MINIMAL ATLASES OF CLOSED SYMPLECTIC MANIFOLDS

YU. B. RUDYAK AND FELIX SCHLENK

ABSTRACT. We study the number of Darboux charts needed to cover a closed connected symplectic manifold (M, ω) and effectively estimate this number from below and from above in terms of the Lusternik–Schnirelmann category of M and the Gromov width of (M, ω).

1. INTRODUCTION AND MAIN RESULTS

A symplectic manifold is a pair (M, ω) where M is a smooth manifold and ω is a non-degenerate and closed 2-form on M. The non-degeneracy of ω implies that M is even-dimensional, $\dim M = 2n$. (We refer to [11] and [23] for basic facts about symplectic manifolds.) The most important symplectic manifold is $\mathbb{R}^{2n}$ equipped with its standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Indeed, a basic fact about symplectic manifolds is Darboux's Theorem which states that locally every symplectic manifold (M^{2n}, ω) is diffeomorphic to $(\mathbb{R}^{2n}, \omega_0)$. More precisely, for each point $p \in M$ there exists a chart

$$\varphi\colon B^{2n}(a) \to M$$

from a ball

$$B^{2n}(a) := \left\{ z \in \mathbb{R}^{2n} \mid \pi \, |z|^2 < a \right\}$$

to M such that $\varphi(0) = p$ and $\varphi^*\omega = \omega_0$. We call such a chart $\left(B^{2n}(a), \varphi\right)$ a Darboux chart. In this paper we study the following question:

> *Given a closed symplectic manifold (M, ω), how many Darboux charts does one need in order to parametrize (M, ω)?*

In other words, we study the number $\mathrm{S_B}(M, \omega)$ defined as

$$\mathrm{S_B}(M, \omega) := \min\{k \mid M = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k\}$$

where each $\mathcal{B}_i$ is the image $\varphi_i\left(B^{2n}(a_i)\right)$ of a Darboux chart.

An obvious lower bound for $\mathrm{S_B}(M, \omega)$ is the diffeomorphism invariant

$$\mathrm{B}(M) := \min\{k \mid M = B_1 \cup \cdots \cup B_k\}$$

Date: July 21, 2005.

2000 *Mathematics Subject Classification.* Primary 53D35, Secondary 55M30,57R17.

where each B_i is diffeomorphic to the standard open ball in $\mathbb{R}^{2n}$.

The volume associated with a symplectic manifold $\left(M^{2n},\omega\right)$ is

$$\operatorname{Vol}(M,\omega) \;=\; \frac{1}{n!}\int_M \omega^n.$$

In particular, $\operatorname{Vol}\left(B^{2n}(a)\right) = \frac{1}{n!}\,a^n$, as it should be. The volume of any symplectically embedded ball in (M,ω) is at most

$$\gamma\,(M,\omega) \;=\; \sup\left\{\operatorname{Vol}\left(B^{2n}(a)\right) \mid B^{2n}(a) \text{ symplectically embeds into } M\right\}.$$

Another lower bound for $\mathrm{S_B}(M,\omega)$ is therefore

$$\Gamma(M,\omega) \;:=\; \left\lfloor \frac{\operatorname{Vol}(M,\omega)}{\gamma(M,\omega)} \right\rfloor + 1$$

where $\lfloor x \rfloor$ denotes the maximal integer which is smaller than or equal to x. Notice that $\gamma(M,\omega) = \frac{1}{n!}\left(\operatorname{Gr}(M,\omega)\right)^n$ where

$$\operatorname{Gr}(M,\omega) = \sup\left\{a \mid B^{2n}(a) \text{ symplectically embeds into } (M,\omega)\right\}$$

is the Gromov width of (M,ω). The symplectic invariant $\Gamma(M,\omega)$ is therefore strongly related to the Gromov width. We abbreviate

$$\lambda(M,\omega) \;:=\; \max\left\{\mathrm{B}(M),\Gamma(M,\omega)\right\}.$$

Summarizing we have that

$$\lambda(M,\omega) \;\le\; \mathrm{S_B}(M,\omega). \tag{1}$$

Before we state our main result, we consider two examples.

1) For complex projective space ^{n} equipped with its standard Kähler form ω_{SF} we have $\mathrm{B}(\mathbb{CP}^n) = n+1$ and $\Gamma(\mathbb{CP}^n,\omega_{SF}) = 2$. In particular,

$$\lambda(\mathbb{CP}^n,\omega_{SF}) \;=\; \mathrm{B}(\mathbb{CP}^n) \;>\; \Gamma(\mathbb{CP}^n,\omega_{SF}) \quad \text{if } n \ge 2.$$

It will turn out that $\mathrm{S_B}(\mathbb{CP}^n,\omega_{SF}) = \lambda(\mathbb{CP}^n,\omega_{SF}) = n+1$ if $n \ge 2$.

2) We fix an area form σ on the 2-sphere $\mathbb{S}^2$, and for $k \in \mathbb{N}$ we abbreviate $\mathbb{S}^2(k) = (\mathbb{S}^2, k\sigma)$. Then $\mathrm{B}\left(\mathbb{S}^2\times\mathbb{S}^2\right) = 3$ and $\Gamma\left(\mathbb{S}^2(1)\times\mathbb{S}^2(k)\right) = 2k+1$. In particular,

$$\lambda\left(\mathbb{S}^2(1)\times\mathbb{S}^2(k)\right) \;=\; \Gamma\left(\mathbb{S}^2(1)\times\mathbb{S}^2(k)\right) \;>\; \mathrm{B}\left(\mathbb{S}^2\times\mathbb{S}^2\right) \quad \text{if } k \ge 2.$$

It will turn out that $\mathrm{S_B}\left(\mathbb{S}^2(1)\times\mathbb{S}^2(k)\right) \;=\; \lambda\left(\mathbb{S}^2(1)\times\mathbb{S}^2(k)\right) \;=\; 2k+1$ if $k \ge 2$.

We refer to Examples 2 and 4 in Section 5 for more details.

Our main result is

Theorem 1. *Let (M,ω) be a closed connected $2n$-dimensional symplectic manifold.*

(i) *If $\lambda(M,\omega) \ge 2n+1$, then $\mathrm{S_B}(M,\omega) = \lambda(M,\omega)$.*

(ii) *If $\lambda(M,\omega) < 2n+1$, then $n+1 \le \lambda(M,\omega) \le \mathrm{S_B}(M,\omega) \le 2n+1$.*

Remarks. 1. The assumption in (i) is met if $[\omega]|_{\pi_2(M)} = 0$, see Proposition 1 (ii) below. It is also met for various symplectic fibrations, see Section 5.

2. Theorem 1 implies that

$$n+1 \le \lambda(M,\omega) < \mathrm{S_B}(M,\omega) \le 2n+1 \quad \text{if } \lambda(M,\omega) \ne \mathrm{S_B}(M,\omega).$$

The following question is based on the examples described in Section 5.

Question. *Is it true that* $\lambda(M,\omega) = \mathrm{S_B}(M,\omega)$ *for all closed symplectic manifolds* (M,ω)*?*

Theorem 1 essentially reduces the problem of computing the number $\mathrm{S_B}(M,\omega)$ to two other problems, namely computing $\mathrm{B}(M)$ and $\Gamma(M,\omega)$. As we shall explain next, the diffeomorphism invariant $\mathrm{B}(M)$ can often be computed or estimated very well.

Recall that the *Lusternik–Schnirelmann category* of a finite CW-space X is defined as

$$\operatorname{cat} X := \min\{k \mid X = A_1 \cup \ldots \cup A_k\},$$

where each A_i is open and contractible in X, [20, 4]. Clearly,

$$\operatorname{cat} M \le \mathrm{B}(M)$$

if M is a compact smooth manifold. It holds that $\operatorname{cat} X = \operatorname{cat} Y$ whenever X and Y are homotopy equivalent. However, the Lusternik–Schnirelmann category is very different from the usual homotopical invariants in algebraic topology and hence often difficult to compute. Nevertheless, $\operatorname{cat} X$ can be estimated from below in cohomological terms as follows. Let H^* be singular (or Čech, or Alexander–Spanier) cohomology theory, with any coefficient ring, and let $\tilde{H}^*$ be the corresponding reduced cohomology. The *cup-length* of X is defined as

$$\mathrm{cl}(X) := \sup\{k \mid u_1 \cdots u_k \ne 0, u_i \in \tilde{H}^*(X)\}.$$

It then holds true that

$$\operatorname{cat} X \ge \mathrm{cl}(X) + 1, \tag{2}$$

see [7]. If X is connected, an estimate of $\operatorname{cat} X$ from above is given by

$$\operatorname{cat} X \le \dim X + 1.$$

This inequality can be substantially improved as follows. Recall that X is said to be p-connected if it is path connected and its homotopy groups $\pi_i(X)$ vanish for $1 \le i \le p$. It turns out that

$$\operatorname{cat} X \le \frac{\dim X}{p+1} + 1 \tag{3}$$

for every p-connected and finite CW-space X. Another useful property of the LS-category is

$$\max\{\operatorname{cat} X, \operatorname{cat} Y\} \le \operatorname{cat}(X \times Y) < \operatorname{cat} X + \operatorname{cat} Y \tag{4}$$

for any CW-spaces X and Y. Proofs of all the above statements and much additional information on LS-category can be found in [4, 12, 13]. $\diamond$

Summarizing we have that

$$\operatorname{cl}(M) + 1 \ \le\ \operatorname{cat} M \ \le\ \mathrm{B}(M) \tag{5}$$

for any smooth manifold. Furthermore, if M^n is closed then $\mathrm{B}(M) \le n+1$, see [19, 26, 36]. Hence,

$$\operatorname{cl}(M) + 1 \le \operatorname{cat} M \le \mathrm{B}(M) \le n+1 \tag{6}$$

for any closed n-dimensional manifold.

These inequalities may be substantially improved if M is symplectic.

Proposition 1. *Let (M,ω) be a closed connected $2n$-dimensional symplectic manifold. Then*

$$n+1 \le \operatorname{cl}(M) + 1 \le \operatorname{cat} M \le \mathrm{B}(M) \le 2n+1.$$

Moreover, the following assertions hold true.

(i) *If $\pi_1(M) = 0$, then $n+1 = \operatorname{cl}(M)+1 = \operatorname{cat} M = \mathrm{B}(M)$.*

(ii) *If $[\omega]|_{\pi_2(M)} = 0$, then $\operatorname{cat} M = \mathrm{B}(M) = 2n+1$.*

(iii) *If $\operatorname{cat} M < \mathrm{B}(M)$, then $n \ge 2$, $n+1 = \operatorname{cl}(M)+1 = \operatorname{cat} M$ and $\mathrm{B}(M) = n+2$.*

On the other hand, the computation of the Gromov width and hence of the number $\Gamma(M,\omega)$ is often a very delicate matter. Fortunately, there has recently been some remarkable progress in this problem, see Section 5.

The paper is organized as follows. In Section 2 we prove Theorem 1. In Section 3 we study the minimal number $\mathrm{S}_{\mathrm{B}}^{=}(M,\omega)$ of *equal* symplectic balls needed to cover (M,ω) as well as the minimal number $\mathrm{S}(M,\omega)$ of symplectic charts diffeomorphic to a ball needed to parametrize (M,ω). In Section 4 we prove Proposition 1, and in the last section we compute the numbers $\mathrm{S}(M,\omega)$, $\mathrm{S}_{\mathrm{B}}(M,\omega)$ and $\mathrm{S}_{\mathrm{B}}^{=}(M,\omega)$ for various closed symplectic manifolds.

Acknowledgements. The idea behind the proof of Theorem 1 belongs to Gromov. We are very grateful to Leonid Polterovich for explaining it to us. The first author was supported by NSF, grant 0406311.

2. Proof of Theorem 1

In view of the inequalities (1) and (6), Theorem 1 is a consequence of

Theorem 2.1. *Let (M,ω) be a closed $2n$-dimensional symplectic manifold.*

(i) *If $\Gamma(M,\omega) \ge 2n+2$, then $\mathrm{S}_{\mathrm{B}}(M,\omega) = \Gamma(M,\omega)$.*

(ii) *If $\Gamma(M,\omega) \le 2n+1$, then $\mathrm{S}_{\mathrm{B}}(M,\omega) \le 2n+1$.*

Proof. We start with describing the idea of the proof, which belongs to Gromov and is as simple as beautiful. For each Borel set A in M we abbreviate its volume

$$\mu(A) := \frac{1}{n!}\int_A \omega^n.$$

Moreover, we define the natural number k by

$$k = \begin{cases} \Gamma(M,\omega) & \text{if} \quad \Gamma(M,\omega) \geq 2n+2, \\ 2n+1 & \text{if} \quad \Gamma(M,\omega) \leq 2n+1. \end{cases} \tag{7}$$

By definition of $\Gamma(M,\omega)$,

$$\gamma(M,\omega) > \frac{\mu(M)}{k}. \tag{8}$$

By definition of $\gamma(M,\omega)$ we find a Darboux chart $\varphi\colon B^{2n}(a) \to \mathcal{B} \subset M$ such that

$$\mu(\mathcal{B}) > \frac{\mu(M)}{k}.$$

In view of this inequality, and since $\dim M + 1 \leq k$, we shall find a cover of M by k sets $\mathcal{C}^1, \dots, \mathcal{C}^k$ where each set $\mathcal{C}^j$ is essentially a disjoint union of small cubes, and where

$$\mu\left(\mathcal{C}^j\right) < \mu(\mathcal{B}) \quad \text{for each } j.$$

Using this and the specific choice of the sets $\mathcal{C}^j$ we shall then be able to construct for each j a symplectomorphism Φ^j of M such that $\Phi^j(\mathcal{C}^j) \subset \mathcal{B}$. The k Darboux charts

$$\left(\Phi^j\right)^{-1} \circ \varphi\colon B^{2n}(a) \to M$$

will then cover M, and so Theorem 2.1 follows.

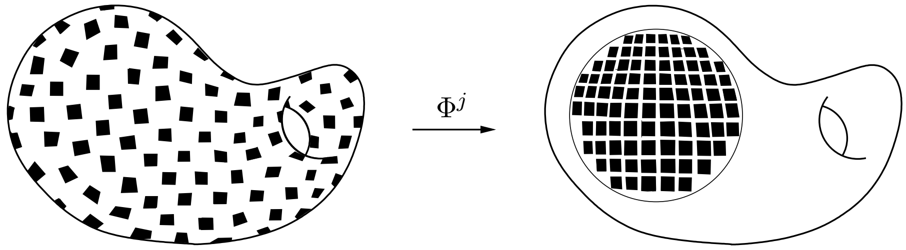

FIGURE 1. The idea behind the map Φ^j.

Notice that $\mu\left(\mathcal{C}^j\right)$ might be very close to $\mu(\mathcal{B})$. In order that the "cubes" in $\mathcal{C}^j$ all fit into the ball $\mathcal{B}$, the map Φ^j should therefore not distort the cubes too much. We shall be able to find such a map Φ^j by constructing an appropriate atlas for (M,ω) and by constructing the set $\mathcal{C}^j$ carefully.

Step 1. Construction of a good atlas of (M,ω)

Let k be the natural number defined in (7). In view of the estimate (8) the real number ε defined by

$$\gamma(M,\omega) \;=\; \frac{\mu(M)}{k} + 2\varepsilon$$

is positive. By definition of $\gamma(M,\omega)$ we can choose a Darboux chart

$$\varphi_0 \colon B^{2n}(a_0) \to \mathcal{B}_0 \subset M$$

such that

$$\mu(\mathcal{B}_0) \;>\; \frac{\mu(M)}{k} + \varepsilon.$$

Since M is compact, we find m other Darboux charts $\varphi_i \colon B^{2n}(a_i) \to \mathcal{B}_i \subset M$ such that

$$M \;=\; \bigcup_{i=0}^{m} \mathcal{B}_i. \tag{9}$$

We can assume that

$$\mathcal{B}_i \;\not\subset\; \bigcup_{j\neq i} \mathcal{B}_j, \qquad i = 0,\dots,m. \tag{10}$$

Given open subsets $U \subset V$ of $\mathbb{R}^{2n}$ we write $U \Subset V$ if $\overline{U} \subset V$, and we say that a symplectic chart $\bigl(\widetilde{U},\widetilde{\varphi}\bigr)$ is larger than a symplectic chart (U,φ) if $U \Subset \widetilde{U}$ and $\varphi = \widetilde{\varphi}|_U$. Using this terminology we can also assume that each chart $\bigl(B^{2n}(a_i),\varphi_i\bigr)$ is the restriction of a larger chart. Then the boundaries of the images $\mathcal{B}_0,\mathcal{B}_1,\dots,\mathcal{B}_m$ are smooth. Using that M is a normal space we next choose for $i = 0,\dots,m$ numbers $a_i' < a_i$ so large that with $\mathcal{B}_i' = \varphi_i\left(B^{2n}(a_i')\right)$ we have

$$\mu\left(\mathcal{B}_0'\right) \;>\; \frac{\mu(M)}{k} + \varepsilon \tag{11}$$

and

$$M \;=\; \bigcup_{i=0}^{m} \mathcal{B}_i'. \tag{12}$$

After renumbering the charts $\bigl(B^{2n}(a_1),\varphi_1\bigr),\dots,\bigl(B^{2n}(a_m),\varphi_m\bigr)$ we can then assume that $\mathcal{B}_1 \cap \mathcal{B}_0' \neq \emptyset$. In view of (10) and since the boundaries of $\mathcal{B}_1$ and $\mathcal{B}_0'$ are smooth, the open set

$$\mathcal{B}_1 \setminus \overline{\mathcal{B}_0'} \;=\; \coprod_{i=1}^{I_1} \mathcal{U}_i$$

is non-empty and consists of finitely many connected components $\mathcal{U}_i$ with piecewise smooth boundaries. For each $i \in \{1,\dots,I_1\}$ we choose a point

$$p_i \;\in\; \partial\mathcal{B}_0' \cap \partial\mathcal{U}_i.$$

Here, ∂A denotes the boundary of a subset A of M. We let $\mathcal{T}_1$ be the rooted tree whose vertices are the root p_0 and the points p_i and whose edges are $[p_0, p_i]$, $i = 1, \dots, I_1$. For notational convenience we set $\mathcal{U}_0 = \mathcal{B}_0$ and $\mathcal{U}_0' = \mathcal{B}_0'$ as well as

$$\mathcal{U}_i' = \mathcal{U}_i \cap \mathcal{B}_1', \quad i = 1, \dots, I_1.$$

It might well be that $\mathcal{U}_i' = \emptyset$ for some i. Clearly,

$$\bigcup_{i=0}^{1} \mathcal{B}_i = \bigcup_{i=0}^{I_1} \mathcal{U}_i \quad \text{and} \quad \bigcup_{i=0}^{1} \overline{\mathcal{B}_i'} = \bigcup_{i=0}^{I_1} \overline{\mathcal{U}_i'}, \tag{13}$$

cf. Figure 2.

