Fourier-Gauss Transform and Quantization

Boris Sternin and Victor Shatalov

Moscow State University
Dept. of Comp. Math. and Cybern.
119899 Moscow

Russia

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-53225 Bonn

Germany

MPI / 94-102

Fourier-Gauss Transform and Quantization*

Boris Sternin and Victor Shatalov
Moscow State University
Dept. of Comput. Math. and Cybern.
119899 Moscow, RUSSIA
e-mail: boris@sternin.msk.su

Abstract

We present a universal procedure of quantization based on a certain integral transform named by the authors a Fourier-Gauss transform. This procedure coincides with Schrödinger quantization for observables, with Maslov's quantization for Lagrangian modules and with Fock quantization for canonic transforms of the phase space.

1 Introduction

1. As it follows from the title, the theme of this paper is quantization. Let us first try to explain this notion in more detailed manner.

As it is well-known, in physics by quantization of the classical mechanics one mean the assignment of quantum objects to the corresponding classical ones. The main of these notions are state of the system and observables. We recall that the state of a system in classical mechanics is determined by a point in the phase space (q, p) (the space of coordinates and momenta), and observables are functions $f(q, p)$ on this space (for example, Hamiltonians). In the same time in quantum mechanics the state of the system is described by the so-called ψ-function (or wave function) and observables are described by operators acting in the state space. For Schrödinger quantization the correspondence of classical and quantum objects (on the level of observables) is given in the following way:

$$q \mapsto \widehat{q} = q,$$

*Supported by Max-Plank-Arbeitsgruppe "Partielle Differentialgleichungen und Komplexe Analysis", Potsdam University, by Laboratoire Jean-Alexandre Dieudonné URA au CNRS N 168, Université de Nice-Sophia Antipolis, and by the Chair of Nonlinear Dynamics Systems and Control Processes of Moscow State University.

$$p \mapsto \widehat{p} = -i\hbar \frac{\partial}{\partial q}$$

so that the (pseudodifferential) operator corresponding to the observable $f(q,p)$ reads

$$\widehat{f} = f\left(\overset{2}{q}, \overset{1}{\widehat{p}}\right) \tag{1}$$

(the indices over the operators determine the order of action of these operators – the so-called Feynman ordering).

However, the correspondence between classical and quantum states is not so simple. The matter is that the measurements of coordinates and momenta of a quantum particle can be give different result with certain probability for one and the same quantum state. Thus, there exist states in which the quantum particle can be found during some experiment (in the semi-classical limit) with the probability equal to zero. The set of classical states in which the particle can be found with 'positive' probabilitiy occurs to be the so-called *isotropic* submanifold of the phase space, that is, the submanifold on which the Cartan form pdq is closed.

¿From the viewpoint of a quantum particle, such submanifolds will be nothing more than the front of oscillations corresponding to the ψ-function. Now the quantization rule for state space becomes quite evident: quantization must assign a ψ-function (more precisely, a class of ψ-functions) to the given isotropic manifold in such a way that these functions have their fronts of oscillations in this isotropic manifold.

It can be shown that the dimension of any isotropic submanifold cannot exceed the dimension of the corresponding configuration space. In this paper we restrict ourselves to the case of isotropic manifolds of maximal dimension named Lagrangian manifolds.

Let us present now the mathematical treatment of the above physical reasons. It will be convenient for us to use the language of the category theory.

Let us fix a phase space, for example, the cotangent space to a smooth real manifold M with the canonical symplectic structure $dp \wedge dq$.

i) Consider the category C whose *objects* are modules $C^\infty(\Lambda)$ of smooth complex-valued functions on Lagrangian manifolds Λ over the ring of classical observables. Thus, an object in this category is the abelian group $C^\infty(\Lambda)$ for some Lagrangian manifold (Lagrangian module) with the following action of the ring $C^\infty(T^*M)$ of classical observables:

$$f \cdot \varphi = f(p,q)|_\Lambda \varphi,$$

where in the right-hand side of the latter equality the usual pointwise multiplication is used.

Morphisms in this category are determined by *symplectic transforms* of the phase space. Namely, if for two given Lagrangian manifolds Λ_1 and Λ_2

$$g : T^*M \to T^*M \tag{2}$$

is a symplectic transform such that $\Lambda_2 = g(\Lambda_1)$, then we assign to (2) the module homomorphism

$$g^* : C^\infty(\Lambda_2) \to C^\infty(\Lambda_1)$$

over the ring homomorphism

$$g^* : C^\infty(T^*M) \to C^\infty(T^*M).$$

ii) Consider also the category $\mathcal{Q}$ whose *objects* are modules $C_\hbar^\infty(M,\Lambda)$ of smooth functions $\psi(q,\hbar)$ depending on a parameter $\hbar$ with the fronts of oscillations in Λ over the ring of quantum observables (that is, pseudodifferential operators). We remark that pseudodifferential operators preserve fronts of oscillations and, hence, the above definition is a correct one.

Morphisms in this category are given by mappings

$$T : C_\hbar^\infty(M,\Lambda_1) \to C_\hbar^\infty(M,\Lambda_2)$$

such that the operator $T\widehat{H}T^{-1}$ is a pseudodifferential operator for any pseudodifferential operator $\widehat{H}$.

More precisely we consider the mapping T as the mapping of quotient spaces $C_\hbar^\infty(M,\Lambda_j)/S^1$ modulo the group of complex numbers λ with $|\lambda| = 1$. It is natural for quantum mechanics, because states corresponding to ψ-functions which differ by such a factor are physically identical.

iii) *The semi-classical quantization* is a contravariant functor from the category $\mathcal{C}$ to the category $\mathcal{Q}$:

$$\mathcal{F} : \mathcal{C} \to \mathcal{Q}$$

such that the module of smooth functions on M with front of oscillations on the given Lagrangian manifold is assigned to the module of smooth functions on this Lagrangian manifold, and observables correspond to pseudodifferential operators on M obtained, for example[1], by Schrödinger quantization (1).

As it is shown in the paper, such a functor exists and, what is more, we present its explicit construction based on a new integral transform which we call *Fourier-Gauss transform.* This is *an invertible* transform taking functions $f(x,\hbar)$ determined on the physical space $\mathbf{R}^n$ to some subspace of functions $\widetilde{f}(x,p,\hbar)$ determined on the phase

[1] We use, for such a correspondence, the Schrödinger quantization. Certainly, it is possible to use another quantization, for example

$$f(q,p) \mapsto f\left(\overset{1}{q},\overset{2}{\widehat{p}}\right)$$

or the Weil quantization. All these rules coincide in principal terms, so, from the viewpoint of the leading terms the corresponding semi-classical approximations will coincide.

space $T^*\mathbf{R}^n$. As we shall see below, using such a transform one can carry out a unified construction of quantization of all classical objects (at least, in principal terms).

Moreover, this procedure coincides with the Schrödinger quantization for observables [1], with the Fock quantization for canonical (symplectic) transforms [2] and with the Maslov quantization for Lagrangian modules [3].

Realizing this construction we obtain $1/\hbar$-pseudodifferential operators as quantization of observables, Fourier integral operators as quantization of symplectic transforms and Maslov's canonical operator as quantization of Lagrangian modules (the reader can find the notions used here, for example, in [4]).

2. To conclude these preliminary considerations we shall make some remarks. First of all, it is clear that to obtain the correspondence between classical and quantum objects, that is, to construct a quantization procedure, we shall try to decompose any quantum state $f(x,\hbar)$ into the 'sum' of elements corresponding to points (x,p) of the phase space $T^*\mathbf{R}^n$, that is, to the classical states. Such a decomposition is *a microlocalization procedure*.

The localization of the function $f(x,\hbar)$ in the phase space can be decomposed in the localization along fibers of $T^*\mathbf{R}^n$ and the localization along the base space. The localization along the base space can be done in different ways. First (and simplest) of these ways is multiplying the initial function $f(x,\hbar)$ by a cut-off function supported in a sufficiently small neighbourhood of the point x_0. Unfortunately, this method gives localization not at point x_0 itself but in a neighbourhood (though small) of this point. That is why we use here another localization procedure based on the 'integral partiton of unity' of the form

$$1 = \left(\frac{1}{2\pi\hbar}\right)^{n/2} \int e^{-\frac{1}{2\hbar}(x-x_0)^2}\, dx.$$

The *localization along the base space* is therefore obtained with the help of multiplication by

$$\delta_\hbar(x-x_0) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} e^{-\frac{1}{2\hbar}(x-x_0)^2};$$

note that

$$\delta_\hbar(x-x_0) \to \delta(x-x_0)$$

as $\hbar \to 0$.

