Length of Geodesics and Quantitative Morse Theory on Loop Spaces.

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Abstract

Let M^n be a closed Riemannian manifold of diameter d. Our first main result is that for every two (not necessarily distinct) points $p, q \in M^n$ and every positive integer k there are at least k distinct geodesics connecting p and q of length $\leq 4nk^2d$.

We demonstrate that all homotopy classes of M^n can be represented by spheres swept-out by "short" loops unless the length functional has "many" "deep" local minima of a "small" length on every space of based loops (or even paths with fixed endpoints) on M^n. For example, one of our results asserts that for every two points $p, q \in M^n$ and every positive integer k there are two possibilities: Either the length functional on the space $\Omega_{pq}M^n$ of paths connecting p and q has k distinct non-trivial local minima with length $\leq 2kd$ and "depth" $\geq 2d$; or for every m every map of S^m into $\Omega_{pq}M^n$ is homotopic to a map of S^m into the subspace $\Omega_p^{4(k+1)(m+1)d}M^n$ of $\Omega_{pq}M^n$ that consists of all paths of length $\leq 4(k+1)(m+1)d$.

1 Main results.

One of the goals of this paper is to prove an effective version of a famous theorem published by J.P. Serre in 1951 ([Se]) that asserts that for every pair of points on a closed Riemannian manifold there exist infinitely many distinct geodesics connecting these points. Here and below two geodesics or geodesic loops are regarded as distinct if they do not differ by a reparametrization.

In our paper [NR0] we have conjectured that there exists a function $f(k, n)$ such that for every positive integer k and every pair of points p, q on a closed n-dimensional Riemannian manifold of diameter d there exist at least k distinct geodesics connecting p and q of length $\leq f(k, n)d$.

In the present paper we prove this conjecture for $f(k,n)=4k^2n$. We first prove it in the case of simply connected manifolds. The general case will then easily follow.

The starting point will be a proof of Serre's theorem by Albert Schwarz ([Sc]). In this paper Schwarz also demonstrates that the length of kth geodesic can be bounded above by $C(M^n)k$, where $C(M^n)$ does not depend on k but only on the Riemannian manifold M^n. (This estimate was later improved by M. Gromov in section 1.4 of [Gr0] in the situation, when p and q are not conjugate allong any geodesic. Gromov proved that in this case the number of geodesics of length $\leq x$ connecting p and q is not less than the sum of Betti numbers $b_i(\Omega_p M^n)$ over i ranging from 1 to $[c(M^n)x]$ for an appropriate constant $c(M^n)$. Whenever for some manifolds (e.g. S^n) this still provides only a linear in k upper bound for the length of a kth shortest geodesic between p and q, for "many" manifolds the sum of the Betti numbers of the loop space grows exponentially in x, and one obtains a logarithmic in k upper bound for the length of a k shortest geodesic.)

The proof of Serre's theorem given by Albert Schwarz, roughly, goes as follows:

Let us consider the space $\Omega_p M^n$ of loops based at p on a closed simply-connected Riemannian manifold M^n. One would like to show that the sum of its Betti numbers is infinite. Then the existence of infinitely many geodesic loops based at p would follow from a standard Morse-theoretic argument.

In fact, Schwarz notes that the Cartan-Serre theorem (cf. [FHT], Theorem 16.10) implies that there exists an even-dimensional real cohomology class u of the loop space $\Omega_p M^n$ such that all its cup powers u^i are non-trivial, thus implying that the sum of Betti numbers of $\Omega_p M^n$ is infinite. Applying Morse theory one obtains a critical point of the length functional corresponding to each power of u. If the critical points are not distinct, i.e. there is a critical point corresponding to u^i and u^j for $i \neq j$, the standard Lyusternik-Schnirelman argument, (see [Kl]), implies that the critical level that corresponds to u^i contains a set of critical points of dimension $\geq dim\ u > 0$, implying the existence of infinitely many geodesic loops based at p. (Schwarz also noted such a degenerate situation cannot occur at all if $dim\ u \geq n$ as the dimension of the set of all geodesics between p and q cannot exceed $n-1$.)

Thus, it is enough to consider the situation when the critical points are distinct. Note also, that an easy argument involving the basics of rational homotopy theory implies that this cohomology class u exists in a dimension $\leq 2n-2$.

Now recall that the Pontryagin product in the rational homology group of the loop space is the product induced by the geometric product $\Omega_p M^n \times \Omega_p M^n \longrightarrow \Omega_p M^n$. (By the geometric product of two loops α and β we just mean their join $\alpha * \beta$.) To estimate the length of the geodesics corresponding to u^i Schwarz defines a "dual", (meaning $< u, c >= 1$), homology class c of u of the same dimension. Then he proves that for every positive i the ith Pontryagin power of c and a rational multiple of u^i are dual. So, the critical point corresponding to u^i also corresponds to c^i.

One can see now that in order to estimate lengths of geodesic loops based at p it is enough to find a representative of c that is is contained in the set of loops based at p of length $\leq L$ for some L. Then c^i can be represented by a chain contained in the set of loops of length $\leq iL$.

To obtain an upper bound for geodesics connecting distinct points $p, q \in M^n$, one considers an explicit homotopy equivalence $h : \Omega_p M^n \longrightarrow \Omega_{p,q} M^n$ that is constructed by fixing a minimizing geodesic between p and q and attaching it at the end of each loop based at p. Then $h_*(u^i)$ can be represented by a chain contained in the set of paths of length $\leq iL + d$ between p and q, whence the length of the ith shortest geodesic between p and q does not exceed $iL + d$.

It is natural to make a conjecture that the length of a "kth-shortest" geodesic between two arbitrary points p, q on an arbitrary closed Riemannian manifold M^n should not exceed kd, where d is the diameter of M^n. Indeed, this conjecture is obviously true for round spheres. On the other end of the spectrum the conjecture is true for closed Riemannian manifolds with torsion-free fundamental groups (Proposition 2 in [NR0]). Yet this conjecture was disproved by a recent example of F. Balacheff, C. Croke, M. Katz ([BCK]). They have proved the existence of a Zoll metric on a 2-sphere for which the length of a shortest periodic geodesic, (and thus, trivially, a shortest non-trivial geodesic loop based at any point) is greater than twice the diameter of the Zoll sphere. As the shortest non-trivial geodesic loop is a second shortest geodesic from its base point to itself, this example shows that the conjecture is false even if $n = k = 2$, the Riemannian manifold is convex and arbitrarily close to a round 2-sphere, and $p = q$ is an arbitrary point of the manifold.

Our proof of the quadratic in k upper bound works as follows. We demonstrate that for every l there are two classes of Riemannian metrics on each closed manifolds: "nice" metrics, where for every m every m-dimensional homotopy class of the manifold can be "swept-out" by "short" loops (of length $\sim lmd$), and "bumpy" metrics, where the length functional on every space of all paths connecting a pair of points has l ("deep") local minima of a controlled length. If the metrics is very "nice" one obtains a linear in $lmdN$ upper bound for the lengths of N distinct geodesic loops immediately from the proof of Serre's theorem by Schwarz. If the metric is very "bumpy", one obtains many short geodesic loops immediately from the definition of "bumpiness".

The case, when our estimate becomes quadratic in k, is the case of Riemannian metrics that are neither "bumpy" enough, nor "nice" enough, so that there are approximately $l = \frac{k}{2}$ "deep" local minima of the length functional on $\Omega_p M^n$ with lengths $\leq 2kd$. These $\frac{k}{2}$ local minima could prevent us from sweeping-out the cycle of interest for us by loops of length smaller than $c(n)kd$ (for an appropriate $c(n)$). As the result the bound for the length of the longest of remaining $\frac{k}{2}$ geodesic loops that we need that follows from the proof of J.-P. Serre's becomes quadratic in k.

Whenever we do not have any actual examples of families of Riemannian metrics demonstrating that the quadratic dependence of our estimate on k is optimal, we believe that they exist - at least in dimensions > 3. So, we conjecture that in general case there is no *curvature-free* upper bound for the lengths of the k shortest geodesics of the form $f(n,k)d$, where f grows slower than a quadratic function of k.

Note also that even in the case of a 2-sphere one cannot hope to find a sweep-out of the cycle c from Schwarz's proof of Serre's theorem by "short" loops due to a counterexample of S. Frankel and M. Katz ([FK]), who found a family of Riemannian metrics on the 2-disc with uniformly bounded diameter and the length of the boundary but such that for every fixed value of τ it is impossible to contract boundaries of each of these 2-discs via closed curves of length $\leq \tau$. Taking the doubles of these 2-discs one obtains a family of Riemannian metrics on S^2 with uniformly bounded diameter that do not admit sweep-outs into loops with uniformly bounded lengths.

We will, however, demonstrate that sweep-outs by short loops can only be obstructed by the existence of many short geodesic loops at each point of a manifold.

To state the first of our main results denote the space of loops of length $\leq L$ based at p on M^n by $\Omega_p^L M^n$.

Theorem 1.1 *Let M^n be a closed Riemannian manifold of dimension n and diameter d, p a point of M^n, k a positive integer number. Then either: 1) There exist non-trivial geodesic loops based at p with lengths in every interval $(2(i-1)d, 2id]$ for $i \in \{1, 2, ..., k\}$. Moreover these geodesic loops*

are local minima of the length functional on $\Omega_p M^n$;
or
2) For every positive integer m every map $f : S^m \longrightarrow \Omega_p M^n$ is homotopic to a map $g : S^m \longrightarrow \Omega_p^{((4k+2)m+(2k-3))d} M^n$. Further, every map $f : (D^m, \partial D^m) \longrightarrow (\Omega_p M^n, \Omega_p^{((4k+2)m+(2k-3))d} M^n)$ is homotopic to a map $g : (D^m, \partial D^m) \longrightarrow \Omega_p^{((4k+2)m+(2k-3))d} M^n$ relative to ∂D^m. If for some L the image of f is contained in $\Omega_p^L M^n$, then the homotopy between f and g can be chosen so that its image is contained in $\Omega_p^{L+2d} M^n$. Also, in this case for every L every map f from S^0 to $\Omega_p^L M^n$ is homotopic to a map g from S^0 to $\Omega_p^{(2k-1)d} M^n$ by a homotopy such that its image is contained in $\Omega_p^{L+2d} M^n$.

This theorem immediately leads to a quadratic bound for the lengths of geodesic loops based at p. Suppose that for some $s < k$ there are $s-1$ non-trivial geodesic loops based at p with lengths in the intervals $(0, 2d], (2d, 4d], ..., (2(s-2)d, 2(s-1)d]$, but no geodesic loops based at p of length in the interval $(2(s-1)d, 2sd]$. Then there exists a representation of an even-dimensional cycle c in the loop space that appears in the proof of Serre's theorem given by A. Schwarz by a spherical cycle that can be formed only by loops of length at most $((8n-6)s + (4n-7))d$ based at p. Thus, we obtain at least s geodesic loops based at p of length $\leq 2(s-1)d$ (including the trivial loop), $s+1$ loops of length $\leq ((8n-6)s + (4n-7))d$, $s+2$ loops of length $\leq 2((8n-6)s + (4n-7))d$,..., $s+i$ loops of length $\leq i((8n-6)s+(4n-7))d$,..., k loops of length $\leq (k-s)((8n-6)s+(4n-7))d$. This expression attains its maximum at $s = \lfloor \frac{k}{2} \rfloor$. The maximal value is $((2n-\frac{3}{2})k^2 + (2n-\frac{7}{2})k - (1+(-1)^k))d$. Note that none of the cycles c^i from the prooof of Serre's theorem by A. Schwarz can "hang" at a local minimum of the length functional on $\Omega_p M^n$ (at least, unless there is a critical level of a dimension $\geq dim\ c$ bu $\leq n-1$ at this local minimum. In such a case one of the local minima will be "lost" due the coincidence, but we will immediately get infinitely many distict geodesics of the same length, which results in a much better upper bound for the length.) Therefore wilthout any loss of generality we can assume that these geodesic loops are distinct. Thus, we are guaranteed to have at least k distinct geodesic loops based at a point p of length $(2n - \frac{3}{2})k^2 + (2n - \frac{7}{2})k - (1 + (-1)^k))d$ (including the trivial loop). (The new geodesic loops are distinct from each other and from the local minima because they have distinct positive indices, when regarded as the critical points of the length functional.) Thus, one obtains the following theorem in the case when M^n is simply-connected, and $p = q$.

Theorem 1.2 *Let M^n be a closed Riemannian n-dimensional manifold with diameter d. Then for every point $p \in M^n$ there exist at least k distinct geodesic loops of length at most $((2n-1.5)k^2+(2n-3.5)k-(1+(-1)^k))d < 2n(k^2+k)d$. More generally, for each pair points $p, q \in M^n$ there exist at least k geodesics starting at p and ending at q of length $\leq ((2n-1.5)k^2+(2n-3.5)k)d+(2n-1.5)kd(p,q)$, if k is even, and $\leq ((2n-1.5)k^2+(2n-3.5)k-2)d+(2n-1.5)(k+1)d(p,q)$, if k is odd. In both cases this upper bound does not exceed $((2n-1.5)k^2+(4n-5)k+(2n-3.5))d < 2n(k+1)^2d$.*

Remark. Denote the smallest *odd* number l such that there exists a non-trivial *rational* homotopy class of M^n by l. An elementary rational homotopy theory (cf. [FHT]) implies that $l \leq 2n-1$. Our proof of 1.2 implies upper bounds $((l-0.5)k^2+(l-2.5)k-(1-(-1)^k))d$ for the lengths of k distinct geodesic loops based at an arbitrary point p of M^n. Similarly for arbitrary $p, q \in M^n$ and arbitrary k there exist at least k distinct geodesics of length not exceeding $((l-0.5)k^2+(l-2.5)k)d+(l-0.5)kd(p,q)$, if k is even, and $((l-0.5)k^2+(l-2.5)k-2)d+(l-0.5)(k+1)dist(p,q)$, if k is odd. These estimates do not exceed $((l-0.5)k^2+(2l-3)k+(l-2.5))d < l(k+1)^2d$. Also note, that in [NR3] we proved a version of Theorem 1.2 in the case $l=3$ but with a worse upper bound that depended factorially on k.

Also, note that $2n(k+1)^2d < 4nk^2d$ for all $k \geq 3$, and that we have a better bound $2nd (< 4nk^2)$, when $k = 2$, proven in [NR1], Therefore, if desired, one can replace the upper bounds for the lengths of k shortest geodesics between p and q in M^n provided by Theorem 1.2 by a simpler looking expression $4nk^2d$.

To prove Theorem 1.2 in the case when M^n is simply-connected but $p \neq q$ we prove a generalization of Theorem 1.1 where Ω_pM^n is replaced by the space $\Omega_{p,q}M^n$ (Theorem 5.3). It immediately yields Theorem 1.2 in the case, when $p \neq q$, exactly as Theorem 1.1 implied the case $p=q$.

