has } P \Leftrightarrow \theta \text{ has } P)$$

and it is said to be a *range down property* if

$$\operatorname{range}\theta \subseteq \operatorname{range}\varphi \Longrightarrow (\varphi \text{ has } P \Rightarrow \theta \text{ has } P).$$

Obviously, a range down property is a range property.

Throughout the paper, orbit closure always means positive orbit closure. The orbit of φ in the system $(S(X), F_f)$ has the form

$$\mathrm{orb}_F(\varphi) := \{\varphi, f \circ \varphi, f^2 \circ \varphi, \dots\}$$

and since $S(X)$ is in general not compact, the orbit closure $\overline{\mathrm{orb}_F(\varphi)}$ may not be compact either. (Since compactness is a topological property, we may equally well use either of the two metrics.) The following proposition gives a criterium for the compactness of an orbit closure. Before stating it, we need to localize the definition of equicontinuity. Given a metric space (X, d) and a family $\mathcal{G}$ of continuous selfmaps of X we say that $\mathcal{G}$ is *equicontinuous on a set* $A \subseteq X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in A$ with $d(x, y) < \delta$ and for every $g \in \mathcal{G}$ we have $d(g(x), g(y)) < \varepsilon$. If $A = X$ we simply say that $\mathcal{G}$ is equicontinuous. By saying that a map g is equicontinuous (on A) we will mean that the family of all its

[1] In the case of a homeomorphism, if one considers the full orbits, a discrete full orbit can be minimal.

iterates $\{g, g^2, \dots\}$ is equicontinuous (on A). Of course, if $B \subseteq A$ and G is equicontinuous on A then it is equicontinuous on B.

Note that if A is compact then for the equicontinuity of $\mathcal{G}$ on A it is sufficient and necessary that $\mathcal{G}$ be equicontinuous (relative to the set A) at every point $a \in A$. This means that, given $\varepsilon > 0$, there is a $\delta > 0$ (possibly depending on a) such that $d(g(x), g(a)) < \varepsilon$ whenever $d(x, a) < \delta$, $x \in A$ and $g \in \mathcal{G}$.

Lemma 4.2. *Let X and Y be metric spaces, X compact, and let $\varphi : X \to Y$ be continuous and surjective. Let $a \in Y$ and let $\delta > 0$. Then there is an $\eta > 0$ such that $B_\eta(a) \subseteq \cup_{x\in\varphi^{-1}(a)}\varphi(B_\delta(x))$.*

Proof. This is just a reformulation of the obvious fact that if U is a neighborhood of the set $\varphi^{-1}(a)$ in the compact space X then $\varphi(U)$ is a neighborhood of the point a. □

Theorem 4.3. *Let $\varphi \in S(X)$. Then the following are equivalent:*

(a) *the orbit closure $\overline{\mathrm{orb}_F(\varphi)}$ is compact,*
(b) *the family of maps $\{\varphi, f \circ \varphi, f^2 \circ \varphi, \dots\}$ is equicontinuous on X,*
(c) *the map f is equicontinuous on* $\mathrm{range}(\varphi)$.

Proof. The equivalence of (a) and (b) follows from Arzelà - Ascoli theorem and the implication (c) $\implies$ (b) is trivial. Assume (b). To get (c) it is sufficient to show that the family of iterates of f is equicontinuous (relative to the set $\mathrm{range}(\varphi)$) at every point of $\mathrm{range}(\varphi)$. So let $a \in \varphi(X)$. Let $\varepsilon > 0$, and let $\delta > 0$ be such that if $d(x_1, x_2) < \delta$, then $d(f^k(\varphi(x_1)), f^k(\varphi(x_2))) < \varepsilon$ for $k = 0, 1, 2, \dots$. Now let $\eta > 0$ (we may assume $\eta < \varepsilon$) correspond to δ as in Lemma 4.2 (with $Y = \varphi(X)$). Let $y \in \varphi(X)$ with $d(a, y) < \eta$. By the lemma $a = \varphi(x)$, $y = \varphi(x')$ with $d(x, x') < \delta$. Then $d(f^k(\varphi(x)), f^k(\varphi(x'))) < \varepsilon$, that is, $d(f^k(a), f^k(y)) < \varepsilon$ for all $k = 0, 1, 2, \dots$. □

Corollary 4.4. *The compactness of an orbit closure is a range down property. In particular, if the orbit closure of* id_X *is compact then the orbit closures of all maps are compact.*

By combining Corollary 4.4 with Proposition 3.3 we immediately get

Corollary 4.5. *The following are equivalent:*

(a) *f is equicontinuous,*
(b) *F is equicontinuous,*
(c) *orbit closure of* id_X *is compact,*
(d) *orbit closures of all maps are compact.*

Equicontinuous systems are of course very particular ones and have nice properties. It is a folk result (see, e.g., [1] and [17]) that if (X, f) is a compact surjective and equicontinuous system, then f is a homeomorphism, all orbit closures are minimal and (X, f^{-1}) is equicontinuous. In the case of a not necessarily compact metric space Y and a not necessarily surjective equicontinuous map $g : Y \to Y$ we have (see, e.g., [25]) that all nonempty

ω-limit sets in (Y, g) are minimal sets and the orbit closure of a point is a minimal set if and only if that point is recurrent for g. Thus, in view of the previous corollary we immediately get

Corollary 4.6. *If f is equicontinuous then*

(a) *all nonempty ω-limit sets in $(S(X), F)$ are minimal sets,*
(b) *the orbit closure $\overline{\mathrm{orb}_F(\varphi)}$ is a minimal set if and only if φ is a recurrent point for F.*

Note also that a full characterization of equicontinuous selfmaps f on a real compact interval and, more generally on a connected graph is known, see [25] and references therein.

The proof of Theorem 4.3 can be easily adapted to show that non-empty ω-limit sets are tied to equicontinuity of some sequence of powers of f:

Theorem 4.7. *Let $\varphi \in S(X)$. Then the following are equivalent*

(a) $\omega_F(\varphi) \neq \emptyset$,
(b) *there is a sequence $n_i \to \infty$ such that the family $\{f^{n_1} \circ \varphi, f^{n_2} \circ \varphi, \dots\}$ is equicontinuous on X.*
(c) *there is a sequence $n_i \to \infty$ such that the family of iterates f^{n_i} is equicontinuous on* range(φ).

Proof. The condition (a) implies that for some n_i we have $f^{n_i} \circ \varphi \to \psi \in S(X)$. Thus, the sequence $f^{n_i} \circ \varphi$ uniformly converges which easily implies (b). Similarly as in the proof of Theorem 4.3 we get that (b) implies (c). Conversely, (c) trivially implies (b). Finally, (b) gives that the closure of the set $\{f^{n_1}\varphi, f^{n_2}\varphi, \dots\}$ is compact in $S(X)$. Hence the sequence $f^{n_i}\varphi$ contains a convergent subsequence and so we get (a). □

Corollary 4.8. *Having a nonempty ω-limit set is a range down property. In particular, if $\omega_F(\mathrm{id}_X) \neq \emptyset$ then all ω-limit sets are nonempty.*

We have two important range down properties: compactness of an orbit closure and nonemptyness of an ω-limit set. The former trivially implies the latter but the converse is not true. In fact, even a minimal set in $(S(X), F)$ may not be compact, see Corollary 4.21.