The *localization along the fibers* can be done with the help of the quantum Fourier transform, in other words, by means of p-representation, at the point x_0:

$$F_{x\to p_0}[f] = \left(\frac{1}{2\pi i\hbar}\right)^{n/2} \int e^{-\frac{i}{\hbar}p_0(x-x_0)} f(x)\, dx.$$

By composition of these two localizations we obtain the *microlocal element* corresponding to the function $f(x,\hbar)$ in the form

$$f_{(x_0,p_0)} = F_{x\to p_0}\{\delta_\hbar(x-x_0)f(x,\hbar)\} = \left(\frac{1}{2\pi\hbar}\right)^{n/2}\left(\frac{1}{2\pi i\hbar}\right)^{n/2}\times$$
$$\times\int \exp\left\{\frac{i}{\hbar}\left[-p_0(x-x_0)+\frac{i}{2}(x-x_0)^2\right]\right\} f(x)\,dx.$$

The latter formula determines an integral transform[2] which we call *the Fourier-Gauss transform* of the function $f(x,\hbar)$. As it will be shown below, the inverse transform is given by

$$\widetilde{f}(x_0,p_0)\mapsto\left(\frac{i}{\pi\hbar}\right)^{n/2}\int e^{\frac{i}{\hbar}\left[p_0(x-x_0)+\frac{i}{2}(x-x_0)^2\right]}\widetilde{f}(x_0,p_0)\,dx_0\,dp_0.$$

It is convenient to renormalize the obtained transform in such a way that the Parseval identity takes place for this transform. This normalization is presented below.

Acknowledgement. This paper was written in the Laboratory of Jean Alexandre Dieudonné, University Nice-Sophia Antipolis during our stay there in summer of 1994. We are grateful to Prof. Frédéric Pham for very fruitful discussions on the topic of the paper.

2 Front of Oscillations

The notion of wave front describes the localization of a function in the phase space in a way similar to that in which the notion of support describes the localization of a function in the configuration space.

Consider first the latter notion. Given a function $u(x)$, its support is the closure of the set of points where $u(x)$ does not vanish,

$$\operatorname{supp} u = \overline{\{x\in\mathbf{R}^n | u(x)\neq 0\}}.$$

Evidently, the support $\operatorname{supp} u$ may be defined also as follows: a point x_0 belongs to supp u if and only if for any smooth finite function $\varphi(x)$ the identity $\varphi(x)u(x)\equiv 0$ implies $\varphi(x_0)=0$. (This variant of definition works for distributions as well.)

For functions depending on the small parameter h we may introduce also the notion of asymptotic support. We say that a function

$$u(x,h) = O(h^\infty) \quad\text{or}\quad u(x,h)\equiv 0 \pmod h$$

[2]This transform can be considered as the "quantum" version of integral transform introduced in the paper [5].

if

$$\forall\alpha\forall N\,\exists C_{\alpha N}: |\widehat{p}^{\alpha}u(x)| \le C_{\alpha N}h^{N}.$$

Later on, we write

$$u(x,h) \equiv 0 \pmod{h^k}$$

if the inequalities

$$||\widehat{p}^{\alpha}u|| \le C_{\alpha}h^{k}$$

are valid for any multiindex α with some constant C_{α}.

Definition 1 The point x_0 belongs to an *oscillatory support* of the function u,

$$x_0 \in \text{osc-supp}\, u$$

if and only if for any smooth finite function $\varphi(x)$ the estimate $\varphi u = O(h^{\infty})$ implies $\varphi(x_0) = 0$.

Equivalently, the point x_0 does not belong to osc-supp$u$ if and only if there exists a smooth finite function $\varphi(x)$ such that $\varphi(x_0) \neq 0$ and $\varphi u = O(h^{\infty})$.

Let $H(x,\hat{p})$ be an pseudodifferential operator.

Proposition 1

a) *If* $H(x,p) = 0$ *over* $U \subset \mathbf{R}^n_x$ *then* $H(x,\hat{p})u = O(h^{\infty})$ *in arbitrary* $U' \subset U$ *for any* u *satisfying the estimates*

$$\|\widehat{p}^{\alpha}u(x)\|_{L_2} \le C_{\alpha}h^{m} \tag{3}$$

for some m.

b) *Pseudodifferential operators respect osc-supports, i.e.*

$$\text{osc-supp}\, H(x,\hat{p})u \subset \text{osc-supp}\, u$$

for any u *satisfying estimates* (3).

Proof. a) When $x \in U'$, $y \in \text{supp}H(y,p)$, we have $|x-y| \ge \varepsilon > 0$. Consequently,

$$H(x,\hat{p})\,u = \left(\frac{1}{2\pi h}\right)^n \int e^{\frac{i}{h}px-y} H(y,p)u(y)\,dp\,dy$$

$$= \left(\frac{1}{2\pi h}\right)^n \int H(y,p)u(y)\left\{\frac{-ih}{|x-y|^2}(x-y)\frac{\partial}{\partial p}e^{\frac{i}{h}px-y}\right\}\,dp\,dy$$

$$= ih\left(\frac{1}{2\pi h}\right)^n \int e^{\frac{i}{h}px-y}\left\{\frac{1}{|x-y|^2}(x-y)\frac{\partial}{\partial p}H(x,p)\right\}u(y)\,dp\,dy$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$= O(h^{\infty}).$$

b) Exercise.

Now let us define a front of oscillations. Since we intend to localize in the phase space, it is not surprising that pseudodifferential operators with finite symbols appear instead of finite cutoff functions.

Definition 2 A point (x_0, p_0) of the phase space belongs to the *front of oscillations* $OF[u]$ of the function u,

$$(x_0, p_0) \in OF[u],$$

if and only if for any pseudodifferential operator $H(x, \hat{p})$ with finite symbol the estimate $H(x, \hat{p})\, u = O(h^\infty)$ implies $H(x_0, p_0) = 0$.

This definition may be reformulated as follows: the point (x_0, p_0) does not belong to $OF[u]$ if for some finite symbol $H(x,p)$ such that $H(x_0,p_0) \neq 0$ we have $H(x, \hat{p})u = O(h^\infty)$.

We may give another description of OF. Namely, $(x_0, p_0) \notin OF[u]$ if and only if there exist finite functions $\varphi(x)$, $\psi(p)$ such that

$$\varphi(x_0) \neq 0, \quad \psi(p_0) \neq 0, \quad |\psi(p) F_{x \to p}[\varphi(x) u(x)]| \leq C_N h^N$$

for any N; in other words,

$$F_{x \to p}[\varphi(x) u(x)] = O(h^\infty)$$

in a neighbourhood of the point p_0. Since

$$\psi(p) F_{x \to p}[\varphi(x) u(x)] = F_{x \to p}[\psi(\hat{p}) \varphi(x) u],$$

it is clear that in comparison with original definition we restrict ourselves to pseudodiffertential operators whose symbols are products of the form $\psi(p)\varphi(x)$.

Both descriptions are in fact equivalent. To show this, suppose that for some $H(x,p)$ we have

$$H(x_0, p_0) \neq 0, \quad H(x, \hat{p})u = O(h^\infty).$$

Find $G(x,p)$ such that

$$G(x, \hat{p}) H(x, \hat{p}) \equiv \psi(\hat{p}) \varphi(x) \; (\mathrm{mod} h^\infty),$$

where $\mathrm{supp}\psi \times \mathrm{supp}\varphi \subset \{H \neq 0\}$ and $\varphi(x_0)\psi(p_0) \neq 0$ (by composition theorem the principal symbol of $G(x, \hat{p})$ should be equal to

$$\frac{\psi(p)\varphi(x)}{H(x,p)};$$

we leave further details to the reader as an exercise). Then

$$G(x,\hat{p})H(x,\hat{p})u = O(h^\infty) = \psi(\hat{p})\varphi(x)u + O(h^\infty),$$

Thus we obtain

$$\psi(\hat{p})\varphi(x)u = O(h^\infty),$$

which completes the proof.

