What are G-functions ?

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INTRODUCTION. G-functions appeared in Siegel's paper [17] about diophantine approximation, and led in the context to an extensive literature (see [2] for a small list). Classically, they are convergent Taylor series $y = \sum_{n \geq 0} a_n x^n$, with rational coefficients a_n such that the common denominator of $a_0, a_1, \ldots, a_n$ grows at most geometrically in n. To our eyes, their interest arises mainly from the following conjecture (anonymous, but "in the air"):

CONJECTURE: G-functions which satisfy linear homogeneous differential equations with coefficients in $\mathbb{Q}(x)$ come from geometry. The last expression means that such a function satisfies some differential equation which belongs to the smallest class stable by standard constructions (subfactor, $\otimes$, ...) and extension, containing all Picard-Fuchs equations associated to the cohomology of algebraic varieties over $\mathbb{Q}(x)$.

The present paper is only a presentation of G-functions, and will form the first chapter of a forthcoming book on these topics. We first define (over number fields) three basic invariants of formal power series: the size σ, the stable size τ, and the global radius ρ, and give their yoga. The local-to-global presentation we have adopted is inspired from [2]. We then turn to examples: rational functions, diagonals, polylogarithms and generalized hypergeometric functions. For all these examples the conjecture holds true, although the latter case offers a non-trivial test (we settle it using Lefschetz's theorem). We try to precise the numerical invariants ρ,σ in these examples. Our presentation of diagonals is inspired by Christol's [5]. At last we gather some "pathologies".

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NOTATIONS

GENERAL NOTATIONS. $\mathbb{N}$ is the set of natural numbers; $\mathbb{Z}$ (resp. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) is the ring (resp. the field) of integers (resp. of rational numbers, of real numbers , of complex numbers). If p is a prime number, $\mathbb{F}_p$ denotes the prime field $\mathbb{Z}/_{p\mathbb{Z}}$ and $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) the ring of p-adic integers (resp. the field of p-adic rational numbers). For $t \in \mathbb{R}$, we shall write $\log^+ t$ for $\log \mathrm{Max}(1,t)$; one has $\log^+ t_1 t_2 \leq \log^+ t_1 + \log^+ t_2$. We denote by $[t]$ the integral part of t : $[t] \in \mathbb{Z}$, $[t] \leq t < [t] + 1$. We denote by $\overline{\lim}$ (resp. $\underline{\lim}$) the upper (resp. lower) limit of a sequence of real numbers. If f,g are two functions of a real variable, with $g \geq 0$, we write $f = O(g)$ if there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all sufficiently large x ; we write $f = o(g)$ (resp. $f \sim g$) if $\lim_{x\to\infty} f(x)/_{g(x)} = 0$ (resp. 1).

PLACES

Symbols:

$\overline{\mathbb{Q}}$	a fixed algebraic closure of the field of rational numbers,
K	a number field; that is to say, a subfield of $\overline{\mathbb{Q}}$ which is a finite extension of $\mathbb{Q}$,
$\mathcal{O}_K$	the ring of integers in K ,

$d = [K:\mathbb{Q}]$	the degree of K over $\mathbb{Q}$,
Σ or $\Sigma(K)$	the set of all places of K ,
Σ_f (resp. Σ_∞)	the subset of finite (resp. infinite) places,
$v\|p$ or $p = p(v)$	v lies above the place p of $\mathbb{Q}$,
K_v	a completion of K with respect to $v \in \Sigma$,
$d_v = [K_v:\mathbb{Q}_{p(v)}]$	the local degree at $v \in \Sigma$; one has $d = \sum_{u\|p} d_v$.

Normalization:

$\| \ \|_v$	the absolute value in K_v normalized in the following way:

$$|p(v)|_v = p(v)^{-d_v/d} \quad \text{if} \quad v \in \Sigma_f \quad \text{(ultrametric case)},$$

$$|\xi|_v = |\xi|^{d_v/d} \quad \text{if} \quad v \in \Sigma_\infty \quad \text{(Archimedean case), where}$$

$\| \ \|$	denotes the Euclidean absolute value on K_v , for $v \in \Sigma_\infty$,
$\mathbb{C}_v$	a completion of an algebraic closure of K_v ; $\| \ \|_v$ extends to $\mathbb{C}_v$,

$i_v : K \hookrightarrow \mathbb{C}_v$ or K_v the natural imbedding.

Remarks:

The symbol $\sum_v$ will denote a summation with all $v \in \Sigma(K)$.

For any finite extension K' of K , any $\zeta \in K$ and $v \in \Sigma(K)$, one has $|\zeta|_v = \prod_{\substack{w \in \Sigma(K') \\ w|v}} |\zeta|_{K',w}$, and all factors have the same value; see [15].

RINGS. Let R be a commutative entire ring with unit. We shall use the following entire rings (with standard operations):

- $R[x]$ the polynomial ring over R ; more generally,
- $R[\underline{x}]$ the polynomial ring in several commuting indeterminates $\underline{x} = (x_1,\dots,x_\nu)$ over R ,
- $R(x)$ the fraction field of $R[x]$,
- $R[[x]]$ the ring of formal powers series over R ,
- $R((x))$ the fraction field of $R[[x]]$,
- $M_\mu(R)$ the ring of square matrices of size μ over R ; we shall identify $M_\mu(R((x)))$ with $M_\mu(R)((x))$,
- I or I_μ its unit,
- $\binom{Y}{n}$ for $Y \in M_\mu(R)$, $\binom{Y}{n} = (n!)^{-1} Y(Y-I)\dots(Y-(n-1)I)$ whenever $n!$ is invertible in R .

We shall also denote by $M_{\mu,\nu}(R)$ the abelian group of matrices with μ rows, ν columns, whose entries belong to R. For $Y \in M_{\mu,\nu}(R)$, we shall denote by ${}_{ij}Y \in R$ the (i,j)-entry of Y. Let us assume that R is a field. For $Y \in M_{\mu,\nu}(R((x)))$, we shall denote by $Y_n \in M_{\mu,\nu}(R)$ the coefficient of x^n in Y, and by ${}_{ij}Y_n \in R$ the coefficient of x^n in ${}_{ij}Y \in R((x))$. For $Y,Z \in M_{\mu,\nu}(R((x)))$, the Hadamard product $Y_* Z \in M_{\mu,\nu}(R((x)))$ is defined by ${}_{ij}(Y_* Z)_n = {}_{ij}Y_n \cdot {}_{ij}Z_n$. Then $(M_{\mu,\nu}(R((x))),+,*)$ is a (non entire) ring with unit; the entries of its unit are $\frac{1}{1-x} \in R((x))$.

What are G-functions ?

