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DYNAMIC ASYMPTOTIC DIMENSION FOR ACTIONS OF VIRTUALLY CYCLIC GROUPS

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Abstract. We show that the dynamic asymptotic dimension of a minimal free action of an infinite virtually cyclic group on a compact Hausdorff space is always one. This extends a well-known result of Guentner, Willett, and Yu for minimal free actions of infinite cyclic groups.

1. Introduction

Dynamic asymptotic dimension was introduced by Erik Guentner, Rufus Willett, and Guoliang Yu in [11]. This is a notion of dimension for actions of discrete groups on locally compact spaces, and it was also defined in the (more general) setting of locally compact étale groupoids. It is related to transfer reducibility of Bartels, Lück, and Reich [3] and asymptotic dimension of Gromov [9], and it can be used to bound the corresponding nuclear dimension [4] of Winter and Zacharias [17] or to prove instances of the Baum–Connes conjecture [10].

The main non-trivial example of [11] shows that for minimal $\mathbb{Z}$-actions on infinite compact spaces the dynamic asymptotic dimension of the action is always one [11, Theorem 3.1]. The proof follows closely the ideas of Ian Putnam in building AF-algebras associated to minimal $\mathbb{Z}$-actions on the Cantor set [14]. Note that minimal $\mathbb{Z}$-actions on infinite spaces are automatically free.

The main purpose of this article is to prove the same rigidity result for minimal free actions of infinite virtually cyclic groups on compact spaces. In particular, we show that the dynamic asymptotic dimension of minimal free actions of the infinite dihedral group on compact spaces is always one.

The class of virtually cyclic groups plays a central role in geometric group theory. It appears in two important conjectures: The Juan-Pineda–Leary conjecture states that a discrete group Γ admitting a Γ-finite CW-model for its classifying space for virtually cyclic subgroups must itself be virtually cyclic [13]. The Farrell–Jones conjecture predicts that the map from the above classifying space of Γ to a point induces an isomorphism in equivariant K-theory and L-theory [8]. If the conjecture holds, one could restrict computations of algebraic K-theory and L-theory of group rings of Γ — which could happen to be very complicated for arbitrary finitely generated groups — to virtually cyclic subgroups of Γ. Moreover, if proved for both K- and L-theory, the Farrell–Jones conjecture can be used to deduce the Borel conjecture on topological rigidity of aspherical manifolds [2]. Currently, virtually cyclic groups have attracted additional attention due to recent developments on their classifying spaces [5, 6].

The Juan-Pineda–Leary conjecture has been confirmed for (acylindrically) hyperbolic groups, elementary amenable groups, 3-manifold groups, one-relator groups,

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CAT(0) cube groups, Artin groups, and linear groups. The Farrell–Jones conjecture is known for (relatively) hyperbolic groups, CAT(0) groups, 3-manifold groups, lattices in virtually connected Lie groups, mapping class groups, and free-by-cyclic groups. The latter results involve establishing the finiteness of certain relative versions of equivariant asymptotic dimension (also known as the finite $\mathcal{F}$-amenability), a notion closely related to the dynamic asymptotic dimension.

Calculating the value of dynamic asymptotic dimension has so far proved to be rather difficult (cf. a question of Willett in [15, Question 8.9]). A significant progress was made in [16], where an exponential upper bound was obtained for free actions of nilpotent groups (on compact metric spaces of finite covering dimension), which was later improved to a linear estimate in [15]. These bounds depend on the dimension of the space acted upon. Already in the context of establishing the finiteness of dynamic asymptotic dimension rather than calculating its exact value, the relation between an action and its restriction to a finite index subgroup is not clear, and a different argument was used in order to extend the finiteness result of [16] to the class of *virtually* nilpotent groups [1]. For actions on zero-dimensional spaces, the dynamic asymptotic dimension is known to be equal to certain other notions of dimension [12], including the amenability dimension.

The finiteness of dynamic asymptotic dimension conjecturally implies finite dynamical complexity [10], but at present this is only known if the dynamic asymptotic dimension equals zero or one. Therefore, it follows from our main result that free minimal actions of infinite virtually cyclic groups have finite dynamical complexity.