Conversely, if

$$\psi(\hat{p})\varphi(x)u = O(h^\infty),\ \psi(p_0) \neq 0,\ \varphi(x_0) \neq 0$$

then $(x_0, p_0) \notin OF[u]$ in the sense of our definition (it suffices to take $H(x,p) = \psi(p)\varphi(x)$).

Theorem 1 (localization) *If* $H(x,p) = 0$ *in a neighbourhood of the point* (x_0, p_0) *then* $(x_0, p_0) \notin OF[H(x,\hat{p})u]$.

Proof. If

$$\text{supp} G(x,p) \subset \{H(x,p) = 0\}$$

then the symbol of $G(x,\hat{p})H(x,\hat{p})$ vanishes identically and by Proposition 1

$$G(x,\hat{p})\{H(x,\hat{p})u\} = O(h^\infty).$$

One may choose G such that $G(x_0, p_0) = 1$ since $H \equiv 0$ in the neighbourhood of (x_0, p_0).

Now we present the asymptotic formula for rapidly oscillating integrals with complex phase functions. This formula (referred below as a *stationary phase formula*) will be of use, in particular, for calculations fo oscillatory fronts.

Let us consider the integral

$$I(x,h) = \left(\frac{i}{2\pi h}\right)^{m/2} \int e^{\frac{i}{h}\Phi(x,y)} \varphi(x,y)\, dy. \tag{4}$$

Here $x = (x^1, \ldots, x^n)$ is a point of the space $\mathbf{R}^n$, $y = (y^1, \ldots, y^m)$ is a point of $\mathbf{R}^m$.

We suppose that:

1) The function $\Phi(x,y)$ is a smooth complex-valued function determined in a neighbourhood of the support of the function $\varphi(x,y)$ such that

$$\text{Im}\, \Phi(x,y) \geq 0$$

everywhere on $\text{supp}\varphi$.

2) The function $\varphi(x,y)$ is a smooth complex-valued function with compact support.

We denote by Ω the (not empty) set of points $x \in \mathbf{R}^n$ such that there exists a point $y \in \mathbf{R}^m$ for which the equations

$$\begin{cases} \operatorname{Im}\Phi(x,y) = 0, \\ \dfrac{\partial \Phi(x,y)}{\partial y} = 0 \end{cases} \tag{5}$$

hold. A point $y \in \mathbf{R}^m$, satisfying these equations will be called *a real stationary point* of the phase function Φ. We suppose, for simplicity, that equations (5) have exactly one solution

$$y = y(x)$$

on $\Omega \cap \operatorname{supp}\varphi$ and that the point $y = y(x)$ is nondegenerate. The latter requirement means that

$$\det \operatorname{Hess}_y \Phi(x,y)\big|_{y=y(x)} \neq 0$$

at any point $x \in \Omega$.

For any natural s we introduce the functions

$${}^s\Phi(x,z) = \sum_{|\alpha|\leq s} \frac{1}{\alpha!} \frac{\partial^{|\alpha|}\Phi(x,y)}{\partial y^\alpha} (i\eta)^\alpha, \tag{6}$$

$${}^s\varphi(x,z) = \sum_{|\alpha|\leq s} \frac{1}{\alpha!} \frac{\partial^{|\alpha|}\varphi(x,y)}{\partial y^\alpha} (i\eta)^\alpha, \tag{7}$$

determined for $x \in \mathbf{R}^n$, $z \in \mathbf{C}^m$, $z = y + i\eta$. Then from the implicit function theorem it follows that in some neighbourhood of Ω there exists a unique solution

$$z = z(x)$$

of the equations

$$\frac{\partial}{\partial z} {}^s\Phi(x,z) = 0$$

where the operator $\partial/\partial z$ is defined in a usual way:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial y} - i\frac{\partial}{\partial \eta}\right).$$

The following statement is valid.

Theorem 2 *Under the above assumptions the following asymptotic expansion of integral* (4)

$$I(x,h) \equiv e^{\frac{i}{h}{}^s\Phi(x,z(x))} \sum_{0\leq k<(s-1)/2} h^k \psi_k(x) \pmod{h^{(s-1)/2}} \tag{8}$$

is valid for any natural s. Here $\psi_k(x)$ are smooth finite functions,

$$\psi_0(x) = \frac{{}^{s-1}\varphi(x, z(x))}{\sqrt{\det \text{Hess}\,(-{}^s\Phi(x,z))|_{z=z(x)}}} \tag{9}$$

and in the above formula we assume

$$\arg\det\left\{\text{Hess}\,(-{}^s\Phi(x,z))|_{z=z(x)}\right\} = \sum_k \arg\lambda_k,$$

where λ_k are eigenvalues of the matrix $\text{Hess}_z\,(-{}^s\Phi(x,z))|_{z=z(x)}$,

$$\arg\lambda_k \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right].$$

In formula (8) (and later on) the comparison (mod $h^{(s-1)/2}$) means that the remainder $R(x,h)$ can be estimated in the following way:

$$||\hat{p}^\alpha R(x,h)||_{L_2} \le C_\alpha h^{(s-1)/2}$$

for any multiindex α.

We do not present here the proof of this Theorem, which is rather a cumbersome one. The reader can find this proof, for example, in the book [4].

Remark 1 In the case when the function Φ can be *analytically continued* up to the function $\Phi(x,z)$, $z \in \mathbf{C}^m$ to a neighbourhood of the real space $\mathbf{R}^m$, one can use this analytic continuation in (8), (9) instead of ${}^s\Phi$.

Remark 2 In the case when the function Φ is *a real-valued* function, formula (8) can be also used with Φ, φ instead of ${}^s\Phi$, ${}^s\varphi$ given by (6) (7).

Examples

Example 1 Consider a plane wave,

$$u(x,h) = e^{\frac{i}{h}xp_0}.$$

We have

$$F_{x\to p}\{\varphi(x)u(x,h)\} = \left(\frac{-i}{2\pi h}\right)^{n/2}\int e^{\frac{i}{h}xp-p_0}\varphi(x)\,dx = \frac{1}{h^{n/2}}\,\tilde{\varphi}'\left(\frac{p-p_0}{h}\right),$$

where $\widetilde{\varphi}'(p)$ is the usual (not quantum) Fourier transform of the function φ. For any N

$$|\widetilde{\varphi}'(p)| \leq \frac{C_N}{|p|^N}$$

as $p \to \infty$ provided that $\varphi(x) \in C_0^\infty$. Thus $F_{x\to p}\{\varphi u\} = O(h^\infty)$ for $p \neq p_0$, while $F_{x\to p}\{\varphi u\}(p_0) \sim h^{-n/2}$. We conclude that

$$OF\left\{e^{\frac{i}{h}xp_0}\right\} = \{p = p_0\} \subset \mathbf{R}^n \oplus \mathbf{R}_n.$$

The direction p_0 is a (co)normal direction to surfaces $xp_0 = \text{const}$ (surfaces of constant phase), and the vector $p_0 = \operatorname{grad} xp_0$ itself is the velocity of phase change.

Example 2 Now let consider a function of the form

$$u(x,h) = e^{\frac{i}{h}S(x)}\varphi(x)$$

(a wave with non-plane front). Note that multiplication by finite functions doesn't change the form of $u(x,h)$.