§ 1. HEIGHTS AND SIZES

1.1 Height of algebraic numbers [18]

Let $\zeta \in \bar{\mathbb{Q}}$ an algebraic number, lying in some number field K. If $\zeta \neq 0$, the following "product formula" holds:

$$\sum_{v \in \Sigma(K)} \log|\zeta|_v = 0 \ .$$

The (logarithmic absolute) height of ζ is defined to be

$$\sum_{v \in \Sigma(K)} \log^+|\zeta|_v =: h(\zeta) \ .$$

Thanks to our normalizations $h(\zeta)$ depends only on ζ but not on K. Thus the height is well-defined over $\bar{\mathbb{Q}}$. Let $p = a_0\prod(x-\zeta_i) \in \mathbb{Z}[x]$ the minimal polynomial of ζ over $\mathbb{Z}$. Then the so-called Mahler measure of ζ, defined as $M(\zeta) := |a_0| \prod \mathrm{Max}(1,\zeta_i)$, is related to the height via the formula

$$\begin{aligned}[\mathbb{Q}(\zeta) : \mathbb{Q}]\, h(\zeta) &= \log M(\zeta) \\ &= \int_0^1 \log|p(e^{2\pi t\sqrt{-1}})|dt \quad \text{(Jensen's formula)} \\ &= \overline{\lim_{n\to\infty}} \frac{1}{n} \log|\mathrm{Resultant}(p, \sum_{i=0}^{n} x^i)| \quad \text{(Langevin's formula)} \ .\end{aligned}$$

For a finite family $(A_k)_k$ of matrices, such that all entries belong to K, we set

$$h((A_k)_k) := \sum_{v\in\Sigma(K)} \log^r \max_{i,j,k} |{}_{ij}A_k|_v .$$

Once again, this quantity does not depend on the choice of the number field which contains the entries ${}_{ij}A_k$ of the A_k's . The following classical inequality holds:

$$h(AB) \leq h(A) + h(B) + \log \nu , \text{ for any}$$
$$A \in M_{\mu,\nu}(\bar{\mathbb{Q}}) , B \in M_{\nu,\rho}(\bar{\mathbb{Q}}) .$$

1.2. Height of polynomials

Let $Y \in M_{\mu,\nu}(\bar{\mathbb{Q}}[x])$, $Y = \Sigma Y_n x^n$. We write as usual $\deg Y = \text{Max}\ \{n\ /\ Y_n \neq 0\}$ for $Y \neq 0$. We shall set:

$$h(Y) := (1 + \deg Y)^{-1}\ h((Y_n)_n) .$$

1.3 Height of formula power series; G-functions

Let $Y \in M_{\mu,\nu}(\bar{\mathbb{Q}}[[x]])$, $Y = \sum_{n\geq c} Y_n x^n$. We denote by $Y \leq N$ the truncated series $\sum_{n=0}^{N} Y_n x^n \in M_{\mu,\nu}(\bar{\mathbb{Q}}[x])$. We set:

$$h(Y) := \overline{\lim_{N\to\infty}}\ h(Y \leq N) .$$

This is a well-defined quantity in $[0,\infty]$. One checks immediately that this definition reduces to the previous one when Y has only finitely many (actually $1 + \deg Y$) non-zero coefficients.

DEFINITION. A G-function is a formal power series y , whose coefficients belong to some number field and whose height $h(y)$ is finite.

EXPLANATION. This is equivalent to the classical definition (Siegel [17]): $y = \sum_{n\geq 0} y_n x^n \in K[[x]]$ is a G-function if and only if

i) for every $v \in \Sigma_\infty$; $\sum_{n\geq 0} i_v(y_n)x^n \in \mathbb{C}_v[[x]]$ defines an analytic function around 0 ,

ii) there exists a sequence of natural integers $(d_n)_{n\in\mathbb{N}}$ which grows at most geometrically, such that $d_n y^m \in \mathcal{O}_k$ for $m = c,\ldots,n$. This equivalence will be proved in 2.3.

1.4 Size of Laurent series

Let $Y \in M_{\mu,\nu}(\bar{\mathbb{Q}}((x)))$, $Y = \sum_{n\geq -N} Y_n x^n$. We set:

$$\sigma(Y) := \begin{cases} 0 & \text{if } Y \text{ is a Laurent polynomial (i.e. if almost all coefficients are } 0 \text{)} \\ h(x^N Y) & \text{otherwise .} \end{cases}$$

One checks immediately that this definition depends only on Y , and not on N . The generalization to the case of a finite family of matrices is immediate.

We shall also use constantly the convenient notation:

$$h_{v,n}(Y) := \frac{1}{n} \operatorname*{Max}_{\substack{i\leq\mu \\ j\leq\nu \\ k\leq n}} \log^+ |{}_{ij}Y_k|_v \text{ ; here } v \text{ denotes a place}$$

of some number field K which contains the coefficients ${}_{ij}Y_k$ of the (i,j) - entries of Y for $i \leq \mu$, $j \leq \nu$, $k \leq n$.

However the non-negative real number $\sum_v h_{v,n}(Y)$ does not depend on the choice of K (by the remark made in the index of notations).

LEMMA 1. $\sigma(Y) = \overline{\lim_{n\to\infty}} \sum_v h_{v,n}(Y)$.

Proof: if $Y \in M_{\mu,\nu}(\bar{\mathbb{Q}}[x,1/x])$, we clearly have $\lim_{n\to\infty} \sum_v h_{v,n}(Y) = 0$, so that it is enough to assume that the sequence $(1/_{\varphi(l)})_{l\geq 0}$ of non-zero coefficients of Y is infinite. We then have

$$\sigma(Y) = \overline{\lim_{l\to\infty}}\ 1/_{\varphi(l)}\cdot h(Y_0,\ldots,Y_{\varphi(l)}) = \overline{\lim_{l\to\infty}} \frac{1}{\varphi(l)} \sum_v \operatorname*{Max}_{\substack{i\leq\mu\\ j\leq\nu\\ k\leq\varphi(l)}} \log^+|{}_{ij}Y_k|_v$$

$$= \overline{\lim_{n\to\infty}} \sum_v \frac{1}{n} \operatorname*{Max}_{\substack{i\leq\mu\\ j\leq\nu\\ m\leq n}} \log^+|{}_{ij}Y_n|_v \ .$$

□

REMARK. We could everywhere replace the indexing set of summation $\Sigma(K)$ by Σ_f (resp. Σ_∞). Denoting by h_f, σ_f (resp. h_∞, σ_∞) the corresponding notions - finite (resp. infinite) part of the height or size - the above proof shows that $\sigma_f(Y) = \overline{\lim_{n\to\infty}} \sum_{v_i\Sigma_p} h_{v,n}(Y)$. Assume that all coefficients of the entries of Y lie in a fixed number field K . Let d_n the common denominator in $\mathbb{N} \setminus \{0\}$ of the entries of $Y_0,\ldots,Y_n$. One has $\sigma_f(Y) \leq \log \overline{\lim_{n\to\infty}} d_n^{1/n} \leq d\sigma_f(Y)$. The elementary proof is omitted.