Corollaries 4.4 and 4.8 (and, as we will see later, also other consequences) follow easily also from the following theorem. In its proof we use the fact that there are topologically transitive uniformly rigid non-minimal (hence non-equicontinuous) systems. Examples are provided by systems which are almost equicontinuous but not equicontinuous. (Almost equicontinuous systems were introduced in [18] and analyzed in [1].) On the other hand, any equicontinuous surjective system on a compact metric space is uniformly rigid. This follows from the classical theorem of Gottschalk, proved for instance in [2, p. 36], that for surjective systems, equicontinuity is equivalent with the uniform almost periodicity (if $\varepsilon > 0$ there is a syndetic set A such that $d(f^a(x), x) < \varepsilon$ for all $a \in A$ and all $x \in X$).

Theorem 4.9. *Let $\varphi, \theta \in S(X)$ with* $\mathrm{range}(\theta) \subset \mathrm{range}(\varphi)$. *Then the orbit map $F^j(\varphi) \mapsto F^j(\theta)$, $j = 0, 1, 2, \dots$ is uniformly continuous with respect to the uniform metric and therefore extends to a continuous map π from $\overline{\mathrm{orb}_F(\varphi)}$ into $\overline{\mathrm{orb}_F(\theta)}$. The map π is equivariant, that is $\pi \circ F = F \circ \pi$. The map π is not necessarily onto. If $\overline{\mathrm{orb}_F(\varphi)}$ is compact then π is onto and so the system $(\overline{\mathrm{orb}_F(\theta)}, F)$ is (also compact and) a factor of the system $(\overline{\mathrm{orb}_F(\varphi)}, F)$.*

Proof. Let $x \in X$ and let $y \in X$ be a point with $\theta(x) = \varphi(y)$. Then $d(f^n(\theta(x)), f^m(\theta(x))) = d(f^n(\varphi(y)), f^m(\varphi(y))) \le d_U(F^n(\varphi), F^m(\varphi))$. Since this holds for all x we have $d_U(F^n(\theta), F^m(\theta)) \le d_U(F^n(\varphi), F^m(\varphi))$, which proves uniform continuity.

Trivially, $\pi \circ F = F \circ \pi$ on $\mathrm{orb}_F(\varphi)$. By continuity, $\pi \circ F = F \circ \pi$ on $\overline{\mathrm{orb}_F(\varphi)}$. We are going to give an example showing that π need not be surjective (in particular, $\pi(\overline{\mathrm{orb}_F(\sigma)})$, the continuous image of a closed set, is an (invariant) subset of $\overline{\mathrm{orb}_F(\theta)}$ that need not be closed).

Consider a topologically transitive uniformly rigid non-minimal system. Let $x_0 \in X$ be such that its orbit closure is not minimal. Let $\theta \in S(X)$ be the constant map $\theta(x) = x_0$. Then $\mathrm{range}(\theta) \subset \mathrm{range}(\mathrm{id})$. As it will be seen from Corollary 4.19, $\overline{\mathrm{orb}_F(\mathrm{id})}$ is minimal. Since $\overline{\mathrm{orb}_F(\theta)}$ is not minimal the map $\pi : \overline{\mathrm{orb}_F(id)} \to \overline{\mathrm{orb}_F(\theta)}$ cannot be surjective. In fact a factor of a minimal flow is minimal (even if the phase spaces are not compact).

Finally assume that $\overline{\mathrm{orb}_F(\varphi)}$ is compact. Then $\pi(\overline{\mathrm{orb}_F(\varphi)})$ is also a compact set and since it contains $\mathrm{orb}_F(\theta)$, it equals $\overline{\mathrm{orb}_F(\theta)}$. Hence we get surjectivity of π. □

Corollary 4.10. *Recurrence is a range down property. In particular, if* id_X *is recurrent then all maps are recurrent.*

Proof. Suppose $\varphi, \theta \in S(X)$ with $\mathrm{range}(\theta) \subset \mathrm{range}(\varphi)$, and $\lambda \in \omega_F(\varphi)$. Then $F^{n_i}(\varphi) \to \lambda$ for some sequence $n_i \to \infty$. Therefore $\pi(F^{n_i}(\varphi)) \to \pi(\lambda)$. By equivariance, $F^{n_i}(\theta) \to \pi(\lambda)$ so $\pi(\lambda) \in \omega(\theta)$. (Notice that this gives another proof of Corollary 4.8.) In particular, if $\varphi \in \omega_F(\varphi)$ then $\theta = \pi(\varphi) \in \omega_F(\theta)$. □

Theorem 4.11. *Let $\varphi, \theta \in S(X)$ with* $\mathrm{range}(\theta) = \mathrm{range}(\varphi)$. *Then $\overline{orb_F(\varphi)}$ and $\overline{orb_F(\theta)}$ are flow isomorphic.*

Proof. We have continuous equivariant maps $\pi : \overline{orb_F(\varphi)} \to \overline{orb_F(\theta)}$ and $\psi : \overline{orb_F(\theta)} \to \overline{orb_F(\varphi)}$ such that $\pi(\varphi) = \theta$ and $\psi(\theta) = \varphi$. It follows that $\pi\psi$ and $\psi\pi$ are the identity maps on their domains, and therefore that π and ψ are bijections. □

Corollary 4.12. *Minimality of orbit closures is a range property.*

Note that minimality of an orbit closure in $S(X)$ is *not* a range *down* property. To see this, consider a system $(S(X), F)$ in which the identity map has a minimal orbit closure (see Corollary 4.21) and take into account that the orbit closure of a constant map need not be minimal. On the other hand, if we add compactness to the minimality of an orbit closure we get a range down property:

Theorem 4.13. *The simultaneous compactness and minimality of an orbit closure is a range down property. In particular, if the orbit closure of* id_X *is compact and minimal then the orbit closures of all maps are compact.*

Proof. Suppose $\varphi, \theta \in S(X)$ with $\mathrm{range}(\theta) \subset \mathrm{range}(\varphi)$ and $\overline{\mathrm{orb}_F(\varphi)}$ is a compact minimal set. Since the factor of a minimal system is minimal, it suffices to use Theorem 4.9. □

We omit simple proofs of the two following lemmas:

Lemma 4.14. *Let $K(\alpha)$ be the set of all maps in $S(X)$ which commute with a given map $\alpha \in S(X)$. Then $K(\alpha)$ is closed and $K(\alpha) \subseteq K(\alpha^k)$ for all $k \geq 1$.*

Lemma 4.15. *Let φ, $\theta \in S(X)$, and suppose $f \circ \varphi = \varphi \circ f$, $f \circ \theta = \theta \circ f$. Then $d_U(f(\varphi), f(\theta)) \leq d_U(\varphi, \theta)$, with equality if f is surjective.*

Let (X, f) be a dynamical system, and let $Y \subset X$. We say that f *is uniformly rigid on* Y if f^{n_i} converges uniformly to the identity on Y for some $n_i \to \infty$. If $Y = X$ this coincides with the usual definition of uniform rigidity.