We have

$$H(x,\hat{p})\,u(x,h) = \left(\frac{1}{2\pi h}\right)^n \int e^{\frac{i}{h}p(x-y)+S(y)}\,H(y,p)\varphi(y)\,dp\,dy.$$

Let us apply the stationary phase formula to this integral. Its phase equals

$$\Phi(x,y,p) = p(x-y) + S(y).$$

Thus the equations for stationary points read

$$\begin{cases} \dfrac{\partial\Phi}{\partial p}(x,y,p) = x - y = 0, \\ \dfrac{\partial\Phi}{\partial y}(x,y,p) = -p + \dfrac{\partial S}{\partial y} = 0, \end{cases}$$

so that

$$\begin{cases} y = x, \\ p = \dfrac{\partial S}{\partial x}(x). \end{cases}$$

Next, we have

$$\operatorname{Hess}_{p,y}(-\Phi)\big|_{y=x,p=\frac{\partial S}{\partial x}(x)} = \left\| \begin{matrix} 0 & 1 \\ 1 & -\dfrac{\partial^2 S}{\partial x^2}(x) \end{matrix} \right\|$$

this yields

$$\det\ \mathrm{Hess}(-\Phi) = (-1)^n \neq 0.$$

Finally, we obtain

$$H(x,\hat{p})u(x,h) \equiv e^{\frac{i}{h}S(x)}H\left(x, \frac{\partial S(x)}{\partial x}\right)\varphi(x) \pmod{h}.$$

Consequently, if

$$H(x,\hat{p})u(x,h) = O(h^\infty),$$

then

$$H(x,p) = 0 \ \text{ on } \ \left\{p = \frac{\partial S(x)}{\partial x}\right\}.$$

On the other hand, if

$$\mathrm{supp} H \cap \left\{p = \frac{\partial S(x)}{\partial x}\right\} = \emptyset$$

then

$$H(x,\hat{p})u(x,h) = O(h^\infty)$$

(the proof employs integration by parts, as in Proposition 1). Thus

$$OF\{e^{\frac{i}{h}S(x)}\varphi(x)\} = \left\{p = \frac{\partial S(x)}{\partial x}\right\} \cap \mathrm{supp}\varphi.$$

Again the direction

$$p = \frac{\partial S(x)}{\partial x}$$

is a (co)normal to surfaces $S = \mathrm{const}$ of constant phase, and the vector itself is the velocity of phase change.

Example 3 Consider a wave packet of the Gaussian form (Gaussian beam in the sequel), given by

$$u(x,h) = \exp\{\frac{i}{h}[S_0 + (x - x_0)p_0 + \frac{i}{2}(x - x_0)^2]\}\varphi(x),$$

where $\varphi(x_0) \neq 0$, $S_0 = \mathrm{const}$.

We have

$$\begin{aligned} & H(x,\hat{p})\,u(x,h) = \\ = \ & \left(\frac{1}{2\pi h}\right)^n \int \exp\{\frac{i}{h}[(x-y)p + S_0 + (y - x_0)p_0 + \frac{i}{2}(y - x_0)^2]\} \\ \times \ & H(x,p)\varphi(y)\,dy\,dp. \end{aligned} \tag{10}$$

Using the stationary phase formula, we obtain for the phase function

$$\Phi = (x-y)p + S_0 + (y-x_0)p_0 + \frac{i}{2}(y-x_0)^2$$

of integral (10) the equations of the (complex) stationary point:

$$\begin{cases} \dfrac{\partial \Phi}{\partial p} = x - y = 0, \\ \dfrac{\partial \Phi}{\partial y} = -p + p_0 + i(y - x_0) = 0. \end{cases}$$

The principal term of the asymptotic expansion of integral (10) is

$$\begin{aligned} H(x,\widehat{p})u(x,h) \;&\equiv\; \exp\{[S_0 + (x-x_0)p_0 + \frac{i}{2}(x-x_0)^2]\} \\ &\times\; {}^{s-1}H(x, p_0 + i(x-x_0))\varphi(x) \pmod{h}. \end{aligned}$$

Expanding the function ${}^{s-1}H(x, p_0 + i(x-x_0))$ into Taylor's series in $(x-x_0)$ and taking into account that for any natural k we have

$$(x-x_0)^k\, e^{-\frac{1}{2h}(x-x_0)^2} = O(h^{k/2}),$$

we obtain the relation

$$\begin{aligned} &H(x,\widehat{p})u(x,h) \equiv \\ &\equiv \exp\{[S_0 + (x-x_0)p_0 + \frac{i}{2}(x-x_0)^2]\}H(x_0,p_0)\varphi(x_0) \pmod{\sqrt{h}}. \end{aligned}$$

It follows that if

$$x_0 \notin \operatorname{supp}\varphi \quad \text{or} \quad (x_0,p_0) \notin \operatorname{supp} H(x,p)$$

then

$$H(x,\hat{p})u = O(h^\infty).$$

On the other hand, if

$$\varphi(x_0) \neq 0,\; H(x_0,p_0) \neq 0$$

then

$$|H(x,\hat{p})u(x_0)| = O(1).$$

Thus

$$OF\{\exp\{\frac{i}{h}[S_0 + (x-x_0)p_0 + \frac{i}{2}(x-x_0)^2]\}\varphi(x)\} = \{(x_0,p_0)\}.$$

We see that *the Gaussian beam is a function with OF consisting of a single point.*

3 Fourier-Gauss Transform and its Properties

Let us consider the family of functions

$$G_{(x,p)}(x',\hbar) = \exp\left\{\frac{i}{\hbar}\left[p(x'-x) + \frac{i}{2}(x'-x)^2\right]\right\} \tag{11}$$

of the variables $(x',\hbar)$ with parameters $(x,p) \in T^*\mathbf{R}^n$.

Definition 3 The function

$$\widetilde{f}(x,p,\hbar) = Uf(x,p,\hbar) = \frac{1}{2^{n/2}(\pi\hbar)^{3n/4}} \int \overline{G_{(x,p)}(x',\hbar)} f(x',\hbar)\, dx' \tag{12}$$

is called a *Fourier-Gauss transform* (or, briefly, *U-transform*) of the function $f(x,\hbar)$. Here the bar means complex conjugation.

The first remarkable fact is that transform (12) is invertible from the left (that is, on its image) and, what is more, the (left) inverse for (12) is given by the explicit formula

$$U^{-1}\widetilde{f}(x,\hbar) = \frac{1}{2^{n/2}(\pi\hbar)^{3n/4}} \int G_{(x',p')}(x,\hbar)\widetilde{f}(x',p',\hbar)\, dx'\, dp'. \tag{13}$$

The following statement is valid.

Theorem 3 *The inversion formula*

$$U^{-1} \circ Uf = f \tag{14}$$

takes place.

Proof. Due to formulas (12) and (13) the left-hand side of formula (14) can be represented in the form[3]

$$U^{-1} \circ Uf(x) = \frac{1}{2^n(\pi\hbar)^{3n/2}} \int G_{(x',p')}(x) \left\{ \int \overline{G_{(x',p')}(x'')} f(x'')\, dx'' \right\} dx'\, dp',$$

or, taking into account the definition (11) of the function $G_{(x',p')}(x)$,

$$\begin{aligned} &U^{-1} \circ Uf(x) = \\ &= \frac{1}{2^n(\pi\hbar)^{3n/2}} \int \exp\left\{\frac{i}{\hbar}\left[p'(x-x'') + \frac{i}{2}(x-x')^2 + \frac{i}{2}(x''-x')^2\right]\right\} \\ &\times f(x'')\, dx''\, dx'\, dp'. \end{aligned}$$

[3]Here and below , for shortness, we omit the explicit indication of the dependence on $\hbar$ of functions under consideration if it does not lead to misunderstanding.

In the latter formula the integral over the variable x' can be computed explicitely and we are led to the formula

$$U^{-1} \circ Uf(x) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{\frac{i}{\hbar}p'(x-x'')} \left\{ e^{-\frac{1}{4\hbar}(x-x'')^2} f(x'') \right\} dx''\, dp'. \qquad (15)$$

¿From the other hand, the inversion formula for (quantum) Fourier transform gives

$$\left(\frac{1}{2\pi\hbar}\right)^n \int e^{\frac{i}{\hbar}p'(x-x'')} \left\{ e^{-\frac{1}{4\hbar}(x_0-x'')^2} f(x'') \right\} dx''\, dp' = e^{-\frac{1}{4\hbar}(x_0-x'')^2} f(x)$$

for any $x_0 \in \mathbf{R}^n$. Substituting $x_0 = x$ into the latter formula, we see that formula (15) can be rewritten in the form

$$U^{-1} \circ Uf(x) = f(x).$$

The latter equality proves the theorem.

The next important property of the introduced transform is the fact that the Parseval identity is valid for it.