To obtain Theorem 1.2 in the nonsimply-connected case we will consider the universal covering of M^n constructed from the space of all paths starting at p via the standard identification and endowed with the pull back Riemannian metric. According to the standard argument that can be found in textbooks on Riemannian geometry one can choose the fundamental domains so that their interiors are all isometric to the complement of the cut locus of the base point p, and, therefore, their diameter does not exceed $2d$. If the cardinality of $\pi_1(M^n)$ is infinite or finite but $\geq k$, we will connect the base point $\tilde{p}$ in the universal covering $\tilde{M}^n$ of M^n with k closest liftings of q by shortest geodesics. The projections of these geodesics to M^n will have lengths $\leq (2k-1)d$, and the theorem follows. If the cardinality of $\pi_1(M^n)$ is less than k, then we observe that $\tilde{M}^n$ is a simply-connected manifold of diameter $\tilde{d} \leq 2|\pi_1(M^n)|d$ (as the diameter of each fundamental domain does not exceed $2d$). Let k_s denote the smallest integer number which is not less than $\frac{k}{|\pi_1(M^n)|}$. We are going to connect $\tilde{p}$ with each lifting of q by k_s or $k_s - 1$ distinct geodesics, so as to obtain the required number k of distint geodesics between p and q after projecting down to M^n. (Obviously, if we need to connect $\tilde{p}$ with some points in the lifting of q to $\tilde{M}^n$ with k_s geodesics, and with some other points in the lifting of q with $k_s - 1$ geodesics, we choose points that we connect with $\tilde{p}$ by k_s geodesics to be the closest to $\tilde{p}$.) If one knows how to prove Theorem 1.2 in the simply-connected case, then one can get a slightly worse upper bound (but still with the leading term $\frac{2}{|\pi_1(M^n)|}(2n-1.5)k^2 d \leq (2n-1.5)k^2 d$) in the nonsimply-connected case. (Indeed, asymptotically k^2 will be divided by $|\pi_1(M^n)|^2$, and multiplied by $2|\pi_1(M^n)vert.$) To prove a better upper bound we will need the following:

Theorem A *Let M be a closed Riemannian manifold of diameter d with a finite fundamental group of cardinality C. The the diameter of the universal covering space $\tilde{M}$ of M endowed with the pull back metric does not exceed Cd.*

It is hard to believe that Theorem A is not known, yet we were not able to find any mention of it in the literature. Therefore we will prove it in Section 6 of this paper. There we also present a proof the following generalization of Theorem A:

Theorem B *If the fundamental group of a closed Riemannian manifold M of diameter d is either infinite or finite of order $\geq k$, then for every pair of points $p, q \in M$ and every k there exist at least k geodesics connecting p and q of length $\leq kd$ that represent different path homotopy classes.*

Using Theorem A we were able to verify that the described simple procedure that allows one to reduce Theorem 1.2 in for a nonsimply-connected M^n to Theorem 1.2 to its universal covering $\tilde{M}^n$ leads to the upper bounds that are not worse than the estimates in the simply-connected case. As it was already mentioned, the verification mostly involves checking what happens for small values of k. We are not going to present here the details of these straightforward and elementary but tedious calculations.

The proof of Theorem 1.1 is based on a seemingly new curve shortening process. Before introducing this process in the proof of the following theorem recall that a path homotopy between two curves β and γ is a homotopy that preserves the end points. In other words, it is a family of curves $\alpha_\tau(t)$

continuously depending on $\tau \in [0,1]$ such that $\alpha_0 = \beta$, $\alpha_1 = \gamma$, and for every $\tau \in [0,1]$ $\alpha_\tau(0) = \alpha_0(0)$ and $\alpha_\tau(1) = \alpha_0(1)$.

Theorem 1.3 *Let M^n be a closed Riemannian manifold of diameter d, and p, q be two arbitrary points of M^n. Let $\gamma(t)$ be a curve of length L connecting points p and q. Assume that there exists an interval $(l, l+2d]$, such that there are no geodesic loops based at p on M^n of length in this interval that provide a local minimum of the length functional on $\Omega_p M^n$. Then there exists a curve $\tilde{\gamma}(t)$ of length $\leq l+d$ connecting p and q and a path homotopy between γ and $\tilde{\gamma}$ such that the lengths of all curves in this path homotopy do not exceed $L+2d$.*

Observe that this theorem immediately implies Theorem 1.1 for $m=0$. Indeed, S^0 consists of two points, so , if $m=0$, then f is just a set of two loops. In the absence of k short geodesic loops providing local minima for the length functional each of these two loops can be shortened as in Theorem 1.3.

Now we would like to give a brief review of some existing results related to Theorem 1.2. The first curvature-free upper bounds for the length of a shortest non-trivial geodesic loop on a closed Riemannian manifold in terms of either diameter or the volume of the manifold were proven by S. Sabourau in [S]. However, Sabourau considered the situation when the minimization of the length of the geodesic loop was performed also over all possible choices of the base point of the loop. R. Rotman ([R]) demonstrated that for every point p on every closed n-dimensional manifold of diameter d the length of the shortest geodesic loop based at p does not exceed $2nd$. (It is easy to see that if the base point is prescribed, then there is no upper bound for the length of the shortest geodesic loop in terms of the volume of the manifold, even if the manifold is diffeomorphic to the 2-sphere.) Note also that a shortest non-trivial geodesic loop based at p is a second shortest geodesic starting and ending at p. In [NR1] it was proven that the same estimate $2nd$ holds for the length of the second shortest geodesic between two arbitrary points p and q of arbitrary n-dimensional Riemannian manifold of diameter d. This is the best known upper bound in the case, when $k=2$ (for every simply-connected manifold M^n).

If $n=2$, one can also produce better estimates than the estimates provided by Theorem 1.2. In [NR2] we proved that if $n=2$ and M^n is diffeomorphic to the 2-sphere , then two arbitrary points can be connected by at least k distinct geodesics of length $\leq (4k^2-2k-1)d$. (In the same paper we also have shown that this estimate can be improved $4k^2 - 6k + 2$ if these points coincide). Making an almost obvious observation that the cycles corresponding to powers of u from the proof of A. Schwarz do not "hang" on local minima of the length functional and, therefore, geodesics corresponding to cup powers of u are different from the geodesics that are local minima of the length functional we can immediately improve these upper bounds for $k > 2$ to $(k^2 + 3k + 3)d$ in the case of geodesics connecting two distinct points of M^2 and $(k^2 + k)d$ in the case of geodesic loops based at any prsecribed points of M^2. (One of these $(k^2 + k)d$ geodesic loops can be trivial.) The simple argument used above to reduce Theorem 1.2 in the non-simply conected case to its simply-connected version (for the univeral covering) can be combined with these estimates for S^2 to deduce that the same upper bound $(k^2 + k)d$ for the lengths of k shortest geodesic loop based at a prescribed point of M^2 and $(k^2 + 3k + 3)d$ for the lengths of k shortest geodesics connecting every prescribed pair of points of M^2 in the case, when M^2 is diffeomorphic to RP^2. Note that Theorem B yields even better upper bounds (namely, kd), in the case, when a closed two-dimensional Riemannian manifold M^2 is not diffeomorphic to S^2 or RP^2. Thus, Theorem 1.2 should only be used in the case when $n, k \geq 3$ (and $|\pi_1(M^n)|$ is finite and "small").

In section 7 we discuss generalizations of Theorems 1.1, 1.3 and 5.3 that involve the notion of the depth of local minima of the length funactional. The formal definition of the depth will be given in section 7. Informally, the depth of a non-trivial local minimum γ of the length functional on $\Omega_p M^n$ is the difference between the maximal length of a loop during an "optimal" path homotopy contracting γ and the length of γ.

First, we observe that Theorem 1.3 remains valid if instead of assuming that there are no local minima of the length functional with length in the interval $(l, l + 2d]$ we assume that there are no local minima of depth $\geq 2d$ with the length in this interval. As a corollary, one can strengthen Theorem 1.1 by requiring in 1) that the geodesic loops with lengths in the intervals $(2(i-1)d, 2id]$ are not only local minima of the length functional, but local minima of depth $\geq 2d$. Then we generalize this stronger version of Theorem 1.1 (as well as of Theorem 5.3) by requiring in case 1) that the depth of these local minima is not only $\geq 2d$ but $\geq S$ for some $S \geq 2d$. The "price" is a corresponding increase of the lengths of loops in case 2) that is proportional to $S - 2d$.

We finish section 7 by observing that for a specific sufficiently large value of S the condition 1) in Theorem 1.1 is violated for $k = 1$, and the generalized form of condition 2) holds unconditionally.

As the result, we obtain a different proof of a well-known theorem first proven by M. Gromov (see section 1.4 in [Gr0] or ch. 7 in [Gr]) that asserts the existence of a constant C such that for every m the inclusion $\Omega_p^{Cm}M^n \subset \Omega_p M^n$ induces the surjective homomorphisms of homotopy groups in all dimensions up to m. (Here M^n is a closed simply-connected Riemannian manifold.) Our proof yields a good estimate for the constant C that seem to be better than the value that one can extract from the proof by Gromov. The comparison between our results and the results by M. Gromov is done in the last section of the paper.

2 A simple lemma and its multidimensional generalization.

The proof of 1.3 uses the following well-known lemma. To state this lemma we are going to introduce the following notations that we will be widely using further below in this paper. We are going to use $^-$ above a letter denoting a path to denote the same path travelled in the opposite direction from its endpoint to its beginning. If a is a path from x to y, amd b is a path from y to z, then we will denote by $a * b$ the join of a and b, that is a path from x to z that first follows a from x to y, and then b from y to z. Observe, that if e_1, e_2 are two paths from p to q, then $e_1 * \bar{e}_2$ is a loop based at p.

Lemma 2.1 *Let e_1, e_2 be two paths starting at q_1 and ending at q_2 on a complete Riemannian manifold M^n. Denote the length of $e_i, i = 1, 2,$ by l_i.*

If loop $\alpha_0 = e_1 * \bar{e}_2$ can be connected with to a (possibly trivial) loop $\alpha = \alpha_1$, (see fig. 1 (a)), by a path homotopy that passes via loops α_τ, $\tau \in [0,1]$, of length $\leq l_1 + l_2$, then there is a path homotopy $h_\tau(t), \tau \in [0,1]$, such that $h_0(t) = e_1(t), h_1(t) = \alpha * e_2(t)$ and the length of the paths during this homotopy is bounded above by $l_1 + 2l_2$.

Proof. For the proof see fig. 1. Note that e_1 is path homotopic to $e_1 * \bar{e}_2 * e_2$ along the curves of length $\leq l_1 + 2l_2$, see fig. 1 (b,c). (We just insert longer and longer segment of $\bar{e}_2$ travelled twice in the opposite directions.) Now observe that as $e_1 * \bar{e}_2$ is path homotopic to α via the curves α_τ of length $\leq l_1 + l_2$, path $e_1 * \bar{e}_2 * e_2$ is path homotopic to $\alpha * e_2$ along the curves $\alpha_\tau * e_2$ of length at most $l_1 + 2l_2$, see fig. 1 (d,e). □

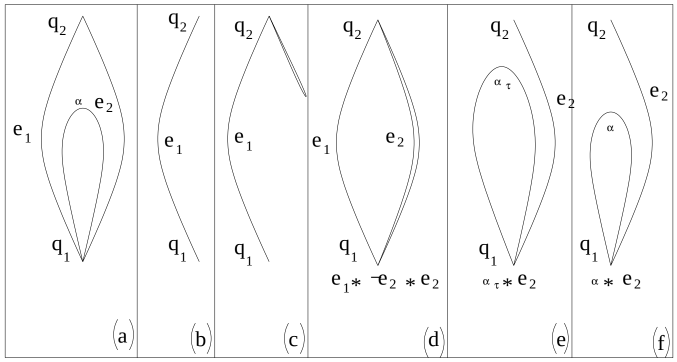

Figure 1: Illustration of the proof of Lemma 2.1.

Note that the above lemma has the following higher-dimensional generalization: Let $f : S^m \longrightarrow \Omega^L_{q_1,q_2}M^n, i = 1,2,\ m \geq 1$, be a continuous map from the m-sphere into a space of (piecewise differentiable) paths on a complete Riemannian manifold M^n between points $q_1, q_2 \in M^n$ of length at most L. Let $L_0 = \min_{s\in S^m}$ length$(f(s))$, and $s_0 \in S^m$ be such that length$(f(s_0)) = L_0$. One can define a new map $F : S^m \longrightarrow \Omega^{L+L_0}_{q_1}M^n$ by the formula $F(s) = f(s) * \bar{f}(s_0)$. Assume that there exists a homotopy $F_t : S^m \longrightarrow \Omega^{L+L_0}_{q_1}M^n$ contracting F. (Here $t \in [0,1], F_0 = F, F_1$ is the constant map to the trivial loop based at q_1.) Then we have the following simple lemma:

Lemma 2.2 *There exists a homotopy $f_t : S^m \longrightarrow \Omega^{L+2L_0}_{q_1q_2}M^n,\ t \in [0,1]$, between $f = f_0$ the constant map f_1 of S^m identically equal to $f(s_0)$ that takes values in the space of paths of length $\leq L + 2L_0$ connecting q_1 and q_2.*

Proof. The homotopy is constructed into two stage. During the first stage we connect f with $f_{\frac{1}{2}}$ defined by the formula $f_{\frac{1}{2}}(s) = f(s) * \bar{f}(s_0) * f(s_0) = F(s) * f(s_0)$. At this stage we take a join of $f(s)$ with longer and longer segments of $\bar{f}(s_0)$ travelled twice with opposite orientations.

During the second stage we contract F using the homotopy $F_t,\ t \in [0,1]$ leaving intact $f(s_0)$ at the end of each loop $F_t(s) * f(s_0)$. $\square$

In the next section we will present a proof of Theorem 1.3.

3 Curve-shortening process in the absence of geodesic loops.

Proof of Theorem 1.3.

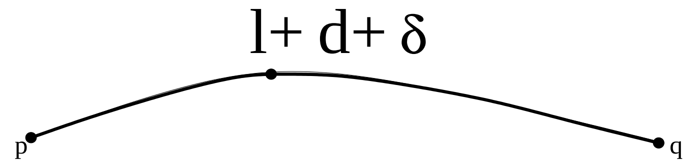

Figure 2: Curve shortening process (i).

Assume that there are no geodesic loops based at p with the length in the interval $(l, l+2d]$. Using an obvious compactness argument we observe that there exists a positive δ_0 such that there are no geodesic loops based at p with the length in the interval $(l, l+2d+\delta]$. Obviously, one can choose the value of δ_0 to be arbitrarily small if desired. Without loss of generality we are assuming that the length L of the curve $\gamma : [0,\ L] \longrightarrow M^n$ parametrized by its arclength is greater than $l+d$. Let $\delta = \delta_0$, if $L \geq l+d+\delta_0$, and $\delta = L-l-d$, if $\delta \in (l+d, l+d+\delta_0)$. Consider the segment $\gamma|_{[0,l+d+\delta]}$ of γ, which we will denote $\gamma_{11}(t)$ and the segment $\gamma|_{[l+d+\delta,L]}$ denoted $\gamma_{12}(t)$ (see fig. 2).

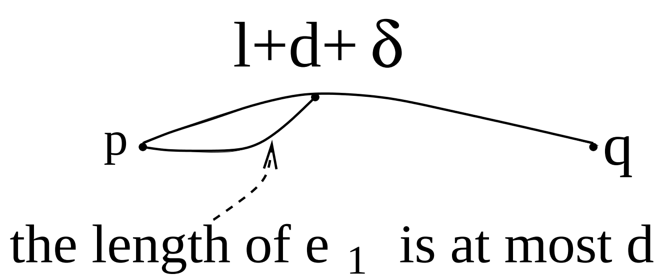

Figure 3: Curve shortening process (ii).

Let us connect points p and $\gamma(l+d+\delta)$ by a minimal geodesic, e_1 (of length $\leq d$), (see fig. 3). Then the pair of curves γ_{11} and e_1 form a loop $\gamma_{11} * \bar{e}_1$ of length $\leq l+2d+\delta$ based at p.

Consider a (possibly trivial) shortest loop α_1 that can be connected with $\gamma_{11} * \bar{e}_1$ by a length non-increasing path homotopy. (Its existence follows from the Ascoli-Arzela theorem.) Obviously, α_1 is a geodesic loop based at p that

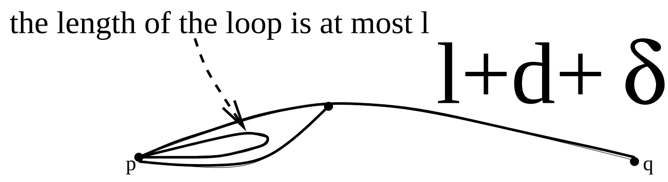

Figure 4: Curve shortening process (iii).

provides a local minimum for the length functional on $\Omega_p M^n$. Therefore our assumptions imply that the length of α_1 is at most l.

