NON-STABILITY OF AK-INVARIANT

TATIANA BANDMAN , LEONID MAKAR-LIMANOV

Abstract. We provide an example of a surface with non-stable $AK-$invariant and non-trivial fundamental group. We also compute the invariant for the cylinders over surfaces endowed with a $\mathbb{C}-$action which has only one singular fiber, and this fiber has only reduced components.

1. Introduction

In this note we again try to understand (see [BML1], [BML2]), what are the properties of a surface which are related to the stability of its AK-invariant. We call $AK-$invariant stable if

$$AK(S \times \mathbb{C}) = AK(S).$$

We provide an example of a surface with non-stable $AK-$invariant. We also compute the invariant for the cylinders over surfaces endowed with $\mathbb{C}-$action which has one singular fiber, and this fiber has reduced components.

Till now the surfaces $S_n = \{x^n y = p(x, z)\}$, where $p(0, v)$ does not have multiple roots were the only examples of smooth surfaces with unstable invariant ([D], [F], [Wi], [BML1]). In particular, the Danielewski surfaces

$$S_{n,m} = \{x^n y = z^m - 1\}$$

are of this kind. The following features unite all these surfaces:

1) $\pi_1(S_n) = 1$;
2) S_n admits a fixed-point-free $\mathbb{C}^+-$action;
3) fibering of S_n which corresponds to $\mathbb{C}^+-$action has only one nonconnected fiber;
4) all components of any fiber are reduced.

As far as non-stability of the invariant is concerned we show that 1) may be replaced by

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1') $\pi_1(S_{n,m})$ is finite cyclic. Condition 2) is not important. However, 3) seems to be important (see section 3) and we believe that it cannot be relaxed. 4) is not crucial: we provide an example where 4) is not satisfied.

On the other hand if 3) and 4) are satisfied then the invariant is non-stable.

Here is a more detailed description of the paper.

Let us call the fundamental group "small" if it is finite cyclic, and "large" otherwise.

In section 1 we review some known facts about surfaces with large fundamental group and remind that their invariant is stable.

In section 2 we give an example of a surface with small (but not trivial) fundamental group and unstable invariant. To obtain this example we describe homological $\mathbb{Q}-$plane S with a $\mathbb{C}^+-$action and with $\pi_1(S) = \mathbb{Z}/m\mathbb{Z}$ (where m is prime) as a quotient of a surface S_n by the cyclic transformation group. In case $AK(S) = \mathbb{C}$ it was done by M. Miyanishi and K. Masuda ([MiMa1][MiMa2]).

In section 3 we prove that for a surface S with properties 3)and 4)

$$AK(S \times \mathbb{C}^n) = \mathbb{C}.$$

2. Surfaces with large fundamental group

Definition 1. Let V be an affine variety and let $G(V)$ be the group generated by all $\mathbb{C}^+$-actions on V. Then $AK(V) \subset \mathcal{O}(V)$ is a subring of all regular $G(V)-$ invariant functions on V.

We want to know when

$$AK(S \times \mathbb{C}) = AK(S). \qquad (1)$$

If logarithmic Kodaira dimension $\overline{k}(S) \geq 0$ it is true due to a Theorem of T. Fujita and S. Iitaka ([Ii-Fu]). So, let S be a surface with logarithmic Kodaira dimension $\overline{k}(S) = -\infty$,

Miyanishi and Sugie ([MiSu], [Su]) proved that the equality $\overline{k}(S) = -\infty$ is equivalent to the existence of a cylinder-like subset (c.l.s.).

Definition 2. A surface S has a cylinder-like subset U (c.l.s. U) if it has a Zariski open subset $U \subset S$ isomorphic to the product $C \times \mathbb{C}$ of a smooth affine curve C and complex line.

Any c.l.s. provides ([Ber], [Mi1], [MiSu]) a line pencil on a surface S i.e. morphism $\rho : S \to C$ into a smooth curve C, such that the fiber $\rho^{-1}(z)$ for a general $z \in \mathbb{C}$ is isomorphic to $\mathbb{C}$. The pencils are different if their general fibers do not coincide. Any line pencil ρ over an affine curve C on a surface S corresponds to a $\mathbb{C}^+$-action φ_ρ on S, such that its general orbit coincides with the general fiber of the pencil (see Lemma 1), and corresponds to a locally nilpotent derivation (lnd) ∂_ρ of the ring $\mathcal{O}(S)$ of regular functions on S, such that $\partial_\rho f = 0$ if and only if f is φ_ρ-invariant ([ML1],[KML], [S], [Mi1]).

Definition 3. Two $\mathbb{C}^+$actions are equivalent, if they induce the same fibering ρ, i.e. they have the same general orbits.

Definition 4. An affine surface is rational, if there exists a dominant regular map from $\mathbb{C}^n$ into S for some $n \in \mathbb{N}$.

If in Definition 2 the surface S is rational then $C \sim \mathbb{C}$ or $C \sim \mathbb{P}^1$.

For a pencil ρ over C one can find a closure $\overline{S}$ of S and the extension $\overline{\rho} : \overline{S} \to \overline{C}$, such, that the general fiber $\overline{\rho}^{-1}(z) \cong \mathbb{P}^1$ and any (-1) curve, contained in a fiber intersects S. For any $z \in C$ the fiber $\rho^{-1}(z) = \cup C_i'$ is a union of smooth disjoint curves $C_i' \cong \mathbb{C}^1$ ([Mi1], Lemma 4.4.1).

Definition 5. ([Fu]) Let $F_z = \overline{\rho^{-1}(z)} = \sum_{i=1}^{i=m} n_i C_i$, where C_i are connected (and irreducible) components. If $m = 1, n_1 = 1$, then the fiber is called non-singular. A singular fiber is either non-connected or has $m = 1, n_1 > 1$. If $F_z = \sum_{i=1}^{i=m} C_i$ (i.e. $n_i = 1$) the fiber is called reduced. The number $\mu(z) = gcd(n_i)$ is called the multiplicity of a fiber. If a fiber has no points in S, by definition $\mu = \infty$.

The topology of surfaces with fibrations was studied by T. Fujita. Let k be the number of fibers with multiplicity $\mu > 1$ (see Definition 5). We shall use the following facts ([Fu]);

I. If the fundamental group $\pi_1(S)$ is infinite, then there is no dominant morphism $\phi : \mathbb{C}^n \to S$.

II. If the fundamental group $\pi_1(S)$ is finite, then $k \leq 3$.

III. If $\pi_1(S)$ is finite non-abelian, then $k = 3$ and for all three fibers $1 < \mu < \infty$;

IV. If $\pi_1(S)$ is finite abelian and non-trivial, then either $k = 3,\ \pi_1(S) \sim \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},\ \mu = 2$ for all three singular fibers, or $k = 2,\ \pi_1(S) \sim \mathbb{Z}/m\mathbb{Z}$, where m is gcd of two multiplicities.