Theorem 4 *For each two functions $f(x)$ and $g(x)$ the equality*

$$(Uf, Ug) = (f, g)$$

is valid. Here the scalar products on the phase and the physical spaces are given by the formulas

$$\left(\tilde{f}(x,p), \tilde{g}(x,p)\right) = \int \tilde{f}(x,p)\overline{\tilde{g}(x,p)}\, dx\, dp$$

and

$$(f(x), g(x)) = \int f(x) g(x)\, dx$$

correspondingly.

Proof is given by the straightforward calculation:

$$\begin{aligned} &(Uf, Ug) = \\ &= \int \left\{ \frac{1}{2^{n/2}(\pi\hbar)^{3n/4}} \int \overline{G_{(x,p)}(x')} f(x')\, dx' \right\} \overline{Ug(x,p)}\, dx\, dp \\ &= \int f(x') \overline{\left\{ \frac{1}{2^{n/2}(\pi\hbar)^{3n/4}} \int G_{(x,p)}(x') Ug(x,p)\, dx\, dp \right\}}\, dx' \end{aligned}$$

$$= \int f(x')\overline{U^{-1} \circ Ug(x')}\, dx'$$

$$= \int f(x')\overline{g(x')}\, dx' = (f, g).$$

Unfortunately, the formula

$$U \circ U^{-1} = \mathrm{id} \tag{16}$$

is *not valid* for the introduced transform. This fact can be easily understood if one remembers that the transform U takes functions of n variables x to functions of $2n$ variables (x,p). Thus, there arises a problem of description of the image of the transform U.

Theorem 5 *The image of transform* (12) *considered on functions* $f(x) \in L_2(\mathbf{R}^n)$ *is the set of functions* $F(x,p)$ *possessing the following two properties:*

1) $\int |F(x,p)|^2\, dx\, dp < \infty$;

2) *the function* $\exp\left[\frac{1}{2\hbar}p^2\right] F(x,p)$ *is an antianalytic function of the variable* $z = x + ip$, *that is,*

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial p}\right) e^{\frac{1}{2\hbar}p^2} F(x,p) = 0.$$

Proof. First, let us check that the function

$$\widetilde{f}(x,p) = Uf(x,p)$$

possesses the properties 1) and 2), if $f \in L_2(\mathbf{R}^n)$. Evidently, the first property is a consequence of the Parseval identity. To prove that the function $\widetilde{f}$ possesses the second property, we remark that

$$e^{\frac{1}{2\hbar}p^2}\widetilde{f}(x,p) = \frac{1}{2^{n/2}(\pi\hbar)^{3n/4}} \int e^{-\frac{1}{2\hbar}(x'-\overline{z})} f(x')\, dx',$$

where $\overline{z} = x - ip$, which proves the required assertion.

To finish the proof, one must check that formula (16) is valid on the set of functions possessing properties 1) and 2). The proof of this fact, based on the Bargman representation of an analytic function (see [6]) has purely technical character and we omit it.

Remark 3 The transform U^{-1} can be extended in a usual way to *generalized* functions (distributions) of variables (x,p). We leave to the reader the verification of this procedure.

4 Quantization of Observables and Lagrangian Modules

To begin with, we show that any pseudodifferential operator can be written down as a conjugation with the help of the transform U of the multiplication by its symbol $H(x,p)$. More exactly, the following statement takes place.

Theorem 6 *For any smooth function $H(x,p)$ with the compact support the comparison*

$$U^{-1}H(x,p)U \equiv H\left(x,\widehat{p}\right) \pmod{\sqrt{\hbar}} \tag{17}$$

is valid.

Proof. Let us write down the value of the left-hand side of formula (17) on the function $f(x)$ in the form of the integral

$$\begin{aligned} &U^{-1}H(x,p)Uf(x) = \\ &= \frac{1}{2^n(\pi\hbar)^{3n/2}}\int \exp\left\{\frac{i}{\hbar}\left[p'(x-x'') + \frac{i}{2}(x-x')^2 + \frac{i}{2}(x''-x')^2\right]\right\} \\ &\times H(x',p')f(x'')\,dx''\,dx'\,dp \end{aligned} \tag{18}$$

(the calculations leading to this formula are quite similar to those used in the proof of Theorem 3 and we omit them). Later on, expanding $H(x',p')$ to the Taylor series in powers of $(x'-x)$ and using the estimate

$$(x'-x)e^{-\frac{1}{2\hbar}(x-x')^2} = O(\sqrt{\hbar})$$

we obtain

$$\begin{aligned} &U^{-1}H(x,p)Uf(x) \equiv \\ &\equiv \left(\frac{1}{2\pi\hbar}\right)^n \int \exp\left\{\frac{i}{\hbar}\left[p'(x-x'') + \frac{i}{4}(x-x'')^2\right]\right\} \times \\ &\times H(x,p')f(x'')\,dx''\,dp' = H\left(x,\widehat{p}\right)f(x) + \widehat{R}f(x) \pmod{\sqrt{\hbar}}, \end{aligned} \tag{19}$$

where the operator $\widehat{R}$ is given by

$$\widehat{R}f(x) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{\frac{i}{\hbar}p'(x-x'')}\left\{e^{-\frac{1}{4\hbar}(x-x'')^2} - 1\right\} H(x,p')f(x'')\,dx''\,dp'.$$

The integral over x' included into the right-hand part of (18) was computed similar to the proof of Theorem 3.

Let us now estimate the remainder $\widehat{R}f$ on the right of (19). Integrating by parts, we obtain

$$\widehat{R}f(x) = -i\hbar \int e^{\frac{i}{\hbar}p'(x-x'')} \frac{e^{-\frac{1}{4\hbar}(x-x'')^2} - 1}{x - x''} \frac{\partial H}{\partial p'}(x, p')\, dx''\, dp'.$$

Since

$$\left| \frac{e^{-\frac{1}{4\hbar}(x-x'')^2} - 1}{x - x''} \right| \leq \frac{C}{\sqrt{\hbar}}$$

with some positive constant $C > 0$, the operator $\widehat{R}$ included into the right-hand part of relation (19) has order $\sqrt{\hbar}$, q.e.d.

The proved statement together with the Parseval identity leads to the following description of the front of oscillations in terms of the U-transform.

Proposition 2 *A point (x_0, p_0) of the phase space does not belong to the front of oscillations $OF\,[f]$ of a function $f(x)$ iff*

$$Uf(x,p) \equiv 0 \pmod{\hbar^\infty}$$

in a neighbourhood of the point (x_0, p_0).

Proof. Actually, due to Theorem 6 for any ψDO $H\,(x, \widehat{p})$ we have

$$U\,[H\,(x, \widehat{p})]\,(x,p) \equiv H(x,p)Uf(x,p) \pmod{\sqrt{\hbar}}.$$

Taking into account the Parseval identity, we see that the function $H\,(x, \widehat{p})\,f$ is of order $O(\hbar^\infty)$ iff the support of the function $H(x,p)$ does not intersect osc-supp$[Uf]$. The latter affirmation proves the Proposition.

Thus, we had obtained the description of the front of oscillations of a function $f(x)$ as the oscillatory support of its U-transform $Uf(x,p)$:

$$OF\,[f] = \text{osc-supp}\,[Uf]\,. \tag{20}$$

Hence, to construct a function $f(x, \hbar)$ with its front of oscillations on a given (Lagrangian) manifold, it is natural to apply the transform U^{-1} to a special function of the form

$$\widetilde{f}(x,p) = (\pi\hbar)^{n/4}\, e^{\frac{i}{\hbar}S} \varphi \delta_{(\Lambda, d\sigma)}(x,p) \tag{21}$$

where S and φ are smooth functions on the Lagrangian (or, more generally, isotropic) manifold Λ, and $\delta_{(\Lambda,d\sigma)}$ is the delta function corresponding to the measure $d\sigma$ and concentrated on the manifold Λ.

The function S in expression (21) must be chosen equal to a non-singular action on Λ, since only in this case the front of oscillations of function (21) exactly coincides with the support of the function φ on the manifold Λ.