2. Definitions and Results

In the sequel, Γ will denote a discrete group acting on a compact Hausdorff space X by homeomorphisms. We shortly denote such an action by $\Gamma \curvearrowright X$. For a finite subset E of Γ we use the notation $E \Subset \Gamma$, and we denote the identity element of Γ by e.

Guentner, Willett, and Yu define $\Gamma \curvearrowright X$ to have dynamic asymptotic dimension at most d if for any $E \Subset \Gamma$ the space can be divided into $d+1$ pieces in such a way that on each piece the "action" has "finite complexity" with respect to E [11]. More precisely, we have the following definition.

Definition 2.1. For $E \subseteq \Gamma$ and an open subset $U \subseteq X$, the equivalence relation $\sim_{U,E}$ on U generated by E is defined as follows: for $x, y \in U$, $x \sim_{U,E} y$ if there is $n \in \mathbb{N}$ and a finite sequence $x = x_0, x_1, \ldots, x_n = y$ in U such that for each $1 \le j \le n$, there exists $g \in E \cup E^{-1}$ such that $gx_{j-1} = x_j$.

The *dynamic asymptotic dimension* of a free action $\Gamma \curvearrowright X$ (denoted $\operatorname{dad}(\Gamma \curvearrowright X)$) is the smallest integer $d \in \mathbb{N}$ with the following property: for each finite subset $E \Subset \Gamma$, there is an open cover $\{U_0, \ldots, U_d\}$ of X such that for each $0 \le i \le d$ the equivalence relation $\sim_{U_i,E}$ on U_i has uniformly finite equivalence classes. If no such d exists, we say that the dimension is infinite.

When U is the whole of X, the relation $\sim_{U,E}$ is the equivalence relation of being in the same $\langle E \rangle$-orbit, where $\langle E \rangle$ denotes the subgroup generated by E. If $\operatorname{dad}(\Gamma \curvearrowright X) = 0$, then there is no choice but to take $U_0 = X$, and hence $\operatorname{dad}(\Gamma \curvearrowright X) = 0$ implies that $\langle E \rangle$-orbits are finite for every $E \Subset \Gamma$, i.e. Γ is locally finite. However, if U_i is a proper subset of X, then equivalence classes of $\sim_{U_i,E}$ may be smaller than the intersection of U_i with $\langle E \rangle$-orbits, and Definition 2.1 requires a uniform bound on their cardinality (depending only on E).

Recall that an action $\Gamma \curvearrowright X$ is *free* if only the identity element e fixes some point in X. When this is not the case, in the above definition the uniform finiteness of equivalence classes has to be replaced by the finiteness of the set of all $g \in \Gamma$ for which there exist $0 \le i \le d$, $x \in U_i$, and a finite sequence $g_1, \dots, g_n \in E$ such that $g = g_n \cdots g_2 g_1$, and $(g_j \cdots g_1)x \in U_i$ for all $1 \le j \le n$ (in this formulation, it is perhaps most natural to consider symmetric $E \Subset \Gamma$).

An action $\Gamma \curvearrowright X$ is *minimal* if X has no proper closed (equivalently, open) Γ-invariant subsets. Minimality is the same as requiring that all Γ-orbits are dense in X. By the Kuratowski–Zorn lemma, every compact Γ-space contains a closed Γ-invariant subset on which Γ acts minimally.

A discrete group Γ is called *virtually cyclic* (or *cyclic by finite*) if it has a cyclic subgroup of finite index. When Γ is infinite, this means that there is a finite index copy of $\mathbb{Z}$ inside Γ. It follows that infinite virtually cyclic groups are residually finite and finitely presented.

An infinite virtually cyclic group Γ is known to have a finite normal subgroup N such that Γ/N is either infinite cyclic or infinite dihedral (see e.g. Theorem 6.12 in Chapter IV of [7]); depending on this one sometimes classifies Γ into *type I* or *type II*, which is reflected in the proof of our main result. Moreover, in an infinite virtually cyclic group Γ, every infinite cyclic subgroup of Γ has finite index. Indeed, if H, K are two infinite cyclic subgroups of Γ and H is of finite index, then $H \cap K$ must be an infinite subgroup of H, so $[H : H \cap K] < \infty$. Since H is of finite index, we get $[\Gamma : H \cap K] < \infty$, and hence $[\Gamma : K] < \infty$.