V. If $\pi_1(S)$ is trivial, $k \leq 1$.

In order to use this information for computing $AK(S \times \mathbb{C})$, we need the following Lemmas.

Lemma 1. *(see also* [MiMa1], *Lemma 1.1) Assume that on a smooth affine surface S there is a $\mathbb{C}^+$-action ϕ. Let $\rho : S \to C$ be a regular map of S onto a smooth curve C such, that the general fiber is an orbit of ϕ. Then C is affine.*

Proof of Lemma 1. Since any $\mathbb{C}^+$action on S defines an lnd in $\mathcal{O}(S)$ ([S]), the Lemma follows from Lemma 1 of [ML2]. According to this Lemma there should be a regular non-constant ϕ-invariant function in $\mathcal{O}(S)$. This function is constant along the fibers of ρ, thus it belongs to $\mathcal{O}(C)$. But then C cannot be compact. $\square$

Lemma 2. *Let A be an affine commutative domain. If $AK(A) = A$ then $AK(A[x]) = AK(A)$.*

Proof of Lemma 2. Let us assume that ∂ is a non-zero locally nilpotent derivation of $A[x]$ which is not identically zero on A. Let $\mathrm{def}(r) = \deg(\partial(r)) - \deg(r)$ where deg stands for the degree relative to x. Let $m = \max(\mathrm{def}(r) | r \in A[x])$. Then $m < \infty$.

Indeed, A is finitely generated. Let $y_1, ..., y_k$ be a generating set of A. If $a \in A$ then $a = p(y_1, ..., y_k)$ where p is a polynomial. So $\partial(a) = \sum_i \partial(y_i) p_i$ where p_i are partial derivatives of p relative to y_i. Since all p_i are elements of A, we may conclude that $\mathrm{def}(a) = \deg(\partial(a)) \leq \max(\deg(\partial(y_i)))$ is uniformly bounded. Now let $r \in A[x]$. Then $r = \sum_{i=0}^{n} r_i x^i$ where $r_i \in A$. So $\partial(r) = \sum_{i=0}^{n} \partial(r_i) x^i + \sum_{i=0}^{n} r_i x^{i-1} \partial(x)$. Therefore $\deg(\partial(r)) \leq \max(n + \max(\deg(\partial(r_i)), n - 1 + \deg(\partial(x)))$ and so $\mathrm{def}(r) = \deg(\partial(r)) - n$ is uniformly bounded. By our definition of m we can write $\partial(a) = \sum_{i=0}^{m} \partial_i(a) x^i$ for any element $a \in A$. We also can write $\partial(x) = \sum_{i=0}^{m+1} e_i x^i$ where $e_i \in A$. Let us define a new derivation ϵ on $A[x]$ by $\epsilon(a) = \partial_m(a) x^m$ and $\epsilon(x) = e_{m+1} x^{m+1}$. It is easy to check that it is indeed a derivation and that this derivation must be locally nilpotent because otherwise ∂ cannot be locally nilpotent. Let $d(r)$ denote degree of $r \in A[x]$ relative to ϵ (see [FLN], [KML]; this degree is defined by the following formula: $d(r) = max\{d | \epsilon^d(r) \neq 0\}$ and is non-negative for non-zero r). Then $d(x) - 1 = d(e_{m+1}) + (m+1)d(x)$. Since this equality is not possible for non-negative $d(x)$ and $d(e_{m+1})$ we can conclude that $e_{m+1} = 0$. So x is a constant for ϵ and ∂_m is a locally nilpotent derivation on A, which implies $AK(A) \neq A$. $\square$

All these facts give us the following

Proposition 1. *Let S be an affine smooth surface with rational closure and $W = S \times \mathbb{C}$. Then:*

I. $AK(W) = AK(S)$, *if* $\pi_1(S)$ *is infinite;*

II. $AK(W) = AK(S) = \mathcal{O}(S)$ *if* $\pi_1(S)$ *is finite and non-abelian;*

III. $AK(W) = AK(S) = \mathcal{O}(S)$ *if* $\pi_1(S) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$;

Proof of Proposition 1. We prove Proposition case by case.

I. If $\pi_1(S)$ is infinite, then S is not rational [Fu]. On the other hand, if $AK(S) = \mathbb{C}$, then the orbit $G(S)$ contains an image of $\mathbb{C}^2$ under a dominant map([BML1]). Hence, $AK(S) \neq \mathbb{C}$ and only two cases are possible.

a) If $AK(S) = \mathcal{O}(S)$, (no actions at all), by Lemma 2, $AK(W) = AK(S) = \mathcal{O}(S)$.

b) If $AK(S) \neq \mathcal{O}(S)$, then $AK(S) = \mathcal{O}(C)$, where C is a curve in Definition 2.

Since the general orbit O_W of $G(W)$ contains an image of $\mathbb{C}^2$ under a dominant map ([BML1]) and there is no dominant map from $\mathbb{C}^2$ into S , the image of O_W under the projection of W onto S is not dense, i.e. it coincides with an orbit of a $\mathbb{C}-$ action on S. Hence, $AK(W) = \mathcal{O}(C) = AK(S)$.

II. If $\pi_1(S)$ is finite and non-abelian, then, due to [Fu], $k = 3$ and there is no fiber with $\mu = \infty$ for any linear pencil. By Lemma 1, we get that there are no $\mathbb{C}^+$ actions on such surfaces [1] and, by Lemma 2, $AK(W) = AK(S) = \mathcal{O}(S)$.

III. $\pi_1(S) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ can happen only if $k = 3$ and $\mu = 2$, i.e. there are no $\mathbb{C}^+$ actions on such surfaces as well. □

3. Surfaces with small fundamental group

The surfaces with finite cyclic fundamental group are much more complicated from $AK-$point of view. We start with three examples of surfaces S_i, $i = 1, 2, 3$, with $\pi_1(S_i) = \mathbb{Z}/2\mathbb{Z}$ and $AK-$invariant of all three possible types.