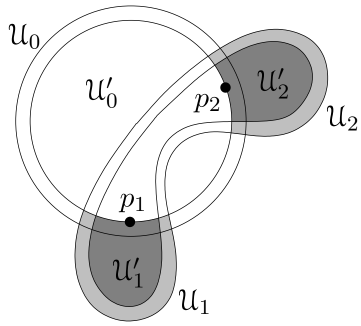

FIGURE 2. The sets $\mathcal{U}_1' \subset \mathcal{U}_1$ and $\mathcal{U}_2' \subset \mathcal{U}_2$ and the points $p_1 \in \partial\mathcal{U}_0' \cap \partial\mathcal{U}_1$ and $p_2 \in \partial\mathcal{U}_0' \cap \partial\mathcal{U}_2$.

The tree $\mathcal{T}_1$ corresponding to Figure 2 is depicted in Figure 4. We also set $U_0 = B^{2n}(a_0)$ and $\phi_0 = \varphi_0 \colon U_0 \to \mathcal{U}_0$ and define the symplectic charts

$$U_i = \varphi_1^{-1}(\mathcal{U}_i), \quad \phi_i = \varphi_1|_{U_i} \colon U_i \to \mathcal{U}_i, \quad i = 1, \dots, I_1.$$

Notice that each chart (U_i, ϕ_i) is the restriction of a larger chart.

If $m \geq 2$, the assumption (12) implies that we can renumber the charts $\left(B^{2n}(a_2), \varphi_2\right), \dots, \left(B^{2n}(a_m), \varphi_m\right)$ such that $\mathcal{B}_2 \cap \bigcup_{i=0}^{1} \mathcal{B}_i' \neq \emptyset$. In view of (10) and since the boundaries of $\mathcal{B}_2$, $\mathcal{B}_0'$ and $\mathcal{B}_1'$ are smooth, the open set

$$\mathcal{B}_2 \setminus \bigcup_{i=0}^{1} \overline{\mathcal{B}_i'} = \coprod_{i=I_1+1}^{I_2} \mathcal{U}_i \tag{14}$$

is non-empty and consists of finitely many connected components $\mathcal{U}_i$ with piecewise smooth boundaries. In view of the second identity in (13) and the definition (14) of $\mathcal{U}_i$ we find for each $i \in \{I_1 + 1, \dots, I_2\}$ an index $\underline{i} \in \{0, \dots, I_1\}$ such that $\partial\mathcal{U}_{\underline{i}}' \cap \partial\mathcal{U}_i \neq \emptyset$, and we choose a point

$$p_i \in \partial\mathcal{U}_{\underline{i}}' \cap \partial\mathcal{U}_i.$$

We let $\mathcal{T}_2$ be the tree obtained from the tree $\mathcal{T}_1$ by adding the vertices p_i and the edges $\left[p_{\underline{i}}, p_i\right]$, $i = I_1+1, \dots, I_2$. We set $\mathcal{U}_i' = \mathcal{U}_i \cap \mathcal{B}_2'$ for $i = I_1+1, \dots, I_2$.

Then

$$\bigcup_{i=0}^{2} \mathcal{B}_i = \bigcup_{i=0}^{I_2} \mathcal{U}_i \qquad \text{and} \qquad \bigcup_{i=0}^{2} \overline{\mathcal{B}'_i} = \bigcup_{i=0}^{I_2} \overline{\mathcal{U}'_i},$$

cf. Figure 3.

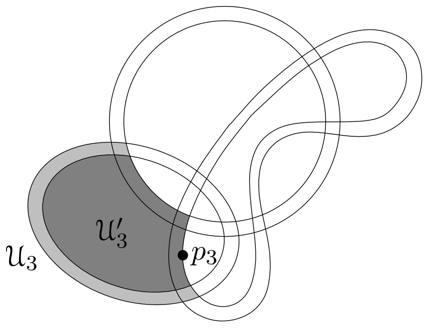

FIGURE 3. The sets $\mathcal{U}'_3 \subset \mathcal{U}_3$ and the point $p_3 \in \partial\mathcal{U}'_1 \cap \partial\mathcal{U}_3$.

The tree $\mathcal{T}_2$ corresponding to Figure 3 is depicted in Figure 4.

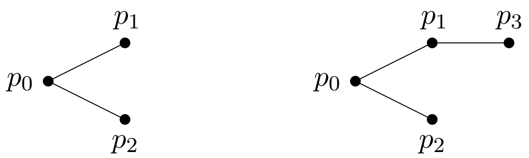

FIGURE 4. The trees $\mathcal{T}_1$ and $\mathcal{T}_2$.

We define the symplectic charts

$$U_i = \varphi_2^{-1}(\mathcal{U}_i), \quad \phi_i = \varphi_2|_{U_i} \colon U_i \to \mathcal{U}_i, \quad i = I_1+1, \dots, I_2.$$

Notice again that each chart (U_i, ϕ_i) is the restriction of a larger chart.

Proceeding this way $m-2$ other times we find a sequence

$$0 =: I_0 < I_1 < \cdots < I_m$$

of integers and $l := I_m + 1$ open connected sets $\mathcal{U}_i \subset M$, $i = 0, \dots, l$, with piecewise smooth boundaries such that for each $j \in \{0, \dots, m-1\}$,

$$\mathcal{B}_{j+1} \setminus \bigcup_{i=0}^{j} \overline{\mathcal{B}'_i} = \coprod_{i=I_j+1}^{I_{j+1}} \mathcal{U}_i. \tag{15}$$

Moreover, defining $j(i)$ by the condition $i \in \left\{I_{j(i)}+1, \dots, I_{j(i)+1}\right\}$ and setting $\mathcal{U}'_i = \mathcal{U}_i \cap \mathcal{B}'_{j(i)+1}$ we have found for each $i \in \{1, \dots, l\}$ an index $\underline{i} \in \{0, \dots, I_{j(i)}\}$ such that $\partial\mathcal{U}'_{\underline{i}} \cap \partial\mathcal{U}_i \neq \emptyset$ and have chosen a point

$$p_i \in \partial\mathcal{U}'_{\underline{i}} \cap \partial\mathcal{U}_i. \tag{16}$$

The vertices of the rooted tree $\mathcal{T} = \mathcal{T}_m$ consist of the root p_0 and the points p_i, and the edges of $\mathcal{T}$ are $\left[p_{\underline{i}}, p_i\right]$, $i = 1, \dots, l$.

The identities (9) and (15) imply that

$$M \;=\; \bigcup_{i=0}^{l} \mathcal{U}_i \tag{17}$$

and that $\sum_{i=0}^{l} \mu\left(\mathcal{U}_i\right) \to \mu(M)$ as $a_j' \to a_j$ for all $j = 0, \dots, m$. Choosing $a_0', \dots, a_m'$ larger if necessary we can therefore assume that

$$\sum_{i=0}^{l} \mu\left(\mathcal{U}_i\right) \;<\; \mu(M) + \varepsilon. \tag{18}$$

We replace the symplectic atlas $\left\{\varphi_i \colon B^{2n}(a_i) \to \mathcal{B}_i,\ i = 0, \dots, m\right\}$ by the symplectic atlas $\{\phi_i \colon U_i \to \mathcal{U}_i,\ i = 0, \dots, l\}$. Here, we still have $(U_0, \phi_0) = \left(B^{2n}(a_0), \varphi_0\right)$, and

$$U_i = \varphi_{j(i)+1}^{-1}\left(\mathcal{U}_i\right), \quad \phi_i = \varphi_{j(i)+1}|_{U_i} \colon U_i \to \mathcal{U}_i, \quad i = 1, \dots, l.$$

Each chart (U_i, ϕ_i) is the restriction of a larger chart $\widetilde{\phi}_i \colon \widetilde{U}_i \to \widetilde{\mathcal{U}}_i$. While $p_i \notin \mathcal{U}_i$ in view of (16), we have $p_i \in \widetilde{\mathcal{U}}_{\underline{i}} \cap \widetilde{\mathcal{U}}_i$ for $i = 1, \dots, l$. Our next goal is to replace the charts $\widetilde{\phi}_i \colon \widetilde{U}_i \to \widetilde{\mathcal{U}}_i$ by charts $\widetilde{\psi}_i \colon \widetilde{V}_i \to \widetilde{\mathcal{U}}_i$ such that for each $i \geq 1$ the transition function

$$\widetilde{\psi}_{\underline{i}}^{-1} \circ \widetilde{\psi}_i \colon\ \widetilde{\psi}_i^{-1}\left(\widetilde{\mathcal{U}}_{\underline{i}} \cap \widetilde{\mathcal{U}}_i\right) \;\to\; \widetilde{\psi}_{\underline{i}}^{-1}\left(\widetilde{\mathcal{U}}_{\underline{i}} \cap \widetilde{\mathcal{U}}_i\right)$$

is the identity near $\widetilde{\psi}_{\underline{i}}^{-1}(p_i)$. We first of all set $\left(\widetilde{V}_0, \widetilde{\psi}_0\right) = \left(\widetilde{U}_0, \widetilde{\phi}_0\right)$. In order to construct $\left(\widetilde{V}_1, \widetilde{\psi}_1\right)$ we first define a symplectic chart $\left(\widehat{V}_1, \widehat{\psi}_1\right)$ by

$$\widehat{V}_1 = \left[d\left(\widetilde{\phi}_1^{-1} \circ \widetilde{\psi}_0\right)(q_1)\right]^{-1}\left(\widetilde{U}_1\right), \quad \widehat{\psi}_1 = \widetilde{\phi}_1 \circ d\left(\widetilde{\phi}_1^{-1} \circ \widetilde{\psi}_0\right)(q_1) \colon \widehat{V}_1 \to \widetilde{U}_1$$

where we abbreviated $q_1 := \widetilde{\psi}_0^{-1}(p_1)$. We then find

$$\left(\widetilde{\psi}_0^{-1} \circ \widehat{\psi}_1\right)(q_1) = q_1 \quad \text{and} \quad d\left(\widetilde{\psi}_0^{-1} \circ \widehat{\psi}_1\right)(q_1) = id. \tag{19}$$

We obtain the desired chart $\left(\widetilde{V}_1, \widetilde{\psi}_1\right)$ from the chart $\left(\widehat{V}_1, \widehat{\psi}_1\right)$ with the help of the following lemma.

Lemma 2.2. *Assume that $\varphi \colon U \to U'$ is a symplectomorphism between two domains U and U' in $\mathbb{R}^{2n}$ such that $\varphi(q) = q$ and $d\varphi(q) = id$ at some point $q \in U$. Then there exist open neighbourhoods $W \subset \widetilde{W} \Subset U$ of q and a symplectomorphism $\rho \colon U \to U'$ such that $\rho|_W = id$ and $\rho|_{U \setminus \widetilde{W}} = \varphi|_{U \setminus \widetilde{W}}$.*

Proof. We can assume that $q = 0$. Following [11, Appendix A.1] we represent the map φ by

$$\begin{aligned} x &= a(\xi, \eta) \\ y &= b(\xi, \eta). \end{aligned}$$

Since $d\varphi(0) = id$, we have $\det(a_\xi(0)) = 1 \neq 0$. According to Proposition 1 in [11, Appendix A.1] we therefore find a smooth function w defined on a neighbourhood $\mathcal{N} \subset \mathbb{R}^{2n}(x,\eta)$ of 0 such that

$$
\left\{ \begin{array}{rcl} \xi & = & x + w_\eta(x,\eta) \\ y & = & \eta + w_x(x,\eta). \end{array} \right. \tag{20}
$$

We can assume that $w(0) = 0$. In view of the identities $\varphi(0) = 0$ and $d\varphi(0) = id$ and the relations (20) we find that all the derivatives of w up to order 2 vanish in 0, i.e.,

$$
w(x,\eta) \,=\, O\left(|(x,\eta)|^3\right). \tag{21}
$$

Choose a smooth function $f\colon [0,\infty[\, \to [0,1]$ such that

$$
f(s) \,=\, \left\{ \begin{array}{ll} 0, & s \leq 1, \\ 1, & s \geq 2, \end{array} \right.
$$

and denote the open ball of radius s in $\mathbb{R}^{2n}(x,\eta)$ by B_s. For each $\varepsilon > 0$ for which $B_{3\varepsilon} \subset \mathcal{N}$ we define the smooth function $w^\varepsilon(x,\eta)\colon B_{3\varepsilon} \to \mathbb{R}$ by

$$
w^\varepsilon(x,\eta) \,=\, f\left(\tfrac{1}{\varepsilon}\,|(x,\eta)|\right) w(x,\eta).
$$

Then

$$
w^\varepsilon|_{B_\varepsilon} = 0 \qquad \text{and} \qquad w^\varepsilon|_{B_{3\varepsilon}\setminus B_{2\varepsilon}} = w|_{B_{3\varepsilon}\setminus B_{2\varepsilon}}. \tag{22}
$$

Abbreviating $\zeta := (x,\eta)$ and $r := |\zeta|$ we compute

$$
\begin{array}{rcl}
w^\varepsilon_{\zeta_i}(\zeta) & = & f'\left(\frac{r}{\varepsilon}\right) \frac{1}{\varepsilon} \frac{\zeta_i}{r}\, w(\zeta) + f\left(\frac{r}{\varepsilon}\right) w_{\zeta_i}(\zeta), \\
w^\varepsilon_{\zeta_i\zeta_j}(\zeta) & = & f''\left(\frac{r}{\varepsilon}\right) \frac{1}{\varepsilon^2} \frac{\zeta_i\zeta_j}{r^2}\, w(\zeta) + f'\left(\frac{r}{\varepsilon}\right) \frac{1}{\varepsilon} \left(\frac{\delta_{ij}}{r} - \frac{\zeta_i\zeta_j}{r^3}\right) w(\zeta) \\
 & & \qquad + f'\left(\frac{r}{\varepsilon}\right) \frac{1}{\varepsilon} \left(\frac{\zeta_i}{r}\, w_{\zeta_j}(\zeta) + \frac{\zeta_j}{r}\, w_{\zeta_i}(\zeta)\right) \\
 & & \qquad + f\left(\frac{r}{\varepsilon}\right) w_{\zeta_i\zeta_j}(\zeta)
\end{array}
$$

where $i,j \in \{1,\dots,2n\}$ and where δ_{ij} denotes the Kronecker symbol. In view of the estimate (21) we therefore find that

$$
w^\varepsilon_{\zeta_i\zeta_j}(\zeta) \,=\, \tfrac{1}{\varepsilon^2} O(r^3) + \tfrac{1}{\varepsilon} O(r^2) + O(r) \,=\, O(r), \quad \zeta \in B_{3\varepsilon},
$$

and so

$$
w^\varepsilon(x,\eta) \,=\, O\left(|(x,\eta)|^3\right), \quad (x,\eta) \in B_{3\varepsilon}. \tag{23}
$$