The main result of this paper extends Theorem 3.1 of [11] from the case of infinite cyclic to that of infinite virtually cyclic groups.

Theorem 2.2. *Let $\Gamma \curvearrowright X$ be a minimal and free action of an infinite virtually cyclic group on a compact Hausdorff space. Then,* $\mathrm{dad}(\Gamma \curvearrowright X) = 1$.

This type of rigidity result has not been known for any concrete example of virtually cyclic groups except for the group of integers. In particular, Theorem 2.2 applies to free minimal actions of the infinite dihedral group.

3. Proofs

The following lemma ensures the existence of open subsets that have disjoint translates by a given finite subset $E \Subset \Gamma$.

Lemma 3.1. *Let $\Gamma \curvearrowright X$ be a free action by homeomorphisms of a group Γ on a non-empty Hausdorff space X. Then, for every $E \Subset \Gamma \setminus \{e\}$ there exists a non-empty open subset $U \subseteq X$ such that $gU \cap U = \emptyset$ for all $g \in E$.*

Proof. Let $x \in X$. By freeness, we have $x \neq gx$ for every $g \in E$. By Hausdoffness, for every $g \in E$ there exist disjoint open neighbourhoods V_g, W_g of the points x, gx. We define $U = \bigcap_{g\in E} V_g \cap \bigcap_{g \in E} g^{-1}W_g$. Each set $g^{-1}W_g$ is open because Γ acts by homeomorphisms, so U is open as a finite intersection of open sets, and it is also non-empty since it contains x. Now, for any $h \in E$ we have:

$$\begin{aligned} U \cap hU &= \Big(\bigcap_{g\in E} V_g \cap \bigcap_{g\in E} g^{-1}W_g\Big) \cap h\Big(\bigcap_{g\in E} V_g \cap \bigcap_{g\in E} g^{-1}W_g\Big) \\ &= \Big(\bigcap_{g\in E} V_g \cap \bigcap_{g\in E} g^{-1}W_g\Big) \cap \Big(\bigcap_{g\in E} hV_g \cap \bigcap_{g\in E} hg^{-1}W_g\Big) \\ &\subseteq V_h \cap hh^{-1}W_h = \emptyset. \end{aligned}$$

because V_h, W_h were chosen to be disjoint. □

Proof of Theorem 2.2. The dynamic asymptotic dimension equals 0 only for actions of locally finite groups, and Γ contains an infinite-order element, so $\mathrm{dad}(\Gamma \curvearrowright X) \geq 1$. Hence, it suffices to prove the opposite inequality.

It is well known that an infinite virtually cyclic group Γ has a maximal finite normal subgroup $H \lhd \Gamma$, and the quotient Γ/H is either $\mathbb{Z}$ or $D_\infty = \langle s, t : s^2 = t^2 = e\rangle$. We will treat both cases simultaneously. Let p denote the quotient map $\Gamma \to \Gamma/H$.

Our proof of the upper bound follows the strategy from [11]. Let a finite $E \Subset \Gamma$ be given. By enlarging it if necessary, one can assume that $E = p^{-1}(B_N)$ for some $N \in \mathbb{N}_{>0}$, where B_N denotes the ball of radius N around the identity element in the group Γ/H with respect to the standard metric on $\mathbb{Z}$ or the word metric on D_∞ associated with the generating set $\{s, t\}$.

By Lemma 3.1, there exists a non-empty open subset $U \subseteq X$ such that $U \cap gU = \emptyset$ for $g \in p^{-1}(B_{5N}) \setminus \{e\}$. By the regularity of compact Hausdorff spaces, there is a smaller non-empty open set V such that $\overline{V} \subseteq U$. Define

$$U_0 = \bigcup_{g\in p^{-1}(B_N)} gU \quad \text{and} \quad U_1 = X \setminus \bigcup_{g\in p^{-1}(B_N)} g\overline{V}.$$

Clearly, $\{U_0, U_1\}$ forms an open cover of X.