In the discussion of Examples 2 and 3 we will be using the following properties of locally nilpotent derivations. Let A be a ring with a filtration and let $\overline{A}$ be the corresponding graded ring. It is known (see section 5 in [KML]) that:

[1]Another proof of this fact was explained to authors to R.V.Gurjar in private communication

1) under some mild conditions (which are satisfied in our examples) a non-zero locally nilpotent derivation ∂ of A induces a non-zero locally nilpotent derivation $\overline{\partial}$ on $\overline{A}$, so that the kernel $\overline{A}^{\overline{\partial}}$ of $\overline{\partial}$ contains $\overline{A^{\partial}}$;

2) a non-zero locally nilpotent derivation ∂ of A allows to define on A a function $d(a) = max\{d|\partial^d(a) \neq 0\}$ (here a is a non-zero element of A). This function is called the degree function relative to ∂ since it possesses all usual properties of a degree function ([FLN]);

3) if a non-zero product is a ∂ constant then all the factors are also ∂ constants (see [FLN]).

Example 1. $S_1 \subset \mathbb{C}^5$ is defined by equations

$$y^2 = x(z-1), \quad t^2 = u(z-1),$$
$$yu = tz, \quad yz = xt, \quad xu = z^2.$$

It has two non-equivalent actions, which correspond to the following locally nilpotent derivations:

$$\begin{array}{rl} \partial x = 0 & \qquad \partial u = 0 \\ \partial y = x & \qquad \partial t = u \\ \partial z = 2y & \qquad \partial z = 2t \\ \partial t = 3z - 2 & \qquad \partial y = 3z - 2 \\ \partial u = 4t & \qquad \partial x = 4y \end{array}$$

Therefore, $AK(S_1) = \mathbb{C}$.

Example 2.

$$S_2 = \{x'(x'y' + 1) = z'^2\} \subset \mathbb{C}^3.$$

It has one action $\partial x' = 0$, $\partial y' = 2z'$, $\partial z' = x'^2$.

Let us show that it has just one class of *lnds* up to equivalence (i.e all other *lnds* define the same fibration). The ring $\mathcal{O}(S_2) = R$ can be embedded in $\mathbb{C}[a, b]$ by mapping x' to a^2, z' to $a(1 + a^3b)$, and y' to $ab(2 + a^3b)$. One *lnd* on R is defined by $\partial(a) = 0$ and $\partial(b) = 1$. Assume that there is another nonzero *lnd* ϵ which does not send a to zero. Let f be from its kernel. Let us take lexicographic order $b >> a$ on $\mathbb{C}[a, b]$ and the induced order on R. We denote by $\overline{u}$ the leading form of an element u and by $\overline{R}$ the ring, generated by the leading forms of elements of R.

Clearly $\overline{f}$ is $a^i b^j$ where $j > 0$. Since $R = \mathbb{C}[x', y'] + z'\mathbb{C}[x', y']$ we see that $\overline{R}$ is generated by a^2, a^4b^2, and a^4b.

So ϵ defines a nonzero *lnd* $\overline{\epsilon}$ on $\overline{R}$ and $\overline{f} \in \ker(\overline{\epsilon})$ (see property 1). Replacing f by f^2 if necessary, we may assume that j is even and so $\overline{f} = (a^4b^2)^k$ (since $\overline{f}$ is $\overline{\epsilon}$-constant, if a factorization of $\overline{f}$ contains two different generators of $\overline{R}$ then $\overline{\epsilon}$ is the zero derivation, due to property 3)).

Let us consider the degree function which is induced by $\overline{\epsilon}$ on $\overline{R}$. Since $\deg(a^4b^2) = 0$ we may assume that $\deg(a) = d$ and $\deg(b) = -2d$. Therefore $\deg(a^2) = 2d$, $\deg(a^4b) = 2d$ and elements of $\overline{R}$ have only even degrees. Since application of $\overline{\epsilon}$ decrees degree by 1 it is impossible. So ϵ must be the zero derivation.

It follows, that $AK(S_2) = \mathbb{C}[x]$.

The same conclusion we get if we consider the "minimal" complement of this surface : the divisor in the complement is not a "zigzag" i.e. is not a linear chain (see [Giz], [Ber] and [BML2]).

Example 3. $S_3 \subset \mathbb{C}^5$ is defined by

$$vu = z^2,$$
$$(u^2 - 1)u = vt,$$
$$(v^2 - 1)v = uw,$$
$$(v^2 - 1)(u^2 - 1) = tw.$$

Again we denote by R the ring $\mathcal{O}(S_3)$.

First of all, $R = R_0 + zR_0$ where R_0 is generated by u, v, t, w. Next, R_0 is the linear span of monomials u^iv^j, $u^it^j, j > 0$, and $v^iw^j, j > 0$ since vt, uw, and tw can be replaced by a combination of monomials of smaller degree relative to t and w.

Since $t = (u^2 - 1)uv^{-1}$, $w = (v^2 - 1)vu^{-1}$, if we consider weights $\omega(u) = 1$ and $\omega(v) = 1$, then $\overline{t} = u^3v^{-1}$, $\overline{w} = v^3u^{-1}$. Therefore $\overline{u^it^j} = u^{3j+i}v^{-j}$, $\overline{v^iw^j} = v^{3j+i}u^{-j}$. Since in these monomials $j > 0$ we see that the leading forms of monomials u^iv^j, $u^it^j, j > 0$, and $v^iw^j, j > 0$ are linearly independent, which means that $\overline{R_0}$ is generated by $u, v, \overline{t}, \overline{w}$. We can substitute $u = z^2v^{-1}$ in the monomials u^iv^j, $u^i\overline{t}^j, j > 0$, and $v^i\overline{w}^j, j > 0$. The resulting monomials stay linearly independent. If we add to them their products with z we will get all $\overline{R}$ and see that $\overline{R}$ is generated by $v, z, \overline{u} = z^2v^{-1}, \overline{t} = z^6v^{-4}, \overline{w} = z^{-2}v^4$.

To get rid of the negative degrees, which are confusing, substitute $z = a^2b^2$ and $v = ab^3$. Then $\overline{R} = span\{a^8, a^3b, a^2b^2, ab^3, b^8\}$.

If we prove that $\overline{R}$ does not have nonzero lnds then R itself also does not have nonzero lnds (see property 1))

Assume that ∂ is a nonzero lnd on $\overline{R}$. Let us take the lexicographic ordering $a >> b$ on $\overline{R}$ and let $\overline{\partial}$ be the corresponding lnd induced on $\widehat{R}$ which is obtained from $\overline{R}$ by this ordering. If $f \in \ker(\overline{\partial})$ then f is a monomial. If this monomial contains both a and b then f^8 is also a $\overline{\partial}$-constant which is divisible by a^8 and b^8 in $\widehat{R}$ which means that both a^8 and b^8 are constants of $\overline{\partial}$ and $\overline{\partial} = 0$.