Theorem 4.16. *Let $\sigma \in S(X)$. Then the following are equivalent:*

(i) *σ is a recurrent point of the dynamical system $(S(X), F)$,*
(ii) *f is uniformly rigid on* $\mathrm{range}(\sigma)$.

Moreover, if $f \circ \sigma = \sigma \circ f$ then any of these conditions is equivalent with the condition

(iii) *The (positive) orbit closure of σ is minimal (hence $\omega(\sigma)$ coincides with the whole orbit closure of σ and so is nonempty).*

Proof. The equivalence of (i) and (ii) is clear. Assume (ii). Then there is a sequence $\{k_i\}$ with $k_i \to \infty$ such that $f^{k_i} \to \mathrm{id}$ uniformly on the range of σ. Equivalently, $F^{k_i}(\sigma) \to \sigma$. Now suppose that $\varphi \neq \sigma$ is in the orbit closure of σ, so $F^{s_i}(\sigma) \to \varphi$ for some sequence $s_i > 0$. By choosing a subsequence if necessary, we may suppose that $s_i < k_i$. Let $m_i = k_i - s_i$.

Since f commutes with σ, trivially f^{m_i} commutes with $f^{s_i} \circ \sigma$ for any $r \geq 0$. Moreover, by Lemma 4.14, f^{m_i} commutes also with φ. Hence

$$d_U(F^{m_i}(\varphi), F^{k_i}(\sigma)) = d_U(F^{m_i}(\varphi), F^{m_i+s_i}(\sigma)) \leq d_U(\varphi, F^{s_i}(\sigma)) \to 0$$

(the last inequality follows from Lemma 4.15). Thus

$$d_U(F^{m_i}(\varphi), \sigma) \leq d_U(F^{m_i}(\varphi), F^{k_i}(\sigma)) + d_U(F^{k_i}(\sigma), \sigma) \to 0$$

so σ is in the orbit closure of φ, which proves (iii).

Conversely, assume (iii). Then σ, as an element of a minimal set, is recurrent and so we have (i). □

Notice that Theorem 4.16 also implies Corollary 4.10.

Theorem 4.17. *Suppose f is surjective and $\varphi \in S(X)$ with $f \circ \varphi = \varphi \circ f$ and $\omega(\varphi) \neq \emptyset$. Then*

(i) *φ is recurrent,*
(ii) *all $\theta \in \omega(\varphi)$ are recurrent, and*
(iii) *if $\theta \in \omega(\varphi)$ then $\mathrm{range}(\theta) = \mathrm{range}(\varphi)$.*

Proof. In fact, we first prove (ii) and (iii). The statement (i) then follows from the assumption that $\omega(\varphi) \neq \emptyset$ and from (ii) and (iii) by Corollary 4.10.

To prove (ii) and (iii), fix $\theta \in \omega(\varphi)$. Since f commutes with φ, it commutes with all $f^k\varphi$. Then Lemma 4.14 gives that also $f \circ \theta = \theta \circ f$. Let $\varepsilon > 0$ Then there are n and $k > 0$ such that $d_U(f^n(\varphi), \theta) < \varepsilon$ and $d_U(f^{n+k}(\varphi), \theta) < \varepsilon$. Applying f^k to the first of these, we get, using Lemma 4.15, that $d_U(f^{n+k}(\varphi), f^k(\theta)) < \varepsilon$, so $d_U(f^k(\theta), \theta) < 2\varepsilon$ and we have (ii).

Since $f^{n_k}(\varphi) \to \theta$ uniformly on X (for some sequence $n_k \to \infty$) and f commutes with φ, we have $\mathrm{range}(\theta) \subseteq \mathrm{range}(\varphi)$. On the other hand, fix $a \in X$ and, by surjectivity of f, choose x_k with $f^{n_k}(x_k) = a$, $k = 1, 2, \ldots$. Without loss of generality we may assume that $x_k \to b$. Then $\varphi(f^{n_k}(x_k)) \to \theta(b)$ and so $\varphi(a) = \theta(b)$. Hence $\mathrm{range}(\varphi) \subseteq \mathrm{range}(\theta)$. We get (iii). $\square$

Corollary 4.18. *Suppose f is surjective, $\omega(\varphi) \neq \emptyset$ and there is $\psi \in S(X)$ satisfying $f \circ \psi = \psi \circ f$ and $\mathrm{range}(\psi) = \mathrm{range}(\varphi)$. Then φ is recurrent.*

Proof. By Corollary 4.8 nonempty ω-limit set is a range property. Thus $\omega(\psi) \neq \emptyset$ and so, by Theorem 4.17, ψ is recurrent. By Corollary 4.10 recurrence is a range property, so φ is recurrent. $\square$

Note that in Theorem 4.17 we cannot remove the assumption that f is surjective. Here is a counterexample: $X = I = [0, 1]$, $\varphi = \mathrm{id}_I$, $f(x) = x/2$, $x \in I$. Further note that if $A = \mathrm{range}(\varphi)$, a necessary condition that there exist a ψ from Corollary 4.18 is that $f(A) = A$.

Corollary 4.19. *Let φ be an arbitrary but fixed surjective map. The following are equivalent:*

(i) *$(X; f)$ is uniformly rigid,*
(ii) *id_X is recurrent,*
(iii) *$\omega(\mathrm{id}_X) \neq \emptyset$,*
(iv) *the orbit closure of id_X is minimal,*
(v) *φ is recurrent,*
(vi) *$\omega(\varphi) \neq \emptyset$,*
(vii) *the orbit closure of φ is minimal,*
(viii) *all maps are recurrent,*
(ix) *all ω-limit sets are nonempty,*
(x) *the orbit closures of all surjective maps are minimal.*

Proof. By the definition, (i)$\Leftrightarrow$(ii). By Corollary 4.20, (ii)$\Leftrightarrow$(iii). By Theorem 4.16, (ii)$\Leftrightarrow$(iv). To finish the proof, just use the facts that recurrence is a range (down) property (Corollary 4.10), having a nonempty ω-limit set is a range (down) property (Corollary 4.8) and minimality of orbit closures is a range property (Corollary 4.12). $\square$

Suppose f is surjective. Then by putting $\varphi = f$ we get from (vi)$\Rightarrow$(i) that if some sequence of powers of a surjective map f converges uniformly, then (X, f) is uniformly rigid. (This is apparently known.)

Implication weaker than (i) $\Longrightarrow$ (x), namely that (i) implies the minimality of the orbit closures of all homeomorphisms of X onto X, follows also from what has been known on uniformly rigid systems. In fact, if (X, f) is uniformly rigid, the closure of subgroup $\{f^n, n \in \mathbb{Z}\}$ in the group of homeomorphisms $H(X)$ is a Polish non-discrete topological group. The orbit closure of any $\varphi \in H(X)$ can be shown to be minimal by using the theory of rigid systems (see [11]).