LEMMA 2. Let $Y \in M_{\mu,\nu}(\bar{\mathbb{Q}}((x)))$.

a) $\mathrm{Max}_{i,j}\, \sigma({}_{ij}Y) \leq \sigma(Y) = \sigma(\zeta Y) \leq \sum_{i,j} \sigma({}_{ij}Y)$, for any $\zeta \in \bar{\mathbb{Q}}$,

b) $\sigma(d/dx\ Y) \leq \sigma(Y)$, for any $n \in \mathbb{N}$,

c) if the residue Y_{-1} of Y vanishes, $\sigma(\int_0^x Y) \leq \sigma(Y) + 1$,

d) for $\zeta \in \bar{\mathbb{Q}}$, set $Y_{(\zeta)} := \Sigma\, Y_n \zeta^n x^n$. Then $\sigma(Y_{(\zeta)}) \leq \sigma(Y) + h(\zeta)$.

Let $(Y_{[k]})_{k=1}^{N}$ a subset of $M_{\mu,\nu}(\bar{\mathbb{Q}}((x)))$, then:

e) $\sigma(\Sigma\, Y_{[k]}) \leq \sigma((Y_{[k]})_k) \leq \Sigma\sigma(Y_{[k]})$,

f) $\sigma({*}Y_{[k]}) \leq \Sigma\sigma(Y_{[k]})$,

g) if $\mu = \nu$, $\sigma(\prod Y_{[k]}) \leq (1+\log N)\sigma((Y_{[k]})_k)$.

Proof: the proof of a,b,d,e,f is straightforward, using lemma 1. Let us prove c): by direct computation, we find

$$h_{v,n}(\int_0^x Y) \leq \begin{cases} h_{v,n}(Y) & \text{if } v \in \Sigma_\infty \\ h_{v,n}(Y) + \frac{1}{n} \mathrm{Max}_{m \leq n} \log |m|_v^{-1} & \text{if } v \in \Sigma_f , \end{cases}$$

so that $\sigma(\int_0^x Y) \leq \sigma(Y) + \overline{\lim}\, \frac{1}{n} \log \mathrm{G.C.M.}(1,2,\ldots,n)$, and the inequality c) follows from Ichebyshev's theorem. In order to prove g), we use a trick introduced in this context by Shidlovski (see Galochkin [12], lemma 7. First we assume without loss of generality that $Y_{[k]} \in M_\mu(\bar{\mathbb{Q}}[[x]])$. Let K be the extension of $\mathbb{Q}$ generated by the m first coefficients ${}_{ij}Y_{kl}$ of the entries ${}_{ij}Y_{[k]}$ of the $Y_{[k]}$'s , and set $Y = \prod_{k=1}^{N} Y_{[k]}$. We have

$${}_{ij}Y_m = \sum_{\Sigma m_k = m} \sum_{l_k=1}^{\mu} {}_{il_1}Y_{1m_1}\ {}_{l_1 l_2}Y_{2_1 m_1} \cdots {}_{l_{N-1} j}Y_{N_1 m_N}$$

. For a finite place $v \in \Sigma_f$, this gives

$$(*) \qquad \log^+|{}_{ij}Y_m|_v \leq \underset{\substack{m_1+\ldots+m_N=m \\ i_1,\ldots,i_N,\ j_1,\ldots,j_N}}{\mathrm{Max}} \sum_{k=1}^{N} \log^r|{}_{i_k j_k}Y_{k_1 m_k}|_v \ .$$

By reordering $Y_1,\ldots,Y_N$, we may suppose that $m_1 \geq m_2 \geq \ldots \geq m_N$, hence $km_k \leq m$. This yields

$\log^r|{}_{ij}Y_m|_v \leq \sum_{k=1}^{N} \underset{m_k \leq m/k}{\mathrm{Max}}\ \underset{i_k,j_k}{\mathrm{Max}} \log^r|{}_{i_k,j_k}Y_{k,m_k}|_v$, from which we deduce

$$h_{v,m}(Y) \leq \sum_{h=1}^{N} 1/k\ h_{v,m/k}((Y_{[1]})_1) \ .$$

For an infinite place $v \in \Sigma_\infty$, we have to add an extra term to the right hand side of (*), namely $\log\#\{m_1,\ldots,m_k)/\Sigma m_k = m\} + \log \cdot N$ which is $\sigma(m)$; in this case we deduce

$$h_{v,m}(Y) \leq \sum_{h=1}^{N} 1/k\ h_{v,m/k}((Y_{[k]})_k + \sigma(1) \ .$$

By summing over $v \in \Sigma(K)$, we find

$$\sigma(Y) \leq \left(\sum_{k=1}^{N} 1/k\right)\sigma((Y_{[1]})_1) \leq (1 + \log N)\sigma((Y_{[k]})_k) \ .$$

□

1.5 <u>The stable size</u>

Let $Y \in M_{\mu,\nu}(\bar{\mathbb{Q}}((x)))$, and for any $N \in \mathbb{N}^*$, let $Y^{\otimes N}$ be a matrix whose entries are the monomials of degree N in the entries of Y . In view of inequality g , of the previous lemma, one can define the "stable" size $\tau(Y)$ to be

$$\tau(Y) = \overline{\lim_{N\to\infty}} (\log N)^{-1} \sigma((1,Y)^{\otimes N}) \ .$$

LEMMA a) <u>One has</u> $0 \leq \tau(Y) = \tau(Y^{\otimes N}) \leq \sigma(Y)$ <u>for any</u> $N \in \mathbb{N}^*$. <u>Let</u> $Y_1,\dots,Y_N$ <u>be elements of</u> $M_\mu(\bar{\mathbb{Q}}((x)))$, <u>and set</u> $Y = (Y_1,\dots,Y_N)$.

b) $\tau(\Sigma\, Y_k) \leq \tau(Y)$

c) $\tau(\prod Y_k) \leq \tau(Y)$.

<u>Proof</u>: straightforward, taking into account lemma 14.2 g).

□

I hope that property c) justifies the label "stable size". By way of example, one can show that if $y \in \mathcal{O}_K[1/N][[x]]$ for some $N \in \mathbb{N}^\times$, then $\tau(y) = 0$ if $\sigma(y) < \infty$ (see exercise 3 below). This invariant occurs in the work of Chednovski [8].

§ 2. <u>RADII</u>

2.1 <u>Local radii of convergence</u>

Let K be a number field, and let $y = \sum_{n\geq 0} y_n x^n \in K[[x]]$. Then for any $v \in \Sigma_K$, $\Sigma\, i_v(y_n)x^n \in \mathbb{C}_v[[x]]$ defines a v-adic Taylor series $y^{(v)}$; we denote by $R_v(y) \in [0,\infty]$ its radius of convergence. By Hadamard's formula, $R_v(y) = \underline{\lim}_{n\to\infty} |y_n|_v^{-1/n}$. More generally, for any Laurent series $y = \sum_{n\geq -N} y_n x^n \in K((x))$, we set $R_v(y) := R_v(x^N y)$; this definition depends only on y but not on N .