Thus we may assume that either $f = a^8$ or $f = b^8$. Both cases are similar. Let $\overline{\partial}(a^8) = 0$. Then $\deg_{\overline{\partial}}(b) = 1$ since otherwise there are no elements in $\widehat{R}$ with $\overline{\partial}$-degree equal to 1. So $\overline{\partial}(a^3b) = \lambda a^{8k}$ where $\lambda \in \mathbb{C}$ and $\overline{\partial}(b) = \lambda a^{8k-3}$. But then $\overline{\partial}(a^2b^2) = 2\lambda a^{8k-1}b$ which is not in $\widehat{R}$ if $\lambda \neq 0$.

Thus, $AK(S_3) = \mathcal{O}(S_3)$.

The surfaces S_1 and S_2 both have fibration over $\mathbb{C}$ (with fibers $\{x = const\}$ and $\{x' = const\}$ respectively) with only one singular fiber $\{x = 0\}$ (respectively $\{x' = 0\}$), which is isomorphic to $\mathbb{C}$ and has multiplicity 2. The surface S_3 has fibering over $\mathbb{P}^1$, defined by the rational function $u/z = z/v$. There are two singular fibers, both isomorphic to $\mathbb{C}$ and both of multiplicity 2. Thus, (see Section 2) $\pi_1(S_i) = \mathbb{Z}/2\mathbb{Z}$, $i = 1, 2, 3$.

Next, we want to build a surface S, such that

1. $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$,
2. $AK(S) = \mathbb{C}[x]$,
3. $AK(S \times \mathbb{C}) = \mathbb{C}$.

We start with a description of the surfaces with only two singular fibers: one fiber of multiplicity $\mu = \infty$ and one fiber of multiplicity m, which consists of the single curve $C \cong \mathbb{C}^1$. These surfaces are $\mathbb{Q}-$planes (see, for example, [Mi3]). It was proved in [MiMa1] that any $\mathbb{Q}-$plane S with $AK(S) = \mathbb{C}$ is a quotient of Danielewski surface $S_{1,m}$ by an action of the group $\mathbb{Z}/m\mathbb{Z} = \pi_1(S)$. We want to use a similar description for $\mathbb{Q}-$planes S with $AK(S) = \mathbb{C}[x]$.

Proposition 2. *Let S be a $\mathbb{Q}$-plane admitting a $\mathbb{C}^+$-action. Let $\pi_1(S) = \mathbb{Z}/m\mathbb{Z}$, where m is prime. Then $S \cong V/\mathcal{G}$, where*

- *V is a hypersurface in $\mathbb{C}^3$ with coordinates (u, y, v) : $V = \{u^s y = v^m - v_1^m + u^m q(u^m, u^\alpha v)\}$, where $q(u, v)$ is a polynomial of degree less then m relative to v;*
- *$\mathcal{G}$ is a group of transformations, generated by $g(u, y, v) = (u\varepsilon, y\varepsilon^{-s}, v\varepsilon^{-\alpha})$ for some $\alpha \in \mathbb{N}$ and $\varepsilon = e^{\frac{2i\pi}{m}}$.*

Proof of Proposition 2. Let ∂ be an *lnd*, corresponding to a $\mathbb{C}^+$-action on S. From the general properties of $\mathbb{Q}$-planes (see, for example, [Mi3] and Prop.1 of [BML2]), it follows that there are two functions $x \in \mathcal{O}(S)$ and $z \in \mathcal{O}(S)$ such that:

a) a fiber $\{x = c \neq 0\} = F_c \sim \mathbb{C}$ is reduced;
b) the fiber $\{x = 0\} = F_0 \sim \mathbb{C}$ has multiplicity m;
c) for $c \neq 0$ the function $z \big|_{F_c}$ is linear (i.e., takes any finite value exactly at one point);
d) $z \big|_{F_0} = 0$;
e) $\partial x = 0$, $\partial z = x^n$, and n is minimal possible for z with these properties.

Consider a surface U which is defined as the normalization of $S \times_{\mathbb{C}} \Gamma$, where Γ stands for the m-ramified cover of $\mathbb{C}$ with the only branch point at $\gamma = 0$ over $x = 0$ (similar to Theorem 3.2 of [MiMa1]). Let $\pi : U \to S$ and $p : U \to \Gamma$ be the natural projections.

The surface U has the following properties:

a) there is a function $u \in \mathcal{O}(U)$ such that $\pi^* x = u^m$;
b) U is an unramified m-sheeted cover of S;
c) fiber $\mathcal{E}_0 = \{u = 0\} = \bigcup_{i=1}^{m} E_i$ is a union of m reduced components $E_i \simeq \mathbb{C}^1$;
d) U is affine ([Sh], Ch.II, Th.4);
e) the general fiber $E_c = \{u = c\} \simeq \mathbb{C}$.

Indeed,

- In property a) $u = p^*\gamma$.
- The properties b) and c) are proved in [BPV], chapter III, §9 for proper maps. Since both properties are local, the same proof is valid in our case as well. The only modification one needs to make in the proof is to use $\pi^*\phi_f$, where $\phi_f = 0$ is a local equation of F_0 in an appropriate neighborhood of a point $f \in F_0$ instead of function g (the global equation of the multiple fiber there).
- e) is valid because $E_c \simeq F_{c^m}$ by construction .

The property e) provides the existence of a $\mathbb{C}^+$-action φ on U with general orbit $\{u = \text{const}\}$. Let ∂_1 be a corresponding *lnd*: $\partial_1(\pi^*(f)) = \pi^*(\partial f)$ for $f \in \mathcal{O}(S)$; $\partial_1(u) = 0$. Since all the fibers $\{u = const\}$ are reduced, we have $\pi_1(U) = 0$. It follows that U is the universal covering of S and $S = U/\mathcal{G}$, where $\mathcal{G}$ is the group of deck transformations. Any transformation $g \in \mathcal{G}$ sends the fiber $E_{u_0} = \{u = u_0\}$ to the fiber $E_{u_0\varepsilon^k} = \{u = u_0\varepsilon^k\}$, where $k \in N$ and $\varepsilon = e^{\frac{2i\pi}{m}}$. In particular, it moves a component E_i of $\mathcal{E}_0$ to the component $E_j \subset \mathcal{E}_0$.