We in particular conclude that $\det(\mathbb{1}_n + w_{x\eta}(x,\eta)) \neq 0$ for all $(x,\eta) \in B_{3\varepsilon}$ if $\varepsilon > 0$ is small enough. The relations

$$
\left\{ \begin{array}{rcl} \xi & = & x + w^\varepsilon_\eta(x,\eta) \\ y & = & \eta + w^\varepsilon_x(x,\eta) \end{array} \right. \tag{24}
$$

therefore implicitly define a symplectic mapping φ^ε near 0, see again [11, Appendix A.1]. The C^2-estimate (23) implies that φ^ε is C^1-close to the identity and that for $\varepsilon > 0$ small enough, φ^ε is defined and injective on all of

$$U^\varepsilon_{3\varepsilon} = \left\{(\xi,\eta) \in \mathbb{R}^{2n} \mid (24) \text{ holds for } (x,\eta) \in B_{3\varepsilon}\right\}.$$

In view of the estimate (23) each of the sets

$$U^\varepsilon_s = \left\{(\xi,\eta) \in \mathbb{R}^{2n} \mid (24) \text{ holds for } (x,\eta) \in B_s\right\}, \quad s \le 3\varepsilon,$$

is contained in the domain U of φ and is diffeomorphic to an open ball provided that $\varepsilon > 0$ is small enough. According to the identities (22), the map φ^ε is the identity on $U^\varepsilon_\varepsilon$ and coincides with φ on the "open annulus" $U^\varepsilon_{3\varepsilon} \backslash \overline{U^\varepsilon_{2\varepsilon}}$. It follows that $\varphi^\varepsilon\left(U^\varepsilon_{3\varepsilon}\right) = \varphi\left(U^\varepsilon_{3\varepsilon}\right)$. We smoothly extend $\varphi^\varepsilon \colon U^\varepsilon_{3\varepsilon} \to \mathbb{R}^{2n}$ to a symplectic embedding $\rho\colon U \to \mathbb{R}^{2n}$ by setting $\rho(z) = \varphi(z)$, $z \in U \setminus U^\varepsilon_{3\varepsilon}$. Then $\rho(U) = \varphi(U) = U'$, and setting $W = U^\varepsilon_\varepsilon$ and $\widetilde{W} = U^\varepsilon_{2\varepsilon} \Subset U^\varepsilon_{3\varepsilon} \subset U$ we find that $\rho|_W = \varphi^\varepsilon|_{U^\varepsilon_\varepsilon} = id$ and $\rho|_{U\backslash\widetilde{W}} = \varphi|_{U\backslash\widetilde{W}}$. The proof of Lemma 2.2 is complete. $\square$

In view of the identities (19) we can apply Lemma 2.2 to the symplectomorphism

$$\widetilde{\psi}_0^{-1} \circ \widehat{\psi}_1 \colon \widehat{\psi}_1^{-1}\left(\widetilde{\mathfrak{U}}_0 \cap \widetilde{\mathfrak{U}}_1\right) \to \widetilde{\psi}_0^{-1}\left(\widetilde{\mathfrak{U}}_0 \cap \widetilde{\mathfrak{U}}_1\right)$$

which fixes q_1, and find open neighbourhoods $W_1 \subset \widetilde{W}_1 \Subset \widehat{\psi}_1^{-1}\left(\widetilde{\mathfrak{U}}_0 \cap \widetilde{\mathfrak{U}}_1\right)$ and a symplectomorphism

$$\rho_1 \colon \widehat{\psi}_1^{-1}\left(\widetilde{\mathfrak{U}}_0 \cap \widetilde{\mathfrak{U}}_1\right) \to \widetilde{\psi}_0^{-1}\left(\widetilde{\mathfrak{U}}_0 \cap \widetilde{\mathfrak{U}}_1\right)$$

such that

$$\rho_1|_{W_1} = id \quad \text{and} \quad \rho_1|_{\widehat{\psi}_1^{-1}(\widetilde{\mathfrak{U}}_0\cap\widetilde{\mathfrak{U}}_1)\backslash\widetilde{W}_1} = \widetilde{\psi}_0^{-1} \circ \widehat{\psi}_1. \tag{25}$$

Set $\widetilde{V}_1 = \widehat{V}_1$. In view of the properties (25) of ρ_1 the map $\widetilde{\psi}_1 \colon \widetilde{V}_1 \to \widetilde{\mathfrak{U}}_1$ defined by

$$\widetilde{\psi}_1 = \begin{cases} \widetilde{\psi}_0 \circ \rho_1 & \text{on} \quad \widehat{\psi}_1^{-1}\left(\widetilde{\mathfrak{U}}_0 \cap \widetilde{\mathfrak{U}}_1\right), \\ \widehat{\psi}_1 & \text{on} \quad \widetilde{V}_1 \setminus \widetilde{W}_1 \end{cases}$$

is a well-defined smooth symplectic chart such that

$$\widetilde{\psi}_0^{-1} \circ \widetilde{\psi}_1 \colon \widetilde{\psi}_1^{-1}\left(\widetilde{\mathfrak{U}}_0 \cap \widetilde{\mathfrak{U}}_1\right) \to \widetilde{\psi}_0^{-1}\left(\widetilde{\mathfrak{U}}_0 \cap \widetilde{\mathfrak{U}}_1\right)$$

is the identity on the open neighbourhood W_1 of $q_1 = \widetilde{\psi}_0^{-1}(p_1)$. Assume now by induction that we have already constructed new charts $\widetilde{\psi}_j \colon \widetilde{V}_j \to \widetilde{\mathfrak{U}}_j$ for $j = 1, \dots, i-1$. Since $\underline{i} < i$, the chart $\left(\widetilde{U}_{\underline{i}}, \widetilde{\phi}_{\underline{i}}\right)$ is already replaced by the chart $\left(\widetilde{V}_{\underline{i}}, \widetilde{\psi}_{\underline{i}}\right)$. Applying the two-step construction shown above to the pair $\left(\widetilde{V}_{\underline{i}}, \widetilde{\psi}_{\underline{i}}\right)$, $\left(\widetilde{U}_i, \widetilde{\phi}_i\right)$ we find a new chart $\widetilde{\psi}_i \colon \widetilde{V}_i \to \widetilde{\mathfrak{U}}_i$ such that the transition function

$$\widetilde{\psi}_{\underline{i}}^{-1} \circ \widetilde{\psi}_i \colon \widetilde{\psi}_i^{-1}\left(\widetilde{\mathfrak{U}}_{\underline{i}} \cap \widetilde{\mathfrak{U}}_i\right) \to \widetilde{\psi}_{\underline{i}}^{-1}\left(\widetilde{\mathfrak{U}}_{\underline{i}} \cap \widetilde{\mathfrak{U}}_i\right)$$

is the identity on an open neighbourhood W_i of $q_i = \widetilde{\psi}_{\underline{i}}^{-1}(p_i)$. In this way we construct a new symplectic atlas

$$\widetilde{\mathfrak{A}} = \left\{ \widetilde{\psi}_i \colon \widetilde{V}_i \to \widetilde{\mathcal{U}}_i,\ i = 0, \dots, l \right\}.$$

Recall that $\mathcal{U}_i \Subset \widetilde{\mathcal{U}}_i$. The collection

$$\mathfrak{A} = \{ \psi_i \colon V_i \to \mathcal{U}_i,\ i = 0, \dots, l \}$$

of smaller charts defined by

$$V_i = \widetilde{\psi}_i^{-1}(\mathcal{U}_i), \quad \psi_i = \widetilde{\psi}_i|_{V_i} \colon V_i \to \mathcal{U}_i$$

is the good atlas of (M, ω) we were looking for. We still have $\big(V_0, \psi_0\big) = \big(B^{2n}(a_0), \varphi_0\big)$ and $\mathcal{U}_0 = \mathcal{B}_0$. We also recall that each set $\mathcal{U}_i$ is connected and has piecewise smooth boundary.

Step 2. The dimension cover $\mathfrak{D}(2n, k)$

Let $k \geq 2n + 1$ be the natural number defined in (7). In this step we shall construct a special cover $\mathfrak{D}(2n, k)$ of $\mathbb{R}^{2n}$ by cubes. Our construction is inspired by an idea from elementary dimension theory, see e.g. [5, Figure 7].

We denote the coordinates in $\mathbb{R}^{2n}$ by $x_1, \dots, x_{2n}$, and we let $\{e_1, \dots, e_{2n}\}$ be the standard basis of $\mathbb{R}^{2n}$. Given a point $q \in \mathbb{R}^{2n}$ and a subset A of $\mathbb{R}^{2n}$ we denote the translate of A by q by

$$q + A = \{ q + a \mid a \in A \}.$$

By a cube we mean a translate of the closed cube $C^{2n} = [0, 1]^{2n} \subset \mathbb{R}^{2n}$. We define the $(2n \times 2n)$-matrix $M(2n, k)$ as the matrix whose diagonal is $(k, 1, \dots, 1)$, whose upper-diagonal is

$$\left(\tfrac{k}{2n}, \tfrac{2n}{2n-1}, \tfrac{2n-1}{2n-2}, \dots, \tfrac{4}{3}, \tfrac{3}{2} \right)$$

and whose other matrix entries all vanish. E.g.,

$$M(2,3) = \begin{bmatrix} 3 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}, \quad M(2,4) = \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}, \quad M(4,5) = \begin{bmatrix} 5 & \frac{5}{4} & 0 & 0 \\ 0 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We consider the infinite union of cubes

$$\mathfrak{C}^1(2n, k) = \bigcup_{v \in \mathbb{Z}^{2n}} M(2n, k)v + C^{2n}$$

and its translates

$$\mathfrak{C}^j(2n, k) = (j-1)e_1 + \mathfrak{C}^1(2n, k), \quad j = 2, \dots, k,$$

and we abbreviate

$$\mathfrak{D}(2n, k) := \bigcup_{j=1}^{k} \mathfrak{C}^j(2n, k),$$

cf. Figure 5 and Figure 6.

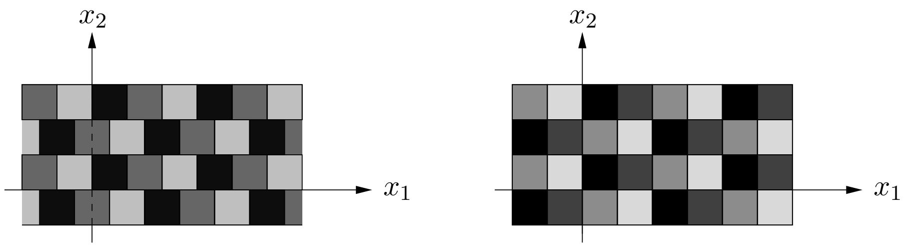

FIGURE 5. Parts of the dimension covers $\mathfrak{D}(2,3)$ and $\mathfrak{D}(2,4)$.

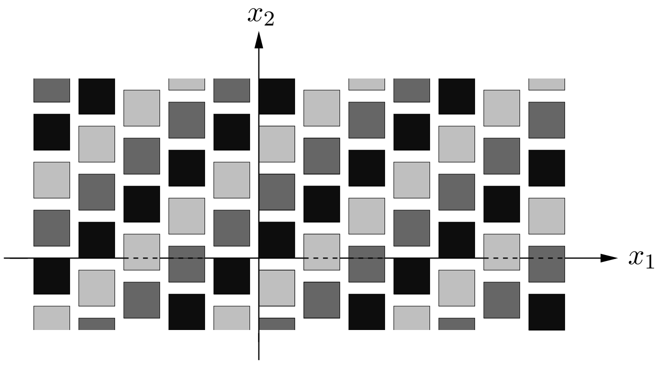

FIGURE 6. A part of the intersections $\mathfrak{C}^1(4,5) \cap \{(x_1,x_2,x_3,x_4) \mid x_3 = i - \frac{1}{2}, x_4 = 0\}$, $i = 1,2,3$.

Finally, we define for each subset A of $\mathbb{R}^{2n}$ and each $m \in \{1,\dots,2n\}$ the cylinder $Z_m(A)$ over A by

$$Z_m(A) \;=\; \{a + \lambda e_m \mid a \in A,\, \lambda \in \mathbb{R}\}\,.$$

Recall that the distance between two subsets A and B of $\mathbb{R}^{2n}$ is defined as

$$\operatorname{dist}(A,B) \;=\; \inf\{|a-b| \mid a \in A,\, b \in B\}\,.$$

Given $\nu > 0$ and a subset A of $\mathbb{R}^{2n}$ we denote the ν-neighbourhood of A by

$$\mathcal{N}_\nu(A) \;=\; \left\{z \in \mathbb{R}^{2n} \mid \operatorname{dist}(z,A) < \nu\right\}.$$

We abbreviate the positive number

$$\delta \;:=\; \min\left(\tfrac{k-2n}{2n}, \tfrac{1}{2n-1}\right). \tag{26}$$

Lemma 2.3.

(i) *For each $j \in \{1,\dots,k\}$ and any cube C of $\mathfrak{C}^j(2n,k)$ we have*

$$\operatorname{dist}(C, \mathfrak{C}^j(2n,k) \setminus C) \;=\; \delta.$$

Moreover,

$$Z_1\,(\operatorname{Int} C) \cap \mathfrak{C}^j(2n,k) \;=\; \bigcup_{l\in\mathbb{Z}} kle_1 + \operatorname{Int} C$$

and

$$Z_m\,(\mathcal{N}_\delta(C)) \cap \mathfrak{C}^j(2n,k) \;=\; \bigcup_{l\in\mathbb{Z}} (2n-m+2)le_m + C, \quad m = 2,\dots,2n.$$

(ii) *We have*

$$\mathfrak{D}(2n,k) \;=\; \bigcup_{j=1}^{k} \mathfrak{C}^j(2n,k) \;=\; \mathbb{R}^{2n}$$

and the interiors of the sets $\mathfrak{C}^j(2n,k)$ *are mutually disjoint.*

The proof, which is elementary, is omitted.

Step 3. The cover of M by small cubes

Let $\mathfrak{A} = \{\psi_i \colon V_i \to \mathcal{U}_i,\ i = 0,\dots,l\}$ be the symplectic atlas of (M,ω) constructed in Step 1 and let $\mathfrak{D}(2n,k) = \bigcup_{j=1}^{k} \mathfrak{C}^j(2n,k)$ be the dimension cover of $\mathbb{R}^{2n}$ constructed in the previous step. For any $r>0$ and any subset A of $\mathbb{R}^{2n}$ we set

$$rA \;=\; \{rz \mid z \in A\}$$

and we denote by $|A|$ the Lebesgue measure of A. Fix $i \in \{0,\dots,l\}$. For $d_i > 0$ we define $\mathfrak{C}_i^j(d_i)$ as the set of those cubes C in $d_i\mathfrak{C}^j(2n,k)$ for which

$$C \subset V_i \quad \text{and} \quad \operatorname{dist}\,(C,\partial V_i) \ge d_i \tag{27}$$

and we abbreviate

$$\mathfrak{D}_i(d_i) \;:=\; \bigcup_{j=1}^{k} \mathfrak{C}_i^j(d_i).$$

In view of the identity (17) and since M is a normal space we find open sets $\breve{\mathcal{U}}_i \Subset \mathcal{U}_i$ such that

$$M \;=\; \bigcup_{i=0}^{l} \mathcal{U}_i \;=\; \bigcup_{i=0}^{l} \breve{\mathcal{U}}_i.$$

Choose $d_i > 0$ so small that $\psi_i^{-1}\big(\breve{\mathcal{U}}_i\big) \subset \mathfrak{D}_i(d_i)$. Then

$$M \;=\; \bigcup_{i=0}^{l} \psi_i\left(\mathfrak{D}_i(d_i)\right). \tag{28}$$

Also notice that the homogeneity of the sets $\mathfrak{C}_i^j(d_i)$ implies that

$$\left|\mathfrak{C}_i^j(d_i)\right| \;\to\; \tfrac{1}{k}\,|V_i| \qquad \text{as} \;\; d_i \to 0$$

for all $j \in \{1,\dots,k\}$. Choosing $d_i > 0$ smaller if necessary we can therefore assume that

$$\left|\mathfrak{C}_i^j(d_i)\right| \;<\; \tfrac{1}{k}\left(|V_i| + \tfrac{k-1}{l+1}\varepsilon\right) \tag{29}$$

for all $i \in \{0, \dots, l\}$ and $j \in \{1, \dots, k\}$.

We denote by $\mathcal{C}^j = \mathcal{C}^j(d_0, \dots, d_l)$ the union of cubes "of the same colour"

$$\mathcal{C}^j = \bigcup_{i=0}^{l} \psi_i\big(\mathfrak{C}_i^j(d_i)\big), \quad j = 1, \dots, k.$$

The cubes $\psi_i(C)$ in $\psi_i\big(\mathfrak{C}_i^j(d_i)\big)$ are called i-cubes. For each connected component $\mathcal{K}$ of $\mathcal{C}^j$ we define the *height* of $\mathcal{K}$ as the maximal $h \in \{0, \dots, l\}$ for which $\mathcal{K}$ contains an h-cube. The set $\mathcal{C}^j$ decomposes as

$$\mathcal{C}^j = \coprod_{h=0}^{l} \mathcal{C}_h^j$$

where $\mathcal{C}_h^j$ is the union of the components of $\mathcal{C}^j$ of height h. In view of (28) we have

$$M = \bigcup_{j=1}^{k} \bigcup_{h=0}^{l} \mathcal{C}_h^j. \tag{30}$$

According to the estimates (29) we can choose for each $i \in \{1, \dots, l\}$ a number

$$\nu_i \in \left]0, \tfrac{\delta}{2}\right[\tag{31}$$

such that

$$(1 + 2\nu_i)^{2n} \left|\mathfrak{C}_i^j(d_i)\right| < \tfrac{1}{k}\left(|V_i| + \tfrac{k-1}{l+1}\varepsilon\right) \tag{32}$$

for all $j \in \{1, \dots, k\}$. Since $\nu_i < \frac{\delta}{2} < 1$, the conditions (27) imply that

$$\mathcal{N}_{\nu_i d_i}(C) \subset V_i \tag{33}$$

for any cube C in $\mathfrak{D}_i(d_i)$.