2.2 <u>The global radius</u>

For $Y \in M_{\mu,\nu}(K((x)))$, we set

$$\rho(Y) := \sum_{v} \log^{+}\left(\operatorname*{Min}_{i,j} R_v({}_{ij}Y)\right)^{-1} \in [0,\infty] .$$

LEMMA 1. $\rho(Y) = \sum_{v} \overline{\lim_{n\to\infty}}\, h_{v,n}(Y)$; ρ <u>is invariant under finite extension of</u> K .

<u>Proof</u>: Hadamard's formula yields

$$\rho(Y) = \sum_{v} \operatorname*{Max}_{i,j} \overline{\lim}\, \frac{1}{n} \log^{+}|{}_{ij}Y_n|_v = \sum_{v} \overline{\lim}\, \operatorname*{Max}_{i,j} \log^{+}|{}_{ij}Y_n|_v .$$

Thus it is enough to show that

$$\overline{\lim_{n\to\infty}}\, \frac{1}{n} \operatorname*{Max}_{\substack{i,j\\ m\le n}} \log^{+}|{}_{ij}Y_m|_v = \overline{\lim_{n\to\infty}}\, \frac{1}{n} \operatorname*{Max}_{i,j} \log^{+}|{}_{ij}Y_n|_v .$$

This is a special case, for $t_n = \operatorname*{Max}_{i,j} \log^{+}|{}_{ij}Y_n|_v$, of the well-known inequality

$$\overline{\lim_{n\to\infty}}\, \frac{1}{n} \operatorname*{Max}_{m\le n} t_m \le \overline{\lim_{n\to\infty}}\, \frac{t_n}{n} := 1 .$$

Indeed, for any $\varepsilon > 0$, let $M_\varepsilon \le N_\varepsilon$ such that $\frac{t_m}{m} \le 1 + \varepsilon$ for $m \ge M_\varepsilon$ and $\frac{t_m}{m} \le \frac{N_\varepsilon}{M_\varepsilon} 1$ for $m < M_\varepsilon$. Then

$$\frac{1}{n} \operatorname*{Max}_{m\le n} t_m \le \operatorname{Max}\left(\operatorname*{Max}_{m\le M_\varepsilon} \left(\frac{m}{n}\right) \frac{t_m}{m} , \operatorname*{Max}_{M_\varepsilon\le m\le N_\varepsilon} \left(\frac{m}{n}\right)\frac{t_m}{m}\right) .$$

The second assertion comes readily from the first one.

REMARK. Here again we could replace the indexing set of summation $\Sigma(K)$ by Σ_f (resp. Σ_∞). The above proof yields corresponding formulae $\rho_f(Y) = \sum_{v\in\Sigma_f} \overline{\lim}\, h_{v,n}(Y)$, $\rho_\infty(Y) = \sum_{v\in\Sigma_\infty} \overline{\lim}\, h_{v,n}(Y)$. Furthermore $\rho(Y) = \rho_f(Y) + \rho_\infty(Y)$, and $\sigma_\infty(Y) \leq \rho_\infty(Y)$.

LEMMA 2. Let $Y \in M_{\mu,\nu}(K((x)))$.

a) $\operatorname{Max}_{i,j} \rho({}_{ij}Y) = \rho(Y) = \rho(\zeta Y)$, <u>for any</u> $\zeta \in K$.

b) $\rho(d/dx\ Y) = \rho(Y)$

c) if the residue Y_{-1} vanishes, $\rho(\int_0^x Y) = \rho(Y)$

d) for $\zeta \in K$, $\rho(Y_{(\zeta)}) \leq \rho(Y) + h(\zeta)$.

Let $(Y_{[k]})_{k=1}^N$ a subset of $M_{\mu,\nu}(K((x)))$.

e) $\rho(\Sigma\, Y_{[k]}) \leq \rho((Y_{[k]})_k) = \operatorname{Max}_k \rho(Y_{[k]})$

f) $\rho(*Y_{[k]}) \leq \Sigma_\rho(Y_{[k]})$

g) if $\mu = \nu$, $\rho(\prod Y_{[k]}) \leq \operatorname{Max}_k \rho(Y_{[k]})$.

<u>Proof</u>: straightforward.

□

2.3 We now prove the equivalence stated in 1.3. Let $y \in K[[x]]$. Assume that $h(y) < \infty$. By lemmata 1 of § 1.4 and 2.2, one gets $\rho_\infty(y) < \infty$ and $\sigma_f(y) < \infty$. The first (resp. second) inequality implies condition 1.3 i) (resp. 1.3 ii), taking into account remark 1.4. Conversely, assume that for any $v \in \Sigma_\infty$, $R_\infty(y) > 0$ (condition 1.3 i) and that $\overline{\lim}_{n\to\infty} d_n^{1/n} < \infty$ (condition 1.3 ii) , where d_n denotes the common denomination in $\mathbb{N} \setminus \{0\}$ of $y_0,\ldots,y_n$. Then $\sigma(y) \leq \sigma_\infty(y) + \sigma_f(y) \leq \rho_\infty(y) + \log \overline{\lim\limits_{n\to\infty}}\, d_n^{1/n} < \infty$.

□

§ 3. SEVERAL VARIABLES, DIAGONALIZATION

3.1 All what precedes extends in a straightforward manner to the case of elements of $K[[\underline{x}]] = K[[x_1,\ldots,x_\nu]]$.

For a multi-index $\underline{n} \in \mathbb{N}^\nu$, we denote by $|\underline{n}|$ its length Σn_i ; $\underline{x}^{\underline{n}}$ means $\prod x_i^{n_i}$. Let $y = \sum\limits_{\underline{n}} y_{\underline{n}} \underline{x}^{\underline{n}} \in K[[\underline{x}]]$; for any place v of K , we set

$$h_{v,n}(y) = \frac{1}{n} \operatorname*{Max}_{|\underline{k}|\leq n} \log^+ |y_{\underline{k}}|_v \ .$$

We also define the global radius (resp. size, stable size) by:

$$\rho(y) = \sum_v \overline{\lim_{n\to\infty}}\, h_{v,n}(y)$$

$$\sigma(y) = \overline{\lim_{n\to\infty}} \sum_v h_{v,n}(y)$$

$$\tau(y) = \overline{\lim_{N\to\infty}}\ \overline{\lim_{n\to\infty}} \frac{1}{n \log N} \sum_v \operatorname*{Max}_{\substack{|\underline{k}|\leq n \\ l\leq N}} \log^+ |(y^l)_{\underline{k}}|_v \ .$$

For $\nu = 1$, previous lemmata show the compatibility with original definitions.