Since $z \big|_{F_0} = 0$, we have $\pi^* z \big|_{\mathcal{E}_0} = 0$. But all the components E_i are reduced; hence $\pi^* z = u^\alpha \cdot v$ for some $\alpha \in \mathbb{N}$ and $v \in \mathcal{O}(U)$ such that $v \big|_{\mathcal{E}_0} \not\equiv 0$. Further on we shall use z and x instead of $\pi^* z$ and $\pi^* x$ when no confusion may arise.

Lemma 3.

a) $(\alpha, m) = 1$.

b) *v is linear along each fiber $E_{u_0} = \{u = u_0 \neq 0\}$.*

c) *$v \big|_{E_i} = v_i$ is a constant, and $v_i \neq v_j$ if $i \neq j$.*

d) $\partial_1 v = u^{mn-\alpha}$.

e) *$g^*(\partial_1 h) = \partial_1(g^* h)$ for any $h \in \mathcal{O}(U)$*

Proof of Lemma 3. **a)** Since $z \big|_{F_0} = 0$, there are coprime numbers r and k and a function $\psi \in \mathcal{O}(S)$ such that $z^k = x^r \cdot \psi$ and $\psi \big|_{F_0} \not\equiv 0$. It follows that $u^{\alpha k} v^k = u^{mr} \pi^*(\psi)$, i.e., $\alpha k = mr$. Since m is prime, the following two cases are possible.

(1) $\alpha = mr$, $k = 1$, and $z = x^r \cdot \psi$, $\partial z = x^r \cdot \partial\psi = x^n$, $\partial\psi = x^{n-r}$. Then ψ is linear along a general fiber and constant along F_0 ([BML2], Prop.1), i.e. n is not minimal;

(2) $\alpha = r$ and $k = m$. But then $(\alpha, m) = (k, r) = 1$.

b) $v = z/u^\alpha$ is linear along each fiber $\{u = c \neq 0\}$ since z is linear along a fiber $\{x = const\}$.

c) Take now E_1 and g a generator of G such that $g^*(u) = u\varepsilon$ and denote: $E_2 = g(E_1), E_3 = g(E_2) \dots E_m = g(E_{m-1})$. Then

$$g^*(u) = u \cdot \varepsilon, \quad g^*(v) = g^*(z) \cdot g^*(u)^{-\alpha} = z \cdot u^{-\alpha}\varepsilon^{-\alpha} = v\varepsilon^{-\alpha}.$$

If v were not constant along a component $E_1 \subset \mathcal{E}_0$, it would be non-constant along each E_i, $i = 1, \dots, m$. This means that v would take any value $v_0 \in \mathbb{C}$ at least m times in $\mathcal{E}_0$, which contradicts the fact that it is linear along the general fiber. (Indeed, $\{v = v_0\}$ intersects each fiber at most at one point).

Thus, $v\,\big|_{E_i} = v_i = \mathrm{const}$. Then $v_2 = \varepsilon^{-\alpha}v_1$, $v_3 = \varepsilon^{-2\alpha}v_1, \dots, v_m = \varepsilon^{-(m-1)\alpha}v_1$, i.e., $v_i \neq v_j$.

d) $\partial_1 v = \partial_1(z/u^\alpha) = x^n/u^\alpha = u^{mn-\alpha}$.

e) Any function $h \in \mathcal{O}(U)$ may be represented as $h = q(u,v)/u^l$ for some polynomial q, since U is rational. The property may be obtained by a direct computation. □

Lemma 4. *The surface U admits a fixed-point-free $\mathbb{C}^+$-action ψ, equivalent to φ.*

Proof of Lemma 4. We have to prove that $\partial_1 = u^\beta\tilde{\partial}$, where $\beta \geq 0$ and $\tilde{\partial}$ is an *lnd* of $\mathcal{O}(U)$ such that for any $i = 1,\dots,m$ if $h \in \mathcal{O}(U)$ and $h\,\big|_{E_i} \neq \mathrm{const}$, then $\tilde{\partial}h\,\big|_{E_i} \neq 0$.

Let $\partial_1 h\,\big|_{E_i} = 0$ for all $h \in \mathcal{O}(U)$, i.e., $\partial_1 h = u^{n_i(h)}\psi$, where ψ is rational on U and $\psi\,\big|_{E_i} \not\equiv 0, \infty$. Let n_i be the minimal value of $n_i(h)$ for $h \in O(U)$.

Since the action of group $\mathcal{G}$ permutes the components,

$$n_1 = n_2 \cdots = n_m \qquad \text{and} \qquad \tilde{\partial} = \frac{\partial_1}{u^{n_1}}$$

is defined and corresponds to a fixed-point-free $\mathbb{C}^+$-action. □

Lemma 5. *Let $\tilde{\partial}v = u^k$. Then the there exists a polynomial $p(u,v) = p_0(v) + up_1(u,v)$ such that the function $y = \frac{p(u,v)}{u^k} \in \mathcal{O}(U)$ is linear along each E_i, $i = 1,\dots,m$.*

Proof of Lemma 5. The polynomial $p_0(v) = v^m - v_1^m$ vanishes along $\mathcal{E}_0$, i.e., $p_0(v) = u^{s_1}\cdot y_1$, where $y_1 \in \mathcal{O}(U)$ and $y_1\,\big|_{E_i} \not\equiv 0$ at least for one value of i. Since $g^*(y_1) = y_1\varepsilon^{-s_1}$, y_1 is constant or non-constant along all E_i simultaneously and $y_1\,\big|_{E_i} \not\equiv 0$ for all i.

We have:

$$u^{s_1}y_1 = p_0(v)$$

$$u^{s_1}\tilde{\partial}y_1 = p_0'(v)u^k$$

$$\tilde{\partial}y_1 = p_0'(v)u^{k-s_1} \in \mathcal{O}(U).$$

It follows, that $k \geq s_1$.

Consider two cases.

I. $y_1 \big|_{E_i} \neq const$. Then $\tilde{\partial} y_1 = p'(v)u^{k-s_1} \not\equiv 0$, i.e. $k - s_1 = 0$, $\tilde{\partial} y_1 \big|_{E_i} = const$, and y_1 is linear along each component E_i.

II. $y_1 \big|_{E_1} = c_1 = const$. Then $y_1 \big|_{E_i} = c_1 \varepsilon^{-s_1 i}$. Let $h < m$ be the natural number, such that $s_1 \equiv \alpha h (\text{mod } m)$, and $p_1(v) = c_1(\frac{v}{v_1})^h$. Then $y_1 - p_1(v) \big|_{\mathcal{E}_0} \equiv 0$ and $y_1 - p_1(v) = u^{s_2} y_2$, where $y_2 \in \mathcal{O}(U)$ and $y_2 \big|_{E_i} \not\equiv 0$ at least for one value of i.