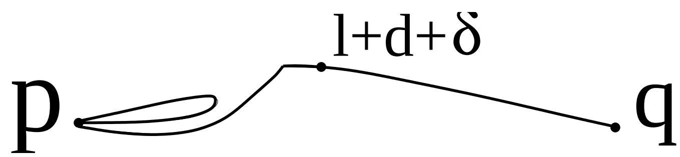

Figure 5: Curve shortening process (iv).

By Lemma 2.1 γ_{11} is path homotopic to the curve $\alpha_1 * e_1 = \tilde{\gamma}_{11}$ along the curves of length at most $l + 3d + \delta$ and, thus, the original curve γ is homotopic to a new curve $\tilde{\gamma}_{11} * \gamma_{12}$ along the curves of length at most $L + 2d$, (see 5).

Note that the length L_1 of this new curve $\gamma_1 = \tilde{\gamma}_{11} * \gamma_{12}$ is at most $L - \delta$. Assuming that L_1 is still greater than $l + d$, we repeat the process again: We parametrize γ_1 by its arclength. Now, let $\gamma_{21} = \gamma_1|_{[0,l+d+\delta]}$ and $\gamma_{22} = \gamma_1|_{[l+d+\delta,L_1]}$. (Here,as before, if $L_1 < l + d + \delta$, then we use $L_1 - l - d$ as a new value of δ, and otherwise take $\delta = \delta_0$.)

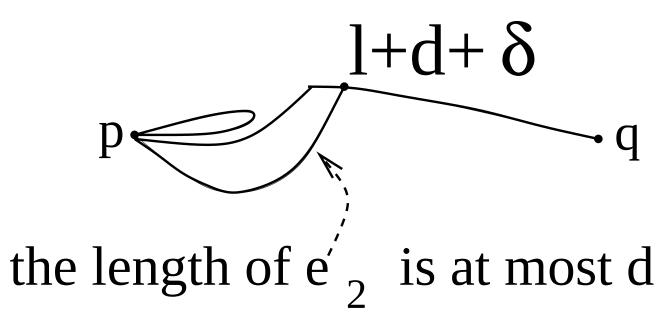

Figure 6: Curve shortening process (v).

Connect the points p and $\gamma_1(l + d + \delta)$ by a minimal geodesic segment

e_2, (see fig. 6). Then γ_{21} and e_2 form a geodesic loop $\gamma_{21} * \bar{e}_2$ based at p of length at most $l + 2d + \delta$.

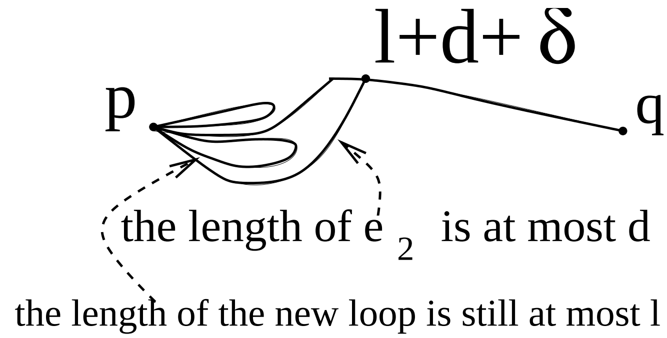

Figure 7: Curve shortening process (vi).

Again, we try to connect this loop with a shortest possible loop α_2 by means of a length non-icreasing path homotopy. The existence of a minimizer α_2 follows from an easy compactness argument, and α_2 is a geodesic loop providing a local minimum of the length functional on $\Omega_p M^n$, (see fig. 7). Therefore, the length of α_2 is at most l.

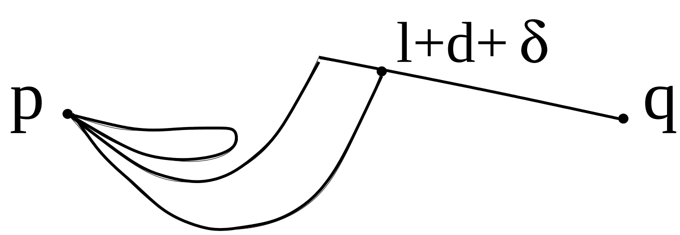

Figure 8: Curve shortening process (vii).

Thus, γ_{21} is path-homotopic to $\tilde{\gamma}_{21} = \alpha_2 * \sigma_2$ along the curves of length at most $l + 3d + \delta$. It follows that γ_1 is homotopic to $\gamma_2 = \tilde{\gamma}_{21} * \gamma_{22}$ along the curves of length at most $L + 2d$, (see fig. 8).

This process will terminate in a finite number of steps with a curve of length $\leq l + d$. $\square$

Note that we have proven a stronger statement. We have shown, assuming the hypothesis of the theorem above, that for any loop $\gamma(t)$ based at p there exists a 1-parameter family of curves C_s^γ of length $\leq l + 3d + \delta$

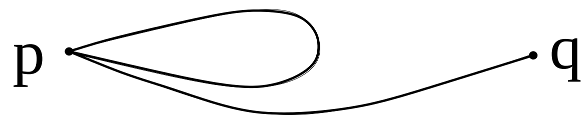

Figure 9: Curve shortening process (viii)

continuously depending on parameter s that connects p with all points of $\gamma(t)$, so that:
A. If we denote $C^\gamma_s(1)$ by $\gamma(\tau(s))$, then $\tau(s)$ is an increasing (but, in general, not strictly increasing) function of s;
B. There exist two partitions: $P^\gamma = \{0 = t^\gamma_0 < t^\gamma_1 < t^\gamma_2 < ... < t^\gamma_{k^\gamma} = 1\}$ and $Q^\gamma = \{0 = s^\gamma_0 < s^\gamma_1 < ... < s^\gamma_{2k^\gamma} = 1\}$, such that
(1) $C^\gamma_s(1) = \gamma(t^\gamma_i)$ for $s \in [s^\gamma_{2i-1}, s^\gamma_{2i}]$.
(2) $C^\gamma_s(1) = \gamma(r)$ for $r \in [r^\gamma_i, r^\gamma_{i+1}]$ and $s \in [s^\gamma_{2i}, s^\gamma_{2i+1}]$.
(3) For all i the length of $C^\gamma_{s_{2i}}$ does not exceed $l + d$.
(4) The curve $C^\gamma_{s_{2k^\gamma}} = C^\gamma_1$ is the final result of the application of the curve-shortening process described in the proof of 1.3 to γ.

The curves C^γ_s are depicted on Fig. 6-10 as the curves connecting p with a variable point that moves from p to q along γ. Note also, that the partition P^γ can be chosen as fine as desired.

Finally, notice that there is a path homotopy H between our original path, γ and C^γ_1 that can be described as follows: At each moment of time t $H(t)$ is the path that first goes along C^γ_t, and then runs along γ from $C^\gamma_t(1)$ to $\gamma(1)$.

Next we will prove the following theorem, which together with Theorem 1.3 immediately implies Theorem 1.1 in the case of $m = 1$.

Theorem 3.1 *Let M^n be a closed Riemannian manifold of diameter d, p a point of M^n, and k a positive integer number. Assume that there exists a positive integer $j \leq k$ such that the length of every geodesic loop that provides a local minimum for the length functional on $\Omega_p M^n$ is not in the interval $(2(j-1)d,\ 2jd]$. Consider a continuous map $f : [0,1] \longrightarrow \Omega_p M^n$ such that the lengths of both $f(0)$ and $f(1)$ do not exceed $2(j-1)d$. Then f is path homotopic to $\tilde{f} : [0,\ 1] \longrightarrow \Omega_p^{(6j-1)d} M^n \subset \Omega_p^{(6k-1)d} M^n$. Moreover, assume that for some L the image of f is contained in $\Omega^L_p M^n$. Then one* *can choose a path homotopy between* f *and* $\tilde{f}$ *so that its image is contained in* $\Omega_p^{L+2d}M^n$*.*

Proof. Choose a partition of $t_0 = 0 < t_1 < t_2 < ... < t_N = 1$ of the interval $[0,1]$, so that $\max_i \max_{\tau\in[0,1]} \text{length}(f(t)|_{t\in[t_{i-1},t_i]}(\tau)) \le \varepsilon$ for a small ε that will eventually approach 0, (see fig. 10. Fig. 10 depicts a situation, when $f(0)$ and $f(1)$ are both constant loops, but the general case is completely analogous.)

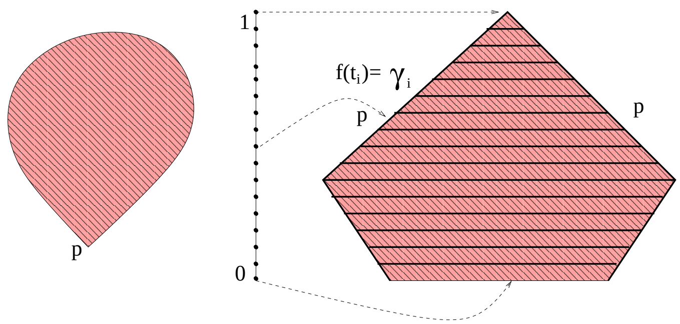

Figure 10: Partition of the map $f : S^1 \longrightarrow M^n$ into "small" intervals

Let us denote loops $f(t_i)$ by $\gamma_i(r)$. (These loops are depicted on fig. 10 as horizontal lines.) We can use Theorem 1.3 to replace all $\gamma_i(r)$ of length that is greater than $(2j-1)d$ by the loops β_i of length $\le (2j-1)d$. The loops that were originally shorter than $(2j-1)d$ will remain as they were. Note that in this case we will also have a family of indiced paths C of controlled length connecting p with all points on the loop as at the end of the proof of Theorem 1.2: namely, the initial segments of the loop.

We will now constract a path between each pair β_{i-1}, β_i that passes through loops of length $\le (6j-1)d + o(1)$. (Here and below $o(1)$ denotes terms that are bounded by a linear function of small parameters of our shortening process denoted δ_0 (see the proof of Theorem 1.3) and ε that can be made arbitrarily small.)

Let us consider a pair of consecutive curves $\sigma = \beta_{i-1}$ and $\alpha = \beta_i$. Recall that these curves were obtained from γ_{i-1} and γ_i respectively. Each curve is obtained through a corresponding curve shortening proces. These curve shortenings correspond to 1-parameter families of curves C_s^σ, C_s^α having properties described after the proof of Theorem 1.3.

Recall that these curves connect p with points on γ_{i-1} and γ_i, and $C_1^{\gamma_{i-1}}$ coincides with σ, and $C_1^{\gamma_i}$ coincides with α.

We will construct a path between σ and α in two steps. In the first step we will consider a loop that is a join of σ and $\bar{\alpha}$, namely, $\sigma * \bar{\alpha}$. We will construct a homotopy that contracts this loop to a point via loops of length $\leq 4jd + o(1)$ (when $\delta + \varepsilon_0 \longrightarrow 0$). The second step will be merely an application of Lemma 2.1. (A summand of $2(j-1)d + d = (2j-1)d$ will be added to the previous upper bound $4jd + o(1)$ for the length of loops during a contracting homotopy on the second step.) The desired estimate $(6j-1)d$ can be obtained by passing to the limit as $\varepsilon + \delta_0 \longrightarrow 0$.

So, we need only to describe the first step to finish our construction. Note that γ_{i-1} and γ_i are very close to each other, and are connected by the *continuous* family of very short curves $f(t)|_{t\in[t_{i-1},t_i]}(\tau)$, $\tau \in [0,1]$. The desired continuous family of loops could be described as the family of all loops $C_{s_1}^{\gamma_{i-1}} * f(t)|_{t\in[t_{i-1},t_i]}(\tau) * \bar{C}_{s_2}^{\gamma_i}$, where $C_{s_1}^{\gamma_{i-1}}(1) = f(t_{i-1})(\tau)$ and $C_{s_2}^{\gamma_i}(1) = f(t_i)(\tau)$ but otherwise s_1, s_2 and τ independently vary over $[0,1]$ interval. It is easy to see that these loops form a continuous one-parametric family that can be naturally parametrized by an interval of length not exceeding 2. Yet observe that whole intervals of values of s_1 and/or s_2 could correspond to some individual values of τ. We have some freedom on how we parametrize this family. We would prefer to parametrize them in a slower way than possible to somewhat improve the bounds on the lengths of these loops. Observe, that some of the curves C have better upper bounds for their length. The reason is that the length of the curves during each path homotopy that shortens the length by δ as described in the proof of Theorem 1.3 could increase up to $2d$ in comparison with the length of one of the curve at the end of the considered step (and up to $2d + \delta$ in comparison with the length of the curve at the beginning of this step). Therefore, we do not want to make these homotopies in families $C^{\gamma_{i-1}}$ and C^{γ_i} simultaneously, but will wait until one of these homotopies ends, and then perform the other.

Here is a more concrete description of the resulting one-parametric family of loops that also takes into account some details of the construction of C_s^γ in the proof of Theorem 1.3 above.

Let $\varepsilon_\tau = f(t)(\tau)$, where τ is fixed and t varies in the interval $[t_{i-1}, t_i]$. Recall that we can insure that the length of ε_τ is arbitrarily small for all $\tau \in [0,1]$ by choosing $t_i - t_{i-1}$ to be sufficiently small.

Let us begin with the loop $\sigma * \bar{\alpha} = C_s^\sigma(1) * \bar{C}_s^\alpha(1)$ that is based at the point

p. Corresponding to C_s^σ and C_s^α consider two pairs of partitions: $\{P^\sigma, Q^\sigma\}$ and $\{P^\alpha, Q^\alpha\}$. Let $P^\sigma = \{0 = r_0^\sigma < r_1^\sigma < ... < r_{k_\sigma-1}^\sigma < r_{k_\sigma}^\sigma = 1\}$ and $P^\alpha = \{0 = r_0^\alpha < r_1^\alpha < ... < r_{k_\alpha-1}^\alpha < r_{k_\alpha}^\alpha\}$. Also let $P = P^\sigma \cup P^\alpha$. Without loss of generality, we can assume that $P = \{0 = r_0^\sigma = r_0^\alpha < r_1^\alpha < r_1^\sigma < r_2^\alpha < r_2^\sigma < ... < r_{k_\alpha-1}^\alpha < r_{k_\sigma-1}^\sigma < r_{k_\alpha}^\alpha = r_{k_\sigma}^\sigma = 1\}$.

We will now present a homotopy that contracts $\sigma * \bar{\alpha}$ to p as a loop over short loops.

(a) $C_1^\sigma * \bar{C}_1^\alpha$ is homotopic to $C^\sigma_{s^\sigma_{2k_\sigma-1}} * C_1^\alpha$ over loops of length at most $4jd$, see fig. 11.

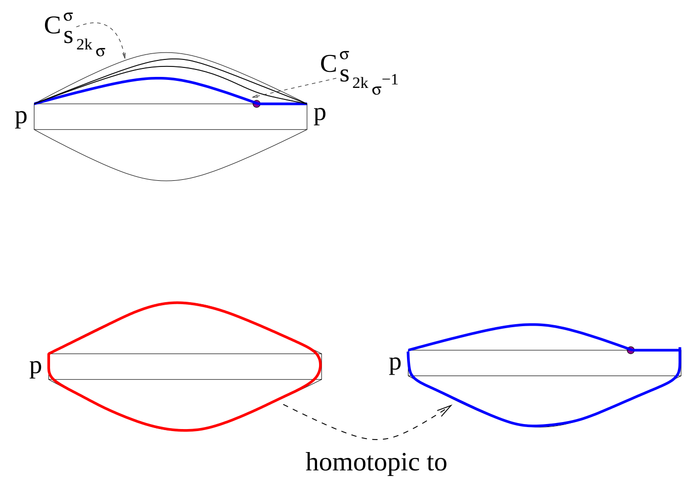

Figure 11: Contracting $\sigma * \bar{\alpha}$ as a loop (i).