3.2 Diagonalization

One defines the diagonalization map Δ_ν from $K[[\underline{x}]]$ to $K[[x]]$ by the formula

$$\Delta_\nu(\Sigma y_{\underline{n}}\underline{x}^{\underline{n}}) = \sum_{n\geq 0} y_{(n,n,\ldots,n)} x^n .$$

This is a useful tool to produce G-functions, through the following lemma (see 4.2):

LEMMA. <u>The following inequalities hold:</u>

$$\rho(\Delta_\nu(y)) \leq \nu\ \rho(y)$$

$$\sigma(\Delta_\nu(y)) \leq \nu\ \sigma(y) .$$

<u>Proof</u>: this follows immediately from the obvious inequality

$$h_{v,n}(\Delta_\nu(y)) \leq h_{v,n\nu}(y) .$$

□

REMARK 1 (Deligne). Assume that for some infinite place v of K , $y^{(v)} := \Sigma\, i_v(y_{\underline{n}})\underline{x}^{\underline{n}}$ is analytic at $\underline{0} \in \mathbb{C}_v^{\ \nu}$, with $\nu > 1$. Then $\Delta_\nu\, y$ is represented by the integral formula

$$(2\pi\sqrt{-1})^{-(\nu-1)} \int_{\substack{|x_2|=\ldots=|x_\nu|=\varepsilon \\ x_1x_2\ldots x_\nu=x}} y\ \frac{dx_2\ldots dx_\nu}{x_2\ldots x_\nu} \quad \text{for } \varepsilon \text{ and } |x| \text{ small enough.}$$

This follows from the residue formula:

$$(2\pi\sqrt{-1})^{-(\nu-1)} \int_{\substack{|x_2|=\ldots=|x_\nu|=\varepsilon \\ x_1x_2\ldots x_\nu=x}} \underline{x}^{\underline{n}}\ \frac{dx_2\ldots dx_\nu}{x_2\ldots x_\nu} \begin{cases} = x^{n_1} & \text{if } n_1 = n_2 = \ldots = n_\nu \\ = 0 & \text{otherwise.} \end{cases}$$

REMARK 2. It seems that diagonals were first introduced in the study of Hadamard product (see e.g. [3]). This relationship is given by the formula:

$$\Delta_\nu(y_1(x_1)\ \dots\ y_\nu(x_\nu)) = y_1 * \dots * y_\nu \ .$$

3.3 Geometric interpretation

Let us set $X = \operatorname{Spec} K(x)[\underline{x}]/_{(x_1x_2\dots x_\nu - x)}$, with $\nu > 1$. Let (E,∇) be a coherent module with integrable connection over some affine open subset U of X , and let σ be some horizontal $K(U)$-linear map from E to $K[[\underline{x}]]$; in other words, $y := \sigma(e)$, for $e \in \Gamma E$, is a solution in $K[[\underline{x}]]$ of an "integrable differential equation".

We consider the $K(x)$-linear map:

$$\Delta_{\nu,\sigma} : e \otimes \frac{dx_2\dots dx_\nu}{x_2\dots x_\mu} \longmapsto \Delta_\nu(\sigma(e)) \ ,$$ for all local sections e of E .

PROPOSITION. The map $\Delta_{\nu,\sigma}$ induces a horizontal map from the algebraic De Rham cohomology group $H^{\nu-1}_{DR}(U,(E,\nabla))$ endowed with Gauss-Manin connection relative to $K(x)$ (see [13]), to $K[[x]]$ endowed with exterior derivative.

Proof: the smooth scheme U is affine, thus there is an isomorphism $H^{\nu-1}_{DR}(U,(E,\nabla)) \simeq E \otimes \Omega^{\nu-1}_{U/K(x)} \Big/ \nabla_{\nu-1}(E \otimes \Omega^{\nu-2}_{U/K(x)})$, where the value at d/dx of the Gauss-Manin connection acts through $\nabla(d/_{d(x_1x_2\dots x_\mu)})$ on E . The statement would follow from

Deligne's integral formula if $\sigma(e)^{(v)}$ were analytic at $\underline{0}$ for some $v \in \Sigma_\infty$. However this can fail if $\underline{0}$ corresponds to an irregular singularity of (E,∇) ; thus we shall rather translate a purely algebraic argument from Christol (see [5]). The relation $\Sigma \frac{dx_i}{x_i} = 0$ in $\Omega^1_{X(K(x))}$, together with the formula $\Delta_\nu\left(x_i \frac{\partial\sigma(e)}{\partial x_i}\right) = x \frac{d}{dx} \Delta_\nu(\sigma(e))$, yields

$$\Delta_{\nu,\sigma}(\nabla_{\nu-1}(e\otimes \frac{dx_2\ldots\widehat{dx_i}\ldots dx_\nu}{x_2\ldots\widehat{x_i}\ldots dx_\nu})) = \Delta_{\nu,\sigma}((x_i\nabla(\partial/\partial x_i)e - x_1\nabla(\partial/\partial x_i)e)\otimes \frac{dx_2\ldots dx_i\ldots dx_\nu}{x_2\ldots x_i\ldots x_\nu})$$

$$= \Delta_\nu\left(x_i \frac{\partial\sigma(e)}{\partial x_i} - x_1 \frac{\partial\sigma(e)}{\partial x_1}\right) = 0 \ .$$

Therefore $\Delta_{\nu,\sigma}$ factors through $H^{\nu-1}_{DR}(U,(E,\nabla))$. In order to prove the horizontality statement, we fix $x_2,\ldots,x_\nu$ and get

$$\Delta_{\nu,\sigma}(x_1\nabla(\partial/\partial x_1)e\otimes \frac{dx_2\ldots dx_\nu}{x_2\ldots x_\nu}) = \Delta_\nu(x_1 \frac{\partial\sigma(e)}{\partial x_1}) = x\ d/dx\ \Delta_{\mu,\sigma}(e\otimes \frac{dx_2\ldots dx_\nu}{x_2\ldots x_\nu}) \ .$$

□

COROLLARY 1. <u>Assume that</u> $H^{\nu-1}_{DR}(U,(E,\nabla))$ <u>is finite-dimensional over</u> $K(x)$ (<u>assume for instance that</u> (E,∇) <u>has only regular singular points</u>) <u>then for</u> $y = \sigma(e)$ <u>as above</u>, $\Delta_\mu(y)$ <u>satisfies an ordinary linear homogeneous differential equation with coefficients in</u> $K(x)$.

COROLLARY 2. <u>Assume that</u> σ <u>is a solution in</u> $K[[\underline{x}]]$ <u>of the Picard-Fuchs system</u> $H^\mu_{DR}(Y/K(\underline{x}))$ <u>of a smooth proper</u> $K(\underline{x})$-<u>variety</u> Y . <u>Then</u> $\Delta_{\mu,\sigma}$ <u>is a solution in</u> $K[[x]]$ <u>of the Picard-Fuchs system</u> $H^{\mu+\nu-1}_{DR}(Z/K(x))$ <u>of a smooth</u> $K(x)$-<u>variety</u> Z .