Lemma 2.4. *If the numbers $d_0, \dots, d_{l-1} > 0$ as well as the ratios d_i/d_{i+1}, $i = 0, \dots, l-1$, are small enough, then the following assertions hold true.*

(i) $\mathcal{C}_h^j \subset \mathcal{U}_h$ *for each $j \in \{1, \dots, k\}$ and $h \in \{0, \dots, l\}$.*

(ii) *Any component $\mathcal{K}$ of $\mathcal{C}_h^j$ contains only one h-cube $\psi_h(C)$, and*

$$\psi_h^{-1}(\mathcal{K}) \subset \mathcal{N}_{\nu_h d_h}(C), \quad h = 1, \dots, l.$$

Proof. We denote by $\mathcal{P}_i^j = \mathcal{P}_i^j(d_0, \dots, d_l)$ the partial union of cubes

$$\mathcal{P}_i^j = \bigcup_{g=i}^{l} \psi_i\big(\mathfrak{C}_i^j(d_i)\big), \quad i = 0, \dots, l;\ j = 1, \dots, k.$$

E.g., $\mathcal{P}_l^j = \psi_l\big(\mathfrak{C}_l^j(d_l)\big)$ and $\mathcal{P}_0^j = \mathcal{C}^j$. Generalizing the above definition we define the *height* of a connected component $\mathcal{K}$ of $\mathcal{P}_i^j$ as the maximal $h \in$

$\{i, \dots, l\}$ for which $\mathcal{K}$ contains an h-cube. The set $\mathcal{P}_i^j$ decomposes as

$$\mathcal{P}_i^j = \coprod_{h=i}^{l} \mathcal{P}_{i,h}^j$$

where $\mathcal{P}_{i,h}^j$ is the union of components of $\mathcal{P}_i^j$ of height h.

Since $\mathcal{P}_l^j$ consists of finitely many disjoint closed cubes, we can choose $d_{l-1} > 0$ so small that each cube of $\psi_{l-1}\big(\mathfrak{C}_{l-1}^j(d_{l-1})\big)$ intersects at most one cube of $\mathcal{P}_l^j$ for each j. Then each component $\mathcal{K}$ of $\mathcal{P}_{l-1,l}^j$ contains only one l-cube. We denote the distinguished cube in $\mathcal{K}$ by $\mathcal{C}(\mathcal{K})$. Since $\mathcal{P}_l^j$ is a compact subset of the open set $\mathcal{U}_l$, we can choose d_{l-1} so small that $\mathcal{P}_{l-1,l}^j \subset \mathcal{U}_l$ for each j. Moreover, choosing d_{l-1} yet smaller if necessary we can assume that

$$\psi_l^{-1}(\mathcal{K}) \subset \mathcal{N}_{\nu_l d_l}\left(\psi_l^{-1}\left(\mathcal{C}(\mathcal{K})\right)\right) \tag{34}$$

for each component $\mathcal{K}$ of $\mathcal{P}_{l-1,l}^j$ and each j.

Since $\mathcal{P}_{l-1}^j$ consists of finitely many disjoint compact components, we can choose $d_{l-2} > 0$ so small that each cube of $\psi_{l-2}\big(\mathfrak{C}_{l-2}^j(d_{l-2})\big)$ intersects at most one component of $\mathcal{P}_{l-1}^j$ for each j. Then each component $\mathcal{K}$ of $\mathcal{P}_{l-2,h}^j$ contains only one h-cube, $h = l, l-1, l-2$. We denote this distinguished cube again by $\mathcal{C}(\mathcal{K})$. If $h \in \{l, l-1\}$, then $\mathcal{C}(\mathcal{K}) = \mathcal{C}\left(\underline{\mathcal{K}}\right)$ where $\underline{\mathcal{K}}$ is the unique component of $\mathcal{P}_{l-1,h}^j$ contained in $\mathcal{K}$, and if $h = l-2$, then $\mathcal{C}(\mathcal{K}) = \mathcal{K}$ is an $(l-2)$-cube. Since $\mathcal{P}_{l-1,l}^j$ is a compact subset of the open set $\mathcal{U}_l$ and since $\mathcal{P}_{l-1,l-1}^j$ is a compact subset of the open set $\mathcal{U}_{l-1}$, we can choose d_{l-2} so small that $\mathcal{P}_{l-2,l}^j \subset \mathcal{U}_l$ and $\mathcal{P}_{l-2,l-1}^j \subset \mathcal{U}_{l-1}$ for each j. Moreover, the compact inclusions (34) imply that we can choose d_{l-2} so small that

$$\psi_l^{-1}(\mathcal{K}) \subset \mathcal{N}_{\nu_l d_l}\left(\psi_l^{-1}\left(\mathcal{C}(\mathcal{K})\right)\right)$$

for each component $\mathcal{K}$ of $\mathcal{P}_{l-2,l}^j$ and each j. Choosing d_{l-2} yet smaller if necessary we can also assume that

$$\psi_{l-1}^{-1}(\mathcal{K}) \subset \mathcal{N}_{\nu_{l-1} d_{l-1}}\left(\psi_l^{-1}\left(\mathcal{C}(\mathcal{K})\right)\right)$$

for each component $\mathcal{K}$ of $\mathcal{P}_{l-2,l-1}^j$ and each j.

Repeating this reasoning $l-2$ other times, we successively find $d_{l-1}, \dots, d_0$ such that assertions (i) and (ii) of the lemma hold true for all $h \in \{1, \dots, l\}$ and all j. Since $\mathcal{C}_0^j \subset \mathcal{U}_0$ by definition of $\mathcal{C}_0^j$, the proof of Lemma 2.4 is complete. □

For $h \geq 1$ the sets $M \setminus \mathcal{C}_h^j$ do not need to be connected. Define the *saturation* $\mathcal{S}(A)$ of a closed subset A of $\mathbb{R}^{2n}$ as the union of A with the bounded components of $\mathbb{R}^{2n} \setminus A$. For a closed subset $\mathcal{A}$ of $\mathcal{U}_h$ for which

$\mathcal{S}\left(\psi_h^{-1}(\mathcal{A})\right) \subset V_h$ we set

$$\mathcal{S}(\mathcal{A}) \;=\; \psi_h\left(\mathcal{S}\left(\psi_h^{-1}(\mathcal{A})\right)\right).$$

By Lemma 2.4 (ii) and the inclusions (33) we have $\mathcal{S}\big(\psi_h^{-1}\big(\mathcal{C}_h^j\big)\big) \subset V_h$ for all $j \in \{1,\dots,k\}$ and $h \in \{0,\dots,l\}$. For $j \in \{1,\dots,k\}$ we can therefore recursively define compact subsets of $\mathcal{U}_h$ by

$$\begin{aligned} \mathcal{S}_l^j &= \mathcal{S}\left(\mathcal{C}_l^j\right), \\ \mathcal{S}_h^j &= \mathcal{S}\left(\mathcal{C}_h^j \setminus \bigcup_{g=h+1}^{l} \mathcal{S}_g^j\right), \qquad h = l-1,\dots,0. \end{aligned}$$

Then each set $M \setminus \mathcal{S}_h^j$ is connected. A component of $\mathcal{S}_h^j$ is just the saturation of a component of $\mathcal{C}_h^j$ which is not enclosed by any component of $\bigcup_{g=h+1}^{l} \mathcal{C}_g^j$. Each component $\mathcal{K}$ of $\mathcal{S}_h^j$ has piecewise smooth boundary, and according to Lemma 2.4 (ii) it contains only one h-cube $\psi_h(C)$, and

$$\psi_h^{-1}(\mathcal{K}) \subset \mathcal{N}_{\nu_h d_h}(C), \qquad h = 1,\dots,l. \tag{35}$$

While a component of $\mathcal{S}_0^j$ is a cube of $\mathcal{C}_0^j$ and a component of $\mathcal{S}_1^j$ is the union of a cube of $\mathcal{C}_1^j$ and the overlapping cubes of $\mathcal{C}_0^j$, a component of $\mathcal{S}_2^j$ might contain a cube of $\mathcal{C}_0^j$ which is disjoint from $\mathcal{C}_1^j \cup \mathcal{C}_2^j$, cf. Figure 7.

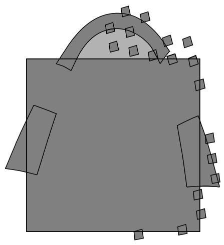

FIGURE 7. A component of $\mathcal{S}_2^j$.

If the ratios d_h/d_{h+1}, $h = 0,\dots,l-1$, are small enough, then Lemma 2.4 (ii) implies that a component of $\mathcal{C}_h^j$ cannot be enclosed by a component of $\mathcal{C}_g^j$ for some $g < h$, and so the sets $\mathcal{S}_h^j$, $h = 0,\dots,l$, are disjoint. We finally abbreviate

$$\mathcal{S}^j := \bigcup_{h=0}^{l} \mathcal{S}_h^j$$

and read off from (30) and the definition of the sets $\mathcal{S}_h^j$ that

$$M = \bigcup_{j=0}^{k} \mathcal{S}^j. \tag{36}$$

Step 4. Moving the cubes of the same colour into $\mathcal{B}_0$

In order to move the sets $\mathcal{S}^j$ into $\mathcal{B}_0$ we will have to choose the d_i yet smaller. We shall then be able to construct for each j a Hamiltonian isotopy Φ^j of M which first moves $\mathcal{S}_0^j$ to a "dense cluster" around the center of $\mathcal{B}_0$ and then successively moves $\mathcal{S}_h^j$ to a "shell" around the already constructed cluster $\bigcup_{g=0}^{h-1} \Phi^j\big(\mathcal{S}_g^j\big)$, $h = 1, \dots, l$.
The main tool for the construction of the maps Φ^j is the following elementary lemma.

Lemma 2.5. *Let K be a compact subset of $\mathbb{R}^{2n}$ and let q be a point in $\mathbb{R}^{2n}$. Denote by $\mathcal{K}$ the convex hull of the union $K \cup (q+K)$. For any open neighbourhood U of $\mathcal{K}$ there exists a symplectomorphism τ of $\mathbb{R}^{2n}$ which is supported in U and which translates K to $q+K$.*

Proof. We follow [11, p. 73]. We choose a smooth function $f\colon \mathbb{R}^{2n} \to \mathbb{R}$ such that $f|_{\mathcal{K}} = 1$ and $f|_{\mathbb{R}^{2n}\setminus U} = 0$. Define the Hamiltonian function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ by

$$H(z) \;=\; f(z)\langle z, -Jq\rangle$$

where $\langle \cdot,\cdot\rangle$ denotes the Euclidean scalar product on $\mathbb{R}^{2n}$ and where J denotes the standard complex structure on $\mathbb{R}^{2n}$ defined by

$$\omega_0(z,w) \;=\; \langle z, -Jw\rangle, \qquad z,w \in \mathbb{R}^{2n}.$$

Recall that the Hamiltonian vector field X_H of H is given by $X_H(z) = J\nabla H(z)$. We conclude that the time-1-map τ of the flow generated by X_H is a symplectomorphism of $\mathbb{R}^{2n}$ which is supported in U. Moreover, for $z \in \mathcal{K}$ we have

$$X_H(z) \;=\; J\nabla H(z) \;=\; J(-Jq) \;=\; q,$$

and so $\tau(z) = z + q$ for all $z \in K$. $\square$

We denote by B_r the open ball of radius r in $\mathbb{R}^{2n}$. We recursively define the open ball B_{r_0} and the open "annuli" $A_{r_{h-1}}^{r_h} = B_{r_h} \setminus \overline{B_{r_{h-1}}}$ by

$$|B_{r_0}| \;=\; \tfrac{1}{k}\left(|V_0| + \tfrac{k-1}{l+1}\varepsilon\right), \tag{37}$$

$$\left|A_{r_{h-1}}^{r_h}\right| \;=\; \tfrac{1}{k}\left(|V_h| + \tfrac{k-1}{l+1}\varepsilon\right), \qquad h = 1,\dots,l. \tag{38}$$

The definitions (37) and (38), the identities $|V_h| = \mu(\mathcal{U}_h)$ and the estimate (18), and the estimate (11) and the identity $\left|B^{2n}(a_0')\right| = \mu(\mathcal{B}_0')$ imply that

$$\begin{aligned}
|B_{r_0}| + \sum_{h=1}^{l} \left|A_{r_{h-1}}^{r_h}\right| &= \frac{1}{k}\sum_{h=0}^{l}\left(|V_h| + \frac{k-1}{l+1}\varepsilon\right) \\
&< \frac{\mu(M)}{k} + \frac{\varepsilon}{k} + \frac{k-1}{k}\varepsilon \\
&= \frac{\mu(M)}{k} + \varepsilon \\
&< \left|B^{2n}(a_0')\right|
\end{aligned} \tag{39}$$

and so

$$B_{r_0} \cup \bigcup_{h=1}^{l} A_{r_{h-1}}^{r_h} \subset B^{2n}(a_0'). \tag{40}$$

Consider again the symplectic atlas $\mathfrak{A} = \{\psi_h \colon V_h \to \mathcal{U}_h,\ h = 0,\dots,l\}$ of (M,ω). Recall that $\psi_0 \colon V_0 \to \mathcal{U}_0$ is the Darboux chart $\varphi_0 \colon B^{2n}(a_0) \to \mathcal{B}_0$ and that the sets $\mathcal{U}_h$ and V_h are connected and have piecewise smooth boundaries. Also recall that there exist larger charts $\widetilde{\psi}_h \colon \widetilde{V}_h \to \widetilde{\mathcal{U}}_h$. We can assume that the sets $\widetilde{\mathcal{U}}_h$ and $\widetilde{V}_h$ are also connected and have piecewise smooth boundaries. We fix $j \in \{1,\dots,k\}$. The construction of the map Φ_0^j will somewhat differ from the one of the maps Φ_h^j for $h \geq 1$ since $\Phi_0^j(\mathcal{S}_0^j)$ will not be disjoint from $\mathcal{S}_0^j$. We start with constructing Φ_0^j.

Proposition 2.6. *If the numbers $d_0,\dots,d_l > 0$ as well as the ratios d_i/d_{i+1}, $i = 0,\dots,l-1$, are small enough, then there exists a symplectomorphism Φ_0^j of M whose support is disjoint from $\bigcup_{h=1}^{l}\mathcal{S}_h^j$ and such that $\Phi_0^j(\mathcal{S}_0^j) \subset \psi_0(B_{r_0})$.*

Proof. We recall that $\mathcal{S}_0^j$ is the set of "free" cubes in $\mathcal{C}_0^j$, i.e., each component of $\mathcal{S}_0^j$ is a cube of $\mathcal{C}_0^j$ which is not enclosed by any component of $\bigcup_{h=1}^{l}\mathcal{C}_h^j$. We abbreviate $\mathfrak{S}_0 := \psi_0^{-1}(\mathcal{S}_0^j)$. Since $\mathfrak{S}_0$ is contained in $\mathfrak{C}_0^j(d_0)$, we deduce from the estimate (29) for $i = 0$ and the definition (37) that

$$|\mathfrak{S}_0| < |B_{r_0}|. \tag{41}$$

We denote by $\mathfrak{Q}$ the standard decomposition of $\mathbb{R}^{2n}$ into closed cubes,

$$\mathfrak{Q} := \bigcup_{v\in\mathbb{Z}^{2n}} v + [0,1]^{2n},$$

and for each $\nu > 0$ and each subset A of $\mathbb{R}^{2n}$ we denote by $\mathfrak{Q}(\nu, A)$ the set of cubes in $\nu\mathfrak{Q}$ which are contained in A. Let s_0 be the number of cubes in $\mathfrak{S}_0$. The estimate (41) implies that after choosing $d_0 > 0$ smaller if necessary we find $\varepsilon_0 > 0$ such that $\mathfrak{Q}(d_0 + \varepsilon_0, B_{r_0})$ contains at least s_0 cubes.