It now suffices to prove that for $i \in \{0, 1\}$ the following set is finite

$$F_i = \left\{ g \in \Gamma \;\middle|\; \begin{array}{l} \text{there exist } g_1, \dots, g_n \in E \text{ and } x \in U_i \text{ such that} \\ g = g_n \cdots g_2 g_1 \text{ and } g_j \cdots g_1 x \in U_i \text{ for all } j \in \{1, \dots, n\} \end{array} \right\},$$

or equivalently that the images of F_i inside Γ/H are finite.

We begin with $i = 0$, claiming that $p(F_0) \subseteq B_{3N}$. Suppose for contradiction that there exist $g_1, \dots, g_n \in E$ and $x \in U_0$ such that $p(g_n \cdots g_1) \notin B_{3N}$, and $x_j := g_j \cdots g_1 x \in U_0$ for all $j \in \{0, \dots, n\}$ (for $j = 0$ one just gets $x_0 := x$). There exists $k \in \{3, \dots, n\}$ such that $p(g_k \cdots g_1) \in B_{3N} \setminus B_{2N}$. By the definition of U_0, the point x can be expressed as gx_U for some $g \in p^{-1}(B_N)$ and $x_U \in U$. Similarly, $x_k = g' x'_U$ for some $g' \in p^{-1}(B_N)$ and $x'_U \in U$. But then

$$x'_U = (g')^{-1} x_k = (g')^{-1} g_k \cdots g_1 x = (g')^{-1} g_k \cdots g_1 g x_U,$$

which is a contradiction because $h := (g')^{-1} g_k \cdots g_1 g$ belongs to $p^{-1}(B_{5N} \setminus B_0) \subseteq p^{-1}(B_{5N}) \setminus \{e\}$, and we assumed that $U \cap hU = \emptyset$ for such h.

We are done with F_0, so let us now consider F_1.

Definition of Case I. If $\Gamma/H \simeq \mathbb{Z}$, we will denote $Z := \Gamma$, and divide it into the positive and negative "halves":

$$Z^+ := p^{-1}(\mathbb{Z}_{\geq 0}) \quad \text{and} \quad Z^- := p^{-1}(\mathbb{Z}_{<0}). \tag{1}$$

Definition of Case II. If $\Gamma/H \simeq D_\infty$, then there is still the cyclic subgroup $\langle st\rangle < D_\infty$ of index 2, and we can define $Z := p^{-1}(\langle st\rangle)$ as the corresponding index 2 subgroup of Γ and $Z^\pm$ by the same formula (1) (in this case, fixing an explicit isomorphism $\mathbb{Z} \simeq \langle st\rangle$ will be needed, and we pick the one sending 1 to st). Let "Case IIa" denote the situation when the Z-action on X is minimal, and "Case IIb" the remaining case.

Cases I and IIa. Note that not only in Case IIa but also in Case I the Z-action on X is minimal. We will now show that it follows that the set of limit points of Z^+x (and alike for Z^-x) is the whole of X for every $x \in X$. Note that for every $g \in Z$, the symmetric difference $gZ^+ \triangle Z^+$ is finite (by the fact that the same clearly holds for $n + \mathbb{Z}_{\geq 0} \triangle \mathbb{Z}_{\geq 0}$ with any $n \in \mathbb{Z}$), and hence the set of limit points of Z^+x and gZ^+x is the same. Thus, the set of limit points of Z^+x is a non-empty (by compactness) closed Z-invariant subset, so it must equal X.

By its density, the "half-orbit" Z^+x of every x must intersect V. In other words, the "inverse half-orbit" $(Z^+)^{-1}V$ of V covers X. By compactness, there exists $M \in \mathbb{N}$ such that already $(p^{-1}(B_M) \cap Z^+)^{-1}V$ covers X. That is, for every $x \in X$

one can pick $g_x^+ \in p^{-1}(B_M) \cap Z^+$ such that $g_x^+ x \in V$. After increasing M if necessary, we have analogous elements $g_x^- \in p^{-1}(B_M) \cap Z^-$.