<u>Proof</u>: let V be an open dense subset of Spec $K\left[\underline{x}, \frac{1}{x_1 \ldots x_\nu}\right]$ such that Y extends to a smooth proper morphism $Y_V \xrightarrow{f} V$, and let us denote by g the obvious smooth morphism $V \longrightarrow \operatorname{Spec} K\left[x_1 \ldots x_\nu, \frac{1}{x_1 \ldots x_\nu}\right]$. Let us consider the Cartesian squares:

$$\begin{array}{ccc} Z & \longrightarrow & Y_V \\ \downarrow & & \downarrow f \\ U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & \operatorname{Spec} K\left[\underline{x}, \frac{1}{x_1 \ldots x_\nu}\right] \end{array} .$$

According to the proposition, $\Delta_{\nu,\sigma}$ is a solution in $K[[x]]$ of $H_{DR}^{\nu-1}(U/K(x), H_{DR}^{\mu}(Z/U))$.

On the other hand, there is the Leray spectral sequence

$$(*) \qquad H_{DR}^{\nu-1}(U/K(x), H_{DR}^{\mu}(Z/U)) \Rightarrow H_{DR}^{\mu+\nu-1}(Z/K(x)) .$$

Let us extend the scalars K to $\mathbb{C}$; since $f_{\mathbb{C}}$ is <u>proper</u> and smooth, the Leray spectral sequence of local systems $R^{\nu-1}g_{\mathbb{C}*}\, R^{\mu}f_{\mathbb{C}*}(\mathbb{C}) \Rightarrow R^{\mu+\nu-1}(gf_{\mathbb{C}})_*(\mathbb{C})$ degenerates [9] 2.4. It follows from the comparison theorem that (*) also degenerates as a spectral sequemce of $K(x)$-vector spaces with connection. Thus $\Delta_{\nu,\sigma}$ is a solution of $H_{DR}^{\mu+\nu-1}(Z/K(x))$.

□

REMARK 3. Combining corollary 2 with remark 2, we get that if $\Sigma a_n x_n$ satisfies a Picard-Fuchs equation from projective geometry, then for any N $\Sigma a_n^N x^n$ satisfies a Picard-Fuchs equation.

§ 4. EXAMPLES

We shall study four typical classes of G-functions, each of which is stable under Hadamard product; namely: rational functions, diagonals of rational functions in several variables, polylogarithms and hypergeometric functions (Geometric and hypergeometric series, were already put forward by C.L. Siegel [17], and G-functions borrow their generic name from these special cases). Each of these series satisfies some linear homogeneous differential equation, which turns out to come from geometry.

4.1 Rational functions

Let $y \in K(x)$, and let us write $\mathrm{pol}(y)$ for the set of poles of y. We may write y as the quotient p/q of two polynomials in $\mathcal{O}_K[x]$. Let us write N for the norm of the first non-zero coefficient of q; then $y \in \mathcal{O}_K[1/N]((x))$. On the other hand, it is immediate that $\rho_\infty(y) < \infty$. Since such series occur frequently, we state a

DEFINITION (Christol). A Laurent series $y \in K((x))$ is globally bounded if and only if

i) for any $v \in \Sigma_\infty$, $R_v(y) > 0$,

ii) there exists $N \in \mathbb{N}^\times$ such that $y \in \mathcal{O}_K[1/N]((x))$.

LEMMA. Any $y \in K(x)$ satisfyies $\rho(y) = \sigma(y) = p(\mathrm{pol}(y))$.

Proof: we have $R_v(y) = \min_{\zeta \in \mathrm{pol}(y)} |\zeta|_v$ for any $v \in \Sigma(K)$, whence the equality $\rho(y) = h(\mathrm{pol}(y))$.

On the other side, the fact that y is globally bounded implies that $h_{v,n}(y) = 0$ for almost all v , and all n . Using lemmata 1 of §§ 1.3 and 2.2, we come by the inequality $\sigma(y) \leq \rho(y)$. In order to show that it is an equality, it suffices to establish the existence of the limit $\lim_{n\to\infty} h_{v,n}(y)$ for any $v \in \Sigma(K)$; but this follows from the fact that coefficients of y satisfy linear reccurence equations for $n >> 0$ (see next remark).

□

REMARK. The lemma generalizes immediately to the case of a matrix $Y \in M_{\mu,\nu}(K(x))$. The stability of $M_{\mu,\nu}(K(x))$ under Hadamard product is easily seen using the characterization of rational series: $y \in K(x) \Longleftrightarrow \exists N \in \mathbb{N}^x$, $\exists Y,Z \in M_N(K)$ such that $Y_n = \mathrm{tr}\, Y.Z^n$ (existence of recurrence relations); we have the formula $(Y_1 * Y_2)_n = \mathrm{tr}(Y_1 \otimes Y_2)(Z_1 \otimes Z_2)^n$, with obvious notations.

4.2 Diagonals of rational functions

We shall denote by $K[\underline{x}]_{(\underline{x})}$ the localization of the ring $K[\underline{x}] = K[x_1,\ldots,x_\nu]$ at the ideal generated by $x_1,\ldots,x_\nu$, and by $K\{\{x\}\}$ the henselization of $K[x]$ at the ideal generated by x (i.e. the subring of $K[[x]]$ of algebraic elements over $K(x)$.)

DEFINITION. <u>Elements in the target</u> $\Delta_\nu(K[\underline{x}]_{(\underline{x})})$ <u>of the diagonalization map restricted to</u> $K[\underline{x}]_{(\underline{x})}$ <u>are called diagonals of rational functions</u> (over K).

REMARK 1. Let us consider again the geometric interpretation of Δ_ν in § 3.3. In the present case, let $p/q \in K[\underline{x}]_{(\underline{x})}$, with $p,q \in K[\underline{x}]$. We may take for U the subset of X where q does not vanish; $E = \mathcal{O}_U$, endowed with exterior derivative ∇ ; σ : the standard horizontal map $\mathcal{O}_U \longrightarrow K[[\underline{x}]]$, where x is replaced by $x_1x_2\ldots x_\nu$; $e := p/q$. We have $H^{\nu-1}_{DR}(U,(E,\nabla)) = H^{\nu-1}_{DR}(U)$, the ordinary algebraic De Rham cohomology of the smooth affine scheme U. This is a finite-dimensional $K(x)$-vector space; see [16] for an algebraic proof which does not use resolution of singularities. According to corollary 3.3, diagonals of rational functions satisfy "Picard-Fuchs" differential equations associated to smooth affine $K(x)$-schemes.

LEMMA. <u>Let</u> $y \in K[[x]]$, $y = \Delta_\nu(p/q)$ <u>be a diagonal of rational function. Then</u> y <u>is a globally bounded G-function, and</u> $\sigma(y) \leq \rho(y) < \infty$.