On the other hand, in (i) $\Longrightarrow$ (x) we cannot remove the assumption of surjectivity in the condition (x). In fact, if we assume (i) and φ is not surjective then its orbit closure may not be minimal. For instance, we claim that for any uniformly rigid transitive, but not minimal dynamical system (X, f) there exists a continuous map $\psi \in S(X)$ the orbit closure of which is *not* minimal. For if $a \in X$ is transitive then the orbit closure of the constant map $\mathrm{con}_a \in S(X)$ is the set of all constant maps $\{\mathrm{con}_x : x \in X\}$. Since (X, f) is not minimal, neither is its isomorphic copy $(\{\mathrm{con}_x : x \in X\}, F)$.

In what follows a subset of X is said to be *discrete* if it has no accumulation point in X.

Corollary 4.20. *Suppose f is surjective. Let also φ be surjective. Then the orbit closure of φ is either minimal or discrete depending on whether $\omega(\varphi)$ is nonempty or empty.*

Proof. If $\omega(\varphi) \neq \emptyset$ then, by Corollary 4.8 and Corollary 4.19, $\overline{\mathrm{orb}_F(\varphi)}$ is minimal. If $\omega(\varphi) = \emptyset$ then $\overline{\mathrm{orb}_F(\varphi)} = \mathrm{orb}_F(\varphi)$ is discrete by definition. $\square$

We finish this section by showing that a minimal set in $(S(X), F)$ may not be compact. Recall that systems which are almost equicontinuous but not equicontinuous provide examples of systems considered in the following theorem:

Corollary 4.21. *If (X, f) is uniformly rigid non-equicontinuous system then the orbit closure of any surjective map is a minimal non-compact set.*

Proof. Let $\varphi \in S(X)$ be surjective. The orbit closure of φ is minimal by Corollary 4.19 and non-compact by Theorem 4.3. $\square$

5. Dense orbits and transitivity

A point with dense orbit is called a *transitive point.* The set of all such points in a system (Y, g) will be denoted by $\mathrm{Trans}(g)$. If $\mathrm{Trans}(g) = Y$, the system is called *minimal.* A system (Y, g) is called *(topologically) transitive* if for every pair of nonempty open sets U and V in X, there is a positive integer n such that $g^n(U) \cap V \neq \emptyset$. Neither transitivity nor density of an orbit in $S(X)$ depend on the choice of an equivalent metric.

If Y has no isolated point then the existence of a dense orbit implies transitivity and if Y is separable and second category then transitivity implies the existence of a dense orbit (see [36]). In particular, if the functional envelope $(S(X), F)$ of a compact system (X, f) is transitive, then it has a dense orbit. In fact, the compactness of X gives the separability of $S(X)$ and the space $S_U(X)$ is also complete and hence second category (thus, due to the equivalence of the uniform and Hausdorff metrics in $S(X)$, also $S_H(X)$ is second category though it is not complete).

We know (see the beginning of Section 4) that the systems $(S_U(X), F)$ and $(S_H(X), F)$ are minimal only if X is a singleton. On the other hand, for every compact metric space X (separable and metrizable is sufficient) the spaces $S_U(X)$ and $S_H(X)$ are separable and so it is not apriori excluded that $(S_U(X), F)$ and $(S_H(X), F)$ have *some* dense orbits even in the case $\operatorname{card} X \geq 2$.

The following example shows that dense orbits in a functional envelope may exist.

Example 5.1. *Functional envelope of the full shift on $A^{\mathbb{N}}$ contains dense orbits.*

Consider a dynamical system $(A^{\mathbb{N}}, \sigma)$ where A is a compact metric space and σ is the shift map, $\sigma(x_1, x_2, x_3, \dots) = x_2, x_3, \dots$, and its functional envelope $(S(A^{\mathbb{N}}), F_\sigma)$. Since $A^{\mathbb{N}}$ is a compact metric space, $S(A^{\mathbb{N}})$ is separable. Fix a dense sequence $\varphi_1, \varphi_2, \varphi_3, \dots$ in $S(A^{\mathbb{N}})$. For every k, express φ_k by using coordinate functions:

$$\varphi_k(x_1, x_2, x_3, \dots) = \left(g_1^{(k)}(x_1, x_2, x_3, \dots), g_2^{(k)}(x_1, x_2, x_3, \dots), \dots\right)$$

or, shortly, $\varphi_k = (g_1^{(k)}, g_2^{(k)}, \dots)$. Define the following element of $S(A^{\mathbb{N}})$:

$$\varphi = \left(g_1^{(1)}, g_1^{(1)}, g_2^{(1)}, g_1^{(2)}, g_2^{(2)}, g_1^{(1)}, g_2^{(1)}, g_3^{(1)}, g_1^{(2)}, g_2^{(2)}, g_3^{(2)}, g_1^{(3)}, g_2^{(3)}, g_3^{(3)}, \dots\right) .$$

Then obviously $\overline{\operatorname{orb}_{F_\sigma}(\varphi)} \supseteq \{\varphi_1, \varphi_2, \varphi_3 \dots\}$ and so the orbit of φ is dense. $\square$

So, a functional envelope $(S(X), F_f)$ of a system (X, f) may contain a dense orbit – the above example shows that this is possible for instance if X is a Cantor set (choose $A = \{0, 1\}$) or a Hilbert cube (choose $A = [0, 1]$). In spite of this fact, in the rest of this section we show that for many spaces X there are no dense orbits in the functional envelope $(S(X), F_f)$ regardless of the choice of f (see Theorem 5.6).

First notice that definitely there are such spaces X that $S(X)$ do not contain any dense orbit. In fact, from what we said in the introduction, we trivially have

Proposition 5.2. *Let X be a nondegenerate metric space with the density index of the semigroup $S(X)$ larger than two. Then there are no dense orbits in the functional envelopes $(S_U(X), F)$ and $(S_H(X), F)$. In particular, this is true for all Euclidean N-spheres.*

Still, what about if $D(S(X)) = 2$, for instance, if $X = I^N$? To give an answer, we start with

Theorem 5.3. *Let X be a compact metric space with* $\operatorname{card} X \geq 2$. *Let the orbit*

$$\operatorname{orb}_F(\varphi) := \{\varphi, f \circ \varphi, f^2 \circ \varphi, \dots\}$$

be dense in $S_U(X)$. Then

(a) *φ is injective but not surjective,*
(b) *f is surjective,*
(c) *$\varphi(X) \subseteq \operatorname{Trans}(f)$, hence f has a dense orbit,*
(d) *f is topologically transitive,*
(e) *if $\varphi(X)$ has nonempty interior in X then f is minimal.*

Proof. Suppose that there are $a \neq b$ in X with $\varphi(a) = \varphi(b)$. Then $f^k \circ \varphi(a) = f^k \circ \varphi(b)$ for all $k \geq 0$. Since every map in $\operatorname{orb}_F(\varphi)$ sends both the points a, b to the same point while the identity sends them to different points, the closure of $\operatorname{orb}_F(\varphi)$ in $S_U(X)$ does not contain the map $\operatorname{id}_X \in S_U(X)$. This proves the injectivity of φ.