(b) $C^\sigma_{s^\sigma_{2k_\sigma-1}} * C_1^\alpha$ is homotopic $C^\sigma_{s^\sigma_{2k_\sigma-1}} * C^\alpha_{s^\alpha_{2k_\alpha-1}}$ over the loops of length $4jd$, (see fig. 12).

(c)$C^\sigma_{s^\sigma_{2k_\sigma-1}} * C^\alpha_{s^\alpha_{2k_\alpha-1}}$ is homotopic to $C^\sigma_{s^\sigma_{2k_\sigma-2}} * \bar{\varepsilon}_{r^\sigma_{k-1}} * \bar{C}^\alpha_{s^\alpha}$ for $s^\alpha \in [s^\alpha_{2k-2}, s^\alpha_{2k-1}]$ over the curves of length at most $(4j-2)d + 2\delta + \varepsilon$, (see fig. 13).

(d) $C^\sigma_{s^\sigma_{2k_\sigma-2}} * \bar{\varepsilon}_{r^\sigma_{k-1}} * \bar{C}^\alpha_{s^\alpha}$ is homotopic to $C^\sigma_{s^\sigma_{2k_\sigma-3}} * \bar{\varepsilon}_{r^\sigma_{k-1}} * \bar{C}^\alpha_{s^\alpha}$ over the curves of length $4jd + 2\delta + \varepsilon$, (see fig. 14).

(e) $C^\sigma_{s^\sigma_{2k_\sigma-3}} * \bar{\varepsilon}_{r^\sigma_{k-1}} * \bar{C}^\alpha_{s^\alpha}$ is homotopic to $C^\sigma_{s^\sigma} * \bar{\varepsilon}_{r^\alpha_{k-1}} * \bar{C}^\alpha_{s^\alpha_{2k_\alpha-2}}$ for $s^\sigma \in [s^\sigma_{2k_\sigma-3}, s^\sigma_{2k_\sigma-4}]$ over the curves of length at most $(4j-2)d + 2\delta + \varepsilon$, (see fig. 15).

Proceeding in the above manner we will contract the loop to p over curves of length at most $4jd + o(1)$.

Observe that after an appropriate reparametrization of all families $C_s^{\gamma_i}$

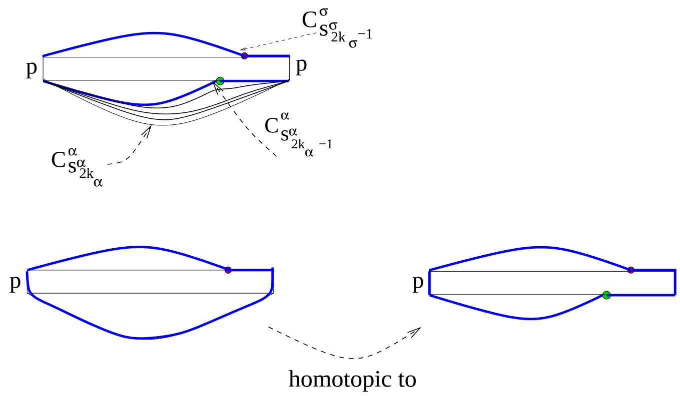

Figure 12: Contracting $\sigma * \bar{\alpha}$ as a loop (ii).

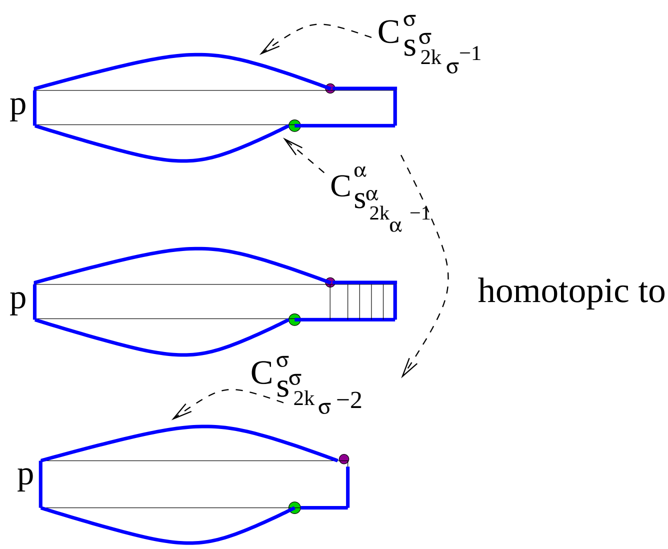

Figure 13: Contracting $\sigma * \bar{\alpha}$ as a loop (iii).

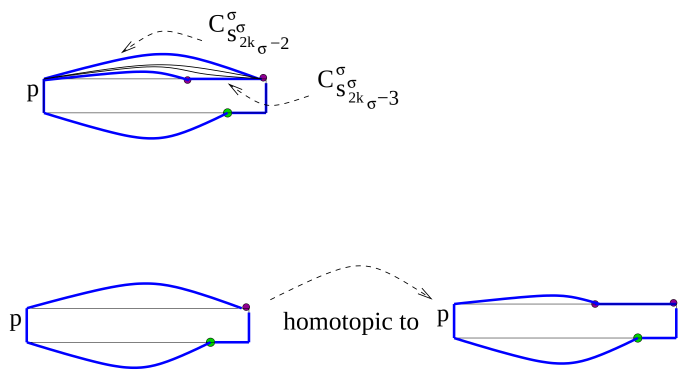

Figure 14: Contracting $\sigma * \bar{\alpha}$ as a loop (iv).

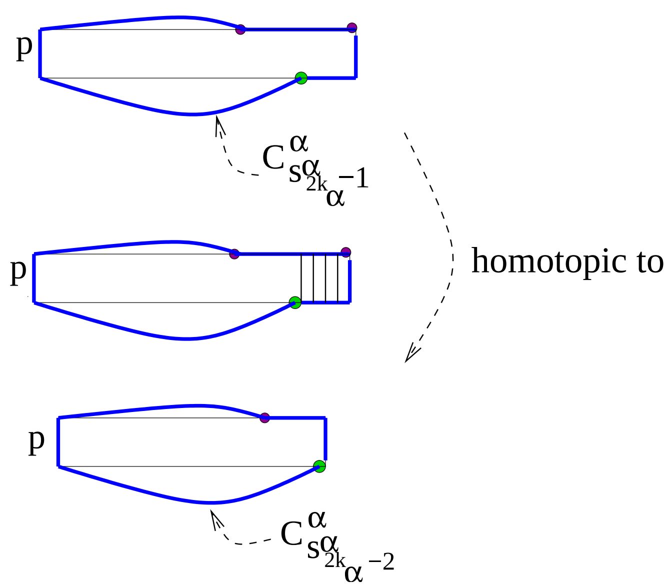

Figure 15: Contracting $\sigma * \bar{\alpha}$ as a loop (v).

by s ("synchronization") all the constructed loops have the form $C_s^{\gamma_{i-1}} * f(t)|_{t\in[t_{i-1},t_i}(\tau(s)) * \bar{C_s^{\gamma_{i-1}}}$ for an appropriate increasing but not necessarily strictly increasing function $\tau(s)$. It is easy to see that one can choose these synchronizations so that the same parametrization of the family C^{γ_i} will work for constructions of path homotopies between β_{i-1} and β_i and between β_i and β_{i+1}. Observe, that for every s the length of one of curves $C_s^{\gamma_{i-1}}$ and $C_s^{\gamma_i}$ does not exceed $(2j-2)d+d+o(1) = (2j-1)d+o(1)$, and the length of the other does not exceed $(2j-1)d+o(1)+2d = (2j+1)d+o(1)$, as $\delta \longrightarrow 0$. Combining the constructed path homotopies between β_{i-1} and β_i for all i we obtain a path in $\Omega_p M^n$ starting at $f(0)$ and ending at $f(1)$, which can be interpreted as a map of $[0,\ 1]$ into $\Omega_p M^n$.

It remains to prove that this path $\tilde{f}$ in $\Omega_p M^n$ is path homotopic to f. Here is the construction of a path homotopy between $\tilde{f}$ and f: At each moment of time we do not shorten $f(t_i)$ for all i using the construction in the proof of Theorem 1.3, as we did above. Instead we use homotopies H_i between γ_i and its shortening that are similar to homotopies described after the proof of Theorem 1.3. (In other words, at every moment we shorten only a part of curves $\gamma_i = f(t_i)$ for all i.) The path $H_i(\lambda)$ consists of two arcs. The first arc is the path $C_\lambda^{\gamma_i}$ (where the parametrization of the one-parametric family $C_s^{\gamma_i}$ is the same as the one that has been used to construct contractions of $\beta_{i-1} * \bar{\beta}_j$ and $\beta_j * \bar{\beta}_{j+1}$ above). The second arc is the arc of γ_i that starts at $C_\lambda^{\gamma_j}(1)$ and ends at the end of γ_j. To construct the desired path homotopy of paths in $\Omega_p M^n$ at the moment λ we replace all long curves γ_i not by $\beta_i = H_i(1)$ but by $H_i(\lambda)$. As we synchronized the parametrizations of different one-parametric families C, $C_s^{\gamma_i}(1)$ and $C_s^{\gamma_{i-1}}(1)$ can be connected by a (very short) arc $f(t)|_{t\in[t_{i-1},t_i]}(\tau)$ for a fixed value of $\tau = \tau(s)$. Now we can form loops $C_r^{\gamma_i} * f(t)|_{t\in[t_{i-1},t_i]}(\tau(r)) * \bar{C_r}^{\gamma_{i-1}}$ for all $r \leq \lambda$ as it was done above. These loops provide a contracting homotopy for the loop $C_\lambda^{\gamma_i} * f(t)|_{t\in[t_{i-1},t_i]}(\tau(\lambda)) * \bar{C_\lambda}^{\gamma_{i-1}}$. Unising Lemma 2.1 we can transform this homotopy into a path homotopy between $C_\lambda^{\gamma_{i-1}}$ and $C_\lambda^{\gamma_i} * \bar{f}(t)|_{t\in[t_{i-1},t_i]}(\tau(\lambda))$. Joining all paths in this path homotopy with the arc γ_{i-1}^λ of γ_{i-1} between $C_s^{\gamma_{i-1}}(1)$ and $\gamma_{i-1}(1) = p$, we obtain a path homotopy between $H_{i-1}(\lambda)$ and $C_\lambda^{\gamma_i} * \bar{f}(t)|_{t\in[t_{i-1},t_i]}(\tau(\lambda)) * \gamma_{i-1}^\lambda$. It remains to construct a path homotopy between $\bar{f}(t)|_{t\in[t_{i-1},t_i]}(\tau(\lambda)) * \gamma_{i-1}^\lambda$ and γ_i^λ. But that path homotopy can be formed by paths $\bar{f}(t)|_{t\in[\varrho,t_i]}(\tau(\lambda)) * f(\varrho,\tau),\ \tau \in [\tau(\lambda),1]$, parametrized by τ, where ϱ is the parameter of the path homotopy, ($\varrho \in [t_{i-1},t_i]$).

It is easy to see from this description that the resulting path homotopy passes through paths of length $\leq L+2d$. (The increase of length takes place already on the first stage, when we shorten curves γ_i using Theorem 1.3. If $L \leq (6k-1)d$, we can just take $\tilde{f}=f$, so without any less of generality we can assume that $L > (6k-1)d$. It is easy to see that the just described path homotopy between $H_{i-1}(\lambda)$ and $H_i(\lambda)$, $\lambda \in [0,1]$ does not lead to any further increase of length of paths above $(6k-1)d$.)

We would like to provide the following less formal explanation (or reinterpretation) of how we constructed the path homotopy between $H_{i-1}(\lambda)$ and $H_i(\lambda)$. These two paths consist of curves $C_\lambda^{\gamma_{i-1}}$ and $C_\lambda^{\gamma_i}$ joined with nearly identical "tails" that are arcs of $\gamma_{i-1}(\tau)$ and $\gamma_i(\tau)$ between $\tau=\tau(\lambda)$ and $\tau=1$. The "tails" form a part of one parametric family of "tails" $f(\varrho,\tau), \tau \in [\tau(\lambda),1]$, where the parameter ϱ ranges in $[t_{i-1},t_i]$. The idea was to "fill" the "digon" formed $C_\lambda^{\gamma_{i-1}}$ and $C_\lambda^{\gamma_i}$ in exactly the same way as we filled the digon formed by $\beta_{i-1}=C_1^{\gamma_{i-1}}$ and $\beta_i=C_1^{\gamma_i}$ by a one-parametric family of paths of controlled length and to attach to each of these paths the corresponding "tail" $f(\varrho,\tau), \tau \in [\tau(\lambda),1]$, for an appropriate value of ϱ. The idea needs a small corrections as 1) $C_\lambda^{\gamma_{i-1}}$ and $C_\lambda^{\gamma_i}$ end at very close but still different points, and will form a digon only after we attach (a very short) path $f(t)|_{t\in[t_{i-1},t_i]}(\tau(\lambda))$ connecting their endpoints to one of them; 2) We will be able to attach an appropriate "tail" $f(\varrho,\tau)$ only after attaching (a very short) arc of $f(t)|_{t\in[t_{i-1},t_i]}(\tau(\lambda))$. As we will see below, this idea directly generalizes to situation, when one deals with maps of higher dimensional spheres to $\Omega_p M^n$.

$\square$

Remark 3.1. Here we would like to summarize some important features of our construction of $\tilde{f}$ that will be used to prove Theorem 1.3 for larger values of m. The constructed homotopies contracting $\beta_{i-1}*\bar{\beta}_i$ that were the main part of the construction of $\tilde{f}$ in the proof of Theorem 3.1 consist of loops containing the images of rectilinear arcs $[t_{i-1},t_i]\times\{\tau\}$ for a variable $\tau \in [0,1]$ that monotonously depends on the parameter of the homotopy. The loops can be naturally diveded into two arcs, such that the length of one of these arcs does not exceed $(2j-1)d+o(1)$, and the length of the other does not exceed $(2j+1)+o(1)$, as $\delta,\varepsilon \longrightarrow 0$. The path homotopy between β_{i-1} and β_i is obtained from this path homotopy contracting $\beta_{i-1}*\bar{\beta}_i$. simply using the construction in Lemma 2.1. The desired map $\tilde{f}$ was constructed by combining the path homotopies between β_{i-1} and β_i for all i.

4 Small spheres in the loop space.

In this section we will demonstrate that in the absence of a great number of short geodesic loops, every homotopy class of $\Omega_p M^n$ can be represented by a sphere that passes through short loops (thus proving Theorem 1.1). We assume that there exists k such that there are no geodesic loops based at p on M^n with the length in the interval $(2(k-1)d, 2kd]$. We are going to prove that f is homotopic to a map $\tilde{f}$ with the image in $\Omega_p^L M^n$, where $L = ((4k+2)m + (2k-3))d$.

The proof of the theorem is done by induction on m. The base step of the induction, $(m = 1)$, had been proven in the previous section. Before presenting the proof of the induction step, we would like to explain how the proof will work to pass from $m = 1$ to $m = 2$. (We will explain the proof in the case, when f is a map of S^2. It will be clear from our explanations that the case, when f is a map of the pair $(D^2, \partial D^2)$ can be treated exactly the same.)