We claim that $p(F_1) \subseteq B_{M+N}$. Indeed, suppose for contradiction that there exist $g_1, \ldots, g_n \in E$ and $x \in U_1$ such that $g_n \cdots g_1 \notin p^{-1}(B_{M+N})$, and $x_j := g_j \cdots g_1 x \in U_1$ for all $j \in \{0, \ldots, n\}$.

In Case I, if $p(g_n \cdots g_1) > 0$, we put $g_x := g_x^+$, and we put $g_x := g_x^-$ if $p(g_n \cdots g_1) < 0$. This way, one has respectively either

$$p(g_x^{-1}) \leq 0 \quad \text{and} \quad p(g_n \cdots g_1 g_x^{-1}) > N$$

or

$$p(g_x^{-1}) > 0 \quad \text{and} \quad p(g_n \cdots g_1 g_x^{-1}) < -N.$$

In both situations, there exists $k \in \{0, \ldots, n-1\}$ such that $p(g_k \cdots g_1 g_x^{-1}) \in B_N$. However, this yields a contradiction: on the one hand, we assumed that $x_k \in U_1$, and on the other hand $x_k = (g_k \cdots g_1 g_x^{-1})(g_x x) \in p^{-1}(B_N)V \subseteq X \setminus U_1$.

In Case IIa, if the minimal word representing $p(g_n \cdots g_1)$ ends with t, then we put $g_x := g_x^+$, and we put $g_x := g_x^-$ if it ends with s. (Recall that $p(g_x^+) \in \mathbb{Z}_{\geq 0}$, so $p(g_x^+) = (st)^l$ for some $l \geq 0$, and $p(g_x^-) \in \mathbb{Z}_{<0}$, so $p(g_x^-) = (st)^{-l} = (ts)^l$ for some $l > 0$.) The respective consequences are as follows:

- the minimal word representing $p(g_n \cdots g_1 g_x^{-1})$ ends with t[1], and its length is more than N[2], and the minimal word representing $p(g_x^{-1})$ ends with s (or it is trivial); or
- the minimal word representing $p(g_n \cdots g_1 g_x^{-1})$ ends with s, and its length is more than N, and the minimal word representing $p(g_x^{-1})$ ends with t.

In either case, there exists $k \in \{0, \ldots, n-1\}$ such that $p(g_k \cdots g_1 g_x^{-1}) \in B_N$. This again leads to a contradiction.

Case IIb. Let us now treat the case when the action $Z \curvearrowright X$ is not minimal, i.e. there exists $x \in X$ such that $A := \overline{Zx} \subsetneq X$. By minimality, for any $g \in \Gamma \setminus Z$ one has

$$X = \overline{\Gamma x} = \overline{(Z \sqcup gZ)x} = \overline{Zx} \cup \overline{gZx},$$

and in fact the last union is also disjoint: the intersection $\overline{Zx} \cap \overline{gZx}$ must be empty because it is a proper Γ-invariant closed subset of X. That is, X splits as a disjoint union of Z-invariant clopen subsets $A, X \setminus A$, and elements of $\Gamma \setminus Z$ switch the two subsets.

We claim that the Z-action on A is minimal. Indeed, if there were $y \in A$ such that $\overline{Zy} \subsetneq A$, then $\overline{\Gamma y}$ would be a non-empty closed Γ-invariant proper subset of X:

$$X = A \sqcup gA \supsetneq \overline{Zy} \sqcup g\overline{Zy} = \overline{\Gamma y}$$

(here, again g is any element of $\Gamma \setminus Z$). Hence, the Z-action on A is minimal, and we conclude the same for its complement gA because the minimality of $Z \curvearrowright A$ is equivalent to the minimality of $gZg^{-1} \curvearrowright gA$ and $gZg^{-1} = Z$. That is, the roles of A and gA are symmetric, and the choice of the point $x \in X$ did not matter for the obtained decomposition.