Now suppose that f is not surjective. Then for all $k \geq 1$ the range of $f^k \circ \varphi$ is a subset of the closed set $f(X) \neq X$. Hence there is $\delta > 0$ such that $d_U(f^k \circ \varphi, \operatorname{id}_X) > \delta$ for $k = 1, 2, \dots$. Since id_X belongs to the closure of $\operatorname{orb}_F(\varphi)$, the only possibility is that $\operatorname{id}_X = \varphi$. In other words,

$$\operatorname{orb}_F(\operatorname{id}_X) := \{\operatorname{id}_X, f, f^2, \dots\}$$

is dense in $S_U(X)$ and for all $k \geq 1$, $f^k(X) \subseteq f(X) \neq X$. Take any $a \in X \setminus f(X)$. Then $d_U(f^k, \operatorname{con}_a) \geq d(a, f(X)) > 0$, $k = 1, 2, \dots$. Since also $d_U(\operatorname{id}_X, \operatorname{con}_a) > 0$ (note that $\operatorname{card} X \geq 2$), the constant map con_a cannot belong to the closure of $\operatorname{orb}_F(\operatorname{id}_X)$, a contradiction.

Since f is surjective, we claim that φ cannot be surjective. Otherwise all the maps in $\operatorname{orb}_F(\varphi)$ would be surjective and so the constant maps would not belong the closure of this orbit (again note that $\operatorname{card} X \geq 2$).

We are going to prove (c). If X is a singleton, this is trivial. So let $\operatorname{card} X \geq 2$. Fix $a \in X$ and $\varepsilon > 0$. Then there is k_a with $d_U(f^{k_a} \circ \varphi, \operatorname{con}_a) < \varepsilon$. Here $k_a \geq 1$ if ε is sufficiently small, because $d_U(\varphi, \operatorname{con}_a) > 0$ (note that φ is injective by (a) while con_a is not injective). Hence $f^{k_a}(\varphi(X)) \subseteq B(a, \varepsilon)$. Since a and ε were arbitrary, this implies that for every $x \in X$ the point $\varphi(x)$ has dense f orbit.

It is known that (b) and (c) imply (d) (see, e.g., [22]). Finally, (e) follows from the fact that if a transitive map is not minimal then it has a dense set of points with non-dense orbits, see [19] or [22]. ☐

Corollary 5.4. *If $(S(X), F)$ has a dense orbit then (X, f) has a dense orbit. If $(S(X), F)$ is transitive then (X, f) is transitive.*

Proof. The first part follows from Theorem 5.3(c). To prove the second part, take into account that transitivity of $(S(X), F)$ implies the existence of a dense orbit (see the beginning of this section) and use Theorem 5.3(d). □

It will be seen from Theorem 5.6 that the converse to this corollary is not true.

Lemma 5.5. *Let X be an infinite compact metric space. Let*

$$\mathrm{orb}_F(\varphi) := \{\varphi, f \circ \varphi, f^2 \circ \varphi, \dots\}$$

be dense in $S_U(X)$. Then

(a) *the sets $\varphi(X), f(\varphi(X)), f^2(\varphi(X)), \dots$ are proper subsets of X, homeomorphic to X, and f is injective on each of them,*
(b) *all the maps in $\mathrm{orb}_F(\varphi)$ are injective but not surjective on X.*

Proof. By Theorem 5.3 the map φ is injective but not surjective and so (b) follows from (a). We are going to prove (a).

We already know that φ injectively maps X onto $\varphi(X) \neq X$. We want to show that f is injective on $\varphi(X)$ and $f(\varphi(X)) \neq X$.

First suppose, on the contrary, that f is surjective on $\varphi(X)$. Then obviously all the maps $f \circ \varphi, f^2 \circ \varphi, \dots$ are surjective on X. Moreover, φ is injective and so different from constant maps. Hence no constant map belongs to the closure of $\mathrm{orb}_F(\varphi)$, a contradiction.

Now suppose, on the contrary, that f is not injective on $\varphi(X)$. Then there are $a \neq b$ in X such that $f(\varphi(a)) = f(\varphi(b))$ and hence also $f^k(\varphi(a)) = f^k(\varphi(b))$ for $k = 1, 2, \dots$. Then $d_U(f^k \circ \varphi, \mathrm{id}_X) \geq 1/2d(a, b) > 0$ for all $k = 1, 2, \dots$. Since φ is not surjective, we have also $d_U(\varphi, \mathrm{id}_X) > 0$ and so id_X does not belong to the closure of $\mathrm{orb}_F(\varphi)$, a contradiction.

Assume, by induction, that f is injective on each of the sets $\varphi(X)$, $f(\varphi(X))$, $\dots$, $f^{k-1}(\varphi(X))$ and that each of the sets $\varphi(X)$, $f(\varphi(X))$, $\dots$, $f^k(\varphi(X))$ is different from X. We want to prove that f is injective but not surjective on $f^k(\varphi(X))$.

First suppoose that f is surjective on $f^k(\varphi(X))$. Then all the maps in the set $S = \{f^{k+1} \circ \varphi, f^{k+2} \circ \varphi, \dots\}$ are surjective on are surjective on X. No constant map belongs to the closuer of S. Since $\mathrm{orb}_F(\varphi) \setminus S$ is finite and we have infinitely many constant maps, $\mathrm{orb}_F(\varphi)$ is not dense, a contradiction.

Now suppose that f is not injective on $f^k(\varphi(X))$. Then there are $a \neq b$ in X such that $f^{k+1}(\varphi(a)) = f^{k+1}(\varphi(b))$ and hence also $f^n(\varphi(a)) = f^n(\varphi(b))$ for $n = k+1, k+2, \dots$. Then $d_U(f^n \circ \varphi, \mathrm{id}_X) \geq 1/2d(a, b) > 0$ for all $n = k+1, k+2, \dots$. Since the maps $f^n \circ \varphi$ are not surjective for $n = 0, 1, \dots, k$, we have also $d_U(f^n \varphi, \mathrm{id}_X) > 0$ for all $n = 0, 1, \dots, k$ and so id_X does not belong to the closure of $\mathrm{orb}_F(\varphi)$, a contradiction. □

By saying that a continuous map $\eta \in S(X)$ is *stably noninjective* we mean that not only η but also all continuous selfmaps of X which are sufficiently close to η are noninjective. The equivalence of the uniform and Hausdorff metrics on $S(X)$ implies that the stable noninjectivity does not depend on the choice of the metric.

The following theorem shows that in many functional envelopes there are no dense orbits.

Theorem 5.6. *Let X be a nondegenerate compact metric space satisfying (at least) one of the following conditions:*

(a) *X admits a stably noninjective continuous selfmap.*

(b) *X contains no homeomorphic copy of X with empty interior in X.*

Then there are no dense orbits in the functional envelope $(S(X), F)$ (and hence the functional envelope is not transitive, see the beginning of this section).

Proof. We know that it is sufficient to work in $(S_U(X), F)$. We are going to prove the theorem if we have (a). First assume that X is finite. Then $S(X)$ is finite and so if $\mathrm{orb}_F(\varphi) := \{\varphi, f \circ \varphi, f^2 \circ \varphi, \dots\}$ is dense in $S_U(X)$, it must be $\mathrm{orb}_F(\varphi) = S(X)$. Moreover, if $f^k \circ \varphi$ is a constant map for some $k \geq 0$ then also the maps $f^n \circ \varphi$, $n \geq k$ are constant. Thus the map id_X appears in the trajectory $\varphi, f \circ \varphi, f^2 \circ \varphi, \dots$ *before* all the constant maps. But this is impossible because if $f^r \circ \varphi = \mathrm{id}_X$ for some $r \geq 0$, then the surjectivity of f (see Theorem 5.3(b)) implies that $f^i \circ \varphi$ is surjective for all $i \geq r$.