To prove this affirmation, we write down the expression of the operator

$$K_{(\Lambda,d\sigma)}(\varphi) = U^{-1}\left\{e^{\frac{i}{\hbar}S}\varphi\delta_{(\Lambda,d\sigma)}(x,p)\right\}$$

in the form of an integral

$$\begin{aligned} K_{(\Lambda,d\sigma)}(\varphi) &= \left(\frac{1}{2\pi\hbar}\right)^{n/2}\int\limits_{\Lambda} G_{(x(\alpha),p(\alpha))}(x,\hbar)e^{\frac{i}{\hbar}S(\alpha)}\varphi(x,\alpha,\hbar)d\sigma(\alpha) \\ &= \left(\frac{1}{2\pi\hbar}\right)^{n/2}\int\limits_{\Lambda} e^{\frac{i}{\hbar}\Phi(x,\alpha)}\varphi(x,\alpha,\hbar)d\sigma(\alpha), \end{aligned} \tag{22}$$

where $\alpha \mapsto (x,(\alpha),p(\alpha))$ is the embedding $\Lambda \subset \mathbf{R}^n \times \mathbf{R}_n$ and we used the notation

$$\Phi(x,\alpha) = S(\alpha) + (x - x(\alpha))\,p(\alpha) + \frac{i}{2}(x - x(\alpha))^2. \tag{23}$$

Now we shall investigate the front of oscillations of the function $u(x,h)$ given by (22). Since the front of oscillations of $G_{(x(\alpha),p(\alpha))}$ is the point $\{(x(\alpha),p(\alpha))\}$, it is clear (and will be proved rigorously below) that one has

$$OF[u(x,\hbar)] \subset \Lambda\bigcap \operatorname{supp}\varphi \tag{24}$$

and generally the equality in (24) does not hold since the oscillations might be "cancelled out" by the integration. Let us derive the necessary conditions for the point (x_0,p_0) to belong to $OF[u]$. For this purpose, take a pseudodofferential operator $H(x,\widehat{p})$ with the finite symbol vanishing outside a small neighborhood of the point (x_0,p_0) and apply it to the function u. We have

$$\begin{aligned} &H(x,\widehat{p})\,u = \\ &= \left(\frac{1}{2\pi\hbar}\right)^{3n/2}\int\int\int e^{\frac{i}{\hbar}[p(x-y)+\Phi(y,\alpha)]}H(y,\alpha)\varphi(x,\alpha,\hbar)d\mu(\alpha)\,dy\,dp. \end{aligned} \tag{25}$$

The phase in the integral (25) is equal to

$$\Psi(x,y,p,\alpha) = p(x-y) + (y - x(\alpha))p(\alpha) + S(\alpha) + \frac{i}{2}(y - x(\alpha))^2. \tag{26}$$

The stationary point equations for the phase (26) have the form

$$\begin{cases} \dfrac{\partial \Psi}{\partial p} = x - y = 0, \\ \dfrac{\partial \Psi}{\partial y} = p(\alpha) - p + i(y - x(\alpha)) = 0, \\ d_\alpha \Psi = dS(\alpha) + (y - x(\alpha))dp(\alpha) - p(\alpha)dx(\alpha) - i(y - x(\alpha))dx(\alpha) = 0, \end{cases}$$

so in the stationary point one has

$$\begin{cases} x = y = x(\alpha), \quad p = p(\alpha), \\ dS(\alpha) = p(\alpha)dx(\alpha). \end{cases} \tag{27}$$

Should the equalities (27) be violated for $y = x_0, \quad p = p_0$ and every $\alpha \in \mathrm{supp}\varphi$, they will not be valid for close (y, p) as well (recall that $\mathrm{supp}\varphi$ is compact), so the integral (32) equals $O(\hbar^\infty)$ provided that the diameter of the support of the function $H(x,p)$ is small enough (integrating by parts yields the proof).

Therefore, the following assertion is valid.

Lemma 1 *For $(x_0, p_0) \in OF[u]$ to be valid it is necessary that $(x_0, p_0) = (x(\alpha), p(\alpha))$ for some $\alpha \in \mathrm{supp}\varphi$ (i. e., that $(x_0, p_0) \in \Lambda \bigcap \mathrm{supp}\varphi$) and the relation (27) be satisfied.*

Thus, if we want (24) to be a precise equality for any finite function φ, we must require that (27) be satisfied everywhere on Λ. Hence, the form

$$p(\alpha)\, dx(\alpha) = p\, dx|_\Lambda$$

must be exact. In particular, it must be closed, i. e., its differential must be equal to zero:

$$dp \wedge dx|_\Lambda = 0.$$

The proved affirmation shows that, first, Λ must be a Lagrangian manifold and, second, that the function S must coincide with the action on the manifold Λ for (24) to be a precise equality.

Remark 4 The constructed operator

$$K_{(\Lambda, d\sigma)} : \ C_0^\infty(\Lambda) \to \ C_\hbar^\infty(\mathbf{R}^n, \Lambda) \tag{28}$$

in mod $\hbar$ coincides with *Masolv's canonical operator* on the Lagrangian manifold Λ with the measure $d\sigma$.

The simple proof of this fact can be carried out, for example, by the stationary phase method; this proof can be found in [7]. The representation (22) of operator (28) is often very useful (see, for example, [8]).

Thus, the quantization of a Lagrangian module $C_0^\infty(\Lambda)$ can be carried out into two stages. First, for any function $\varphi \in C_0^\infty(\Lambda)$ we construct the function of the form (21) and then apply to the obtained function the transform U^{-1}.

In what follows we shall see also that the kernel of the quantized (with the help of the U-transform) symplectic transform is also automatically representable as application of the canonical operator to some function on the graph of the corresponding symplectic transform, that is, as a Fourier integral operator in the usual sense (see, for example, [4]).

5 Quantization of Symplectic Transforms

In this section we shall show that the quantization of some symplectic transform

$$g : T^*\mathbf{R}^n \to T^*\mathbf{R}^n \tag{29}$$

is essentially the conjugation with the help of the U-transform of canonical change of variables (29). More exactly, the following affirmation is valid.

Theorem 7 *The operator*

$$T_g = U^{-1} e^{\frac{i}{\hbar}S(x,p)} g^* U,$$

or, in another form,

$$f(x) \mapsto U^{-1}\left\{\left(\frac{1}{2i}\right)^{n/2} e^{\frac{i}{\hbar}S(x,p)} Uf\left[g(y,q)\right]\right\}(x), \tag{30}$$

where the function $S(x,p)$ is determined by the relation

$$dS = p\,dx - g^*(q\,dy),$$

is the Fourier integral operator $T(g,1)$ with the symbol equal to 1 corresponding to symplectic transform (29).

Remark 5 We emphasize that for each symplectic transform (29) the mapping

$$g^! : f \mapsto e^{\frac{i}{\hbar}S} g^* f$$

is a module homomorphism

$$g^! : \mathcal{D}'_\hbar(T^*\mathbf{R}^n, \Lambda_2) \to \mathcal{D}'_\hbar(T^*\mathbf{R}^n, \Lambda_1)$$

if the Lagrangian manifolds Λ_1 and Λ_2 are due to the relation

$$g(\Lambda_1) = \Lambda_2.$$

This clarifies the appearence of the factor $e^{\frac{i}{\hbar}S}$ in formula (30).

Remark 6 The same assertion (as in Theorem 7) is valid for the operator

$$f \mapsto U^{-1}\left\{\left(\frac{1}{2i}\right)^{n/2} e^{\frac{i}{\hbar}S(x,p)}\varphi(x,p)Uf\left[g(y,q)\right]\right\}(x)$$

which coincides with the Fourier integral operator $T(g,\varphi)$ defined with the help of the canonical operator $K_{(\Lambda_g,d\sigma)}\varphi$.