By switching the roles of A and its complement, one can assume that the intersection $V \cap A$ is non-empty, bringing us back to a situation very similar to that previously considered in Cases I and IIa: the action $Z \curvearrowright A$ is minimal, and $V \cap A$

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[1]Because the minimal word representing $p(g_n \cdots g_1)$ has the form $tstst \cdots stst = t(st)^m$ or $stst \cdots stst = (st)^m$, and the minimal word representing $p(g_x)$ has the form $stst \cdots stst = (st)^l$ for some $0 \leq l < m$, we conclude that $p(g_n \cdots g_1 g_x^{-1}) = p(g_n \cdots g_1)(p(g_x))^{-1}$ has the form $t(st)^{m-l}$ or $(st)^{m-l}$.

[2]Because the length of $p(g_n \cdots g_1)$ is more than $M + N$, and the length of $p(g_x^{-1})$ is at most M.

is a non-empty open subset of A. Hence, there exists $M \in \mathbb{N}$ such that for every $x \in A$ one can pick $g_x^+ \in p^{-1}(B_M) \cap Z^+$ and $g_x^- \in p^{-1}(B_M) \cap Z^-$ such that $g_x^+ x, g_x^- x \in V \cap A$.

Again, we claim that $p(F_1) \subseteq B_{M+N}$. Suppose for contradiction that there exist $g_1, \ldots, g_n \in E$ and $x \in U_1$ such that $g_n \cdots g_1 \notin p^{-1}(B_{M+N})$, and $x_j := g_j \cdots g_1 x \in U_1$ for all $j \in \{0, \ldots, n\}$.

Subcase IIb.1. If $x \notin A$, we pick $\sigma \in p^{-1}(\{s,t\})$ and define $y := \sigma^{-1}x$, $h_1 := g_1\sigma$, and $h_j := g_j$ for $j \in \{2, \ldots, n\}$: this guarantees that $h_j \cdots h_1 y = x_j \in U_1$ for $j > 0$, but it may happen that $y \notin U_1$. By choosing σ such that $p(\sigma)$ equals either s or t, one can ensure that $p(h_1) = p(g_1\sigma)$ is shorter than $p(g_1)$ (unless of course $p(g_1) = e$), and in particular we have $p(h_j) \in B_N$ for $j \in \{1, \ldots, n\}$. It may however happen that the length of $p(h_n \cdots h_1)$ is shorter than that of $p(g_n \cdots g_1)$, but we know that $p(h_n \cdots h_1) \notin B_{M+N-1}$.

Subcase IIb.2. If $x \in A$, then we just put $y := x$ and $h_j := g_j$ for $j \in \{1, \ldots, n\}$, in which case one has $h_j \cdots h_1 y = x_j \in U_1$ for all $j \geq 0$.

Having defined y and (h_j) in both subcases of Case IIb, we finish as in Case IIa. If the minimal word representing $p(h_n \cdots h_1)$ ends with t, then we put $g_y := g_y^+$, and otherwise we put $g_y := g_y^-$. Then, respectively either

- the minimal word representing $p(h_n \cdots h_1 g_y^{-1})$ ends with t, and its length is more than $N - 1$, and the minimal word representing $p(g_y^{-1})$ ends with s (or it is trivial); or
- the minimal word representing $p(h_n \cdots h_1 g_y^{-1})$ ends with s, and its length is more than $N - 1$, and the minimal word representing $p(g_y^{-1})$ ends with t.

In both cases, there exists $k \in \{1, \ldots, n\}$ such that $p(h_k \cdots h_1 g_y^{-1}) \in B_N$ (it is important that we can find such $k \neq 0$ because one knows that $h_k \cdots h_1 y \in U_1$ only for $k \neq 0$). This yields a contradiction as before: we have both $h_k \cdots h_1 y \in U_1$ and $h_k \cdots h_1 y = (h_k \cdots h_1 g_y^{-1})(g_y y) \in p^{-1}(B_N)(V \cap A)$, while U_1 and $p^{-1}(B_N)V$ are disjoint. □

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