Now assume that X is infinite and $\mathrm{orb}_F(\varphi)$ is dense in $S_U(X)$. By Lemma 5.5 all the maps in $\mathrm{orb}_F(\varphi)$ are injective and so they cannot approach the stably noninjective map in $S(X)$.

Now assume that (b) is fulfilled and that $\mathrm{orb}_F(\varphi)$ is dense in X. By Theorem 5.3(d), f is topologically transitive and $\varphi(X)$ is a homeomorphic copy of X with all its points being transitive for f. Thus, by (b), f is a topologically transitive map with $\mathrm{int}(\mathrm{Trans}(f)) \neq \emptyset$ and so it is minimal (see [19] or [22]). Since $\varphi(X) \subseteq \mathrm{Trans}(f)$ has nonempty interior, there are x and $n > 0$ with $x, f^n(x)$ in $\varphi(X)$. Since f is minimal, the pair $(x, f^n(x))$ is distal for f (the proximality of this pair would easily imply the existence of a periodic point). Hence there is $\delta > 0$ such that for any $k \geq 0$, $\mathrm{range}(f^k \circ \varphi)$ has diameter $> \delta$. So, the closure of $\mathrm{orb}_F(\varphi)$ does not contain any constant map. $\square$

Not all nondegenerate spaces satisfy (a) or (b). Example 5.1 shows that for every nondegenerate compact metric space A, the space $A^{\mathbb{N}}$ satisfies neither (a) nor (b). In particular, this is true for the Cantor set and the Hilbert cube.

Notice that Theorem 5.6 applies to I^N and even to most of the 'usual' spaces. In fact, we have

Proposition 5.7. *Let X be a metric space satisfying (at least) one of the following:*

(a) *X is a nondegenerate space having an isolated point (this includes finite sets with more than one point),*

(b) *X is any space containing a homeomorphic copy J of $\mathbb{R}^n$ for some $n \geq 1$ such that J is open in X (this includes all manifolds),*
(c) *X is a rigid continuum.*

Then $S(X)$ contains a stably noninjective map.

Proof. In the case (a), the constant map sending all the points to a fixed isolated point is stably noninjective.

The proof in the case (b) follows with a little care from three facts: 1)J contains a homeomorphic copy K of an n-dimensional compact interval I^n, 2) I^n is an absolute retract, 3) from [29, Lemma 7] it follows that I^n admits a stably noninjective selfmap η.

In (c), obviously every constant map is stably noninjective. □

6. Topological entropy

Recall the Bowen-Dinaburg definition of topological entropy (see [38]).

Let (Z, ϱ) be a metric space and $f : Z \to Z$ be uniformly continuous. For any $n \geq 1$ the function

$$\varrho_n(x, y) := \max_{0 \leq j \leq n-1} \varrho(f^j(x), f^j(y))$$

defines a metric on Z equivalent with ϱ.

Fix $n \geq 0$ and $\varepsilon > 0$ and let K be a compact set in Z. A subset $E \subset K$ is called (n, f, ε)-*separated*, if for any two distinct points $x, y \in E$, $\varrho_n(x, y) > \varepsilon$. We say that a subset $F \subset Z$ (n, f, ε)-*spans* the set K, if for every point $x \in K$ there exists a point $y \in F$ such that $\varrho_n(x, y) \leq \varepsilon$. Note that since K is compact, the sets E and F are finite.

Denote by $\operatorname{sep}(n, f, \varepsilon; K)$ the maximal cardinality of a (n, f, ε)-separated set in K, and by $\operatorname{span}(n, f, \varepsilon; K)$ the minimal cardinality of a set which (n, f, ε)-spans K.

For every $\varepsilon > 0$ and $n \geq 0$ it holds

$$\operatorname{span}(n, f, \varepsilon; K) \leq \operatorname{sep}(n, f, \varepsilon; K) \leq \operatorname{span}(n, f, \varepsilon/2; K) .$$

The quantities $\log \operatorname{sep}(0, f, \varepsilon; K)$ and $\log \operatorname{span}(0, f, \varepsilon; K)$ are called ε-capacity and ε-entropy, respectively, of (K, ϱ) in the sense of Kolmogorov and Tikhomirov [20]. Thus, $\log \operatorname{sep}(n, f, \varepsilon; K)$ and $\log \operatorname{span}(n, f, \varepsilon; K)$ are the ε-capacity and ε-entropy of (K, ϱ_n).

The *topological entropy* $\operatorname{ent}(f, K)$ *on a compact set* $K \subset Z$, is defined by

$$\operatorname{ent}(f, K) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \operatorname{sep}(n, f, \varepsilon; K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \operatorname{span}(n, f, \varepsilon; K),$$

or, in other words,

$$\operatorname{ent}(f, K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} [\varepsilon\text{−capacity}(K, \varrho_n)] = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} [\varepsilon\text{−entropy}(K, \varrho_n)] .$$

Then the *topological entropy* $\operatorname{ent}(f)$ *of a map* $f : Z \to Z$ is defined by

$$\operatorname{ent}_{\varrho}(f) = \operatorname{ent}(f) := \sup\{\operatorname{ent}(f, K) : \ K \subset Z \text{ and } K \text{ is compact}\} .$$

If $\mathrm{span}^K(n, f, \varepsilon; K)$ denotes the minimal cardinality of subsets of the compact set K which (n, f, ε)-span K then $\mathrm{span}^K(n, f, \varepsilon; K) \le \mathrm{sep}(n, f, \varepsilon; K) \le \mathrm{span}^K(n, f, \varepsilon/2; K)$ and $\mathrm{ent}(f, K) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{span}^K(n, f, \varepsilon; K)$ (see [38]).

Recall that for uniformly equivalent metrics ϱ_1 and ϱ_2, $h_{\varrho_1}(X, f) = h_{\varrho_2}(X, f)$. If ϱ_1 is stronger than ϱ_2 then $\mathrm{ent}_{\varrho_1}(X, f) \ge \mathrm{ent}_{\varrho_2}(X, f)$ (see [38]).

We know that the map $F : \varphi \mapsto f \circ \varphi$ is uniformly continuous on both the spaces $S_U(X)$ and $S_H(X)$ and so we can study its topological entropy.

Recall that $d_U(\varphi_1, \varphi_2) \ge d_H(\varphi_1, \varphi_2)$ for all $\varphi_1, \varphi_2 \in S(X)$ and that these two metrics are equivalent on $S(X)$, hence uniformly equivalent on compact subsets of $S(X)$. Thus, if we denote the topological entropies of the systems $(S_U(X), F)$ and $(S_H(X), F)$ by $\mathrm{ent}_U(F)$ and $\mathrm{ent}_H(F)$, respectively, then

$$\mathrm{ent}\,_U(F) \ge \mathrm{ent}\,_H(F)\ .$$

We start with the investigation of the topological entropy of the system $(S_U(X), F)$.