<u>Proof</u>: we may assume that $p,q \in \mathcal{O}_K[\underline{x}]$; let us denote by N the norm of $q(\underline{0}) \neq 0$. Then it is clear that $p/q \in \mathcal{O}_K$ and $y \in \mathcal{O}_K[1/N][[x]]$. On the other side, the v-adic radius of convergence $R_v(p/q)$ is non zero for every $v \in \Sigma(K)$, and the same holds for $R_v(y)$ according to Hadamard's formula. This shows that y is a globally bounded G-function. The

deduction $\sigma(y) \leq \rho(y)$ is made as in lemma 4.1. In fact, it could be shown that $\sigma_f(y) = \rho_f(y) \leq \nu h_f(q(\underline{0})^{-1}) \leq \nu\, h(q(\underline{0}))$.

□

It happens that diagonal of rational functions occur very frequently, even though it is often difficult to find the (non-unique) relevant rational function. To explain this fact, G. Christol [6] has set the following conjecture up:

CONJECTURE. Every globally bounded solution in $K[[x]]$ of a linear homogeneous differential equation with coefficients in $K[x]$ is a diagonal of a rational function.

We now prove that algebraic functions are diagonals of rational functions in two variables (Christol-Furstenberg [4][11]).

PROPOSITION. The equality $\Delta_2(K[x_1,x_2]_{(x_1,x_2)}) = K\{x\}$ holds.

Sketch of proof: in fact we shall only consider the inclusion $\supseteq$. Let $y \in K\{x\}$ and let $r(y,x) := 0$ be a polynomial equation for y . Assuming that $r(0,0) = 0$, $\frac{\partial r}{\partial y}\Big|_{(0,0)} \neq 0$, $\frac{\partial r}{\partial x}\Big|_{(0,0)} \neq 0$, we shall exhibit a rational function p/q such that $\Delta_2(p/q) = y$. We set $q(x_1,x_2) = \frac{1}{x_1}\, r(x_1,x_1x_2)$, so that $1/q \in K[x_1,x_2]_{(x_1,x_2)}$, and $\frac{\partial q}{\partial x_2}\Big|_{(0,0)} \neq 0$.

Let us consider the following diagram (where X and U have the same meaning as in remark 1, and $Z = X \setminus U$):

$$\begin{array}{ccccccccc}
0 & \xrightarrow{=} & H^1_{DR}(X\cup\{0\}) & \longrightarrow & H^1_{DR}(U) & \xrightarrow{Res_{Z\cup\{0\}}} & H^0(Z\cup\{0\}) & \longrightarrow & 0 \\
 & & \downarrow & & \| & \nwarrow \varphi & \uparrow & & \\
0 & \longrightarrow & H^1_{DR}(X) & \longrightarrow & H^1_{DR}(U) & \xrightarrow{Res_Z} & H^0(Z) & \longrightarrow & 0
\end{array}$$

where all arrows are horizontal maps, and where the horizontal rows are the residue exact sequences: Res_Z is the "coefficient of dq/q ", given at the stage of differential forms by $Res_Z(p/q \; {}^{dx_2}/_{x_2}) = \left(\frac{\partial q}{\partial x_2}\right)^{-1} p/x_2 \big|_{q(x_1,x_2)=0}$.

Now the derivation d/dx extends in a unique way to $K(x,y)$, whence a connection on this space, which can be identified with Gauss-Manin connection on $H^0(Z)$. It follows that the image of $y \in K(x,y) \simeq H^0(Z)$ under φ is given by the class of $p/q \cdot {dx_2}/_{x_2}$, where $p = x_1 x_2 \; \partial q/\partial x_2$.

The following diagram of horizontal maps

$$\begin{array}{ccccc}
H^1_{DR}(U) & \xleftarrow{\varphi} & H^0(Z) & \xleftarrow{\approx} & K(x,y) \\
\Delta_{2,\sigma}\downarrow & & & & \downarrow \\
K[[x]] & = & = & = & K[[x]]
\end{array}$$

(where σ is defined in the above remark) shows that $(\Delta_{2,\sigma}\circ\varphi)(y)$ satisfies the same differential equation as y , and $(\Delta_{2,\sigma}\circ\varphi)(y)\big|_0 = x \; \Delta_2\left(\frac{1}{q} \; {}^{\partial q}/\partial x_2\right)\Big|_0 = 0$. It follows that $y = \Delta_2\left(\frac{x_1 x_2}{q} \cdot {}^{\partial q}/\partial x_2\right)$.

□

For a proof of the reversed inclusion $\subseteq$, with an argument from linguistics, see [10] 5.

REMARK 3: the stability of diagonals of rational functions under Hadamard product is immediate from the formula:

$$\Delta_{\nu_1+\nu_2}(r_1(x_1,\dots,x_{\nu_1})r_2(x_{\nu_1+1},\dots,x_{\nu_1+\nu_2})) = \Delta_{\nu_1}r_1 * \Delta_{\nu_2}r_2 .$$

However the subclass of algebraic functions is not stable under $*$; by way of counterexample, one may take (Jungen, 1931) :

$$(1-x)^{1/2} * (1-x)^{-1/2} = \Delta_4(4/(2-x_1-x_2)(2-x_3-x_4)) = {}_2F_1(1/2,1/2,1,x)$$
$$= \sum_{n\geq 0}\binom{2n}{n}^2 \left({}^{x}/_{16}\right)^n \text{ , which is transcendental.}$$

4.3 Polylogarithms

We turn back to more down-to-earth examples. Let $L_k = \sum_{n\geq 0} x^n/_{n^k}$ be the k^{th}-polylogarithmic series. It satisfies the "unipotent" differential equation: $d/dx\ \frac{1-x}{x}\ (xd/dx)^k\ L_k = 0$ obtained from the chain rule $xd/dx = L_{k-1}$, $L_0 = x/_{1-x}$; the other solutions can be expressed by means of the functions $1, \log x,\dots,\log^{k-1} x$.

LEMMA. One has $\rho(L_k) = 0,\ \sigma(L_k) = k$.

Proof: this is a straightforward consequence of Tchebyshev's theorem. Moreover, we shall show elsewhere that $\tau(L_1) = 1$.

□

REMARK. Integration of any formal power series y is nothing but the Hadamard product $xy * L_1$.