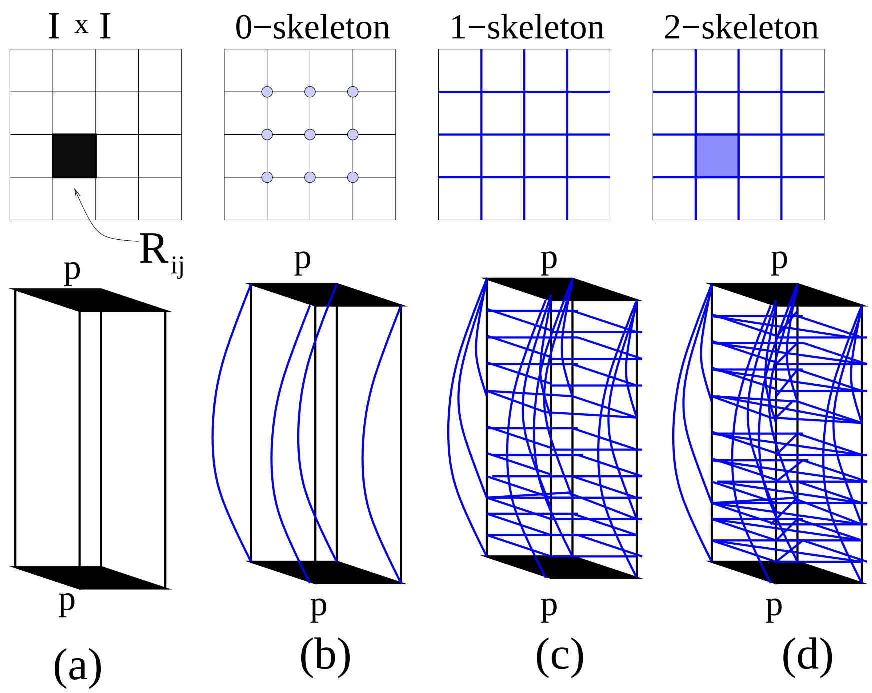

Figure 16: Replacing the map

Let $I = [0, 1]$. Consider a map $f : I \times I \longrightarrow \Omega_p M^n$, where

$\partial(I \times I)$ is mapped to p. Let us subdivide $I \times I$ into small squares, of size that will be specified later. Consider a square R_{ij} with vertices $(x_{i-1}, y_{j-1}), (x_{i-1}, y_j), (x_i, y_{j-1}), (x_i, y_j)$. Each of these vertices corresponds to a loop in $\Omega_p M^n$. Each of these loops that is too long, (i.e. of length greater than $(2k-1)d$) will be replaced by a shorter one as in Theorem 1.3 via a path homotopy described in the proof of Theorem 1.3. Now, we will replace the edges that connect these vertices as in the proof of Theorem 3.1 in the previous section (see fig. 16 (a)-(c)). Next, we have to "fill" the interiour of the square, (fig. 16 (d)).

The boundary of this square corresponds to a 2-sphere in M^n. It is convenient to consider it as a CW-complex with the following sturcture: a boundary of a parallelipiped in which two opposite faces have been identified with a point p, (see 17). Note that two copies of p at the beginning and the end of considered loops will be sometimes depicted on our figures as two different points (fig. 17). This convention will make our figures easier to draw and comprehend, and will also make clearer the fact that our proof can be easily adapted to prove the generalization of the theorem, where $\Omega_p M^n$ in the conclusion of the theorem is replaced by $\Omega_{pq} M^n$ for two arbitrary points p, q.

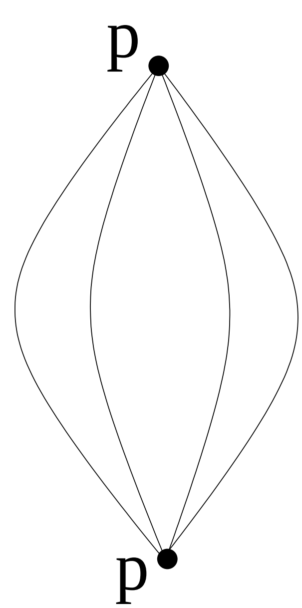

Figure 17: Sphere decomposition.

Each of the four "large" cells of the sphere has a natural decomposition into "short" loops based at p. (Here "short" means of length at most $4kd + o(1)$; recall the proof of Theorem 3.1 in the previous section). The filling will be done in four steps.

Step 1. We will construct a homotopy between the sphere and the the point p, such that for every $\tau \in [0,1]$ the (map of the) sphere S^2_τ in the homotopy will have a similar decomposition into short loops based at p. This means that each sphere S^2_τ will be given a CW-structure of the cell complex of the boundary of a parallelipiped in which two opposite faces are mapped into p and into another point depending on τ and denoted $q(\tau)$, respectively, and where each of the remaining four ("large") faces are swept-out by loops of length at most $4kd + o(1)$ based at p. We will be using the proof of Theorem 3.1 and, in particular, the facts summarized in Remark 3.1 to do this construction.

Step 2. For each τ we can construct a *vertical sweep-out* of S^2_τ by curves of length of at most $(6k+1)d+o(1)$, which will vary continuously with respect to τ. Here by the vertical sweep-out we mean a continuous 1-parametric family of curves joining a pair p, q_τ of points of S^2_τ. In fact, this family can be parametrized by a circle. This step essentially consists in an application of Lemma 2.1 to each of four "large" faces of S^2_τ. The resulting path homotopy for each face corresponds to one quarter of the circle parametrizing the whole family.

Step 3. Each of these paths can be paired with a fixed path of length $\leq (2k+1)d+o(1)$ to obtain a sweep-out of each sphere S^τ_2 by loops of length $\leq (8k+2)d+o(1)$. This path is the join of $C^{f(v)}_\tau$ (from the proof of Theorem 1.3, where v denotes an arbitrary vertex of the considered rectangle R_{ij}) with a very short segment connecting its endpoint with q_τ in the image of $R_{ij} \times \{\tau\}$. But for reasons of continuity one needs to consider the same vertex v for all values of τ. (It might seem that we could ensure a somewhat smaller length of the loops: For every value of τ we could find a vertex v (that would depend on τ) such that $C^{f(v)}_\tau$ has length $\leq (2k-1)d + o(1)$. But in this case the resulting circle in $\Omega_p M^n$ representing S^2_τ does not need to depend continuously on τ, and is not suitable for our purposes.)

Step 4. We can now apply Lemma 2.2 to obtain a 3-disc filling the original 2-sphere, S^2_1, with the disc being swept-out by paths connecting p and q of length at most $(8k+2)d+(2k-1)d+o(1) = (10k+1)d+o(1)$. This results in a map $F : S^3 \longrightarrow M^n$ with a vertical sweep-out (by paths connecting p and q) of a controlled length, which in turn induces a map $\tilde{f} : S^2 \longrightarrow \Omega_p M^n$

that passes through loops of length at most $(10k+1)d+o(1)$.

Note that the only step in this construction which is not immediate and requires a more detailed description is the first one. We will therefore describe it in more details than the other three steps. Let us once again consider a map $f|_{R_{ij}}$. It induces a map $F : R_{ij} \times [0,1] \longrightarrow M^n$ defined by the formula $F(x,y,t) = f(x,y)(t)$. (see fig. 18).

Figure 18: $f|_{R_{ij}}$

Consider a slicing of $R_{ij} \times I$ into rectangles $R_{ij} \times \{\tau\}$ parallel to R_{ij}. That is, at each fixed time τ, we will be considering the restriction of $f(x,y)(\tau)$ for a fixed value of $t = \tau$. We want the length of the image of each straight line segment of $f(R_{ij} \times \{\tau\})$ under F to be small, (much smaller than some $\varepsilon > 0$, which will eventually go to 0), (see fig. 19(a)). This can be achieved by making the subdivision in the rectangles R_{ij} sufficiently fine. Each of the considered slices can be swept-out by short curves, (i.e. of length at most ε) as in fig. 19 (b) in a continuous canonical way.

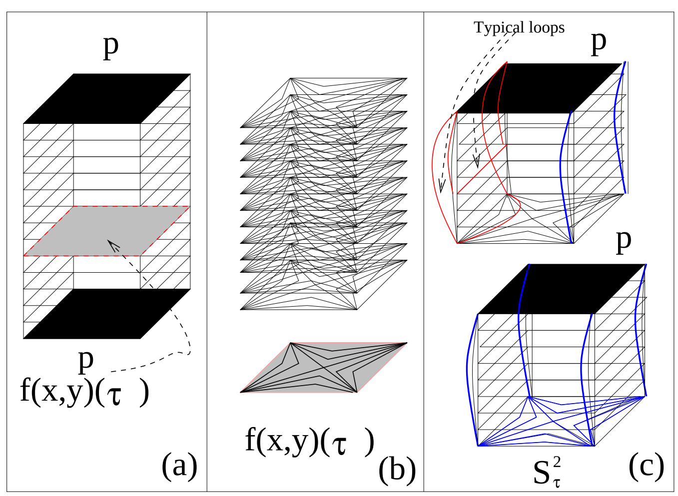

Figure 19: Slicing.

We will construct a map from $\partial(R_{ij} \times I)$ in the following way: the upper face will be mapped to $F(R_{ij} \times \{\tau\})$, the lower face will be mapped to the point p and the side faces will be mapped as follows:

Recall that in the course of the proof of Theorem 3.1 we have replaced the curves corresponding to vertices of R_{ij} by short curves (of length $\leq (2k-1)d + o(1)$), and that we have then constructed path homotopies between the pairs corresponding to the edges of R_{ij}. These homotopies correspond to the edges of R_{ij} and generate 2-discs in M^n. Recall, that they were obtained by an application of Lemma 2.1 to a certain homotopy that contracted the loop formed by joining two paths (in fact, loops) obtained as the "shortening" of images of the vertices. (One of these two paths is taken with the oppostie orientation.) This homotopy consisted of the loops $g^\tau = C_s^{\gamma_{i-1}} * f(x,y)|_{x \ or \ y \in [t_{i-1}, t_i]}(y \ or \ x) * \bar{C_s^{\gamma_i}}$ of length $\leq (2k-1)d + (2k+1)d + o(1) = 4kd + o(1)$. (Recall, that we denote terms involving a linear combination of parameters in our proofs that can be made arbitrarily small by $o(1)$.)

Observe that for every value of $\tau \in [0,1]$ we can restrict these homotopies, so that they end at g^τ instead of g^1 (and contract g^τ to the constant loop g^0). For every value of τ we will map side faces to the discs generated by these path homotopies. Note that we can combine the sweep-out of four side faces into loops with the described sweep-out of the top into curves. Namely, we can extend each of the path homotopies forming the side faces by a stage, when we change only the small middle segement of g^τ: We start from the loop, where the image of a side of $R_{ij} \times \{\tau\}$ is replaced by the image under f of two half-diagonals of $R_{ij} \times \{\tau\}$, meeting at the image of the center of $R_{ij} \times \{\tau\}$ under F, that we will denote q_τ, and go through loops, where the middle segment is replaced by the image under F of two-segment broken lines in $R_{ij} \times \{\tau\}$ shown at the bottom of Fig. refslicing (b). If we combine the natural sweep-outs of the side faces with the sweep-out of the top, then we will obtain (a map of the) 2-sphere S^2_τ swept-out (as described in our description of Step 1) into four families of loops corresponding to four "side" faces of the boundary of a parallelipiped, (see Fig. 19(c)). Each of the four families of loops is formed by a contraction of a loop based at p and passing through q_τ. Recall, that the length of loops during these homotopies is bounded by $4kd + o(1)$.

Now we are going to apply Lemma 2.1 to each of these four families of loops to replace them by four families of paths between p and q_τ. This stage was called Step 2. Its purpose is that then we can combine these four families of loops into one family of paths parametrized by S^1. Moreover, we want to ensure that each of four resulting families of paths depends continuously on τ.

For each of these four families (and for all values of τ) we consider the corresponding edge of R_{ij}. Let v be the vertex of this edge that was used in the last application of Lemma 2.1 during the application of the shortening process described in the proof of Theorem 3.1 to this edge. Consider curves $C_\tau^{f(v)}$ of length $\leq (2k+1)d + o(1)$ constructed in the course of shortening $f(v)$ as in the proof of Theorem 1.3. Each of these curves cconnects p with a point on $f(v)$. Of course, $C_1^{f(v)}$ is the result of the shortening of $f(v)$. Note that $C_\tau^{f(v)}$ is an arc in the corresponding loop g^τ (as $f(v)$ is either γ_{i-1} or γ_i). So, we can apply Lemma 2.1 to turn the homotopy between g^0 and g^τ into a path homotopy between $C_\tau^{f(v)}$ and the other "half" of g^τ that passes through paths of length $(2k+1)d + 4kd + o(1) = (6k+1)d + o(1)$.

Thus, after we apply Lemma 2.1 to each of the four homotopies, we will obtain four families of paths of length $\leq (6k+1)d + o(1)$ connecting p and q_τ, that will together form one family of paths parametrized by S^1 providing a sweep-out of S^2_τ.

Now we are going to reinterprete them as families of loops of length $(8k+2)d + o(1)$ (Step 3) providing a contraction of the family of the loops corresponding to $\tau = 1$ (and to the initial 2-sphere S^2_1).

Recall that the initial sphere S^2_1 is the image in M^n of the circle in the loop space that was obtained as the image of the boundary of R_{ij} under a map obtained from the original map f by shortenings of the loops corresponding to four vertices as in the proof of Theorem 1.3, and then by shortening of one-parametric families of loops corresponding to the four edges of R_{ij} as in the proof of Theorem 3.1. Therefore, we would like to use the proof of Lemma 2.2 to obtain a "shortening" of the restriction of f on R_{ij} (Step 4). Here we are supposed to choose a path between p and $q_1 = p$ of minimal length to be used as "$f(s_0)$" in notations of Lemma 2.2. We choose the shortening $C(v)$ of $f(v)$ for one of the vertices. Its length does not exceed $(2k-1)d$. We attach $\bar{C}(v)$ to all paths in our sweep-out of S^1_τ. So, the length of paths in the resulting vertical sweep-out of the constructed 3-cell filling of S^2_1 does not exceed $(8k+2)d + o(1) + (2k-1)d \leq (10k+1)d + o(1)$. As $q_1 = p$, these paths hapeen to be loops, and we constructed the extension of f from ∂R_{ij} to its interior, such that the images of all points in the interior are in $\Omega_p^{(10k+1)d+o(1)} M^n$.

It remains to demonstrate that when we will change the restriction of f on all vertices, edges and the interiors of the rectangles R_{ij} of the considered fine cell subdivision of S^2 as described, we obtain a map g that is homotopic to f through a homotopy with the image in the space of sufficiently short loops.

The idea of this demonstration is simple. Moreover, it is the same idea that had been used for this purpose in the case, when $m = 1$ at the end of the proof of Theorem 3.1. Namely, at a moment time λ, we do not shorten the loops $f(v)$ that are the images of the vertices of the chosen fine triangulation of S^2 as prescribed by the proof of Theorem 1.3. Instead we shorten only a certain initial arc of the curve $f(v)$, replacing it by a path C_λ from the construction in the proof of Theorem 1.3 applied to $f(v)$. Then we take the join of C_λ with the remaining part of $f(v)$ - exactly, as it was done in the proof of Theorem 3.1.

