A REMARK ON TAUBES' GRAFTING PROCEDURE OF INSTANTONS

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Introduction

In this short note we explain a feasible variation of the grafting procedure introduced by Taubes in [T] showing the existence of anti–self–dual (ASD) connections on a smooth compact oriented negative–definite Riemannian 4–manifold X. The method he used is to, put in roughly, glue a rescaled standard instanton I_λ of radius $\lambda << 1$ on $\mathbb{R}^4$ to the trivial flat connection θ on X before deforming such a connection to an ASD one. This procedure involves cutting off the connection I_λ on an $O(\sqrt{\lambda})$–annulus about the origin of $\mathbb{R}^4$. As speculated by Donaldson however, a smaller annulus of radius some fixed large multiple $N\lambda$ of λ might already suffice the business. The reason backing this idea is as follows. The field strength of I_λ diminishes considerably outside $N\lambda$–balls $B_{N\lambda} \subset \mathbb{R}^4$ for $N >> 0$ and consequently the damage incurred by the cutoff procedure to the ASD connection I_λ could be so small that the framework developed in [T] at least in principle would apply. Our goal here is to explain this is indeed the case and give a modification of [T] that suit the purpose.

This consideration is motived by the problem regarding how close two particle–like "instantons" on X can stay away from each other maintaining their particle–like character. Our discussion here improves the allowable closeness from $0(\sqrt{\lambda})$ to $0(\lambda)$ in this respect. As will be discussed in [M], this helps the understanding of bubbling off "multi–instantons" at a single point of X and hence the compactification of Yang–Mills moduli spaces.

Since our modified construction does not require new techniques, it would be enough for us just to sketch the idea of modification leaving Taubes' original construction to [T] or the exposition [L]. Those interested would not find it too difficult to account for the missing details in this discussion.

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§1 A brief review of Taubes' grafting procedure

We begin with a quick review of the method developed in [T] to solve the anti–self–duality equation in some situation. Let P be an SU(2)–bundle over X and ad P the associated adjoint bundle. Given a connection A on P one can define in a usual way exterior derivatives

$$d_A : \Omega^p(\text{ad } P) \longrightarrow \Omega^{p+1}(\text{ad } P)$$

acting on smooth sections of p–form on X with values in ad P. By the splitting $\Omega^2 \simeq \Omega^2_+ \oplus \Omega^2_-$ of 2–forms into self–dual and anti–self–dual parts Ω^2_+ and Ω^2_- respectively, we can associate to A an elliptic self–adjoint operator

$$d_A^+ d_A^* : \Omega^2_+(\text{ad } P) \longrightarrow \Omega^2_+(\text{ad } P)$$

where d_A^+ denotes $d_A : \Omega^1(\text{ad } P) \longrightarrow \Omega^2(\text{ad } P)$ followed by the projection $\Omega^2 \longrightarrow \Omega^2_+$ while d_A^* is the adjoint of d_A^+. Let

$$\mu(A) = \inf_{u \in \Omega^2_+(\mathrm{ad}\,P)} \frac{\|d^*_A u\|_{L^2}}{\|u\|_{L^2}} \geq 0$$

be the first eigenvalue of $d^+_A d^*_A$ and for those A with $\mu(A) > 0$ we define

(1.1) $$\zeta(A) = \mu(A)^{-1/2}(1 + \mu(A) + \|F_+(A)\|^3_{L^3})^{1/2} \text{ and}$$

(1.2) $$\delta(A) = \|F_+(A)\|_{L^2} + \zeta(A)\|F_+(A)\|_{L^{4/3}}(1 + \|F(A)\|_{L^4})$$

where $F_+(A)$ denotes the self–dual part of the curvature field $F(A)$ of A.

(1.3) Theorem (Taubes) There is a small constant $\epsilon > 0$ such that if $\delta(A) < \epsilon$ then we can solve the anti–self–duality equation

$$F_+(A + d^*_A u) = 0$$

for some $u \in \Omega^2_+(\mathrm{ad}\,P)$ with $\|d^*_A u\|_{L^2_1} \leq \text{const.}\ \delta(A)$.

Note that this theorem is proved by an iterative scheme whose validity relies on whether or not the number $\delta(A)$ is small enough.

Theorem (1.3) is good enough for a family of connections $\{A_\lambda\}_{\lambda<<1}$ on X obtained by a grafting procedure as follows. Recall first on $\mathbb{R}^4$ the standard ASD connection I after suitable rescaling gives an ASD connection I_λ whose curvature has precisely one half of its field strength gathered on the λ–ball $B_\lambda(o) = \{v \in \mathbb{R}^4 : |v| \leq \lambda\}$.