Since $g^*(y_2) = y_1 \varepsilon^{-s_1 - s_2}$, y_1 is constant or non-constant on all E_i simultaneously and $y_2 \big|_{E_i} \not\equiv 0$ for all i.

We have :

$$u^{s_1+s_2} y_2 = p_0(v) + u^{s_1} p_1(v),$$
$$\tilde{\partial} y_2 = (p_0'(v) + u^{s_1} p_1'(v)) u^{-(s_1+s_2)+k}$$

It follows that $k \geq (s_1 + s_2)$.

If $y_2 \big|_{E_i} \neq const$, it is linear along each E_i, if not, we continue the process.

After r steps we get

$$u^{s_1+s_2+\ldots+s_r} y_r = p_0(v) + u^{s_1} p_1(v) + \ldots + u^{s_1+s_2+\ldots+s_{r-1}} p_{r-1}(v),$$

and either

$$s_1 + s_2 + \ldots + s_r < k,\ \ y_r \big|_{E_i} = const$$

or

$$s_1 + s_2 + \ldots + s_r = k,\ \ y_r \big|_{E_i} \neq const,$$
$$\tilde{\partial} y_r = (p_0'(v) + u^{s_1} p_1'(v) + \ldots + u^{s_1+s_2+\ldots+s_{r-1}} p_{r-1}'(v))$$

and y_r is linear along the components E_i.

So, after a finite number of steps we will get a function y, which is liner along each E_i and

$$u^k y = p_0(v) + u\tilde{q}(u, v).$$

Moreover, by construction, $g^* y = y\varepsilon^{-k}$ and $q(u, v)$ has degree less then m relative to v. □

We proceed with the proof of the Proposition 2. Let $\theta : U \to \mathbb{C}^3$ be defined as $\theta(p) = (u, y, v)$ for a point $p \in U$.

This regular map is bijective: indeed, the function u distinguishes the fibers, v is linear along the general fiber and distiguishes components of the singular fiber, and y is linear along the components of the singular fiber.

The surface $V = \theta(U)$ is given by equation

$$\{u^k y = v^m - v_1^m + u\tilde{q}(u,v)\}.$$

This equation is invariant under the action of group $\mathcal{G}$ if only $u\tilde{q}(u,v) = u^m q(u^m, u^\alpha v)$. Since V is a smooth surface bijection θ is an isomorphism. $\square$

Example 4. We use Proposition 2 for the case $m = 2, s = 2$. The Danielewski surface

$$S_{2,2} = \{x^2 y_1 = z_2^2 - 1\}$$

is the universal cover of the surface S_2 (see Example 2). Indeed, consider the following map:

$$(x, y_2, z_2) \to (x' = x^2,\ y' = y_2,\ z' = xz_2).$$

This map glues the points $s = (x, y_2, z_2)$ and $Ts = (-x, y_2, -z_2)$ and is unramified double covering. Obviously,

$$x'^2 y' + x' = z'^2.$$

The surface S_2 has the fundamental group $\mathbb{Z}/2\mathbb{Z}$ and precisely one singular fiber $\{x' = 0\}$, which is a double straight line.

All actions on this surface are equivalent: $AK(S_2) = \mathbb{C}[x]$

Now we want to compute the invariant of its cylinder $Z = S_2 \times \mathbb{C}$. The universal cover of this cylinder is $Y = S_{2,2} \times \mathbb{C} \cong S_{1,2} \times \mathbb{C}$ ([F],[D]), where $S_{1,2} = \{xy_1 = z_1^2 - 1\}$.

Lemma 6. *(communicated by P. Russell).*

$$Y = \begin{cases} xy_1 = z_1^2 - 1 \\ x^2 y_2 = z_2^2 - 1 \\ xu = z_1 - z_2 \\ y_1 - xy_2 = u(z_1 + z_2). \end{cases}$$

The surface $S_{2,2}$ is defined in it as $u = const$ and the surface $S_{1,2}$ as $w = u^3 x + 3u^2 z_2 + 3uxy_2 + z_2 y_2$.

Proof of Lemma 6. Let $A = \mathbb{C}[x, y_1, y_2, z_1, z_2, u]$ where $xy_1 = z_1^2 - 1$, $x^2 y_2 = z_2^2 - 1$, and $xu = z_1 - z_2$. Then $A = \mathbb{C}[x, y_2, z_2][u]$ since $z_1 = xu + z_2$ and $y_1 = u^2 x + 2uz_2 + xy_2$.

Similarly, $A = \mathbb{C}[x, y_1, z_1][w]$, where $w = u^3x + 3u^2z_2 + 3uxy_2 + z_2y_2$ since $xw = y_1z_1 - 2u$ and $wz_1 = y_2 - u^2 + y_1^2$. So $2u = -xw + z_1y_1,\ \ y_2 = wz_1 + u^2 - y_1^2$ and $2z_2 = x^2w - z_1^3 + 3z_1$.

□

Cylinder $Z = S_2 \times \mathbb{C}$ is a factor of Y by the transformation T', which preserves $S_{2,2}$ (i.e. the value of u) and coinsides on $S_{2,2}$ with T:

$$T' : (x, y_2, z_2, y_1, z_1, u) \to (-x, y_2, -z_2, -y_1, -z_1, u).$$

On the other hand $Z = \{(x', y', z', t) : {x'}^2y' + x' = {z'}^2\} \subset \mathbb{C}^4$ and the map $Y \to Z$ is defined as

$$x' = x^2, y' = y_2, z' = xz_2, t = u.$$

Due to Lemma 6

$$u = (1/2)(-xw + z_1y_1),$$

$$y_2 = wz_1 + u^2 - y_1^2,$$

$$z_2 = (1/2)(x^2w - z_1^3 + 3z_1).$$

Any *lnd* defined on $\mathcal{O}(S_{1,2})$ may be extended to an *lnd* on $\mathcal{O}(Y)$. In particular, the *lnd* defined by:

$$\partial y_1 = 0,\ \ \partial z_1 = y_1,\ \ \partial x = 2z_1,\ \ \partial w = 0.$$

Then

$$\partial u = (1/2)(-2z_1w + y_1^2),$$

$$\partial y_2 = (wy_1 + u(y_1^2 - 2wz_1)),$$

$$\partial z_2 = (1/2)(4xwz_1 - 3z_1^2y_1 + 3y_1),$$

and ∂ is invariant under T', i.e.$\partial T' = T'\partial$. Indeed, say,

$$T'(\partial x) = T'(2z) = -2z = \partial T'(x),$$

and we can check it similarly for the remaining generators.