Recall that $k \geq 2n+1$ and recall from the estimate (39) that $r_0 < \sqrt{a_0'/\pi}$. We define $\widetilde{r}_0 > r_0$ by

$$\widetilde{r}_0 \;=\; \min \left\{ \frac{2k}{4n+1} r_0,\; \frac{1}{2}\left(r_0 + \sqrt{a_0'/\pi} \right) \right\} \tag{42}$$

and we denote by $\mathfrak{S}_0^{\text{int}}$ the set of cubes in $\mathfrak{S}_0$ contained in $B_{\widetilde{r}_0}$. Since $B_{\widetilde{r}_0} \subset B^{2n}(a_0')$ and since $\mathcal{B}_0' = \psi_0\left(B^{2n}(a_0')\right)$ is disjoint from $\mathcal{U}_h$ and $\mathcal{S}_h^j \subset \mathcal{U}_h$, $h \geq 1$, the set $B_{\widetilde{r}_0}$ is disjoint from $\psi_0^{-1}\left(\mathcal{S}_h^j\right)$, $h \geq 1$. In particular, $\mathfrak{S}_0^{\text{int}}$ is the set of cubes in $\mathfrak{C}_0^j(d_0)$ contained in $B_{\widetilde{r}_0}$, cf. Figure 9. We abbreviate the set of exterior cubes in $\mathfrak{S}_0$ by

$$\mathfrak{S}_0^{\text{ext}} \;:=\; \mathfrak{S}_0 \setminus \mathfrak{S}_0^{\text{int}}.$$

Lemma 2.7. *For d_0 and ε_0 small enough there exists a symplectomorphism θ of $\widetilde{V}_0$ such that*

(i) *the support of θ is contained in $B_{\widetilde{r}_0}$ and disjoint from $\mathfrak{S}_0^{\text{ext}}$;*
(ii) *θ maps each cube of $\mathfrak{S}_0^{\text{int}}$ into a cube of $\mathfrak{Q}\left(d_0+\varepsilon_0, B_{r_0}\right)$;*
(iii) *the set of cubes in $\mathfrak{Q}\left(d_0+\varepsilon_0, B_{r_0}\right)$ containing a cube of $\theta\left(\mathfrak{S}_0^{\text{int}}\right)$ is contractible.*

Proof. Using Lemmata 2.3 and 2.5 we successively construct symplectomorphisms $\theta_{2n}, \theta_{2n-1}, \dots, \theta_1$ such that θ_{2n} “collapses” $\mathfrak{S}_0^{\text{int}}$ along the x_{2n}-axis and θ_i “collapses” $\theta_{i+1} \circ \cdots \circ \theta_{2n}\left(\mathfrak{S}_0^{\text{int}}\right)$ along the x_i-axis, $i = 2n-1, \dots, 1$, and such that the composite map

$$\theta \;=\; \theta_1 \circ \cdots \circ \theta_{2n}$$

meets assertion (i) as well as assertions (ii) and (iii) with $\mathfrak{Q}\left(d_0+\varepsilon_0, B_{r_0}\right)$ replaced by $\mathfrak{Q}\left(d_0+\varepsilon_0, B_{\widetilde{r}_0}\right)$, cf. Figure 8.

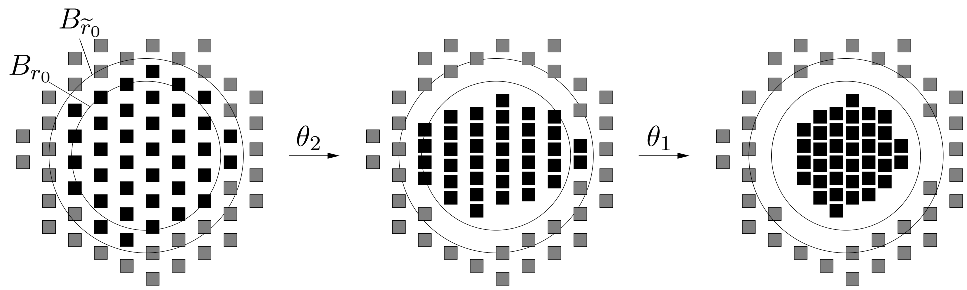

FIGURE 8. The map $\theta = \theta_1 \circ \theta_2$ for $j = 1$.

In order to see that assertions (ii) and (iii) can be fulfilled as stated, we infer from the definition of the set $d_0\mathfrak{C}^j(2n,k) \supset \mathfrak{S}_0^{\text{int}}$ given in Step 2 that

$$\frac{\operatorname{diam} \mathfrak{S}_0^{\text{int}}}{\operatorname{diam} \theta\left(\mathfrak{S}_0^{\text{int}}\right)} \;\to\; \frac{k}{2n} \qquad \text{as } d_0 \to 0 \text{ and } \varepsilon_0 \to 0.$$

In view of the choice (42) of $\widetilde{r}_0$ we can therefore choose d_0 and ε_0 so small that $\theta\left(\mathfrak{S}_0^{\text{int}}\right) \subset \mathfrak{Q}\left(d_0+\varepsilon_0, B_{r_0}\right)$, as desired. □

Lemma 2.8. *If the numbers $d_0, \dots, d_l > 0$ as well as the ratios d_i/d_{i+1}, $i = 0, \dots, l-1$, are small enough, then there exists a symplectomorphism Θ_0 of $\widetilde{V}_0$ such that*

(i) *the support of Θ_0 is compact and disjoint from*

$$\psi_0^{-1}\left(\bigcup_{h=1}^{l} \mathcal{S}_h^j\right) \cup \theta\left(\mathfrak{S}_0^{\text{int}}\right);$$

(ii) *Θ_0 maps each cube of $\mathfrak{S}_0^{\text{ext}}$ into a cube of $\mathfrak{Q}\left(d_0+\varepsilon_0, B_{r_0}\right)$.*

Proof. The set $\mathcal{U}_0 \setminus \bigcup_{h=1}^{l} \mathcal{S}_h^j$ might not be connected for any choice of $d_0, \dots, d_l$, in which case not every cube in $\mathcal{S}_0^j$ can be moved into $\psi_0\left(B_{r_0}\right)$ inside $\mathcal{U}_0 \setminus \bigcup_{h=1}^{l} \mathcal{S}_h^j$. This is the reason why we work in the extended chart $\widetilde{\psi}_0 \colon \widetilde{V}_0 \to \widetilde{\mathcal{U}}_0$. We choose the numbers $d_0, \dots, d_l$ so small that each component of $\bigcup_{h=1}^{l} \mathcal{S}_h^j$ which intersects $\mathcal{U}_0$ is contained in $\widetilde{\mathcal{U}}_0$. The component $\widehat{\mathcal{U}}_0$ of $\widetilde{\mathcal{U}}_0 \setminus \bigcup_{h=1}^{l} \mathcal{S}_h^j$ containing $\mathcal{B}_0'$ then contains $\mathcal{S}_0^j$, and the set $\widehat{V}_0 \widetilde{\psi}_0^{-1}\big(\widehat{\mathcal{U}}_0\big)$ is an open connected set with piecewise smooth boundary which contains $\mathfrak{S}_0$.

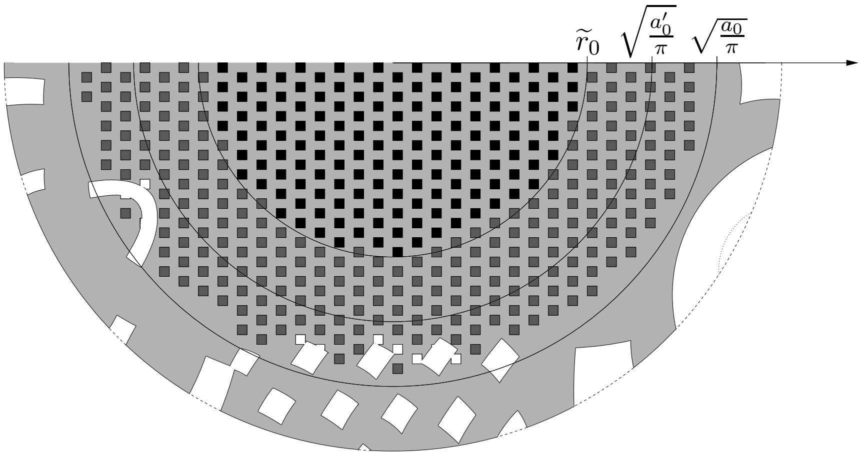

FIGURE 9. Half of the subset $\mathfrak{S}_0 = \mathfrak{S}_0^{\text{int}} \cup \mathfrak{S}_0^{\text{ext}}$ of $\widehat{V}_0$.

In order to move the cubes in $\mathfrak{S}_0^{\text{ext}}$ into B_{r_0} we shall associate a tree with $\mathfrak{S}_0^{\text{ext}}$. Recall that $\mathfrak{S}_0^{\text{ext}}$ is a subset of $d_0\mathfrak{C}^j(2n,k)$. We enlarge $\mathfrak{S}_0^{\text{ext}}$ to the set $\widehat{\mathfrak{S}}_0^{\text{ext}}$ defined as the set of cubes in $d_0\mathfrak{C}^j(2n,k) \setminus \mathfrak{S}_0^{\text{int}}$ which are contained in $\widehat{V}_0$. Abbreviate

$$\lambda_m := \begin{cases} k & \text{if } m = 1, \\ 2n-m+2 & \text{if } m \in \{2, \dots, 2n\}. \end{cases}$$

We say that two cubes C and C' of $\widehat{\mathfrak{S}}_0^{\text{ext}}$ are *m-neighbours* if

$$C' \;=\; C \pm d_0 \lambda_m e_m$$

for some $m \in \{1, \dots, 2n\}$ and if their convex hull is contained in $\widehat{V}_0$. According to Lemma 2.3 (i) the (interior of) the convex hull of two neighbours does not intersect any third cube of $\widehat{\mathfrak{S}}_0^{\text{ext}}$, cf. Figure 5. We define $\mathcal{G}_0'$ to be the graph whose edges are the straight lines connecting the centers of neighbours in $\widehat{\mathfrak{S}}_0^{\text{ext}}$, and we define $\mathcal{G}_0$ to be the graph obtained from $\mathcal{G}_0'$ by declaring the intersections of edges to be vertices, cf. Figure 10.

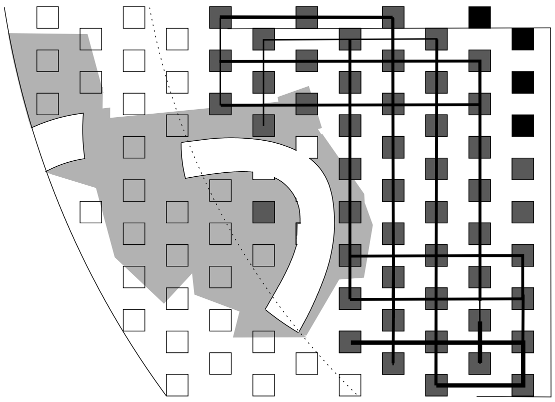

FIGURE 10. Part of the graph $\mathcal{G}_0$ associated with $\widehat{\mathfrak{S}}_0^{\text{ext}}$.

Since $\widehat{V}_0$ is an open connected relatively compact set with piecewise smooth boundary, we can choose d_0 so small that the graph $\mathcal{G}_0$ is connected. Choosing d_0 yet smaller if necessary, we can also assume that

$$\sqrt{2n}\, d_0 \;<\; \frac{\widetilde{r}_0 - r_0}{2} \tag{43}$$

and that the convex hull of any two neighbours in $\widehat{\mathfrak{S}}_0^{\text{ext}}$ is contained in $\widehat{V}_0 \setminus \overline{B_{r_0}}$. We then in particular have that $\mathfrak{S}_0^{\text{ext}}$ is disjoint from $\overline{B_{r_0}}$. Let C_1 be a cube of $\mathfrak{S}_0^{\text{ext}}$ whose distance to B_{r_0} is minimal. We choose a maximal tree $\mathcal{T}_0$ in $\mathcal{G}_0$ which is rooted at the center of C_1. Denote a vertex of $\mathcal{T}_0$ represented by the center of a cube C of $\mathfrak{S}_0^{\text{ext}}$ by $v(C)$ and write $\prec$ for the partial ordering on $\mathfrak{S}_0^{\text{ext}}$ induced by $\mathcal{T}_0$. We number the s_0^{ext} many cubes in $\mathfrak{S}_0^{\text{ext}}$ in such a way that

$$v(C_i) \prec v(C_{i'}) \implies i < i'. \tag{44}$$

We finally recall that $\mathfrak{Q}\,(d_0 + \varepsilon_0, B_{r_0})$ contains at least s_0 cubes. In view of Lemma 2.7 (iii) we can therefore choose cubes $Q_1, \dots, Q_{s_0^{\text{ext}}}$ from the set $\mathfrak{Q}(d_0 + \varepsilon_0, B_{r_0}) \setminus \theta\left(\mathfrak{S}_0^{\text{int}}\right)$ in such a way that each of the sets

$$\theta\left(\mathfrak{S}_0^{\text{int}}\right) \cup \bigcup_{g=1}^{i} Q_i, \tag{45}$$

$i = 1, \dots, s_0^{\text{ext}}$, is contractible.

We are now in a position to move the cubes of $\mathfrak{S}_0^{\text{ext}}$ into B_{r_0}. We shall successively move C_i into Q_i, $i = 1, \dots, s_0^{\text{ext}}$. Define $\widehat{r}_0 \in \,]r_0, \widetilde{r}_0[$ by $\widehat{r}_0 := (r_0 + \widetilde{r}_0)/2$. In view of the assumption (43) we can then estimate the diameter of a cube in $\mathfrak{S}_0^{\text{ext}}$ by

$$\sqrt{2n}\, d_0 \; < \; \widehat{r}_0 - r_0. \tag{46}$$

We first use Lemma 2.5 to construct a symplectomorphism ϑ_1 of $\widetilde{V}_0$ whose support is contained in $\widehat{V}_0$ and disjoint from

$$\bigcup_{g=2}^{s_0^{\text{ext}}} C_g \,\cup\, \theta\left(\mathfrak{S}_0^{\text{int}}\right)$$

and which maps C_1 into Q_1. Indeed, since C_1 is a cube of $\mathfrak{S}_0^{\text{ext}}$ closest to B_{r_0} and in view of the estimate (46) we can first move C_1 into the annulus $B_{\widehat{r}_0} \setminus B_{r_0}$ without touching $\bigcup_{g \geq 2} C_g$, and in view of Lemma 2.7 (iii) we can then move the image cube along a piecewise linear path inside $B_{\widehat{r}_0} \setminus B_{r_0}$ to a position from which it can be moved into B_{r_0} to its preassigned cube Q_1 without touching $\theta\left(\mathfrak{S}_0^{\text{ext}}\right)$.

Assume now by induction that we have already constructed symplectomorphisms ϑ_g which moved the cubes C_g into the cubes Q_g for $g = 1, \dots, i-1$. We are going to construct a symplectomorphism ϑ_i of $\widetilde{V}_0$ whose support is contained in $\widehat{V}_0$ and disjoint from

$$\bigcup_{g=i+1}^{s_0^{\text{ext}}} C_g \,\cup\, \bigcup_{g=1}^{i-1} Q_g \,\cup\, \theta\left(\mathfrak{S}_0^{\text{int}}\right) \tag{47}$$

and which maps C_i into Q_i. Let γ be the piecewise linear path from $v(C_i)$ to $v(C_1)$ determined by the tree $\mathcal{T}_0$. Because of (44), all the cubes of $\mathfrak{S}_0^{\text{ext}}$ on γ except C_i have already been moved into B_{r_0}. Using Lemmata 2.3 (i) and 2.5 we can therefore move C_i along γ to (the "former locus" of) C_1 without touching $\bigcup_{g \geq i+1} C_g$. More precisely, let σ be a segment of γ, i.e., σ is a straight line which is parallel to a coordinate axis and connects two vertices v and v' of $\mathcal{G}_0$. Let R be the convex hull of the cubes C_v and $C_{v'}$ congruent to C_i and centered at v and v', respectively. In view of Lemma 2.3 (i), the closed rectangle R either is disjoint from $\bigcup_{g \geq i+1} C_g$ or it touches some cubes C_j, $j \geq i+1$, along a face. In the first case, we can directly apply Lemma 2.5 to move C_v to $C_{v'}$ without touching $\bigcup_{g \geq i+1} C_g$. In the second case, we first move the touching cubes C_j a bit away from R, then move C_v to $C_{v'}$, and then move the displaced cubes back to their former locus, cf. Figure 11. We can do this in such a way that the support of the resulting map τ_σ which translates C_v to $C_{v'}$ is disjoint from $\bigcup_{g \geq i+1} C_g$. Since R is contained in $\widehat{V}_0 \setminus \overline{B_{r_0}}$ we can also arrange that the support of τ_σ is contained in $\widehat{V}_0 \setminus \overline{B_{r_0}}$. Composing the maps τ_σ corresponding to the segments of γ we obtain a symplectomorphism τ_i whose support is contained in $\widehat{V}_0$ and disjoint from the

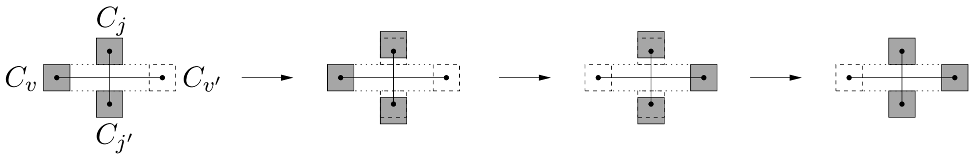

FIGURE 11. How to move C_v to $C_{v'}$ along a path blocked by C_j and $C_{j'}$.

set (47) and which maps C_i to C_1. Since the set (45) is contractible, we can now proceed as in the construction of ϑ_1 and construct a symplectomorphism ϑ_i which moves the image of C_i at C_1 into Q_i without touching the set (47). The composition $\vartheta_i \circ \tau_i$ is as desired.