Identifying such $B_\lambda(\underline{o})$ with geodesic λ–balls $B_\lambda(x_0)$ for some $x_0 \in X$ we can define a connection A_λ on X by specifying A_λ to be trivial outside geodesic ball $B_r(x_0)$ for $r >> \sqrt{\lambda}$ while in $B_r(x_0)$ we put

$$(1.4) \qquad A_\lambda = d + \beta_{\sqrt{\lambda}} \cdot I_\lambda$$

for some cutoff function $\beta_{\sqrt{\lambda}}(x) = \beta(|x - x_0| / \sqrt{\lambda})$ on X where β is a smooth function on $\mathbb{R}^+ = \{y \in \mathbb{R} : y \geq 0\}$ satisfying

$$\beta(y) = \begin{cases} 1 & \text{if } y \leq 1 \\ \text{decreasing} & \text{if } 1 \leq y \leq 2 \\ 0 & \text{if } 2 \leq y . \end{cases}$$

For such connections A_λ on X Taubes obtains

(1.5) $\mu(A_\lambda) \geq \text{const} > 0$,

(1.6) $\|F(A_\lambda)\|_{L^p} \leq \text{const.}\ \lambda^{4/p-2}$, and

(1.7) $\|F_+(A_\lambda)\|_{L^p} \leq \text{const.}\ \lambda^{4/p}$.

Now we may apply Theorem (1.3) to solve $F_+(A_\lambda + d^*_{A_\lambda} u_\lambda) = 0$ for some small $u_\lambda \in L^2_1(\Omega^2_+(\text{ad } P))$ and thereby conclude the existence of particle–like instantons $A_\lambda + d^*_{A_\lambda} u_\lambda$ on X. Notice that in this line of argument the estimates (1.6), (1.7) have never been used whatsoever in establishing Theorem (1.3) and this is the point that we shall exploit.

§2 A modification of Taubes' grafting procedure

As explained in the introduction we are interested to know if Taubes' iterative scheme works for a slightly different family of grafted–in connection $\{\tilde{A}_\lambda\}_{\lambda>>1}$ defined by just about the same method of constructing A_λ differing only in that in place of (1.4) we put

$$\tilde{A}_\lambda = d + \beta_{N\lambda} \cdot I_\lambda, \quad N >> 0, \tag{2.1}$$

on small geodesic balls of X. For this kind of connections we can deduce firstly

$$\mu(\tilde{A}_\lambda) \geq \text{const.} > 0 \tag{2.2}$$

by a small modification of the argument used in showing (1.5) and secondly that

$$\|F(\tilde{A}_\lambda)\|_{L^p} \leq \text{const.}\ \lambda^{4/p-2}, \tag{2.3}$$

$$\|F_+(\tilde{A}_\lambda)\|_{L^p} \leq \text{const.}\ N^{4/p-4}\lambda^{4/p-2}. \tag{2.4}$$

However, in the failure of having a good control on the term $\|F_+(\tilde{A}_\lambda)\|_{L^3}$, we are not able to deduce from these estimates that $\delta(\tilde{A}_\lambda)$ in (1.2) would necessarily be small. This hampers a direct application of Theorem (1.3) to such situations and motivates us to look for other alternatives.

Concerning this problem we find here certain variation of Taubes' iterative scheme will do the business. To be more precise we apply the iterative process rather than

Theorem (1.3) to the family of connections $\{\tilde{A}_\lambda\}$. Taking into account of the estimates (2.3) and (2.4) in the process then, we find the iteration can proceed provided only that

$$\gamma(\tilde{A}_\lambda) = \|F_+(\tilde{A}_\lambda)\|_{L^2} + \|F_+(\tilde{A}_\lambda)\|_{L^{4/3}} \cdot \|F(\tilde{A}_\lambda)\|_{L^4}$$

is small. This (weaker) requirement poses little difficulty for $\tilde{A}_\lambda$ to fulfil as

$$\gamma(\tilde{A}_\lambda) \leq \text{const.}\ (\frac{1}{N^2} + \frac{\lambda}{N} \cdot \frac{1}{\lambda}) \leq \text{const.}\ \frac{1}{N}$$

can be made arbitrarily small for $N >> 0$. Thus we may draw the following desired conclusion for $\tilde{A}_\lambda$ in parallel to Theorem (1.3).

(2.5) Proposition For $\lambda << 1$ and $N >> 0$ we can solve

$$F_+(\tilde{A}_\lambda + d^*_{\tilde{A}_\lambda} \tilde{u}_\lambda) = 0$$

for some $\tilde{u}_\lambda \in \Omega^2_+(\text{ad } P)$ with $\|d^*_{\tilde{A}_\lambda} \tilde{u}_\lambda\|_{L^2_1} \leq \text{const.}\ \gamma(\tilde{A}_\lambda)$.

The proof of this proposition can be settled by arguments used in [T] after suitable modifications and we leave the details to the reader. The main point is to make use of estimates (2.3) and (2.4) in the iterative scheme to do away with the involvement of $\|F_+(\tilde{A}_\lambda)\|_{L^3}$, the term that invalidates the application of Theorem (1.3) to $\tilde{A}_\lambda$.

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