Therefore this *lnd* can be pushed down to $\mathcal{O}(Z)$, which is the factor of Y by T'. This means that in Z there is an *lnd*, for which x' is not invariant. Hence $AK(Z) = \mathbb{C}$, though $AK(S_2) = \mathbb{C}[x]$.

4. Cylinders over surfaces with reduced fibers and primitive Feiseler factor

We want to show in this section that the property 2) of Danielewski surfaces (see Section 1) is not essential for non-stability of $AK-$invariant of a surface.

The Danielewski construction represents a surface with a $\mathbb{C}^+$action with reduced components of fibers as an affine bundle over the "Danielewsky-Fieseler" factor.

For a given c.l.s. $U \subset S$ or a corresponding fibering ρ the "Danielewsky-Fieseler" (DF) factor $X = S/_U = S/_\rho$ is a (non-separated) prevariety X, such that the points of X are in one-to-one correspondence with the connected components of $\rho^{-1}(c)$ for all $c \in C$. The precise and detailed definition was given in [D], [F].

We describe this factor in the following way.

Definition 6. A factor X of a smooth surface S by a pencil ρ is an algebraic prevariety X included into the following commutative diagram

$$\begin{tikzcd} S \arrow[rr, "\pi"] \arrow[dr, "\rho"'] & & X \arrow[dl, "\sigma"] \\ & C & \end{tikzcd}$$

and having the following properties:

1) maps π, σ, ρ are regular;
2) σ is an isomorphism of $X \setminus \{x_{i,j}\},\ i = 1, \cdots r$ onto $C \setminus \{c_1, \ldots, c_r\}$ where the set of points $\{x_{i,j}\} \in X$ is finite;
3) let $F_{c_i} = \rho^{-1}(c_i) = \bigcup_{j=1}^{t} C_{ij}$ be the union of t connected (necessarily irreducible [Mi1],Lemma 4.4.1) components C_{ij}. Then $\pi\big|_{C_{ij}} = x_{ij}$ and $x_{ij} \neq x_{ik}$ if $j \neq k$. There is a one-to-one correspondence between points $x_{ij} \in \sigma^{-1}(c_i)$ and components $C_{ij} \subset \rho^{-1}(c_i)$.

We call X primitive, if $C \cong \mathbb{C}$ and there is only one singular fiber.

Theorem 1. *Let S be a smooth affine rational surface and let $\alpha : S \times \mathbb{C} \to S$ be a C^+-action on S for which all the components of all the fibers are reduced. Assume that the DF factor $X = S/\alpha$ is primitive. Then $AK(\mathcal{O}(S \times \mathbb{C})) \cong \mathbb{C}$*

The proof of the Theorem is based on the following Theorem of Sh. Kaliman and M. Zaidenberg ([KZ]).

Theorem KZ. *Assume that there exists a dominant morphism $f\colon X \to S$ of a smooth quasiprojective variety X to a smooth quasiprojective variety S, and that a general fiber $f^{-1}(s)$, $s \in S$, is isomorphic to $\mathbb{C}^2$. Then there exists a Zariski open subset S_0 of S such that $f^{-1}(S_0) \cong S_0 \times \mathbb{C}^2$. Moreover, if we denote by ϕ the isomorphism $f^{-1}(S_0) \to S_0 \times \mathbb{C}^2$, and by $p\colon S_0 \times \mathbb{C}^2 \to S_0$ the projection to the first factor, then $p^{-1}(s) = \phi(f^{-1}(s))$ for any $s \in S_0$.*

Proof of Theorem 1. Since X is primitive, the base of fibering $C \cong \mathbb{C}$ and we may assume that the multiple point is $c = 0$. Let $\sigma^{-1}(0) = \{x_1, ..., x_k\}$. Consider a surface $S_1 \subset \mathbb{C}^3$ defined by

$$xy = (z-1)(z-2)...(z-k) = p(z),$$

where $\{x, y, z\}$ are coordinates in $\mathbb{C}^3$. S_1 is a smooth affine surface with two C^+-actions β and δ. The orbits of β, δ are the curves $R_x = \{x = const\}, R_y = \{y = const\}$ respectively, and the corresponding *lnds* of $\mathcal{O}(S_1)$ are defined as follows:

I. $\beta : \partial x = 0, \partial z = x, \partial y = p'(z)$;

II. $\delta : \partial y = 0, \partial z = y, \partial x = p'(z)$.

Note that the DF factors $S_1/_\beta \cong S_1/\delta \cong X$.

Now consider a commutative diagram

$$\begin{tikzcd}
S\times\mathbb{C} & \cong & & S\times_X S_1 \arrow[dl, "p"'] \arrow[dr, "p_1"] & & = & W \\
& & S \arrow[dr, "\pi_\alpha"'] & & S_1 \arrow[dl, "\pi_\beta"] \arrow[drr, "\pi_\delta"] & & \\
& & & X \arrow[d, "\sigma"'] & & & X \arrow[d, "\sigma_1"'] \\
& & & \mathbb{C} & & & \mathbb{C}
\end{tikzcd}$$

All the fibers of π_β are reduced , and the action β is fixed-point-free. Following Danielewsky and Feiseler ([D],[F]) it means that $W = S \times_X S_1$ has the following properties:

a) W is a $\mathbb{C}^+$ bundle over S;

b) (W, p) and (W, p_1) are locally-trivial fiber-bundles over S and S_1 respectively.

It follows, that:

a') $W \cong S \times \mathbb{C}$, since any $\mathbb{C}^+$- bundle over affine surface S is trivial ;

b')Consider a map $\mu = \sigma_1 \circ \pi_\delta \circ p_1 : S \times \mathbb{C} \to \mathbb{C}$. The general fiber $P_y = \mu^{-1}(y),\ y \in \mathbb{C}$ is a locally trivial fiber-bundle (restriction of (W, p_1)) over a curve $R_y = (\sigma_1 \circ \pi_\delta)^{-1}(y) \subset S_1$. The fiber of this bundle is reduced and isomorphic to $\mathbb{C}$, and $R_y \cong \mathbb{C}$ for a general y. Thus, $P_y \cong \mathbb{C}^2$ ([MiSu]) for a general y.