After all, the composite map

$$\Theta_0 \;=\; \left(\vartheta_{s_0^{\text{ext}}} \circ \tau_{s_0^{\text{ext}}}\right) \circ \cdots \circ (\vartheta_2 \circ \tau_2) \circ \vartheta_1$$

is a symplectomorphism of $\widetilde{V}_0$ which meets assertions (i) and (ii). □

Let θ and Θ_0 be the symplectomorphisms guaranteed by Lemmata 2.7 and 2.8. The symplectomorphism

$$\widetilde{\psi}_0 \circ \Theta_0 \circ \theta \circ \widetilde{\psi}_0^{-1}$$

of $\widetilde{\mathcal{U}}_0$ smoothly extends by the identity to a symplectomorphism Φ_0^j of M whose support is disjoint from $\bigcup_{h=1}^{l} \mathcal{S}_h^j$ and such that $\Phi_0^j\big(\mathcal{S}_0^j\big) \subset \psi_0\left(B_{r_0}\right)$. The proof of Proposition 2.6 is complete. □

Proposition 2.9. *If the numbers $d_0, \dots, d_l > 0$ as well as the ratios d_i/d_{i+1}, $i = 0, \dots, l-1$, are small enough, then there exists for each $h = 1, \dots, l$ a symplectomorphism Φ_h^j of M whose support is disjoint from*

$$\bigcup_{g=0}^{h-1} \Phi_g^j\big(\mathcal{S}_g^j\big) \;\cup\; \bigcup_{g=h+1}^{l} \mathcal{S}_g^j$$

and such that $\Phi_h^j\big(\mathcal{S}_h^j\big) \subset \psi_0\big(A_{r_{h-1}}^{r_h}\big)$.

Proof. We first explain the construction of Φ_1^j. Recall from the end of Step 3 that $\mathcal{S}_1^j \subset \mathcal{U}_1$ is the union of those components of $\mathcal{C}_1^j$ which are not enclosed by any component of $\bigcup_{h=2}^{l} \mathcal{C}_h^j$. Each component $\mathcal{K}$ consists of a 1-cube $\psi_1(C)$ and some overlapping cubes of $\mathcal{C}_0^j$, and

$$\psi_1^{-1}(\mathcal{K}) \;\subset\; \mathcal{N}_{\nu_1 d_1}(C) \;\subset\; V_1.$$

For any cube C of $\mathfrak{C}_1^j(d_1)$ we denote by C^{ν_1} the closed cube of width $(1 + 2\nu_1)d_1$ concentric to C. This is the smallest closed cube containing the neighbourhood $\mathcal{N}_{\nu_1 d_1}(C)$ of C. We abbreviate

$$\mathfrak{S}_1 \,:=\, \bigcup C^{\nu_1}$$

where the union is taken over those cubes C of $\mathfrak{C}_1^j(d_1)$ that lie in $\psi_1^{-1}(\mathcal{S}_1^j)$. In view of the choice (31) the cubes C^{ν_1} are disjoint. Since the compact subset $\psi_1^{-1}(\mathcal{S}_1^j)$ of V_1 is disjoint from the compact subset $\psi_1^{-1}\left(\bigcup_{h\geq 2}\mathcal{S}_h^j\right)$ of $\overline{V_1}$, we can choose $\nu_1 > 0$ (and for this $d_0 > 0$) so small that $\mathfrak{S}_1$ is disjoint from $\psi_1^{-1}\left(\bigcup_{h\geq 2}\mathcal{S}_h^j\right)$. Since for each cube C^{ν_1} in $\mathfrak{S}_1$ the cube C belongs to $\mathfrak{C}_1^j(d_1)$, we read off from the estimate (32) for $i=1$ and the definition (38) for $h=1$ that

$$|\mathfrak{S}_1| \,<\, \left|A_{r_0}^{r_1}\right|. \tag{48}$$

Let s_1 be the number of cubes in $\mathfrak{S}_1$. The estimate (48) implies that after choosing $d_1 > 0$ and $\nu_1 > 0$ smaller if necessary we find $\varepsilon_1 > 0$ such that $\mathfrak{Q}\left((1+2\nu_1)d_1+\varepsilon_1, A_{r_0}^{r_1}\right)$ contains at least s_1 cubes.

We next choose the numbers $d_0,\dots,d_l$ so small that each component of $\bigcup_{h=2}^l \mathcal{S}_h^j$ which intersects $\mathcal{U}_1$ is contained in $\widetilde{\mathcal{U}}_1$. The component $\widehat{\mathcal{U}}_1$ of $\widetilde{\mathcal{U}}_1 \setminus \bigcup_{h=2}^l \mathcal{S}_h^j$ containing $\mathcal{B}_0'$ then contains $\mathcal{S}_0^j$, and the set $\widehat{V}_1 \widetilde{\psi}_1^{-1}(\widehat{\mathcal{U}}_1)$ is an open connected set with piecewise smooth boundary which contains $\mathfrak{S}_1$.

We enlarge $\mathfrak{S}_1^{\text{ext}}$ to the set $\widehat{\mathfrak{S}}_1^{\text{ext}}$ defined as the set of cubes in $d_1\mathfrak{C}^j(2n,k)$ which are contained in $\widehat{V}_1$. □

In order to complete the proof of Theorem 2 we choose $d_0,\dots,d_l > 0$ such that the conclusions of Propositions 2.6 and 2.9 hold for each $j \in \{1,\dots,k\}$, and we define the symplectomorphism Φ^j of M by

$$\Phi^j \,=\, \Phi_h^j \circ \cdots \circ \Phi_1^j \circ \Phi_0^j.$$

In view of Propositions 2.6 and 2.9 and the inclusion (40) we then have

$$\begin{aligned}
\Phi^j\left(\mathcal{S}^j\right) \;&=\; \Phi^j\left(\bigcup_{h=0}^l \mathcal{S}_h^j\right)\\
&=\; \bigcup_{h=0}^l \Phi_h^j\left(\mathcal{S}_h^j\right)\\
&\subset\; \psi_0\left(B^{2n}(a_0')\right)\\
&\subset\; \mathcal{B}_0.
\end{aligned}$$

This and the identity (36) imply that the k Darboux charts

$$\left(\Phi^j\right)^{-1} \circ \varphi_0 \colon B^{2n}(a) \,\to\, M$$

cover M. The proof of Theorem 2 is finally complete, and so Theorem 1 is also proved. □

3. Variations of the theme

Consider again a closed $2n$-dimensional symplectic manifold (M,ω). In the symplectic packing problem, one usually considers packings of (M,ω) by *equal* balls, see [10, 22, 35, 1, 2, 30]. In analogy to this we study the number

$$\mathrm{S}_{\mathrm{B}}^{=}(M,\omega) \;=\; \min\{k \mid M = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k\}$$

where now each $\mathcal{B}_i$ is the symplectic image $\varphi_i\left(B^{2n}(a)\right)$ of *the same ball* .

Theorem 3.1. *Let (M,ω) be a closed $2n$-dimensional symplectic manifold. Then Theorem 1 holds with $\mathrm{S}_{\mathrm{B}}(M,\omega)$ replaced by $\mathrm{S}_{\mathrm{B}}^{=}(M,\omega)$.*

Proof. In the proof of Theorem 1 we have covered (M,ω) by equal balls and have thus proved Theorem 1 with $\mathrm{S}_{\mathrm{B}}(M,\omega)$ replaced by $\mathrm{S}_{\mathrm{B}}^{=}(M,\omega)$. □

Clearly,

$$\mathrm{S}_{\mathrm{B}}(M,\omega) \;\le\; \mathrm{S}_{\mathrm{B}}^{=}(M,\omega). \tag{49}$$

For every $a>0$ we denote by $\operatorname{Emb}(B(a),M)$ the space of symplectic embeddings of $\left(\overline{B^{2n}(a)},\omega_0\right) \hookrightarrow (M,\omega)$ endowed with the C^∞-topology.

Corollary 3.2. *Assume that $\lambda(M,\omega) \ge 2n+1$ or that $\operatorname{Emb}(B(a),M)$ is path-connected for all $a>0$. Then $\mathrm{S}_{\mathrm{B}}(M,\omega) = \mathrm{S}_{\mathrm{B}}^{=}(M,\omega)$.*

Proof. If $\lambda(M,\omega) \ge 2n+1$, then Theorem 1 and Theorem 3.1 yield $\mathrm{S}_{\mathrm{B}}(M,\omega) = \lambda(M,\omega)$ and $\mathrm{S}_{\mathrm{B}}^{=}(M,\omega) = \lambda(M,\omega)$.

Assume now that $\operatorname{Emb}(B(a),M)$ is path-connected for all $a>0$, and choose $k = \mathrm{S}_{\mathrm{B}}(M,\omega)$ symplectic embeddings $\varphi_i \colon \overline{B^{2n}(a_i)} \hookrightarrow M$ such that $M = \bigcup_{i=1}^k \varphi_i\left(B^{2n}(a_i)\right)$. We choose $\varepsilon>0$ so small that

$$M = \bigcup_{i=1}^k \varphi_i\left(B^{2n}(a_i-\varepsilon)\right),$$

and set $a_i' = a_i - \varepsilon$. We can assume that $a_1' = \max_i a_i'$. The identity $\mathrm{S}_{\mathrm{B}}(M,\omega) = \mathrm{S}_{\mathrm{B}}^{=}(M,\omega)$ follows from

Lemma 3.3. *For each $i \ge 2$ there exists a symplectic embedding*

$$\widetilde{\varphi}_i \colon B^{2n}\left(a_1'\right) \hookrightarrow M$$

such that $\widetilde{\varphi}_i|_{B^{2n}(a_i')} = \varphi_i|_{B^{2n}(a_i')}$.

Proof. By assumption, there exists a smooth family of symplectomorphisms $\varphi_i^t \colon B^{2n}(a_i) \hookrightarrow M$ such that

$$\varphi_i^0 = \varphi_1|_{B^{2n}(a_i)} \quad \text{and} \quad \varphi_i^1 = \varphi_i.$$

Consider the subsets

$$A = \bigcup_{t\in[0,1]} \{t\} \times \varphi_i^t\left(B^{2n}(a_i)\right) \quad \text{and} \quad B = \bigcup_{t\in[0,1]} \{t\} \times \varphi_i^t\left(B^{2n}(a_i')\right)$$

of $[0,1]\times M$. Since each set $\varphi_i^t\left(B^{2n}(a_i)\right)$ is contractible, there exists a smooth time-dependent Hamiltonian function $H\colon A\to\mathbb{R}$ generating the symplectic isotopy $\varphi_i^t\circ\left(\varphi_i^0\right)^{-1}:\varphi_1\left(B^{2n}(a_i)\right)\to M$. By Whitney's Theorem there exists a smooth function $f\colon[0,1]\times M\to[0,1]$ such that $f=1$ on B and $f=0$ on $M\setminus A$. Let Φ be the time-1-map $M\to M$ of the flow generated by Hamiltonian fH. Then

$$\Phi \;=\; \varphi_i^1\circ\left(\varphi_i^0\right)^{-1} \qquad \text{on } \;\varphi_1\left(B^{2n}(a_i')\right).$$

We define the embedding $\widetilde{\varphi}_i\;:=\;\Phi\circ\varphi_1|_{B^{2n}(a_i)}\colon B^{2n}(a_i)\hookrightarrow M$ and find that on $B^{2n}(a_i')$ we have

$$\widetilde{\varphi}_i \;=\; \Phi\circ\varphi_1 \;=\; \varphi_i^1\circ\left(\varphi_i^0\right)^{-1}\circ\varphi_1 \;=\; \varphi_i^1\circ\varphi_1^{-1}\circ\varphi_1 \;=\; \varphi_i^1.$$

The proof of Lemma 3.3 is complete, and so Corollary 3.2 is also proved. □

The spaces $\operatorname{Emb}(B(a),M)$ are known to be path-connected for all $a>0$ for $n=1$ and for a class of symplectic 4-manifolds containing (blow-ups of) rational and ruled manifolds, see [21]. No closed symplectic manifold is known for which $\operatorname{Emb}(B(a),M)$ is not path-connected for some $a>0$. We thus ask

Question 3.4. Is it true that $\mathrm{S_B}(M,\omega)=\mathrm{S}_{\mathrm{B}}^{\equiv}(M,\omega)$ for every closed symplectic manifold (M,ω)?

We next study the "symplectic Lusternik–Schnirelmann category" $\mathrm{S}(M,\omega)$ defined as

$$\mathrm{S}(M,\omega) \;=\; \min\{k \mid M=\mathcal{U}_1\cup\dots\cup\mathcal{U}_k\}$$

where each $\mathcal{U}_i$ is the image $\varphi_i\left(U_i\right)$ of a symplectic embedding $\varphi_i\colon U_i\to\mathcal{U}_i\subset M$ of a bounded subset U_i of $\left(\mathbb{R}^{2n},\omega_0\right)$ diffeomorphic to the open ball in $\mathbb{R}^{2n}$.

Theorem 3.5. *Let (M,ω) be a closed $2n$-dimensional symplectic manifold. Then $\mathrm{S}(M,\omega)\le 2n+1$.*

Theorem 3.5 will follow from a stronger result dealing with coverings by displaceable sets. We say that a subset $\mathcal{U}$ of M is *displaceable* if there exists an autonomous Hamiltonian function $H\colon M\to\mathbb{R}$ whose time-1-map φ_H displaces $\mathcal{U}$, i.e., $\varphi_H\left(\mathcal{U}\right)\cap\mathcal{U}=\emptyset$. Define the invariant $\mathrm{S_{dis}}(M,\omega)$ as

$$\mathrm{S_{dis}}(M,\omega) \;=\; \min\{k \mid M=\mathcal{U}_1\cup\dots\cup\mathcal{U}_k\}$$

where each $\mathcal{U}_i$ is as in the definition of the invariant $\mathrm{S}(M,\omega)$ and is in addition displaceable. Coverings by such subsets $\mathcal{U}_i$ play a role in the recent construction of Calabi quasimorphisms on the group of Hamiltonian diffeomorphisms of (M,ω) in [6], see also [3].

Theorem 3.6. *Let (M,ω) be a closed $2n$-dimensional symplectic manifold. Then $\mathrm{S_{dis}}(M,\omega)\le 2n+1$.*

Of course, $\mathrm{B}(M) \le \mathrm{S}(M,\omega) \le \mathrm{S}_{\mathrm{dis}}(M,\omega)$. Theorem 3.6 thus implies Theorem 3.5, and Proposition 1 and Theorem 3.6 yield

$$n+1 \le \mathrm{cl}(M)+1 \le \operatorname{cat} M \le \mathrm{B}(M) \le \mathrm{S}(M,\omega) \le \mathrm{S}_{\mathrm{dis}}(M,\omega) \le 2n+1$$

and $\mathrm{B}(M) = \mathrm{S}(M,\omega) = \mathrm{S}_{\mathrm{dis}}(M,\omega) = 2n+1$ if $[\omega]|_{\pi_2(M)} = 0$. For the 2-sphere we have $2 = \mathrm{S}\left(\mathbb{S}^2\right) < \mathrm{S}_{\mathrm{dis}}\left(\mathbb{S}^2\right) = 3$.

Question 3.7. Is it true that $\mathrm{B}(M) = \mathrm{S}(M,\omega)$ for every closed symplectic manifold (M,ω)?

Proof of Theorem 3.6: Theorem 3.6 is a consequence of the construction in the previous section and the following

Proposition 3.8. *For every $\varepsilon > 0$ there exists a symplectic embedding $\psi\colon (U,\omega_0) \hookrightarrow (M,\omega)$ of a bounded subset U of $\mathbb{R}^{2n}$ diffeomorphic to a ball such that $\psi(U)$ is displaceable and*

$$|U| > \frac{\mu(M)}{2} - \varepsilon.$$

Indeed, choose $\varepsilon > 0$ so small that

$$\frac{\mu(M)}{2} - \varepsilon > \frac{\mu(M)}{2n+1}.$$

For the set $\psi(U) \subset M$ guaranteed by Proposition 3.8 we then have

$$\mu\left(\psi(U)\right) > \frac{\mu(M)}{2n+1}.$$

Repeating the construction in the proof of Theorem 2.1 with the ball $\mathcal{B} = \varphi\left(B^{2n}(a)\right)$ replaced by $\psi(U)$ and with $k = 2n+1$, we find a covering $\bigcup \mathcal{U}_i$ of M by $2n+1$ domains $\mathcal{U}_i \subset M$ which are diffeomorphic to balls and displaceable.