Now we consider four paths C_λ corresponding to vertices of every small rectangle R_{ij}. They can be considered as "long" edges of the 1-skeleton of a parallelipiped in M^n, and filled by a map of the parallelipiped almost exactly as it was done above in the case, when $\lambda = 1$. The only difference is that we need first to attach to C_λ very short paths connecting them with a point $q_{\tau(\lambda)}$ inside the slice $f(x,y)(\tau(\lambda)), x, y \in R_{ij}$, to make them end at the same point. This (filling map of the) parallelipiped is swept-out by paths of a controlled length connecting p and $q_{\tau(\lambda)}$. Then we would like to attach to each of these paths the "tail" $f(x,y)(\tau)$, where τ varies from $\tau(\lambda)$ to 1 for an appropriate (x, y). Of course, in order to do so we first need to connect $q_{\tau(\lambda)}$ with $f(x,y)(\tau(\lambda))$ by a path of length $\leq \varepsilon$ inside $f(R_{ij} \times \{\tau(\lambda)\})$. As the result of this construction we will obtain for every λ a family of paths starting and ending at p of acceptable for us length. Combining all these families, we will obtain maps from S^2 to $\Omega_p M^n$. For $\lambda = 1$ we will obtain g, and for $\lambda = 0$ we will obtain the original map f.

The same idea of construction of a homotopy between f and g works verbatim for higher dimensions, so we will not repeat it again in the proof of the general case of Theorem 1.1 that we are going to present now. The proof is completely parallel to the proof in the case $m = 2$.

Proof. Before starting our proof we would like to make a comment about our figures. We will be sometimes depicting a copy of the base point p at the end of loops as a separate point. We believe that this convention makes our figures easier to comprehend, and facilitates the understanding of the fact that our proof immediately generalizes to the case, when $\Omega_p M^n$ in the conclusion of the theorem is replaced by $\Omega_{p,q} M^n$ for two arbitrary points $p, q \in M^n$.

We will present a proof only in the case, when f is a map of S^m. The proof in the case, when f is a map of the pair $(D^m, \partial D^m)$ is completely analogous.

Let $f : I^m \longrightarrow \Omega_p M^n$ be a continuous map such that ∂I^m is mapped to p. We can regard it as a sphere in the loop space $\Omega_p M^n$. We can partition I^m into m-cubes $R_{i_1,\ldots,i_m}$, so that the image of each cube has an arbitrarily small diameter.

We will now construct a new map $\tilde{f} : I^m \longrightarrow \Omega_p M^n$ that passes through short loops, and is homotopic to f. This construction is inductive to skeleta of I^m. Theorem 1.3 tells us how to replace the vertices and Theorem 3.1 tells us how to replace the edges.

For every m-tuple $(i_1, \ldots, i_m)$ consider a map $F : R_{i_1,\ldots,i_m} \times [0,1] \longrightarrow M^n$, where $F(x_1, ..., x_m, \tau) = f(x_1, ..., x_m)(\tau)$. For each fixed τ consider a slice $f(x_1, ..., x_m)(\tau)$ (that we are going to call τ-slice below). We want to ensure that the length of the image under F of every straight line segment of every τ-slice is much smaller than some small ε, which can be done by refining the partition. Each slice can be continuously swept out by curves as in Fig. 20.

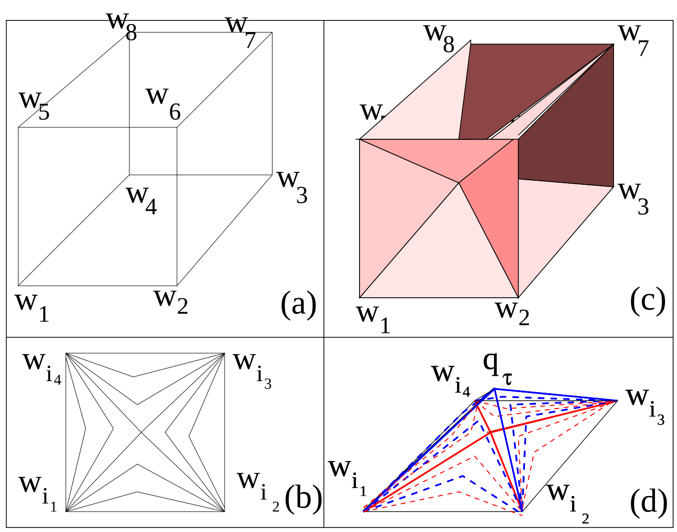

Figure 20: Sweeping out of a slice

Note that each face of the τ-slice $Z_\tau = f(x_1, ..., x_m)(\tau)$ comes with a sweep out by short curves from the previous induction step, (see fig. 20 (b)). Let q_τ be a point in the interiour of Z_τ. Consider the cones with the vertex at q_τ over the boundary of each face, (see fig. 20 (c)). Each of these cones can be continuously deformed to the corresponding face, and we can continuously extend the sweep out of this face to the cone over this face with the vertex at q_τ, (see fig. 20 (d)), which will result in the sweep-out of the whole slice into curves that can be made arbitrarily short.

Next, let us note that there is an m-disc in M^n that corresponds to each face of $Q_\tau = R_{i_1,\ldots,i_m} \times [0,\tau]$, for which this face is also the τ-slice. This disc was constructed during the previous induction step. This disc is swept-out by loops based at a point p of length at most $((m-1)(4k+2)+(2k-1))d+o(1)$ Moroever, it follows from our construction that the interiors of the images of all side faces of $R_{i_1,\ldots,i_m} \times [0,1]$ are sliced into short arcs appearing in the middle of some of these loops. These short arcs are images under f of the broken lines made of two segments (or, sometimes, one segment) connecting the vertices on fig. 20 (b).

For each side face of Q_τ we can consider a homotopy, where only these middle parts of these loops change, as it is depicted on fig. 20 (d) and described above, and the other parts remain intact. Moreover, here we modify only the middle parts of the loop on the "top" of Q_τ, i.e. those that corresponds to points in the considered face of $R_{i_1,\ldots,i_m} \times \{\tau\}$ (Fig. 20 (c), (d) corresponds to the case $m = 3$, but the general picture will be completely similar - and obvious for all values of m.) By doing this, we extend the already constructed map F of the side face to the cone over the top face of the considered side face. This cone is in $R_{i_1,\ldots,i_m} \times \{\tau\}$; its vertex is the center of $R_{i_1,\ldots,i_m} \times \{\tau\}$. Combining these extensions for all side faces we extend F to the whole $R_{i_1,\ldots,i_m} \times \{\tau\}$. Moreover, these extensions can be natuarlly swept-out by $2m$ families of loops corresponding to the side faces.

As the result, for each τ we obtain a map S^m_τ of $\partial(R_{i_1,\ldots,i_m} \times [0,\tau])$ into M^n that can be sliced into $2m$ families of loops described above. The construction done on the previous steps of induction implies that each of these families of loops provides a contraction of the map of S^{m-2} into $\Omega_p M^n$ that was denoted S^{m-2}_1 via spheres S^{m-2}_τ swept-out by loops of length $\leq ((m-1)(4k+2)-2)d + o(1)$.

We would like to apply Lemma 2.2 to this homotopy for every value of τ. To apply Lemma 2.2 (or, rather, its proof) we need to fix a path between p and another point, that we will denote q_τ, and define as the image of the center of the τ-slice under $F = f(x_1,\ldots,x_m)(t)$. (This path was denoted "$f(s_0)$" in the text of Lemma 2.2.) We choose this path as the join of $C^{f(v)}_\tau$ and an extremely short arc that connects the endpoint of $C^{f(v)}_\tau$ (on $f(v)$)

and q_τ along the image of the diagonal of the τ-slice. Here v is a vertex of the considered face of $R_{i_1,\ldots,i_m}$. (The choosen vertex v must be the same as the vertex that we used at the very last stage involving the last application of Lemma 2.2 to construct S_τ^{m-2} corresponding to the considered face of $R_{i_1,\ldots i_m}$ on the previous step of the induction process.) As before, $C_\tau^{f(v)}$ is a path from the continuous family of paths of length $\leq (2k+1)d+o(1)$ connecting p with the points of $f(v)$ obtained in the course of shortening of $f(v)$ as described in the proof of Theorem 1.3. After we apply Lemma 2.2 we obtain a representation of each side face as a $(m-1)$-disc in the space of paths between p and q_τ of length $\leq ((4k+2)(m-1)-2)d+(2k+1)d+o(1)$.

We can combine these $2m$ (maps of) $(m-1)$-discs (that agree on their intersections) regarded as faces of the boundary of a m-dimensional cube C^m into a map of $S^{m-1}=\partial C^m$ to $\Omega_{p,q_\tau}^{((4k+2)(m-1)-2)d+(2k+1)d+o(1)}M^n$. This map will be denoted S_τ^m. When $\tau=1$, $q_\tau=p$, and S_1^m is the sphere that we need to contract in order to extend g from the boundary of $R_{i_1,\ldots,i_m}$ to its interior.

Our next step is the conversion of maps S_τ^m of S^m into spaces of paths $\Omega_{pq_\tau}M^n$ into maps of S^m into Ω_pM^n by choosing a fixed path $p_\tau\in\Omega_{pq_\tau}M^n$ and attaching to each path connecting p and q_τ in the image of S_τ^m $\bar{p}_\tau$ (that is, p_τ the opposite direction from q_τ to p). We define p_τ as follows. We choose a vertex v of $R_{i_1,\ldots i_m}$ and consider the family of paths $C_t^{f(v)}$ corresponding to $f(v)$ (as described in the proof of Theorem 3.1). For every value of τ we use $C_t^{f(v)}$ with the maximal value of t that goes to $f(v)(\tau)$. We take its join with a very short arc that connects $f(v)(\tau)$ with q_τ in the image of the τ-slice, and use the result as p_τ. Observe, that p_τ continuously depends on τ and has length $\leq (2k+1)d+o(1)$. The length of the resulting loops will be bounded by $((4k+2)(m-1)-2+(2k+1)+(2k+1))d+o(1)=((4k+2)m-2)d+o(1)$.

Now we apply Lemma 2.2 again. We use $C_1^{f(v)}$ as "$f(s_0)$" from Lemma 2.2. As the result, we will extend S_1^m to a map of D^{m+1} (identified with $R_{i_1,\ldots,i_m}$) into $\Omega_{pq_1}^{((4k+2)m-2+(2k-1))d+o(1)}M^n=\Omega_p^{((4k+2)m+(2k-3))d+o(1)}M^n$.

The resulting map of a disc that we identify with $R_{i_1,\ldots,i_m}$ into Ω_pM^n constitutes a part of the desired map of S^m into Ω_pM^n. Observe that we did not change the restriction of this map on the boundary of $R_{i_1,\ldots,i_m}$, when we were constructing its extension from the boundary to the interior of $R_{i_1,\ldots,i_m}$. Now we can combine the constructed maps over all m-cells $R_{i_1,\ldots,i_m}$ to obtain the desired map.

The proof of the fact that the constructed map of S^m into Ω_pM^n is homotopic to the initial one through the space of loops of length specified in the theorem is done exactly as in the case of $m = 2$ above (and very similarly to the proof of the same fact in the case $m = 1$). □

5 Short geodesic segments connecting pairs of points

In this section we will prove that for each pair of points on a closed Riemannian manifold there exist "many" "short" geodesic segments that join the points. This fact follows directly from the following lemmas, which are restatements of the similar lemmas for geodesic loops.

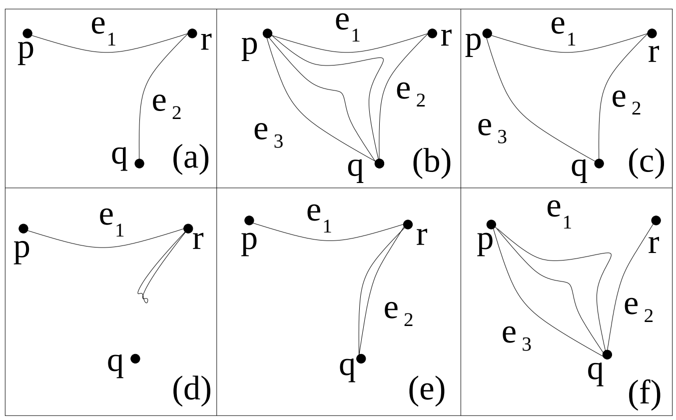

Figure 21: Short path homotopy

Lemma 5.1 *Let M^n be a closed Riemannian manifold of diameter d. Let e_1, e_2 be segments of lengths l_1, l_2 respectively, where e_1 connects a point p with a point r and e_2 connects the point r to a point q, (see fig. 21 (a)). Consider the join $e_1 * e_2$. Assume that it is path homotopic to a path e_3 of length $l_3 \leq l_1 + l_2$ via a length non-increasing path homotopy, (see fig. 21 (b)).*

Then there exists a path homotopy between e_1 and $e_3 * \bar{e_2}$, (fig. 21 (c)) that passes through curves of length at most $l_1 + 2l_2$.

Proof. The proof is essentially demonstrated by fig. 21 (d)-(f). We begin with e_1, (fig. 21 (d)), which is homotopic to $e_1 * e_2 * \bar{e_2}$ over the curves of length $l_1 + 2l_2$, (fig. 21 (e)). Since $e_1 * e_2$ is path homotopic to e_3, and the path homotopy does not increase the length, we can attach $\bar{e_2}$ to all paths in this path homotopy to obtain a homotopy between $e_1 * e_2 * \bar{e_2}$ and $e_3 * \bar{e_2}$, (see fig. 21 (f)). ☐

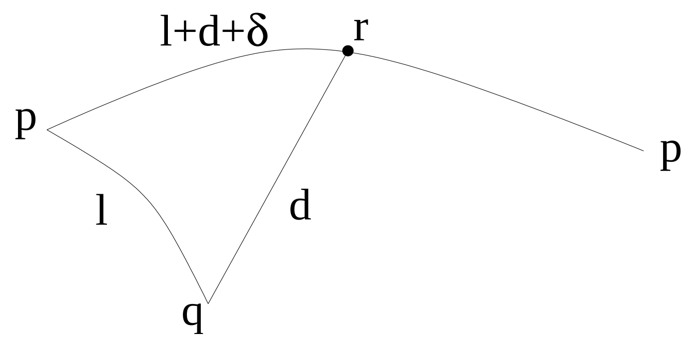

Figure 22: Modified length shortening process

Theorem 5.2 *Let M^n be a closed Riemannian manifold, p, q, x points of M^n. Let $\gamma(t)$ be a curve of length L starting at p and ending at x. Then, if there exists an interval $(l, l+2d]$, such that there is no geodesic connecting p and q on M^n of length in this interval providing a local minimum of the length functional on $\Omega_{pq}M^n$, then there is a path-homotopy between $\gamma(t)$ and a path $\tilde{\gamma}(t)$ of length at most $l+d$ passing through curves of length at most $L+2d$.*

Proof. The proof relies on the previous lemma, but is otherwise analogous to the proof of Theorem 1.3, (see fig. 22).

For example, here is an adaptation of the first step of the curve shortening process described in our proof of Theorem 1.3 By compactness there exists a small δ, such that there are no short geodesics connecting p and q of length in the interval $(l, l+2d+\delta]$. Consider a segment e_1 of the original curve of length $l+d+\delta$ connecting p with some point r. Let us denote a minimizing geodesic connecting the point r with the point q by e_2. The curve $e_1 * e_2$ is path homotopic to e_3 of length at most l. (Here we define e_3 as a shortest path which is path homotopic to $e_1 * e_2$ via a length non-increasing homotopy.) Therefore, by the previous lemma there is a path homotopy between e_1 and $e_3 * \bar{e_2}$ of length at most $l+d$ over the curves of length at most $l+2d$.

☐

The above result has the following corollaries.