According to Theorem KZ, there exist a Zariski open subset $Z \subset S \times \mathbb{C}$ and $U \subset \mathbb{C}$ such that $Z = \mu^{-1}(U),\quad Z \cong U \times \mathbb{C}^2$, and μ is the projection on the first factor in this product. Due to [Mi2] Lemma 1.4, there exist two commuting *lnds* on the ring $\mathcal{O}(S \times \mathbb{C})$, such that the general orbit of the group generated by the corresponding $\mathbb{C}^+$-actions coincides with P_y. The image $\sigma \circ \pi_\beta \circ p_1(P_y) = \sigma \circ \pi_\beta(R_y) = \mathbb{C}$. Thus, there is a $\mathbb{C}^+$-action ϵ on $S \times \mathbb{C}$, whose orbit is not contained in a fiber $Q_z = p_1^{-1} \circ \pi_\beta^{-1} \circ \sigma^{-1}(z),\ z \in \mathbb{C}$.

We obtained three $\mathbb{C}^+$ actions on $S \times \mathbb{C}$: α', induced by α, ω, acting along the fibers of the map $p : W \to S$, and ϵ. The general orbit of the group generated by α' and ω is $Q_z = p_1^{-1} \circ \pi_\beta^{-1} \circ \sigma^{-1}(z)$. The orbit of the third action ϵ is not contained in Q_z. Hence, the general orbit of the group generated by all three actions is three-dimensional, and $AK(\mathcal{O}(S \times \mathbb{C})) \cong \mathbb{C}$. $\square$

Example 5. Consider a surface

$$S_5 = \begin{cases} x^2 y_2 = z_2^2 - 1 \\ (z_2 - 1) y_2 = xt \\ x y_2^2 = (z_2 + 1) t \end{cases}$$

This surface is obtained from $S_{2,2} = \{x^2 y_2 = z_2^2 - 1\}$ by the following affine modification: the point $x = 0, z_2 = -1, y_2 = 0$ is blown up, and the curve $x = 0, z_2 = -1, y_2 \neq 0$ is taken off. Since the graph of the complement of this surface is not linear ([Giz], [Ber]) surface S_5 admits only one class of $\mathbb{C}^+-$actions, i.e. any *lnd* is proportional to the following one:

$$\partial x = 0,\ \partial z_2 = x^3,\ \partial y_2 = 2 z_2 x,\ \partial t = 3 z_2^2 - 2 z_2 - 1.$$

The corresponding fibering has one singular fiber $x = 0$ with two connected reduced components $x = 0, z_2 = 1, t = 0$ and $x = 0, z_2 = -1, y_2 = 0$.

This *lnd* vanishes along the curve $x = 0, z_2 = 1, t = 0$, i.e. there are fixed points for any $\mathbb{C}^+-$action.

According to Theorem 1, $AK(\mathbb{C} \times S_5) = \mathbb{C}$.

We will show it explicitly, using the same formulae as in Example 4

Let $B = \mathbb{C}[x, y_1, y_2, z_1, z_2, u, t]$ where $xy_1 = z_1^2 - 1$, $x^2y_2 = z_2^2 - 1$, $(z_2 - 1)y_2 = xt$, $xy_2^2 = (z_2 + 1)t$ and $xu = z_1 - z_2$. Then $B = \mathbb{C}[x, y_2, z_2, t][u]$ since $z_1 = xu + z_2$ and $y_1 = u^2x + 2uz_2 + xy_2$.

Therefore $B = \mathcal{O}(S_5 \times \mathbb{C})$ and ∂ can be extended on B by $\partial u = 0$.

Substituting into the identity $xt = (z_2 - 1)y_2$ expressions for y_2 and z_2 trough x, y_1, z_1, w, (see Example 4) we get:

$$xt = (z_1 - xu - 1)y_2 = -xuy_2 + (z_1 - 1)[wz_1 - y_1^2 + \frac{1}{4}(z_1y_1 - xw)^2] =$$
$$-xuy_2 + \frac{1}{4}(z_1 - 1)(x^2w^2 - 2xy_1z_1w) + (z_1 - 1)[wz_1 - y_1^2 + \frac{1}{4}z_1^2y_1^2].$$

To simplify this expression take

$$s = t + uy_2 - \frac{1}{4}(z_1 - 1)(xw^2 - 2y_1z_1w).$$

Then

$$xs = \frac{1}{4}(z_1 - 1)(4wz_1 - 4y_1^2 + z_1^2y_1^2).$$

Next,

$$4xs = (z_1 - 1)(4wz_1 - 4y_1^2 + z_1^2y_1^2) = (z_1^2 - 1)(y_1^2(z_1 - 1) + 4w)-$$
$$(z_1 - 1)(4w + 3y_1^2) = xy_1[y_1^2(z_1 - 1) + 4w] - (z_1 - 1)(4w + 3y_1^2).$$

Introducing now

$$r = -4s + y_1[y_1^2(z_1 - 1) + 4w]$$

we obtain $xr = (z_1 - 1)(4w + 3y_1^2)$. Since

$$(z_1 + 1)xr = (z_1^2 - 1)(4w + 3y_1^2) = xy_1(4w + 3y_1^2)$$

we also have $(z_1 + 1)r = y_1(4w + 3y_1^2)$. Clearly, $B = \mathbb{C}[x, y_1, y_2, z_1, z_2, u, r]$, and, as in Example 4, $B = \mathbb{C}[x, y_1, z_1, r, w]$,

Let us define now an *lnd* of B by $\tilde{\partial}y_1 = \tilde{\partial}r = 0$, $\tilde{\partial}z_1 = 4y_1$, $\tilde{\partial}x = 8z_1$, and $\tilde{\partial}w = r$. It acts on the cylinder, and $\tilde{\partial}x \neq 0$.

A third *lnd*, ∂', can be obtained from the presentation $B = \mathbb{C}[x, y_2, z_2, t][u]$:

$$\partial'u = 1, \partial'x = \partial'z_2 = \partial'y_2 = \partial't = 0.$$

Using the notations of Theorem 1, the map $\mu(x, y_2, z_2, u, t) = y_1$ and the fiber of this map is $\mathbb{C}^2$ with coordinates z_1, r. Indeed, for $y_1 = c$ we have

$$x = \frac{z_1^2 - 1}{c},$$
$$w = \frac{(z_1 + 1)r - 3c^2}{4c},$$
$$u = 1/2(-xw + z_1 c),$$
$$y_2 = wz_1 + u^2 - c^2,$$
$$z_2 = z_1 - xu.$$

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T. Bandman, Department of Mathematics and Statistics, Bar-Ilan University, 52900, Ramat Gan, ISRAEL

E-mail address: bandman@macs.biu.ac.il

L. Makar-Limanov, Department of Mathematics, Wayne State University, Detroit, MI, 48202, USA

E-mail address: lml@math.wayne.edu