Proof of Proposition 3.8: We fix $\varepsilon > 0$. Let $k \in \mathbb{N}$ and $d > \delta > 0$. For $j \in \mathbb{N} \cup \{0\}$ we denote by ξ_{jd} the translation by jd in the x_1-direction and by $\eta_{-d/2}$ the translation by $-d/2$ in the y_1-direction. Consider the open subsets $C_j(d) = \xi_{2j}\left(\eta_{-d/2}\left(]0,d[^{2n}\right)\right)$ and

$$\mathcal{N}(k,d,\delta) = \coprod_{j=0}^{k} C_j(d) \cup \left(]0,(2k+1)d[\times]-\delta,\delta[^{2n-1}\right)$$

of $(\mathbb{R}^{2n},\omega_0)$.

Figure 12 shows a set $\mathcal{N}(k,d,\delta) \subset \mathbb{R}^{2n}$ for $k=1$.
According to [31, Section 6.1] there exist k, d and δ and a symplectic embedding $\psi\colon \mathcal{N}(k,d,\delta) \hookrightarrow (M,\omega)$ such that

$$\left|\coprod_{j=0}^{k} C_j(d)\right| > \mu(M) - \varepsilon. \tag{50}$$

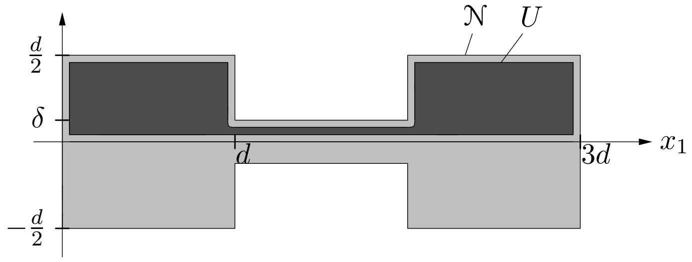

FIGURE 12. The sets $\mathcal{N}$ and U for $k = 1$.

Set $\mathcal{N}^+(k,d,\delta) = \mathcal{N}(k,d,\delta) \cap \{y_1 > 0\}$, and denote by $\partial\mathcal{N}^+(k,d,\delta)$ the boundary of this set. For $\nu > 0$ we set

$$U_\nu = \left\{z \in \mathcal{N}^+(k,d,\delta) \mid \operatorname{dist}\left(z, \partial\mathcal{N}^+(k,d,\delta)\right) > \nu\right\},$$

For $\nu < \delta$ the set U_ν is connected and diffeomorphic to a ball. In view of (50) we can choose $\nu < \delta$ so small that

$$|U_\nu| > \frac{\mu(M)}{2} - \varepsilon.$$

For such a choice of k, d, δ and ν we abbreviate $\mathcal{N} = \mathcal{N}(k,d,\delta)$ and $U = U_\nu$. We shall construct a Hamiltonian isotopy φ_t of $\mathbb{R}^{2n}$ which is generated by an autonomous Hamiltonian function with support in $\mathcal{N}$ and such that $\varphi_1(U) \cap U = \emptyset$. The autonomous Hamiltonian diffeomorphism Φ of (M,ω) defined by

$$\Phi(z) = \begin{cases} \psi \circ \varphi_1 \circ \psi^{-1}(z) & \text{if } z \in \psi(\mathcal{N}) \\ z & \text{if } z \notin \psi(\mathcal{N}) \end{cases}$$

then displaces $\psi(U)$. In order to construct the Hamiltonian isotopy φ_t, we choose a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ such that on $]0,(2k+1)d[$ the graph of f is contained in $\pi(\mathcal{N})$ and lies above $\pi(U)$. Then the Hamiltonian function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ defined by

$$H(x_1, y_1, x_2, \dots, y_n) = -\int_0^{x_1} f(s)\,ds$$

generates the isotopy

$$\phi_t\colon (x_1, y_1, x_2, \dots, y_n) \mapsto (x_1, y_1 - tf(x_1), x_2, \dots, y_n), \quad t \in [0,1],$$

which satisfies $\phi_t(U) \subset \mathcal{N}$ for all $t \in [0,1]$ and $\phi_1(U) \cap U = \emptyset$. Choose now a smooth function $h\colon \mathbb{R}^{2n} \to [0,1]$ which is equal to 1 on $\bigcup_{t\in[0,1]} \phi_t(U)$ and vanishes outside $\mathcal{N}$. The Hamiltonian isotopy φ_t generated by the Hamiltonian function hH is then as required. $\square$

4. Proof of Proposition 1

Since (M, ω) is symplectic, $[\omega]^n \neq 0$, and so $n+1 \leq \mathrm{cl}(M)+1$. The first statement in Proposition 1 follows from this estimate and from (6).

A main ingredient in the remainder of the proof is the following theorem of W. Singhof, who thoroughly studied the relation between $\mathrm{B}(M)$ and $\operatorname{cat} M$.

Theorem 4.1. (Singhof, [33, Corollary (6.4)]) *Let M^m be a closed smooth p-connected manifold with $n \geq 4$ and* $\operatorname{cat} M \geq 3$. *Then*

(a) $\mathrm{B}(M) \,=\, \operatorname{cat} M \quad$ *if* $\operatorname{cat} M \,\geq\, \dfrac{m+p+4}{2(p+1)}$;

(b) $\mathrm{B}(M) \,\leq\, \left\lceil \dfrac{m+p+4}{2(p+1)} \right\rceil \quad$ *if* $\operatorname{cat} M \,<\, \dfrac{m+p+4}{2(p+1)}$.

(*Here, $\lceil x \rceil$ denotes the minimal integer which is greater than or equal to x.*)

Notice that if we consider only symplectic manifolds, the assumptions $\dim M \geq 4$ and $\operatorname{cat} M \geq 3$ in Theorem 4.1 can be dropped. Indeed, if $\dim M = 2$, it is easy to see that we are in the situation of (a) in Theorem 4.1; and if $\operatorname{cat} M = 2$, then $\frac{1}{2} \dim M \leq \mathrm{cl}(M)+1 \leq \operatorname{cat} M = 2$ yields $\dim M = 2$.

(i) If M is simply connected, (3) shows that $\operatorname{cat} M = n+1$, and since $p=1$, we are in the situation of Theorem 4.1, item (a), so $\mathrm{B}(M) = \operatorname{cat} M$.

(ii) It has been proved in [28] that $[\omega]|_{\pi_2(M)} = 0$ implies $\operatorname{cat} M = 2n+1$, and so the claim follows together with $\mathrm{B}(M) \leq 2n+1$.

(iii) As we remarked above, $\mathrm{B}(M) = \operatorname{cat} M$ if $n=1$. So let $n \geq 2$ and assume that $\mathrm{B}(M) > \operatorname{cat} M$. By (i) we have $p=0$. The claim now readily follows from Theorem 4.1. $\square$

Remarks 4.2. 1. The inequality $\mathrm{cl}(M)+1 \leq \operatorname{cat} M$ can be strict: For the Thurston–Kodaira manifold described in [23, Example 3.8] we have $\pi_2(M) = 0$ and hence $\operatorname{cat} M = 5$, but $\mathrm{cl}(M) = 3$, see [27]. More generally, $\mathrm{cl}(M)+1 < \operatorname{cat} M = \dim M + 1$ for any symplectic non-toral nilmanifold, see [29].

2. It follows from [17, Prop. 13] and [4, Prop. 3.6] that there exist closed smooth manifolds with $\operatorname{cat} M < \mathrm{B}(M)$. No symplectic examples are known, however.

Examples 4.3. 1. If $\left(M^{2n}, \omega\right)$ admits a Riemannian metric with nonnegative Ricci curvature and has infinite fundamental group, then

$$\operatorname{cat} M \,\geq\, n+1+\frac{b_1(M)}{2} \quad \text{and} \quad b_1(M) > 0,$$

see [25, Theorem 4.3]. In particular, $\operatorname{cat} M \geq n+2$, and so $\operatorname{cat} M = \mathrm{B}(M)$ by Proposition 1 (iii).

2. Assume that the homomorphism $[\omega]^{n-1} \colon H^1(M; \mathbb{R}) \to H^{2n-1}(M; \mathbb{R})$ (multiplication by the class $[\omega]^{n-1}$ is a non-zero map. Kähler manifolds with $H^1(M; \mathbb{R}) \neq 0$ have this property. Using Poincaré duality we see that $\mathrm{cl}(M) \geq n+1$, and so $n+2 \leq \operatorname{cat} M = \mathrm{B}(M)$.

5. Examples

In this section we compute or estimate the number $\mathrm{S_B}(M,\omega)$ for various closed symplectic manifolds (M,ω). In view of Theorem 1 and Proposition 1, understanding $\mathrm{S_B}(M,\omega)$ is often equivalent to understanding the Gromov width $\mathrm{Gr}(M,\omega)$. Our list of examples therefore resembles the list of manifolds whose Gromov widths are known.

We shall frequently use the following well-known fact.

Lemma 5.1. (Greene–Shiohama, [9]) *Let U and V be bounded domains in $\left(\mathbb{R}^2, dx \wedge dy\right)$ which are diffeomorphic and have equal area. Then U and V are symplectomorphic.*

1. Surfaces. A closed 2-dimensional symplectic manifold is a closed oriented surface equipped with an area form.

Corollary 5.2. *Let (Σ_g,σ) be a closed oriented surface with area form σ. Then*

$$\mathrm{S_B}(\Sigma_g,\sigma) = \begin{cases} 2 & \textit{if } g=0, \\ 3 & \textit{if } g \geq 1. \end{cases}$$

Proof. In view of Lemma 5.1 we have $\mathrm{S_B}(\Sigma_g,\sigma) = \mathrm{B}(\Sigma_g)$, and so the corollary follows in view of Proposition 1. $\square$

2. Minimal ruled 4-manifolds. As before we denote by Σ_g the closed oriented surface of genus g. Recall that there are exactly two orientable $\mathbb{S}^2$-bundles with base Σ_g, namely the trivial bundle $\Sigma_g \times \mathbb{S}^2 \to \Sigma_g$ and the nontrivial bundle $\Sigma_g \ltimes \mathbb{S}^2 \to \Sigma_g$ [23, Lemma 6.25].

a) Trivial $\mathbb{S}^2$-bundles. Fix area forms σ_{Σ_g} and $\sigma_{\mathbb{S}^2}$ of area 1 on Σ_g and $\mathbb{S}^2$, respectively. By the work of Lalonde–Mc Duff and Li–Liu every symplectic form on $\Sigma_g \times \mathbb{S}^2$ is diffeomorphic to $a\sigma_{\Sigma_g} \oplus b\sigma_{\mathbb{S}^2}$ for some $a,b>0$ (see [16]). We abbreviate $\Sigma_g(a) := (\Sigma_g, a\sigma_{\Sigma_g})$ and $\mathbb{S}^2(b) := (\mathbb{S}^2, b\sigma_{\mathbb{S}^2})$.

Corollary 5.3. *For $\mathbb{S}^2(a) \times \mathbb{S}^2(b)$ with $a \geq b > 0$ we have*

$$\mathrm{S_B}\left(\mathbb{S}^2(a) \times \mathbb{S}^2(b)\right) \begin{cases} \in \{3,4,5\} & \textit{if } \ 1 \leq \frac{a}{b} < \frac{3}{2}, \\ \in \{4,5\} & \textit{if } \ \frac{3}{2} \leq \frac{a}{b} < 2, \\ = \left\lfloor \frac{2a}{b} \right\rfloor + 1 & \textit{if } \ \frac{a}{b} \geq 2, \end{cases}$$

cf. Figure 13, *and for $\Sigma_g(a) \times \mathbb{S}^2(b)$ with $g \geq 1$ and $a,b>0$ we have*

$$\mathrm{S_B}\left(\Sigma_g(a) \times \mathbb{S}^2(b)\right) \begin{cases} \in \{4,5\} & \textit{if } \ 0 < \frac{a}{b} < 2, \\ = \left\lfloor \frac{2a}{b} \right\rfloor + 1 & \textit{if } \ \frac{a}{b} \geq 2, \end{cases}$$

cf. Figure 14.

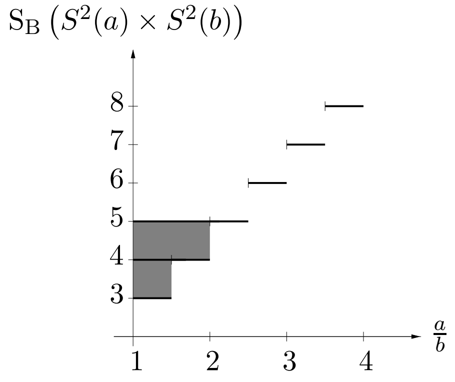

FIGURE 13. What is known about $\mathrm{S_B}\left(\mathbb{S}^2(a) \times \mathbb{S}^2(b)\right)$ and $\mathrm{S_B}\left(\mathbb{S}^2 \ltimes \mathbb{S}^2, \omega_{ab}\right)$.

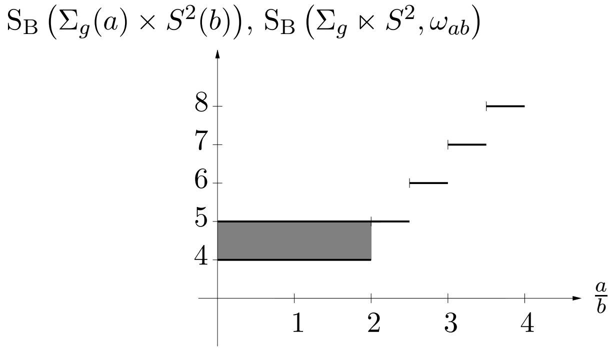

FIGURE 14. What is known about $\mathrm{S_B}\left(\Sigma_g(a) \times \mathbb{S}^2(b)\right)$ and $\mathrm{S_B}\left(\Sigma_g \ltimes \mathbb{S}^2, \omega_{ab}\right)$.

Proof. Proposition 1, item (i) yields $\mathrm{B}(\mathbb{S}^2 \times \mathbb{S}^2) = 3$. Moreover, the Non-Squeezing Theorem implies that $\mathrm{Gr}\,(\mathbb{S}^2(a) \times \mathbb{S}^2(b)) = b$, and so

$$\Gamma\left(\mathbb{S}^2(a) \times \mathbb{S}^2(b)\right) = \left\lfloor \tfrac{2a}{b} \right\rfloor + 1.$$

The first half of the corollary now follows from Theorem 1.

Applying the inequality (2) and the estimate (4) we find that $\mathrm{cat}(\Sigma_g \times \mathbb{S}^2) = 4$, and so $\mathrm{B}(\Sigma_g \times \mathbb{S}^2) = 4$ in view of Proposition 1, item (iii). Moreover, it follows from Theorem 6.1.A in [1] that

$$\Gamma\left(\Sigma_g(a) \times \mathbb{S}^2(b)\right) \;=\; \left\lfloor \max\left(1, \tfrac{2a}{b}\right) \right\rfloor + 1.$$

The second half of the corollary now follows from Theorem 1. □

b) Nontrivial $\mathbb{S}^2$-bundles. Let $A \in H_2(\Sigma_g \ltimes \mathbb{S}^2; \mathbb{Z})$ be the class of a section with self intersection number -1, and let F be the homology class of the fiber. We set $B = A + \frac{1}{2}F$. Then $\{F, B\}$ is a basis of $H_2(\Sigma_g \ltimes \mathbb{S}^2; \mathbb{R})$. For $a, b > 0$ we fix a representative ω_{ab} of the Poincaré dual of $aF + bB$. By [23, Theorem 6.27] and the work of Lalonde–Mc Duff and Li–Liu (see [16]),

1. Every symplectic form on $\mathbb{S}^2 \ltimes \mathbb{S}^2$ is diffeomorphic to ω_{ab} for some $a > \frac{b}{2} > 0$.

2. Every symplectic form on $\Sigma_g \ltimes \mathbb{S}^2$, $g \geq 1$, is diffeomorphic to ω_{ab} for some $a, b > 0$.

Corollary 5.4. *For* $(\mathbb{S}^2 \ltimes \mathbb{S}^2, \omega_{ab})$ *with* $a > \frac{b}{2} > 0$ *we have*

$$\mathrm{S_B}\left(\mathbb{S}^2 \ltimes \mathbb{S}^2, \omega_{ab}\right) \begin{cases} \in \{3,4,5\} & \text{if } \frac{1}{2} \leq \frac{a}{b} < \frac{3}{2}, \\ \in \{4,5\} & \text{if } \frac{3}{2} \leq \frac{a}{b} < 2, \\ = \left\lfloor \frac{2a}{b} \right\rfloor + 1 & \text{if } \frac{a}{b} \geq 2, \end{cases}$$

cf. Figure 13*, and for* $\left(\Sigma_g \ltimes \mathbb{S}^2, \omega_{ab}\right)$ *with* $g \geq 1$ *and* $a, b > 0$ *we have*

$$\mathrm{S_B}\left(\Sigma_g \ltimes \mathbb{S}^2, \omega_{ab}\right) \begin{cases} \in \{4,5\} & \text{if } 0 < \frac{a}{b} < 2, \\ = \left\lfloor \frac{2a}{b} \right\rfloor + 1 & \text{if } \frac{a}{b} \geq 2, \end{cases}$$

cf. Figure 14.