Proof of Theorem 7. Let the functions

$$y = y(x,p), \quad q = q(x,p)$$

determine symplectic transform (29). We write down operator (30) in the integral form using definitions (12) and (13) of the transforms U and U^{-1}:

$$\begin{aligned} &U^{-1}\left[\left(\frac{1}{2i}\right)^{n/2} e^{\frac{i}{\hbar}S(x,p)}Uf(g(y,q))\right](x) = \\ &= \frac{i^{n/2}}{(2\pi\hbar)^{3n/2}}\int G_{(x',p')}(x)e^{\frac{i}{\hbar}S(x',p')} \\ &\times \left\{\int \overline{G_{(y,q)}(y')}f(y')\,dy'\right\}\Bigg|_{y=y(x',p'),q=q(x',p')} dx'\,dp'. \end{aligned}$$

Using formula (11), one can rewrite the latter formula in the form

$$\begin{aligned} T(g,1)f &= \frac{(-i)^{n/2}}{(2\pi\hbar)^{3n/2}}\int \exp\left\{\frac{i}{\hbar}\left[S(x',p') + p'(x-x') + \frac{i}{2}(x-x')^2\right.\right. \\ &- \left.\left. q(x',p')(y'-y(x',p')) + \frac{i}{2}(y'-y(x',p'))^2\right]\right\} f(y')\,dy'\,dx'\,dp' \\ &= \left(-\frac{i}{2\pi\hbar}\right)^{n/2}\int K(x,y')f(y')\,dy', \end{aligned}$$

where the kernel $K(x,y')$ is given by

$$\begin{aligned} K(x,y') &= \left(\frac{1}{2\pi\hbar}\right)^{n}\int \exp\left\{\frac{i}{\hbar}\left[S(x',p') + p'(x-x') + \frac{i}{2}(x-x')^2\right.\right. \\ &\quad - \left.\left. q(x',p')(y'-y(x',p')) + \frac{i}{2}(y'-y(x',p'))^2\right]\right\}\,dx'\,dp'. \end{aligned}$$

The latter expression exactly coincides with the expression for the canonically represented function

$$K(x, y') = K_{(\Lambda_g, d\sigma)}(1)$$

on the Lagrangian manifold $\Lambda_g = \operatorname{graph} g$ with the measure $d\sigma = (dp \wedge dx)^{\wedge n}$, written in the coordinates (x', p') of the manifold Λ_g. This follows from the fact that the non-singular action S on the Lagrangian manifold Λ_g is determined by the formula

$$S = \int (p\,dx - q\,dy)|_{\Lambda_g} = \int p\,dx - g^*(q\,dy).$$

6 Main Theorems

Let us summarize the constructions of this paper. We had defined above the integral transform U and its inverse U^{-1}. With the help of these transforms the following objects can be easily interpreted.

1) *Pseudodifferential operators* (that is, quantization of observables) are described as the conjugation of multiplication by the symbol with respect to the transform U:

$$H(x, \widehat{p}) \equiv \mathrm{Ad}_U\, H(x,p) = U^{-1} H(x,p) U \pmod{\sqrt{\hbar}}.$$

This means that these operators can be determined via the commutative diagram

$$\begin{array}{ccc}
C_\hbar^\infty(T^*\mathbf{R}^n) & \xrightarrow{H(x,p)} & C_\hbar^\infty(T^*\mathbf{R}^n) \\
{\scriptstyle U}\uparrow & & \uparrow{\scriptstyle U} \\
C_\hbar^\infty(\mathbf{R}^n) & \xrightarrow{H(x,\widehat{p})} & C_\hbar^\infty(\mathbf{R}^n)
\end{array}$$

2) *Canonical operator* (that is, the quantization of the module of smooth functions on a Lagrangian manifold Λ), can be described as the application of the transform U^{-1} to functions of the special form

$$K_{(\Lambda, d\sigma)}(\varphi) = U^{-1}\left\{ e^{\frac{i}{\hbar}S} \varphi \delta_{(\Lambda, d\sigma)} \right\},$$

that is, as the composition

$$C_0^\infty(\Lambda) \xrightarrow{e^{\frac{i}{\hbar}S}\delta} \mathcal{D}'_\hbar(T^*\mathbf{R}^n) \xrightarrow{U^{-1}} C_\hbar^\infty(\mathbf{R}^n).$$

3) *Fourier integral operators* (that is, quantization of symplectic transforms), are given as the conjugation of the variable change with multiplication by $e^{\frac{i}{\hbar}S}$:

$$T_g = U^{-1} \left(\frac{1}{2i}\right)^{n/2} e^{\frac{i}{\hbar}S(x,p)} g^* U,$$

that is, can be determined by the commutative diagram

$$\begin{array}{ccc}
C_{\hbar}^{\infty}(T^{*}\mathbf{R}^{n}) & \xrightarrow{e^{\frac{i}{\hbar}S_{g}}\cdot} & C_{\hbar}^{\infty}(T^{*}\mathbf{R}^{n}) \\
{\scriptstyle U}\uparrow & & \uparrow{\scriptstyle U} \\
C_{\hbar}^{\infty}(\mathbf{R}^{n}) & \xrightarrow{T_{g}} & C_{\hbar}^{\infty}(\mathbf{R}^{n})
\end{array}$$

Thus, all main classical objects mentioned in points 1) and 2) in the beginning of this paper occur to be quantized in the framework of one and the same approach connected with the U-transform.

Let us formulate now some statements which can be easily proved with the help of U-transform.

Theorem 8 *The commutation formula*

$$H(x,\widehat{p})\,K_{(\Lambda,d\sigma)}(\varphi) \equiv K_{(\Lambda,d\sigma)}(\,H|_{\Lambda}\,\varphi) \pmod{\sqrt{\hbar}}$$

is valid.

The affirmation of this theorem can be expressed with the help of the following commutative diagram[4]

$$\begin{array}{ccc}
C_{\hbar}^{\infty}(\mathbf{R}^{n},\Lambda) & \xrightarrow{H(x,\widehat{p})} & C_{\hbar}^{\infty}(\mathbf{R}^{n},\Lambda) \\
{\scriptstyle K_{(\Lambda,d\sigma)}}\uparrow & & \uparrow{\scriptstyle K_{(\Lambda,d\sigma)}} \\
C_{0}^{\infty}(\Lambda) & \xrightarrow{H(x,p)|_{\Lambda}} & C_{0}^{\infty}(\Lambda).
\end{array}$$

Here by $C_{\hbar}^{\infty}(\mathbf{R}^{n},\Lambda)$ we denote the space of smooth functions with front of oscillations in Λ.

Theorem 9 *The composition formula*

$$H_1(x,\widehat{p}) \circ H_2(x,\widehat{p}) \equiv H_3(x,\widehat{p}) \pmod{\sqrt{\hbar}}$$

is valid, where $H_3(x,p) = H_1(x,p)H_2(x,p)$.

Theorem 10 *Valid is the following formula*

$$T_{g^{-1}}H(x,\widehat{p})\,T_{g} \equiv H_1(x,\widehat{p}) \pmod{\sqrt{\hbar}},$$

where $H_1(x,p) = g^{*}H(x,p)$.

[4]All diagrams below commute mod $\sqrt{\hbar}$.

The statement of this theorem can be illustrated with the help of the following commutative diagram

$$\begin{array}{ccc} PSD\,(\mathbf{R}^n) & \xrightarrow{Ad\,T_g} & PSD\,(\mathbf{R}^n) \\ {\scriptstyle Ad_U}\uparrow & & \uparrow{\scriptstyle Ad_U} \\ C^\infty\,(T^*\mathbf{R}^n) & \xrightarrow{g^*} & C^\infty\,(T^*\mathbf{R}^n) \end{array}$$

where by $PSD\,(\mathbf{R}^n)$ we denote the set of ψDO's in $\mathbf{R}^n$.

Theorem 11 *The following formula*

$$T_g K_{(\Lambda_2, d\sigma_2)}(\varphi) \equiv K_{(\Lambda_1, d\sigma_1)}(g^*\varphi) \pmod{\sqrt{\hbar}}$$

is valid.

This means that the following diagram

$$\begin{array}{ccc} C_\hbar^\infty\,(\mathbf{R}^n, \Lambda_2) & \xrightarrow{T_g} & C_\hbar^\infty\,(\mathbf{R}^n, \Lambda_1) \\ {\scriptstyle K_{(\Lambda_2, d\sigma_2)}}\uparrow & & \uparrow{\scriptstyle K_{(\Lambda_1, g^*(d\sigma_2))}} \\ C_0^\infty\,(\Lambda_2) & \xrightarrow{g^*} & C_0^\infty\,(\Lambda_1) \end{array}$$

where $\Lambda_2 = g\,(\Lambda_1)$, commutes.