Theorem 5.3 *Let M^n be a closed Riemannian manifold of diameter d, p, q, x be points of M^n. Assume that there exists $k \in N$ such that there is no geodesic of length in the interval in the interval $(dist(p,q) + (2k-2)d, dist(p,q) + 2kd]$ joining p and q that are local minima of the length functional on $\Omega_{pq}M^n$. Then for every positive integer m every map $f : S^m \longrightarrow \Omega_{px}M^n$ is homotopic to a map $\tilde{f} : S^m \longrightarrow \Omega^L_{px}M^n$, where $L = ((4k+2)m+(2k-3))d+(2m+1)dist(p,q)$. Further, every map $f : (D^m, \partial D^m) \longrightarrow (\Omega_{px}M^n, \Omega_{px}^{((4k+2)m+(2k-3))d+(2m+1)dist(p,q)}M^n)$ is homotopic to a map $\tilde{f} : (D^m, \partial D^m) \longrightarrow \Omega_{px}^{((4k+2)m+(2k-3))d+(2m+1)dist(p,q)}M^n$ relative to ∂D^m. If for some R the image of f is contained in $\Omega^R_{px}M^n$, then the homotopy between f and $\tilde{f}$ can be chosen so that its image is contained in $\Omega^{R+2d}_{px}M^n$. Also, in this case for every $R > 0$ every map $f : S^0 \longrightarrow \Omega^R_{px}M^n$ is homotopic to a map $\tilde{f} : S^0 \longrightarrow \Omega_{px}^{(2k-1)d+dist(p,q)}$ by means of a homotopy with the image inside $\Omega^{R+2d}_{px}M^n$.*

This theorem can be proven exactly as Theorem 1.1 but using the previous theorem instead of Theorem 1.3 on the very first step of induction (from $m = 1$ to $m = 2$).

Corollary 5.4 *Let M^n be a closed $(m-1)$-connected Riemannian manifold of diameter d with a non-trivial mth homotopy group. Then if for some pair of points p, q there exists $k \in N$, such that no geodesic connecting p and q has the length in the interval $(2(k-1)d + dist(p,q), 2kd + dist(p,q)]$ and is a local minimum of the length functional on $\Omega_{pq}M^n$, then the length of a shortest non-trivial periodic geodesic on M^n is at most $((4k+2)m + (2k-3))d + (2m+1)dist(p,q) \leq (4km + 2k + 4m - 2)d$.*

Proof. By Theorem 5.3 we can construct a non-contractible sphere of dimension m in the space of loops based at p that is swept-out by closed curves of length at most $((4k+2)m + (2k-3))d + (2m+1)dist(p,q)$. Now the standard proof of the Lyusternik-Fet theorem establishing the existence of a non-trivial periodic geodesic on every closed manifold (cf. [Kl]) will yield the desired upper bound for the length of a shortest periodic geodesic. □

This corollary vastly generalizes previous results by F. Balacheff ([B]) and R. Rotman ([R2]) in some directions. These previous results correspond to the cases $k = 1$, $p = q$ and $m = 1$ ([B]) and $k = 1$, $p = q$ and $m = 2$ ([R2]). Yet the upper bound for the length of a shortest non-trivial periodic geodesic provided by the corollary in these two cases is somewhat worse than the corresponding bounds in the quoted papers ($5d$ versus $4d$ in [B], and $11d$ versus $6d$ in [R2]).

6 Non-simply connected case

Here we prove Theorem A that is required to complete the proof of Theorem 1.2 in the non-simply connected case, as well as its generalization Theorem B.

First, recall a result of Gromov ([Gr]) asserting that for every closed Riemannian manifold M of diameter d and every point $p \in M$ there exists a finite presentation of $\pi_1(M^n)$ such that all its generators can be realized by geodesic loops of length $\leq 2d$ based at p. This result will be repeatedly used in this section.

Definition. *Let G be a finitely presented group. A word in generators of G and their inverses is called minimal if the element of G presented by this word cannot be presented by a word of smaller length. The complexity of an element of G with respect to the considered finite presentation is the length of a minimal word representing this element. The complexity of the trivial element is, by definition, zero.*

Proposition 6.1. *Let G be a finitely presented group. Assume that there exists an element $h \in G$ of complexity $m \geq 1$. Then G has at least $2m$ elements: the trivial element e, h, and at least two elements of complexity i for every $i = 1, 2, \ldots, m-1$.*

This proposition has the following immediate corollary:

Corollary 6.2. *Assume that G is a finitely presented finite group of order l. Then the complexities of elements of G do not exceed $\frac{l}{2}$. If there exists an element of complexity $\frac{l}{2}$, then it is unique.*

Proof of Proposition 6.1: We will start from the following observation that will be repeatedly used in our proof: Any subword of a minimal word is minimal.

Now assume that h can be represented by a minimal word starting from a positive power of a generator a. Among all minimal words representing h and starting from a^j for some j choose a word w, where j is maximal possible.

Case 1. If $w = a^m$, then for every i between 1 and $m-1$ words a^i and a^{-i} are minimal and represent different words of complexity i. (Indeed, if $a^r = a^{-s}$ for $r, s \in \{-(m-1), \ldots, m-1\}$, $r \neq -s$, then $a^{r+s} = a^{-(r+s)} = a^{|r+s|} = e$.

As $|r+s| \leq 2m-2$, a^m can be represented by a shorter word $a^{m-|r+s|}$, which contradicts the minimality of a^m.)
Case 2. $w = a^k b l_1 \ldots l_{m-k-1}$, where $b,\ l_1,\ \ldots, l_{m-k-1}$ are generators of G or their inverses. Moreover, we can assume that b is not equal to a power of a.

Now we can consider $2k$ words a^i, a^{-i} for $i = 1, \ldots, k-1$. We have two distinct words of complexity i for each considered value of i in this set.

For every value of $i \in \{k, \ldots m-1\}$ consider the initial subword of w of length k, and the subword of w of length k starting from the second letter. For example, for $i = k$ we will be considering a^k and $a^{k-1}b$, for $i = k+1$ $a^k b$ and $a^{k-1} b l_1$. These words are minimal and represent elements of G of complexity i. We need only to verify that they are not equal to each other. But if $a^{k-1} b l_1 \ldots l_{i-k} = a^k b l_i \ldots l_{i-k-1}$, then we can replace the subword $a^{k-1} b l_1 \ldots l_{i-k}$ by $a^k b l_i \ldots l_{i-k-1}$ in w and obtain the word $a^{k+1} b l_1 \ldots l_{i-k-1} l_{i-k+1} \ldots l_{m-k-1}$ of length m representing h but starting from a higher power of a than w. This contradicts to the definition of w. QED

Proof of Theorem A: Let $\tilde{p}, \tilde{q}$ be two points of the universal covering $\tilde{M}$ of M. We know that $\tilde{M}$ can be tiled by isometric fundamental domains of radius d centered at points $\tilde{p}_i$ that project to the same point $p \in M$ as $\tilde{p}$. (The interiors of these domains are Voronnoi cells of $\tilde{p}_i$, i.e. sets of points x of $\tilde{M}$ for which $\tilde{p}_i$ is the closest to x point in the inverse image of p.) We can assume that $\tilde{p} = \tilde{p}_1$ is the base point of $\tilde{M}$. The number of these fundamental domains is equal to the cardinality C of $\pi_1(M)$. Correspondingly, each of these fundamental domains contains a point $\tilde{q}_i$ that projects to the same point of M as $\tilde{q}$. Assume that $\tilde{q} = \tilde{q}_j$ for some $j \in \{1, \ldots C\}$.

Note that for every j $\tilde{p}_j$ corresponds to an element g_j of $\pi_1(M)$. Vice versa for every element $g \in G$ we can define the corresponding $\tilde{p}_j$ by lifting a loop representing g; $\tilde{p}_j$ will be the endpoint of the lifted loop and will not depend on the choice of a loop representing g. If a finite presentation of $\pi_1(M)$ is chosen, all its generators represented by loops in M of length $\leq 2d$ based at p, and g_j is represented by a word v in generators of M and their inverses of length l, then the distance between $\tilde{p}$ and $\tilde{p}_j$ cannot exceed $2dl$, as $\tilde{p}_j$ is the endpoint of the result of lifting to $\tilde{M}$ of a join of l loops in M representing generators of $\pi_1(M)$ iand their inverses combined exactly as the corresponding letters in v. Let u be the maximal complexity of an element of $\pi_1(M)$ with respect to the chosen finite presentation of $\pi_1(M)$. Corollary 6.2 implies that either $u < \frac{C}{2}$, and therefore $u \leq \frac{C-1}{2}$, or $u = \frac{C}{2}$, but there is only one element of this complexity, and the complexity of the other elements does not exceed $\frac{C}{2}-1$. In the first case, $dist(\tilde{p}_1,\tilde{p}_j)\leq 2d(\frac{C-1}{2})\leq d(C-1)$, and $dist(\tilde{p},\tilde{q})=dist(\tilde{p}_1,\tilde{q}_j)\leq dist(\tilde{p}_1,\tilde{p}_j)+dist(\tilde{p}_j,\tilde{q}_j)\leq d(C-1)+d=Cd$. In the second case, $dist(\tilde{p},\tilde{q})\leq Cd$ by the same argument unless the element of $\pi_1(M^n)$ corresponding to $\tilde{p}_j$ has complexity $\frac{C}{2}$.

In this last case, we are first going to make the following observation: The distance from a point z in a fundamental domain domain to the boundary of the fundamental domain is at most d. Indeed, we can connect the center of the fundamental domain with z by a geodesic and continue the geodesic to the boundary of the fundamental domain. The length of this geodesic does not exceed d, and the same is obviously true for the length of a segment of this geodesic starting at z.

Now denote one of the closest to $\tilde{q}_j$ points in the boundary of its fundamental domain by ϱ. The ditance between $\tilde{q}_j$ and ϱ does not exceed d. The point ϱ must be in the closure of another fundamental domain centered at $\tilde{p}_m$ for some $m\neq j$. Now we can write

$$dist(\tilde{p},\ \tilde{q})=dist(\tilde{p}_1,\ \tilde{q}_j)\leq dist(\tilde{p}_1,\tilde{p}_m)+dist(\tilde{p}_m,\varrho)+dist(\varrho,\tilde{q}_j)\leq$$

$$\leq 2d(\frac{C}{2}-1)+d+d=Cd.$$

QED.

Corollary 6.2.A *Let G be a (finite or infinite) finitely presented group, and k an integer number greater than 2. Assume that G has at least k elements. Then either*
(1) There exist at least k elements of G with complexity strictly less than $\frac{k}{2}$; or
(2) The number k is even. There is at least one element of complexity $\frac{k}{2}$, and there exist exactly $k-1$ elements of complexity $\leq\frac{k}{2}-1$. Moreover, in this case G is isomorphic to one of the following groups: Z, Z_N for some $N\geq k$, Z_2*Z_2, or $Z_2*Z_2/\{(ab)^N\}$ for some $N\geq\frac{k}{2}$, where a, b are the non-trivial elements in the two copies of Z_2.

Proof: If not all elements of G have complexity $<\frac{k}{2}$, then there exists an element of complexity $\frac{k}{2}$, if k is even, or $\frac{k+1}{2}$, if k is odd. Arguing as in the proof of Proposition 6.1 we see that in the second case G has k elements of complexity $\leq\frac{k-1}{2}$. Assume now that k is even, and there exists an element of complexity $\frac{k}{2}$. Arguing as in the proof of Proposition 6.1 we see that there are at least $k-1$ elements of complexity $\leq\frac{k}{2}-1$. Assume that all elements of G of complexity $\leq\frac{k}{2}-1$ are among these $k-1$ elements constructed in the proof of Proposition 6.1. Then we see that either (a) the element of complexity $\frac{k}{2}$ is a power of a generator a, each other generator of G is equal to a or a^{-1}, and, therefore G is cyclic; or (b) the element of complexity $\frac{k}{2}$ is represented by a word of the form $a^i b \ldots$, where $b \neq a$ or a^{-1}. In case (b) every other generator c must be equal to a, b or their inverses. Furthermore, as there are exactly two elements of complexity one a^{-1} must be equal to a, and b^{-1} to b. Therefore G is isomorphic either $Z_2 * Z_2$ or to some its quotient. It is easy to see that if G is a quotient of $Z_2 * Z_2$, then this quotient must be isomorphic to $< Z_2 * Z_2 | (ab)^N = e >$ for $N \geq \frac{k}{2}$. QED.

Proof of Theorem B. Let $\tilde{p}$ be a fixed lifting of p to the universal covering $\tilde{M}$ of M. Tile M by fundamental domains such that their interiors are the Voronnoi cells of points in the inverse image of p under the covering map. These fundamental domains have radius d. All of them correspond to different elements of G that acts as a group of covering transformations. If $\tilde{p}_i$ corresponds to an element of G of a complexity l, then $dist(\tilde{p}, \tilde{p}_i) \leq 2dl$, and if $\tilde{q}_i$ is the lifting of q that lies in the fundamental domain centered at $\tilde{p}_i$, then $dist(\tilde{p}, \tilde{q}_i) \leq dist(\tilde{p}, \tilde{p}_i) + dist(\tilde{p}_i, \tilde{q}_i) \leq 2ld + d = (2l+1)d$. Assume that there exists k elements of G of complexity $\leq \frac{k-1}{2}$. Then we can connect $\tilde{p}$ with k liftings of q into $\tilde{M}$ at distances $\leq kd$ from $\tilde{p}$ by geodesics. Projecting these geodesics to M we will obtain k distinct geodesics between p and q of length $\leq kd$, which are not even pairwise path homotopic.

Corollary 6.2.A implies that it remains to consider the case, when G is a cyclic group of infinite order or of order $\geq k$, or when G is either $Z_2 * Z_2$ or its quotient $< a, b | a^2 = e, b^2 = e, (ab)^N = e >$, $N \geq \frac{k}{2}$.

The proof of Proposition 2 in [NR0] implies the existence of k pairwise non path-homotopic geodesics of length $\leq kd$ connecting p and q in the case, when G is Z or Z_N, $N > k$. Theorem A implies the desired assertion, when $G = Z_k$ or, when $G =< a, b | a^2 = b^2 = (ab)^{\frac{k}{2}} = e >$. It remains to consider the cases when $G = Z_2 * Z_2$ or $Z_2 * Z_2 / \{(ab)^N\}$, $N > \frac{k}{2}$. This can be easily done using the method of proof of Proposition 2 in [NR0].

To complete the proof in this remaining case first note that there are exactly $k-1$ elements of G of complexity $\leq \frac{k}{2} - 1$, namely $e, a, b, ab, ba, \ldots, (ab)^{\frac{k}{2}-1}a, (ba)^{\frac{k}{2}-1}b$. If we consider the liftings of q into the corresponding fundamental domains, connect them with $\tilde{p}$ by a minimal geodesics, and project these geodesics back to M, the result will be $k-1$ distict geodesics in M between p and q of length $\leq (\frac{k}{2}-1)(2d) + d = (k-1)d$. It remains to construct one more geodesic between p and q of length $\leq kd$. We will prove that $\tilde{p}$ and the lifting of q in either the fundamental domain corresponding tp $(ab)^{\frac{k}{2}-1}a$ or $(ba)^{\frac{k}{2}-1}b$ can be connected by a geodesic of length

$\leq kd$. Note, that we are immediately guaranteed a geodesic between these points (for either of these two domains) of length $\leq \frac{k}{2}(2d) + d = (k+1)d$, but we want to improve one of these two upper bounds by d. Of course, we will immediately obtain the desired improvement if there exists a geodesic between $\tilde{p}$ and a lifting of q in one of the fundamental domains corresponding to elements of G of complexity between 2 and $\frac{k}{2} - 1$ of length $\leq 2d$. Indeed, in this case we can connect $\tilde{p}$ and the center of this fundamental domain by a path of length $\leq 3d$ (instead of at least $4d$), and at least one of these two upper bounds improves by d, as desired.