Proof. Since $\mathbb{S}^2 \ltimes \mathbb{S}^2$ is simply connected, $\mathrm{B}(\mathbb{S}^2 \ltimes \mathbb{S}^2) = 3$ in view of Proposition 1, item (i). Moreover, based on Biran's work [1] it has been computed in [30] that

$$\Gamma\left(\mathbb{S}^2 \ltimes \mathbb{S}^2, \omega_{ab}\right) = \left\lfloor \tfrac{2a}{b} \right\rfloor + 1.$$

The first half of the corollary now follows from Theorem 1.

Using the Leray–Hirsch Theorem, we find that $\mathrm{cl}\left(\Sigma_g \ltimes \mathbb{S}^2\right) = 3$, and so $\mathrm{cat}\left(\Sigma_g \ltimes \mathbb{S}^2\right) \geq 4$. On the other hand, $\Sigma_g \ltimes \mathbb{S}^2$ having a section, and it is not hard to see that $\mathrm{cat}\left(\Sigma_g \ltimes \mathbb{S}^2\right) \leq 4$ (cf. the proof of Proposition 3.3 in [32]). In view of Proposition 1, item (iii) we conclude that $\mathrm{B}(\Sigma_g \ltimes \mathbb{S}^2) = 4$. Moreover, it has been computed in [30] that

$$\Gamma\left(\Sigma_g \ltimes \mathbb{S}^2, \omega_{ab}\right) \;=\; \left\lfloor \max\left(1, \tfrac{2a}{b}\right) \right\rfloor + 1.$$

The second half of the corollary now follows from Theorem 1. □

3. Products of surfaces. As before we denote by Σ_g the closed oriented surface of genus g. In view of the previous example we assume $g \geq 1$. If $g = 1$ we write $T^2 = \Sigma_1$. By a theorem of Moser [24], any two area forms on Σ_g of total area a are diffeomorphic. We write $\Sigma_g(a)$ for this symplectic manifold.

Corollary 5.5.

(i) $\mathrm{S_B}\left(T^2(a) \times \Sigma_g(b)\right) = 5$ *if* $\frac{a}{b} < \frac{5}{2}$.

(ii) $\mathrm{S_B}\left(\Sigma_g(a) \times \Sigma_h(b)\right) = 5$ *if* $\frac{2}{5} < \frac{a}{b} < \frac{5}{2}$.

Proof. By Proposition 1, item (ii) we have that

$$\mathrm{B}\left(\Sigma_g \times \Sigma_h\right) \;=\; 5 \quad \text{for all } g, h \geq 1.$$

Using Lemma 5.1 we see that the discs $B^2(a)$ and $B^2(b)$ symplectically embed into $\Sigma_g(a)$ and $\Sigma_h(b)$, respectively. Therefore, the ball $B^4(\min(a,b)) \subset B^2(a) \times B^2(b)$ symplectically embeds into $\Sigma_g(a) \times \Sigma_h(b)$, and so

$$\Gamma\left(\Sigma_g(a) \times \Sigma_h(b)\right) \leq 5 \quad \text{whenever } \tfrac{2}{5} < \tfrac{a}{b} < \tfrac{5}{2}.$$

Claim (ii) now follows from Theorem 1.

We prove Claim (i) following [14]. For each $c > 0$ we consider the rectangle

$$R(c) = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < c\},$$

and the linear symplectic map

$$\begin{array}{rcl} \varphi\colon (R(c) \times R(c), \omega_0) & \to & (\mathbb{R}^2 \times \mathbb{R}^2, \omega_0) \\ (x_1, y_1, x_2, y_2) & \mapsto & (x_1 + y_2, y_1, -y_2, y_1 + x_2) \end{array}$$

where $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Let $T^2(1) = \left(\mathbb{R}^2/\mathbb{Z}^2, dx_1 \wedge dy_1\right)$ be the standard symplectic torus. Then the projection $p\colon \left(\mathbb{R}^2, dx_1 \wedge dy_1\right) \to T^2(1)$ is symplectic, and so the composition

$$(p \times id) \circ \varphi\colon R(c) \times R(c) \to T^2(1) \times \mathbb{R}^2$$

is also symplectic. It is easy to see that this map is an embedding and that

$$((p \times id) \circ \varphi)\,(R(c) \times R(c)) \;\subset\; T^2 \times]-c, 0[\times]0, c+1[.$$

In view of Lemma 5.1 the ball $B^4(c)$ symplectically embeds into $R(c) \times R(c)$, and $]-c,0[\times]0,c+1[$ symplectically embeds into $\Sigma_g(c(c+1))$. We conclude that the ball $B^4(c)$ symplectically embeds into $T^2(1) \times \Sigma_g(c(c+1))$ for each $c > 0$, i.e.,

$$\operatorname{Gr}\left(T^2(1) \times \Sigma_g(d)\right) \;\geq\; \tfrac{1}{2}\left(\sqrt{4d+1} - 1\right) \qquad \text{for each } d > 0.$$

This estimate and a computation yield

$$\Gamma\left(T^2(a) \times \Sigma_g(b)\right) \;=\; \Gamma\left(T^2(1) \times \Sigma_g\left(\tfrac{b}{a}\right)\right) \;\leq\; 5 \quad \text{whenever } \tfrac{a}{b} < \tfrac{9}{10}.$$

Now, the already proved Claim (ii) and Theorem 1 imply Claim (i). □

Remark 5.6. Assume that $g \geq 1$, $h \geq 2$ and $\frac{a}{b} \geq \frac{5}{2}$. The method used in the proof of 2) in Corollary 5.5 only yields the linear estimate

$$\mathrm{S_B}\left(\Sigma_g(a) \times \Sigma_h(b)\right) \;\leq\; \left\lfloor \tfrac{2a}{b} \right\rfloor + 1.$$

A variant of the method used in the proof of 1), however, yields the estimate

$$\mathrm{S_B}\left(\Sigma_g(a) \times \Sigma_h(b)\right) \;\leq\; C(h)\, \frac{\frac{a}{b}}{\left(\log \frac{a}{b}\right)^2}$$

where $C(h) > 0$ is a constant depending only on h (see [14]) .

4. Complex projective space. Let $\mathbb{CP}^n$ be the complex projective space and let ω_{SF} be the unique $\mathrm{U}(n+1)$-invariant Kähler form on $\mathbb{CP}^n$ whose integral over $\mathbb{CP}^1$ equals 1.

Corollary 5.7. $\mathrm{S_B}\left(\mathbb{CP}^n, \omega_{SF}\right) \;=\; n+1.$

Proof. In view of Proposition 1, we have

$$\mathrm{S_B}\left(\mathbb{CP}^n, \omega_{SF}\right) \;\geq\; B((\mathbb{CP}^n) \;\geq\; n+1.$$

On the other hand, we define for $0 \leq i \leq n$ maps $f_i \colon B^{2n}(1) \to \mathbb{CP}^n$ by

$$f_i \colon \mathbf{z} = (z_1, \dots, z_n) \;\mapsto\; \left[z_1 : \ldots : z_{i-1} : \sqrt{1-|\mathbf{z}|^2} : z_{i+1} : \ldots : z_n \right].$$

It is well known that f_i is a symplectomorphism between $B^{2n}(1)$ and $\mathbb{CP}^n \setminus S_i$, where $S_i = \{[u_1 : \ldots : u_{i-1} : 0 : u_{i+1} : \ldots : u_n]\} \cong \mathbb{CP}^{n-1}$ is the i-th coordinate hypersurface (see e.g. [15]). Since

$$\mathbb{CP}^n \;\subset\; \bigcup_{i=0}^{n} f_i\left(B^{2n}(1)\right),$$

we conclude that also $\mathrm{S_B}(\mathbb{CP}^n, \omega_{SF}) \leq n+1$, and so the corollary follows. $\square$

Remark 5.8. By a theorem of Taubes [34], any symplectic form on $\mathbb{CP}^2$ is diffeomorphic to $a\,\omega_{SF}$ for some $a \neq 0$. In view of Corollary 5.7 we thus have

$$\mathrm{S_B}(\mathbb{CP}^2, \omega) = 3 \quad \textit{for any symplectic form } \omega \textit{ on } \mathbb{C}P^2.$$

5. Complex Grassmann manifolds. Let $G_{k,n}$ be the Grassmann manifold of k-planes in $\mathbb{C}^n$, and let $\sigma_{k,n}$ be the standard Kähler form on $G_{k,n}$ normalized such that $\sigma_{k,n}$ is Poincaré dual to the generator of $H_2(G_{k,n};\mathbb{Z}) = \mathbb{Z}$. Since $(G_{n-k,n}, \sigma_{n-k,n}) = (G_{k,n}, \sigma_{k,n})$, we can assume that

$$k \in \left\{1, \dots, \left\lfloor \tfrac{n}{2} \right\rfloor \right\}.$$

We define the number $p_{k,n}$ by

$$p_{k,n} \;=\; \frac{(k-1)! \cdots 2!\, 1! \cdot (k(n-k))!}{(n-1)! \cdots (n-k+1)!\, (n-k)!}. \tag{51}$$

Notice that $p_{k,n} = \deg(p(G_{k,n}))$ where

$$p \colon G_{k,n} \hookrightarrow \mathbb{CP}^{\binom{n}{k}-1}$$

is the Plücker map [8, Example 14.7.11], and so $p_{k,n}$ is indeed an integer.

Since $(G_{1,n}, \sigma_{1,n}) = (\mathbb{CP}^{n-1}, \omega_{SF})$, we assume $k \geq 2$.

Corollary 5.9.
$\mathrm{S_B}(G_{2,4}, \sigma_{2,4}) \in \{5, \dots, 9\}$, $\quad \mathrm{S_B}(G_{2,5}, \sigma_{2,5}) \in \{7, \dots, 13\}$,
$\mathrm{S_B}(G_{2,6}, \sigma_{2,6}) \in \{15, 16, 17\}$,
$\mathrm{S_B}(G_{2,n}, \sigma_{2,n}) = p_{2,n} + 1$ *for all* $n \geq 7$,
$\mathrm{S_B}(G_{k,n}, \sigma_{k,n}) = p_{k,n} + 1$ *for all* $k \geq 3$.

Proof. Since $G_{k,n}$ is simply connected and since

$$\dim G_{k,n} \;=\; 2k(n-k), \tag{52}$$

we read off from Proposition 1, item (i) that

$$\mathrm{B}\,(G_{k,n}) \;=\; k(n-k)+1. \tag{53}$$

Moreover,

$$\mathrm{Vol}\,(G_{k,n},\sigma_{k,n}) \;=\; \frac{p_{k,n}}{(k(n-k))!} \tag{54}$$

(see [8, Example 14.7.11]), and it has been proved in [18] that

$$\mathrm{Gr}\;(G_{k,n},\sigma_{k,n}) \;=\; 1.$$

Therefore,

$$\Gamma\,(G_{k,n},\sigma_{k,n}) \;=\; p_{k,n}+1. \tag{55}$$

The corollary now follow from the identities (52), (53) and (55), Theorem 1 and a straightforward computation. □

References

[1] P. Biran. Symplectic packing in dimension 4. *Geom. Funct. Anal.* **7** (1997) 420–437.

[2] P. Biran. A stability property of symplectic packing. *Invent. Math.* **136** (1999) 123–155.

[3] P. Biran, M. Entov and L. Polterovich. Calabi quasimorphisms for the symplectic ball. *Commun. Contemp. Math.* **6** (2004), no. 5, 793–802.

[4] O. Cornea, G. Lupton, J. Oprea, D. Tanré,. *Lusternik-Schnirelmann category.* Mathematical Surveys and Monographs, **103**. American Mathematical Society, Providence, RI, 2003. .

[5] R. Engelking. *Dimension theory.* North-Holland Mathematical Library **19**. North-Holland Publishing Co., Amsterdam-Oxford-New York; PWN—Polish Scientific Publishers, Warsaw, 1978.

[6] M. Entov and L. Polterovich. Calabi quasimorphism and quantum homology. *Int. Math. Res. Not.* 2003, no. 30, 1635–1676.

[7] S. Froloff and L. Elsholz. Limite inférieure pour le nombre des valeurs critiques d'une fonction, donnée sur une variété. *Mat. Sbornik* **42(5)** (1935) 637-643.

[8] W. Fulton. *Intersection Theory.* Second edition. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, 1998.

[9] R. Greene and K. Shiohama. Diffeomorphisms and volume preserving embeddings of non-compact manifolds. *Trans. Amer. Math. Soc.* **255** (1979) 403-414.

[10] M. Gromov. Pseudo-holomorphic curves in symplectic manifolds. *Invent. math.* **82** (1985) 307-347.

[11] H. Hofer and E. Zehnder. *Symplectic Invariants and Hamiltonian Dynamics.* Birkhäuser 1994.

[12] I. M. James. On Category, in the sense of Lusternik and Schnirelman. *Topology* **17** (1978) 331-348.

[13] I. M. James. Lusternik–Schnirelmann category. In *Handbook of algebaric topology.* Ed. by I. James, Elsevier Science B. V., Amsterdam, 1995, 1293-1310.

[14] M.-Y. Jiang. Symplectic embeddings from $\mathbf{R}^{2n}$ into some manifolds. *Proc. Roy. Soc. Edinburgh Sect. A* **130** (2000), 53–61.

[15] Y. Karshon. Appendix to [22].

[16] F. Lalonde and D. Mc Duff. J-curves and the classification of rational and ruled symplectic 4-manifolds. Contact and symplectic geometry (Cambridge, 1994), 3–42, *Publ. Newton Inst.* **8**, Cambridge Univ. Press, Cambridge, 1996.

[17] P. Lambrechts, D. Stanley and L. Vandembroucq. Embeddings up to homotopy of two-cones in Euclidean space. *Trans. Amer. Math. Soc.* **354** (2002) 3973–4013.

[18] G. Lu. Gromov-Witten invariants and pseudo symplectic capacities. *Israel J. math* (to appear). Preprint `dg-ga 0103195`.

[19] E. Luft. Covering manifolds with open discs. *Illinois J. Math.* **13** (1969) 321-326.

[20] L. Lusternik and L. Schnirelmann. *Méthodes Topologiques dans les Problèmes Variationelles*. Hermann, Paris 1934.

[21] D. Mc Duff. From symplectic deformation to isotopy. Topics in symplectic 4-manifolds (Irvine, CA, 1996), 85–99, *First Int. Press Lect. Ser., I,* Internat. Press, Cambridge 1998.

[22] D. Mc Duff and L. Polterovich. Symplectic packings and algebraic geometry. *Invent. math.* **115** (1994) 405–429.

[23] D. Mc Duff and D. Salamon. *Introduction to Symplectic Topology.* Second edition. Oxford Mathematical Monographs, Clarendon Press, Oxford University Press, New York 1998.

[24] J. Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.* **120** (1965) 286-294.

[25] J. Oprea. Category bounds for nonnegative Ricci curvature manifolds with infinite fundamental group. *Proc. Amer. Math. Soc.* **130** (2002) 833–839.

[26] P. P. Osborne and J. L. Stern. Covering manifolds with cells. *Pacific. J. Math.* **30** (1969) 201-207.

[27] Yu. B. Rudyak. On analytical applications of stable homotopy (the Arnold conjecture, critical points). *Math. Z.* **230** (1999) 659–672.

[28] Yu. B. Rudyak and J. Oprea. On the Lusternik–Schnirelmann Category of Symplectic Manifolds and the Arnold Conjecture *Math. Z.* **230** (1999) 673–678.

[29] Yu. Rudyak and A. Tralle. On symplectic manifolds with aspherical symplectic form. *Topol. Methods Nonlinear Anal.* **14** (1999) 353–362.

[30] F. Schlenk. Packing symplectic manifolds by hand. To appear in *J. Symplectic Geom.*

[31] F. Schlenk. Embedding problems in symplectic geometry. De Gruyter Expositions in Mathematics. Walter de Gruyter Verlag, Berlin. 2005.

[32] W. Singhof. Generalized higher order cohomology operations induced by the diagonal mapping. *Math. Z.* **162** (1978), 7–26.

[33] W. Singhof. Minimal coverings of manifolds with balls. *Manuscripta Math.* **29** (1979), 385-415.

[34] C. Taubes. $SW \Rightarrow Gr$: from the Seiberg-Witten equations to pseudo-holomorphic curves. *J. Amer. Math. Soc.* **9(3)** (1996) 845-918.

[35] L. Traynor. Symplectic packing constructions. *J. Differential Geom.* **42** (1995) 411–429.

[36] E. C. Zeeman, The Poincaré conjecture for $n \geq 5$. *Topology of 3-manifolds and related topics*. Prentice Hall, Englewood Cliffs, N.J., 1962, 198-204.

(Yu. B. Rudyak) University of Florida, Department of Math., 358 Little Hall, PO Box 118105, Gainesville, FL 32611-8105, USA

E-mail address: `rudyak@math.ufl.edu`

(F. Schlenk) Mathematisches Institut, Universität Leipzig, 04109 Leipzig, Germany

E-mail address: `schlenk@math.uni-leipzig.de`