Theorem 12 *The following composition formula*

$$T(g_2, \varphi_2) \circ T(g_1, \varphi_1) \equiv T(g_2 \circ g_1, g_1^*(\varphi_2)\varphi_1) \pmod{\sqrt{\hbar}}$$

is valid for the corresponding choice of the base point and the signs of square roots in the operator on the right in the latter formula.

Remark 7 The theorems formulated above, being the main ones in the theory of ψDO's and Fourier integral operators, have very hard proves. The comparisons in them are established up to the arbitrary order of $\hbar$. The technique of U-transform allows one to obtain these results as *easy consequences* of the properties of this transforms, however, only (at the moment) in principal terms (up to terms of order $\sqrt{\hbar}$).

Let us formulate this results in slightly other redaction.

Theorem 13 (quantization of observables). *For any Hamilton function $H(x,p)$ the operator*

$$\hat{H} = U^{-1} H(x,p) U$$

coincides up to $O(\hbar)$ with the quantum ovservable

$$H(x,\hat{p}), \quad \hat{p} = -i\hbar \frac{\partial}{\partial x}$$

determined by the Schrödinger quantization.

Moreover, the correspondence

$$H(x,p) \mapsto \hat{H}$$

is (in the leading term) the ring homomorphism, that is,

$$\widehat{(H_1 H_2)} \equiv \hat{H}_1 \cdot \hat{H}_2 \pmod{\hbar}.$$

Theorem 14 (quantization of Lagrangian modules). *Let Λ be a Lagrangian manifold and $C_0^\infty(\Lambda)$ is the module of smooth functions over the ring of (classical) observables. Let S be a nonsingular action on Λ and $\delta_{(\Lambda,d\sigma)}$ be the δ-function concentrated on Λ with a measure $d\sigma$. Then the operator*

$$K_{(\Lambda,d\sigma)}(\varphi) = U^{-1}\{e^{\frac{i}{\hbar}S}\varphi\delta_{(\Lambda,d\sigma)}(x,p)\}.$$

is the Maslov canonical operator on the Lagrangian manifold Λ with the measure $d\sigma$.

Moreover, the operator $K_{(\Lambda,d\sigma)}$ is a module homomorphism over the above mentioned ring homomorphism:

$$K_{(\Lambda,d\sigma)}(H(x,p)\varphi) \equiv \hat{H}\, K_{(\Lambda,d\sigma)}(\varphi) \pmod{\hbar}.$$

Theorem 15 (quantization of canonical transforms). *Let*

$$g : T^*\mathbf{R}^n \to T^*\mathbf{R}^n \tag{31}$$

be a canonical transform and let $S(x,p)$ be a solution to the equation

$$dS = p\,dx - g^*(q\,dy),$$

where (y,q) are coordinates in the image of mapping (31). *Then the operator*

$$T_g = U^{-1}\left(\frac{1}{2i}\right)^{n/2} e^{\frac{i}{\hbar}S(x,p)} g^* U \tag{32}$$

determines the quantization of transform (31), *that is, this operator is a Fourier integral operator corresponding to canonical transform* (31).

Moreover, operator (32) determines a module homomorphism

$$\hat{H}\, T_g f = T_g(g^*\hat{H})\, f$$

over the ring homomorphism

$$\hat{H} \mapsto (g^*\hat{H}) = T_g^{-1}\, \hat{H}\, T_g.$$

The detailed exposition of the above notions the reader can find in [7], as well as the main statements and the bibliography.

7 Concluding Remarks

Here we shall present in a more explicit way the connection between the introduced transform and the notion of microlocalization.

For any $(x_0, p_0) \in T^*\mathbf{R}^n$ we denote by $\mathcal{M}^{\mathrm{osc}}_{(x_0,p_0)}$ the quotient of $H(\mathbf{R}^n)$ by the space of functions whose front of oscillations does not contain the point (x_0, p_0) of the phase space. The space $\mathcal{M}^{\mathrm{osc}}_{(x_0,p_0)}$ we call *the space of oscillatory microfunctions at* (x_0, p_0). Since the projection into $\mathbf{R}^n$ of the front of oscillations of a function is exactly its osc-support, one can see that any function $f(x, \hbar)$ is uniquely determined modulo $O(\hbar^\infty)$ by its image in all $\mathcal{M}^{\mathrm{osc}}_{(x_0,p_0)}$ as (x_0, p_0) runs over $T^*\mathbf{R}^n$.

With this observation one can identify the space $H(\mathbf{R}^n)/O(\hbar^\infty)$ with the global sections of the sheaf $\mathcal{M}^{\mathrm{osc}}$ of oscillatory microfunctions over $T^*\mathbf{R}^n$. Then there arises a problem of description of the sheaf $\mathcal{M}^{\mathrm{osc}}$ by representing its sections as smooth functions on the space $T^*\mathbf{R}^n$. The solution of this problem carried out by means of Fourier-Gauss transform was described above.

Namely, *the space of oscillatory microfunctions* $\mathcal{M}^{\mathrm{osc}}_{(x_0,p_0)}$ *at a point* $(x_0, p_0) \in T^*\mathbf{R}^n$ *of the phase space is isomorphic to the space of germs of functions* $F(x, p, \hbar)$ *on the phase space* $T^*\mathbf{R}^n$ *lying in the image of transform* U*, that is, such that*

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial p}\right) e^{\frac{1}{2\hbar}p^2} F(x, p, \hbar) = 0.$$

The proof of this affirmation which is an easy consequence of the Parseval identity and the description of the front of oscillations in terms of U-transform (Proposition 2) is left to the reader. We only remark that, due to this fact, the Fourier-Gauss transform U deliveres us a solution of the problem of description of $\mathcal{M}^{\mathrm{osc}}$ mentioned above.

We remark here that in smooth situation the problem of description of the sheaf $\mathcal{M}$ of microfunctions was solved in a pioneer paper by A. Weinstein [9] on the order and symbol of distribution. In this paper, with the help of a different localization procedure, the author gives a description of the sheaf of microfunctions as the sheaf of sections of the so-called symbols of distributions and proves that for Hörmander's Fourier integral distributions [10] sections of $\mathcal{M}$ can be treated as sections of the corresponding bundle of symbols.

References

[1] E. Schrödinger. Quantisierung als Eigenwertproblem. *Ann. d. Phys.*, 1926. **79**, 361; **79**, 489; **80**, 437; **81**, 109.

[2] V.A. Fock. On canonical transformation in classical and quantum mechanics. *Vestnik Leningr. Gos. Univ.*, **16**, 1959, 67 – 71. [Russian]. English transl.: Acta Phys. Acad. Sci. Hungar. **27**, 1969, 219 – 224.

[3] V.P. Maslov. *Théorie des Perturbations et Méthods Asymptotiques.* MGU, Moscow, 1965. [Russian]. Traduit de russe: Dunod, Paris, 1972.

[4] A.S. Mishchenko, V.E. Shatalov, and B.Yu. Sternin. *Lagrangian Manifolds and the Maslov Operator.* Springer-Verlag, Berlin – Heidelberg – New York – Tokyo, 1990.

[5] A. Cordoba and C. Fefferman. Wave packets and Fourier integral operators. *Comm. in Partial Differential Equations*, **3**, No. 11, 1978, 979–1005.

[6] V. Bargman. On a Hilbert space of analytic functions and an associated integral transform. *Comm. Pure Appl. Math.*, **14**, 1961, 187–214.

[7] B.Yu. Sternin and V.E. Shatalov. *Ten Lectures on Quantization*, volume 22 of *Publications pedagogiques.* Université de Nice-Sophia Antipolis, Nice, Juin 1994. Preprint.

[8] M.V. Karasev. New global asymptotics and anomalies for the problem of quantization of the adiabatic invariant. *Funct. Anal. Appl.*, **24**, No. 2, 1990, 104–114.

[9] A. Weinstein. The order and symbol of a distribution. *Trans. Amer. Math. Soc.*, **241**, 1977, 1 – 54.

[10] L. Hörmander. Fourier integral operators I. *Acta Math.*, **127**, 1971, 79 – 183.

Nice – Potsdam, Summer 1994