Therefore, we assume that the distances from $\tilde{p}$ to all liftings of q to fundamental domains corresponding to elements of G of complexity between 2 and $\frac{k}{2} - 1$ are greater than $2d$. Now realize a and b by geodesic loops l_a and l_b of length $\leq 2d$ based at p. Denote the midpoint of l_a by A. Connect A and q by a minimizing geodesic γ_a. Denote two halves of l regarded as paths between p and A by l_{1a}, l_{2a}. Consider paths $l_{1a} * \gamma_a$ and $l_{2a} * \gamma_a$ between p and q. Apply a length non-increasing curve shortening process to both these paths. At the end we will obtain two geodesics between p and q of length $\leq 2d$. The liftings of these geodesics to $\tilde{M}$ can connect $\tilde{p}$ only with liftings of q in the fundamental domains corresponding to e, a or b, as our assumption implies that they are too short to reach liftings of q to other fundamental domains. As the join of $l_{1a} * \gamma_a$ and $l_{2a} \bar{*} \gamma_a$ is a loop homotopic to a, the lifting of one of these two paths connects $\tilde{p}$ with the lifting of q to the fundamental domain centered at $\tilde{p}$ and corresponding to e, and the other, say $l_{2a} * \gamma_a$ connect $\tilde{p}$ with the lifting of q into the fundamental domain corresponding to a. Now consider the path $(l_a * l_b)^{\frac{k}{2}-1} * l_{2a} * \gamma_a$ between p and q. Its length does not exceed $(2d)(\frac{k}{2} - 1) + d + d = kd$. This path lifts to a geodesic between $\tilde{p}$ and the lifting of q corresponding to $(ab)^{\frac{k}{2}-1}a$. Applying to this path a curve-shortening process we will obtain the desired geodesic between $\tilde{p}$ and the lifting of q to the fundamental domain corresponding to $(ab)^{\frac{k}{2}-1}a$ of length $\leq kd$. QED.

7 Depth of local minima

We will start from the following definition:

Definition 7.0 *Let γ be a path connecting two (not necessarily) distinct) points p and q in M^n. Assume that γ is a local minimum of the length functional on $\Omega_{pq}M^n$ but not the global minimum of the length functional* *on the connected component of $\Omega_{pq}M^n$ containing γ (and all paths path homotopic to γ). For every path homotopy $F : [0,1] \longrightarrow \Omega_{pq}M^n$ between γ and a path of length that is strictly smaller than the length of γ define the level of F as the maximum of lengths of paths $F(t)$ for $t \in [0,1]$. Define the level of γ as the infimum of levels of all path homotopies between γ and a path of a smaller length. Define the depth of γ as the difference between its level and length.*

If γ is a global minimum of the length functional on its connected component of $\Omega_{pq}M^n$, then we say that the level and the depth of γ are infinite.

We are going to present the following generalizations of Theorems 1.3, 1.1, 5.3:

Theorem 7.1 *Let M^n be a closed Riemannian manifold of diameter d, p and q points in M^n, and $S \geq 2d$ a real number. Let $\gamma(t)$ be a curve of length L connecting points p and q. Assume that there exists an interval $(l, l+2d]$, such that there are no geodesic loops based at p on M^n of length in this interval that provide a local minimum of the length functional on $\Omega_p M^n$ of depth $\geq S$. Then there exists a curve $\tilde{\gamma}(t)$ of length $\leq l+d$ connecting p and q and a path homotopy between γ and $\tilde{\gamma}$ such that the lengths of all curves in this path homotopy do not exceed $L + (S - 2d)$.*

Proof: The proof is essentially the same as the proof of Theorem 1.3 with the following modification: If we get stuck at a geodesic loop of length in the interval $[l, l+d)$ which is a local minimum of the length functional on $\Omega_p M^n$ of depth $\leq S$, we contract this loop to a point by a path homotopy paying the price that the length of curves during the general path homotopy increases by the summand $\leq \max\{S - 2d, 0\}$. QED.

Using Theorem 7.1 in the proof of Theorem 1.1 instead of Theorem 1.3 we obtain

Theorem 7.2 *Let M^n be a closed Riemannian manifold of dimension n and diameter d, p a point of M^n, k a positive integer number, $S \geq 2d$ a real number. Then either:*

1) There exist non-trivial geodesic loops based at p with lengths in every interval $(2(i-1)d, 2id]$ for $i \in \{1, 2, ..., k\}$. Moreover these geodesic loops are local minima of the length functional on $\Omega_p M^n$ of depth $\geq S$;

or

2) For every positive integer m every map $f : S^m \longrightarrow \Omega_p M^n$ is homotopic to a map $g : S^m \longrightarrow \Omega_p^{((4k+2)m+(2k-5))d+(2m-1)S} M^n$. Further, every map $f : (D^m, \partial D^m) \longrightarrow (\Omega_p M^n, \Omega_p^{((4k+2)m+(2k-5))d+(2m-1)S} M^n)$ is homotopic to a map $g : (D^m, \partial D^m) \longrightarrow \Omega_p^{((4k+2)m+(2k-5))d+(2m-1)S} M^n$ relative to ∂D^m. If for some L the image of f is contained in $\Omega_p^L M^n$, then the homotopy between f and g can be chosen so that its image is contained in $\Omega_p^{L+S} M^n$. In addition, for every L every map f from S^0 to $\Omega_p^L M^n$ is homotopic to a map g from S^0 to $\Omega_p^{(2k-1)d} M^n$ by a homotopy with the image inside $\Omega_p^{L+S} M^n$.

Similarly,

Theorem 7.3 *Let M^n be a closed Riemannian manifold of diameter d, p, q, x be points of M^n. $S \geq 2d$ a real number. Assume that there exists $k \in N$ such that there is no geodesic of length in the interval in the interval $(dist(p,q) + (2k-2)d, dist(p,q) + 2kd]$ joining p and q which is local minimum of the length functional on $\Omega_{pq} M^n$ of depth $\geq S$. Then for every positive integer m every map $f : S^m \longrightarrow \Omega_{px} M^n$ is homotopic to a map $\tilde{f} : S^m \longrightarrow \Omega_{px}^L M^n$, where $L = ((4k+2)m + (2k-5))d + (2m+1)dist(p,q) + (2m-1)S$. Further, every map $f : (D^m, \partial D^m) \longrightarrow (\Omega_{px} M^n, \Omega_{px}^{((4k+2)m+(2k-5))d+(2m+1)dist(p,q)+(2m-1)S} M^n)$ is homotopic to a map $\tilde{f} : (D^m, \partial D^m) \longrightarrow \Omega_{px}^{((4k+2)m+(2k-5))d+(2m+1)dist(p,q)+(2m-1)S} M^n$ relative to ∂D^m. If for some R the image of f is contained in $\Omega_{px}^R M^n$, then one can choose the homotopy between f and $\tilde{f}$ so that its image is contained in $\Omega_{px}^{R+S} M^n$. Also, in this case for every R every map $f : S^0 \longrightarrow \Omega_{px}^R M^n$ is homotopic to a map $\tilde{f} : S^0 \longrightarrow \Omega_{px}^{(2k-1)d+dist(p,q)}$ via a homotopy passing through curves of length $\leq R + S$ connecting p and x.*

Definition 7.4 *Let $S_p(M^n)$ denote the maximal depth of a local minimum of the length functional on $\Omega_p^{2d} M^n$. (The maximum exists as the set of all loops of length $\leq 2d$ on M^n parametrized by the arclength is compact.) Equivalently, we can define $S_p(M^n)$ as the minimal number S such that each loop λ based at p of length $\leq 2d$ is contractible via a path homotopy passing through loops of length $\leq$ length$(\lambda) + S$. We will call $S_p(M^n)$ the depth of (M^n, p).*

Apply Theorem 7.2 to $S = S_p(M^n)$ and $l = 0$. By definition of $S_p(M^n)$ there are no local minima of the length functional on $\Omega_p M^n$ with length in the interval $(0, 2d]$ and depth $\leq S_p(M^n)$. Therefore,

Theorem 7.4 *Let M^n be a closed Riemannian manifold with diameter d, p a point of M^n, S the depth of (M^n, p), and m a positive integer number. Then every map $f : S^m \longrightarrow \Omega_p M^n$ is homotopic to a map $\tilde{f} : S^m \longrightarrow \Omega_p^{(6m-3)d+(2m-1)S} M^n$. Further, every map $f : (D^m, \partial D^m) \longrightarrow (\Omega_p M^n, \Omega_p^{(6m-3)d+(2m-1)S} M^n)$ is homotopic to a map $\tilde{f} : (D^m, \partial D^m) \longrightarrow \Omega_p^{(6m-3)d+(2m-1)S} M^n$ relative to ∂D^m. If for some R the image of f is contained in $\Omega_p^R M^n$, then one can choose the homotopy between f and $\tilde{f}$ so that its image is contained in $\Omega_p^{R+S} M^n$. Also, for every $R > 0$ every map $f : S^0 \longrightarrow \Omega_p^R M^n$ is homotopic to a map $\tilde{f} : S^0 \longrightarrow \Omega_p^d M^n$ by a homotopy with the image inside $\Omega_p^{R+S} M^n$.*

8 Quantitative Morse theory on loop spaces.

The quantitative Morse theory on loop spaces was initiated in [Gr0] (see also ch. 7 of [Gr]). It studies injectivity and surjectivity properties of homomorphisms in homology induced by the inclusions of sublevel sets of the length functional on a loop space into the loop space. Here is the main result which is a part of Theorem 7.3 in [Gr]:

Theorem 8.1 (M. Gromov) *For every closed simply-connected Riemannian manifold M^n and a point $p \in M^n$ there exists a constant C such that for every positive integer m the inclusion $\Omega_p^{Cm} M^n$ into $\Omega_p M^n$ induces surjective homomorphisms $H_i(\Omega_p^{Cm} M^n) \longrightarrow H_i(\Omega_p M^n)$ for all $i \in \{0, 1, \dots, m\}$.*

In other words, for every $m \geq 1$ all m-dimensional homology classes of $\Omega_p M^n$ can be realized by cycles “made” out of loops of length $\leq Cm$ based at p. To prove this theorem Gromov constructed for some C and every m an explicit finite dimensional CW-subcomplex $X_m \subset \Omega_p^{Cm} \subset \Omega_p M^n$ such that every map of every m-dimensional CW-complex Y into $\Omega_p M^n$ is homotopic to a map of Y into X_m.

He does not estimate C in his proof. Yet it is easy to see that his proof yields an upper bound for C in terms of the following quantity that we will denote $W_p(M^n)$: This quantity is defined as the minimal w such that every loop of length $\leq 2d$ based at p can be contracted to p by a path homotopy H such that the length of the trajectory $H(*, t), t \in [0, 1]$, of every point of γ during H does not exceed w.

In other words, denote the minimal T such that the inclusion homomorphisms $\pi_i(\Omega_p^T M^n) \longrightarrow \pi_i(\Omega_p M^n)$ are surjective for all $i \in \{0, 1, \dots, m\}$ by

$T_{M^n,p}(m)$. (It is a matter of taste whether to use the homology or the homotopy groups in this definition; the resulting notions are equivalent.) Then the original proof of Gromov implies that $T_{M^n,p}(m) \leq Cm$, and one can use the proof to get an upper bound for C in terms of $W_p(M^n)$. (Observe, that Gromov also notes that $T_{M^n,p}(m) \geq cm$ for some $c > 0$.)

On the other hand our Theorem 7.4 has the following immediate corollary:

Theorem 8.2 *Let M^n be a closed simply-connected Riemannian manifold of diameter d, p a point of M^n, and S the depth of (M^n, p) (see Definition 7.4). Then*
A. *For every positive integer m the inclusion homomorphisms $\pi_i(\Omega_p^{(6m-3)d+(2m-1)S} M^n) \longrightarrow \pi_i(\Omega_p M^n)$ are surjective for all $i \in \{0, 1, \ldots, m\}$. Equivalently, every map of a m-dimensional polyhedron X to $\Omega_p M^n$ is homotopic to a map of X into $\Omega_p^{(6m-3)d+(2m-1)S} M^n$.*
B. *Let m be any non-negative integer number, $R > 0$ a real number. If a map $f : S^m \longrightarrow \Omega_p^R M^n$ is contractible, then it can be contracted within $\Omega_p^{\max\{R,(6m+3)d+2mS\}+S} M^n$.*

Proof: Part A can be proven by a straightforward application of Theorem 7.4. To prove part B we first apply Theorem 7.4 to homotop f to a (contractible) map $\tilde{f}$ of S^m into $\Omega_p^{(6m-3)d+(2m-1)S} M^n$ inside $\Omega_p^{R+S} M^n$, if $m > 0$ and $R > (6m-3)d + (2m-1)S$. If $m > 0$ and $R \leq (6m-3)d + (2m-1)S$, or $m = 0$ and $R \leq d$ we just take $\tilde{f} = f$. If $m = 0$ and $R > d$ we use Theorem 7.4 to homotop f to a map $\tilde{f}$ with the image in $\Omega_p^d M^n$ inside $\Omega_p^{R+S} M^n$.

Then we consider a homotopy F that contracts $\tilde{f}$. We regard F as a map of $(D^{m+1}, \partial D^{m+1}) \longrightarrow (\Omega_p M^n, \Omega_p^{(6m-3)d+(2m-1)S} M^n)$, if $m > 0$, or $(D^{m+1}, \partial D^{m+1}) \longrightarrow (\Omega_p M^n, \Omega_p^d M^n)$ if $m = 0$. Now we again apply Theorem 7.4 to replace F by a homotopy $\tilde{F}$ with the image inside $\Omega_p^{(6(m+1)-3)d+(2(m+1)-1)S} M^n = \Omega_p^{(6m+3)d+(2m+1)S} M^n$. Now we see that the combination of the homotopies from f to $\tilde{f}$ and the contracting homotopy $\tilde{F}$ takes place in $\Omega_p^{\tilde{R}} M^n$, where $\tilde{R} = \max\{R + S, (6m+3)d + (2m+1)S\} = \max\{R, (6m+3)d + 2mS\} + S$. QED.

As $S \geq 2d$, for every $m \geq 1$ $(6m+3)d + (2m+1)S \leq 7.5mS$. Therefore, Theorem 8.2 implies that $T_{M^n,p}(m) \leq 7.5 S_p(M^n) m$. Thus, we obtain the following corollary:

Theorem 8.3

$$T_{M^n,p}(m) \leq 7.5 S_p(M^n) m.$$

To compare our upper bound for $T_{M^n,p}(m)$ with the upper bound that follows from the original proof of Theorem 8.1 given by Gromov (see ch. 7 of [Gr]) observe, that according to [NR] $S_p(M^n) \leq 2W_p(M^n) + 2d$, and as $W_p(M^n)$ is, obviously, greater than or equal to d, $S_p(M^n) \leq 4W_p(M^n)$. (The inequality $S_p(M^n) \leq 2W_p(M^n) + 2d$ immediately follows from the fact that any homotopy contracting a curve γ of length L to a point p such that the length of the trajectory of evry point does not exceed W can be converted into a homotopy where the length of curves does not exceed $2W + l$. The idea is very simple: One first moves only a very small interval of γ, so that only its central part reaches p. Then we gradually expand the "tooth". At every stage only a very short interval of γ is being homotoped towards p.) On the other hand, known upper bounds for $W_p(M^n)$ in terms of $S_p(M^n)$ involve also the injectivity radius of M^n (or the contractibility radius, or, at least, the simply connectedness radius of M^n) - and are also exponential in $\frac{S_p(M^n)}{inj(M^n)}$ (see [NR]). Also, although we did not check the details, the examples constructed in the proof of Theorem 1.2 of [P] seem to demonstrate that $W_p(M^n)$ can, indeed, be exponentially larger than $S_p(M^n)$ even in situations, when the simply-connectedness radius is ~ 1. Thus, our upper bound $7.5S_p(M^n)$ for $\sup_m \frac{T_{M^n,p}(m)}{m}$ in Theorem 8.3 seems to be qualitatively better than an upper bound following from the original proof.

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