Theorem 6.1. *There is a system (X, f) with zero topological entropy (even asymptotic one) such that the corresponding system $(S_U(X), F)$ has infinite topological entropy.*

Proof. The system from Example 3.4 will do. So, $X = \{0\} \cup \{2^{-n} : n = 0, 1, 2, \dots\}$ and $f(0) = f(1) = 0$ and $f(2^{-n}) = 2^{-n+1}$ for $n = 1, 2, 3, \dots$. The system (X, f) has zero topological entropy because the fixed point 0 is its only nonwandering point (the system is even asymptotic).

The set $X \times X$ is naturally embedded in the square $[0, 1]^2$ in the plane. Fix an integer $N \ge 2$. Let K be the set of continuous functions $X \to X$ whose graphs are subsets of $\bigcup_{i=0}^{N-1} L_i$ where L_i is the straight line going through the origin $(0, 0)$ and having the slope 2^{-i}. Then K is a compact subset of $S_U(X)$ [2]. (In fact, let a sequence $(g_n)_{n=1}^{\infty}$ of functions belonging to K be given. Take into account that for any $r \ge 1$, there is an infinite subsequence of $(g_n)_{n=1}^{\infty}$ such that all the maps in this subsequence coincide on the set $\{2^{-(r-1)}, 2^{-(r-2)}, \dots, 1\}$. Using this fact it is easy to see that $(g_n)_{n=1}^{\infty}$ has a uniformly convergent subsequence.)

Fix $\varepsilon < 1/2$ and $n \ge 1$. In each of the points $2^{-(n-1)}, 2^{-(n-2)}, \dots, 1$ we have N possibilities for the values of functions in K (because we have N lines L_i). So, there are N^n functions in K which are mutually different and whose graph on $\{0\} \cup \{2^{-i} :\ i \ge n\}$ is a subset of L_{N-1}. Take any two of these functions and denote them by φ and ψ. There is $k \in \{0, 1, \dots, n-1\}$ such that they differ at 2^{-k}, say, φ is larger than ψ at that point. So, by the definition of K,

$$\varphi\left(\frac{1}{2^k}\right) > \psi\left(\frac{1}{2^k}\right) \quad \text{belong to} \quad \left\{\frac{1}{2^j} :\ j = k, k+1, \dots, k+N-1\right\}\ .$$

[2] and hence also a compact subset of $S_H(X)$, since the two metrics are equivalent on $S(X)$

Then the definition of f shows that $f^i \circ \varphi(2^{-k}) = 1$ for some $0 \le k \le i \le k+N-2 \le n+N-3$. For this i we have $f^i \circ \psi(2^{-k}) \le 1/2$ and so we have proved that in K there are at least N^n functions which are $(n+N-2, \varepsilon)$-separated in $(S_U(X), F)$. for any $\varepsilon < 1/2$ and any $n \ge 1$. So,

$$\begin{aligned} \operatorname{ent}_U(F) \ge \operatorname{ent}_U(F,K) &= \lim_{\varepsilon\to 0} \limsup_{m\to\infty} \frac{1}{m} \log \operatorname{sep}(m,\varepsilon) \\ &\ge \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n+N-2} \log \operatorname{sep}(n+N-2,\varepsilon) \\ &\ge \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n+N-2} \log N^n = \log N \ . \end{aligned}$$

Since N was arbitrary, $\operatorname{ent}_U(F) = +\infty$. □

If $X = I = [0,1]$ we cannot have a proximal system (I, f) such that $(S_U(X), F)$ has positive topological entropy. In fact, we would have (see the discussion after Proposition 3.3) $\bigcap_{n=0}^{\infty} f^n(I) = \{x_0\}$ and so $\bigcap_{n=0}^{\infty} F^n(C(I)) = \{\mathrm{con}_{x_0}\}$. Hence $\operatorname{ent}_U(F) = 0$. Still, we have

Theorem 6.2. *There is a nondecreasing interval map f (hence $\operatorname{ent}(f) = 0$) such that the corresponding system $(S_U(X), F)$ has infinite topological entropy.*

Proof. Take the map f such that $f(x) = 2x$ for $x \in [0, 1/2]$ and $f(x) = 1$ for $x \in [1/2, 1]$. The dynamics of f on the set $X = \{0\} \cup \{2^{-n} : \ n = 0, 1, \dots\}$ is the same as that of the map from the proof of previous theorem except of the value at the point 1. Nevertheless, in the proof of the previous theorem this value was not important at all (the computation of $\operatorname{ent}_U(F)$ would be the same also with $f(1) = 1$).

Consider the family K of all continuous maps $\varphi : \ I \to I$ such that the graph of $\varphi|_X$ is a subset of $\bigcup_{i=0}^{N-1} L_i$ (L_i being the same as in the proof of previous theorem) and φ is linear in between. Then K is a compact subset of $S_U(I)$.

Now in the same way as in the proof of the previous theorem one can show that $\operatorname{ent}_U(F, K) \ge \log N$ and hence $\operatorname{ent}_U(F) = +\infty$. □

Now we are going to investigate the topological entropy on $S_H(X)$.

Contrary to the case of uniform metric, there are natural compact metric spaces containing $S_H(X)$. One can consider the compact metric space 2^{X^2} of all closed subsets of X with the Hausdorff metric or to take the smallest such space, i.e. the closure $\overline{S_H(X)}$ of $S_H(X)$ in 2^{X^2} or, finally, to consider some intermediate compact metric space between 2^{X^2} and $\overline{S_H(X)}$. To describe the latter spaces, we need some definitions.

Recall that if X and Y are topological spaces, a multifunction (i.e., a set-valued map) M from X to Y is called *upper semicontinuous* at $z \in X$ if whenever V is an open subset of Y that contains $M(z)$, then the set $\{x \in X : \ M(x) \subseteq V\}$ contains a neighbourhood of z. It is called *usco* (which stands for usc+co = upper semicontinuous + compact values) if it is upper semicontinuous at every $z \in X$ and for each $z \in X$, the set $M(z)$ is a *nonempty* compact subset of Y. If an usco multifunction has all values $M(x)$, $x \in X$, connected, then it is said to be *uscon* (which stands for usco + con = usco + connected values). If X and Y are compact metric spaces and if we identify multifunctions with their graphs, the Hausdorff metric can be applied to usco multifunctions similarly as to continuous maps. In fact, in this case a multifunction from X to Y with *closed values* is easily seen to be upper semicontinuous if and only if it is *closed*, which means that its graph is closed in $X \times Y$ (the implication from the right to the left uses compactness of Y). In our case $X = Y$ is a compact metric space and so we can consider the spaces $\mathrm{USCO}_H(X)$ and $\mathrm{USCON}_H(X)$ of all usco and uscon, respectively, multifunctions from X to X. The subscript H stands for the Hausdorff metric. Thus, the elements of $\mathrm{USCO}_H(X)$ can be alternatively described as those multifunctions from X to X which have closed graphs and have nonempty values. In the case of elements of $\mathrm{USCON}_H(X)$ we additionally demand that the values be connected.

Since X is a compact metric space, the space 2^X is compact and by [15, Theorem 1], $\mathrm{USCO}_H(X)$ is also compact. Since $S_H(X) \subseteq \mathrm{USCO}_H(X)$, we then have

$$S_H(X) \subseteq \overline{S_H(X)} \subseteq \mathrm{USCO}\,_H(X) \subseteq 2^{X^2} \ .$$

Further, by [14, Theorem 3], for the closure of $S_H(X)$ in 2^{X^2} we have $\overline{S_H(X)} \subseteq \mathrm{USCON}\,_H(X)$ provided X is locally connected. Thus, we have

$$S_H(X) \subseteq \overline{S_H(X)} \subseteq \mathrm{USCON}\,_H(X) \subseteq \mathrm{USCO}\,_H(X) \subseteq 2^{X^2} \qquad \text{if } X \text{ is l.c. .}$$

If $X = I := [0,1]$, then (see, e.g., [16]) *any* element of $\mathrm{USCON}_H(I)$ can be approximated (in the Hausdorff metric) by continuous maps $I \to I$:

$$S_H(I) \subseteq \overline{S_H(I)} = \mathrm{USCON}\,_H(I) \subseteq \mathrm{USCO}\,_H(I) \subseteq 2^{I^2} \qquad \text{if } I = [0,1] \ .$$

So, we want to study the entropy of F on $S_H(X)$ and we know that $S_H(X)$ is a subset of a larger *compact* metric space 2^{X^2}. Note that F naturally extends to $\overline{S_H(X)}$ and even to 2^{X^2}. For a compact set $A \subseteq X^2$ we put $F(A) = \{[x, f(y)] : \ [x,y] \in A\}$. Notice that if $\varphi \in S(X)$ then $F(\mathrm{gr}(\varphi)) = \mathrm{gr}(f \circ \varphi)$. Since we make no distinction between functions and their graphs, this is the same as $F(\varphi) = f \circ \varphi$ and so F defined on 2^{X^2} coincides with F previously defined on $S(X)$.

Therefore to prove that $\mathrm{ent}_H(F) > 0$ it is sufficient to find large enough (n, ε)-separated sets in $\overline{S_H(X)}$ (we use the fact that the entropy on a set and on its closure are the same, see e.g. [21]).

We start with an analog of Theorem 6.1 (we do not know whether (X, f) can be chosen to be asymptotic):

Theorem 6.3. *There is a system (X, f) with zero topological entropy such that the corresponding system $(S_H(X), F)$ has infinite topological entropy.*

Proof. Let us start with the asymptotic countable system (X, f) from the proof of Theorem 6.1. Fix $\varepsilon < 1/2$ and $n \in \mathbb{N}$. We are going to show that in $S_H(X)$ there is a $(2n-3, \varepsilon)$-separated set with cardinality 2^n.

Let S be the set of all functions $\varphi : X \to X$ with the following properties:

- $\varphi(x) = 0$ for $x \leq 1/2^n$ and
- $\varphi(1/2^k) \in \{1/2^{2k}, \dots, 1/2^{2k+1}\}$ for $k = 0, 1, \dots, n-1$.

Then $\operatorname{card} S = 2^n$. Take $\varphi, \psi \in S$, $\varphi \neq \psi$. There is $k \in \{0, 1, \dots, n-1\}$ such that φ differs from ψ at 2^{-k}, say φ is larger than ψ at that point. So,

$$\varphi\left(\frac{1}{2^k}\right) = \frac{1}{2^{2k}} > \frac{1}{2^{2k+1}} = \psi\left(\frac{1}{2^k}\right) .$$

Hence $f^{2k} \circ \varphi(2^{-k}) = 1$ and $f^{2k} \circ \psi(2^{-k}) = 1/2$. On the other hand, for all $x > 2^{-k}$ we have $f^{2k} \circ \varphi(x) = f^{2k} \circ \psi(x) = 0$ and for all $x < 2^{-k}$ we have that both $f^{2k} \circ \varphi(x)$ and $f^{2k} \circ \psi(x)$ are smaller than or equal to $1/4$. Hence the Hausdorff distance between $f^{2k} \circ \varphi$ and $f^{2k} \circ \psi$ is $1/2$. Since $2k \leq 2n-2$, the set S is $(2n-3, \varepsilon)$-separated. This gives

$$\begin{aligned}
\operatorname{ent}_H(F) &= \operatorname{ent}(F, S_H(X)) = \operatorname{ent}(F, \overline{S_H(X)}) \\
&= \lim_{\varepsilon \to 0} \limsup_{m \to \infty} \frac{1}{m} \log \operatorname{sep}(m, \varepsilon) \\
&\geq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{2n-3} \log \operatorname{sep}(2n-3, \varepsilon) \\
&\geq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{2n-3} \log 2^n = \frac{1}{2} \log 2 .
\end{aligned}$$

Once we know that for our zero entropy system (X, f) the entropy of $(S_H(X), F_f)$ is positive, it is easy to find a zero entropy system (Y, g) such that the corresponding system $(S_H(Y), G_g)$ has infinite entropy (here $G_g(\varphi) := g \circ \varphi$ for every $\varphi \in S_H(Y)$). First realize that $F_{f^n} = (F_f)^n$ and so, while all the systems (X, f^n) have zero entropy, all the systems $(S_H(X), F_{f^n})$ have positive entropy tending to infinity as $n \to \infty$.

Choose closed intervals I_n and J_n in such a way that for their lengths it holds $|I_n| = |J_n| \searrow 0$ and $\min I_1 = 0$, $\max I_n = \min J_n$, $\max J_n = \min I_{n+1}$ for all $n = 1, 2, \dots$ and $\bigcup_{n=1}^{\infty}(I_n \cup J_n) = [0, 1)$. For every n choose a system (X_n, f_n) with $X_n \subseteq [0, 1]$ in such a way that it be topologically conjugate to (X, f^n) and the convex hull of X_n be I_n. Then put $Y = \bigcup_{n=1}^{\infty} X_n \cup \{1\}$ and define g by $g(1) = 1$ and $g|_{X_n} = f_n$ for all n. Obviously, $g \in C(Y)$ and (Y, g) has zero entropy. Fix n. For every $\alpha \in S_H(X_n)$ the map β_α defined by $\beta_\alpha|_{X_n} = \alpha$ and $\beta_\alpha(y) = 0$ for all $y \in Y \setminus X_n$ belongs to $S_H(Y)$. It is obvious that the map $S_H(X_n) \to S_H(Y)$ sending each α to β_α is an isometrical embedding. Since $G_g|_{S_H(X_n)} = F_{f_n}$ and $(S_H(X_n), F_{f_n})$ is by Lemma 2.3 topologically conjugate to $(S_H(X), F_{f^n})$ we get $\operatorname{ent}_H(G_g) \geq \operatorname{ent}_H(F_{f_n}) = \operatorname{ent}_H(F_{f^n}) \to \infty$. □

Contrary to the previous theorem, on the interval the situation is probably different. We conjecture that if the topological entropy of an interval system (I, f) is zero then also the corresponding system $(S_H(I), F)$ has zero topological entropy.

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