4.4 Generalized hypergeometric functions

For $a \in \mathbb{Q}$, we set $(a)_0 = 1$, $(a)_{n+1} = (a+n)(a)_n$, and for $\underline{a} := (a_1,\dots,a_\mu) \in \mathbb{Q}^\mu$ we set $(\underline{a})_n = \prod_{m=1}^{\mu} (a_m)_n$. To any couple $(\underline{a},\underline{b})$ in $(\mathbb{Q}-\{-\mathbb{N}\})^\mu \times (\mathbb{Q}-\{-\mathbb{N}\})^\nu$, we associate the hypergeometric function

$$y = F(\underline{a},\underline{b},x) := \sum_{n\geq 0} (\underline{a})_n/(\underline{b})_n \, x^n .$$

LEMMA. The three conditions $\rho(y) < \infty$, $\sigma(y) < \infty$ and $\mu = \nu$ are equivalent. If they are satisfied, one has

$$\rho(y) = \sigma(y) = \sum_{m=1}^{\mu} (h_f(a_m) - h_f(b_m)) .$$

Proof: either of the conditions $\rho(y) < \infty$, $\sigma(y) < \infty$ implies that for $v \in \Sigma_\infty$, $R_v(y) > 0$, which implies in turn that $\mu \leq \nu$, and $R_v(y) \geq 1$ (hence $\rho_\infty(y) = \sigma_\infty(y) = 0$). Let N be the greatest common denominator of the a_m, b_m's ; for $p > N$ and $n \to \infty$, we have:

$$\left| (a_m)_n/(b_m)_n \right|_p = O\left(p^{\log n}\right) ,$$

$$\left| 1/(b_n)_n \right|_p^{1/n} \sim p^{1/p-1} ,$$

$$\operatorname{den}\left(N^n (a_m)_n/(b_m)_n\right) = O\left(e^{1/\log n}\right) ,$$

and $\left(\text{den } N^n/(b_m)_n\right)^{1/n} \sim n/e$ (Stirling), see the appendix. The former two estimates, together with the divergence of $\sum_{p>N} \frac{\log p}{p-1}$, show that $\rho(y) < \infty \Rightarrow \mu \geq \nu$.

The latter two estimates show that $\sigma(y) < \infty \Rightarrow \mu \geq \nu$. Conversely the first and third estimates show that $\mu = \nu$ implies finiteness for ρ and σ, and that

$$\rho(y) = \sum_{p \nmid N} \overline{\lim_{n\to\infty}}\, h_{p,n}$$

$$\sigma(y) = \overline{\lim_{n\to\infty}} \sum_{p \nmid N} h_{p,n} .$$

A straightforward computation (remarking that $|(a_m)_n|_p = |a_p|_p^n$ if $|a|_p > 1$ then leads to the equality $\rho(y) = \sigma(y) = \sum_{m=1} (\log \text{den } a_m - \log \text{den } b_m)$.

□

REMARK 1. We could define hypergeometric series for parameters $(\underline{a},\underline{b})$ in $(K \setminus \{-\mathbb{N}\})^{\mu+\nu}$ for any number field. However it follows from Chudnovski [7] that such a hypergeometric series is a G-function only if $(\underline{a},\underline{b}) \in (\mathbb{Q} \setminus \{-\mathbb{N}\})^{\mu+\nu}$.

REMARK 2. G. Christol [6] has determined all globally bounded hypergeometric functions. The extra condition is the following one: let N as above; then for any M with $0 \leq M < N$ and $(M,N) = 1$, and for any positive integer j with $j \leq \mu$, $\# \{i / Ma_i \alpha\, Mb_j\} \geq \# \{i / Mb_i \alpha Mb_j\}$ (here α is the total ordering of $\mathbb{R}$ defined by

$$y \,\alpha\, z \Longleftrightarrow y + [-y] < z + [-z] \quad \text{or} \quad (y + [-y] = z + [-z] \ \text{and} \ y \geqq z)).$$

Let us now introduce the classical Meijer G-functions, which however are not G-functions in Siegel's sense! These are integrals of Mellin-Barnes type over some suitable loop:

$$G^{m,n}_{\nu,\mu}(\underline{a},\underline{b},x) := \frac{1}{2\pi\sqrt{-1}} \oint \frac{\prod_{\partial=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{m} \Gamma(1-a_j+s)}{\prod_{j=m+1}^{\mu} \Gamma(1-b_j+s) \prod_{j=n+1}^{\nu} \Gamma(a_j - s)} x^s ds ,$$

$$\text{for} \quad 0 \leq m \leq \mu, \ 0 \leq n \leq \nu .$$

In the case $\mu = \nu$, these functions satisfy some fuchsian differential equation. Namely, $z := G^{m,n}_{\mu,\mu}(\underline{a},\underline{b},(-1)^{m+n}x)$ satisfies the equation

$$(*) \quad (-1)^{\mu} x \prod_{j=1}^{\mu} (\partial - a_j + 1) z = \prod_{j=1}^{\mu} (\partial - b_j) z \quad \text{where} \quad \partial = x\, d/dx ,$$

whose singularities are $X = 0,\ (-1)^{\mu}$ and ∞ .

The link with hypergeometric series is given by the formulae

$$F(\underline{a},\underline{b},x) = \frac{\prod_{j=1}^{\mu} \Gamma(b_j)}{\prod_{j=1}^{\mu} \Gamma(a_j)} G^{\mu,1}_{\mu,\mu}(\underline{a},\underline{b},-1/x) = \frac{\prod_{j=1}^{\mu} \Gamma(b_j)}{\prod_{j=1}^{\mu} \Gamma(a_j)} G^{1,\mu}_{\mu,\mu}(\underline{1}-\underline{a},\underline{1}-\underline{b},x)$$

and

$$G^{m,n}_{\mu,\mu}(\underline{a},\underline{b},x) = \sum_{k=1}^{m} \frac{\prod_{\substack{j=1\\ j\neq k}}^{m} \Gamma(b_j-b_k) \prod_{j=1}^{n} \Gamma(1+b_k-a_j)}{\prod_{j=m+1}^{\mu} \Gamma(1+b_k-b_j) \prod_{j=n+1}^{\mu} \Gamma(a_j+b_k)} x^{b_k} F(\underline{-a}+\underline{1+b_k}, \underline{-b}+\underline{1+b_k}, (-1)^{\mu-m-n}x)$$

where we set $\underline{h} = (h,\dots,h)$ for any $h \in \mathbb{Q}$, see [1] 5.5. The latter formula shows that $G^{m,n}_{\mu,\mu}$ is a linear combination (with transcendental constant coefficients) of some Siegel G-functions.

REMARK 3. In the case $\mu = \nu = 1$, we have $F(a,b,x) = {}_2F_1(a,1,b,x)$, the classical hypergeometric function, and it is well-known that equation (*) is a factor of a Picard-Fuchs equation [14]. For higher μ , this is by no means obvious. However it remains that:

PROPOSITION. (for $\mu = \nu$) $F(\underline{a},\underline{b},x)$ <u>satisfies some Picard-Fuchs differential equation</u>.

<u>Proof</u>: according to remarks of § 4.2, we have

$$F(\underline{a},\underline{b},x) = \mathop{*}_{i=1}^{\nu} ({}_2F_1(a_i,1,b_i,x)) = \Delta_\nu(\prod_{i=1}^{\nu} {}_2F_1(a_i,1,b_i,x_i)) .$$

By corollary 2 in § 3.3, it suffices to show that $\prod_{i=1}^{\nu} {}_2F_1(a_i,1,b_i,x_i)$ satisfies a Picard-Fuchs differential equation associated $H^{\mu}_{DR}(Y/_{\mathbb{Q}(\underline{x})})$ for some <u>proper</u> smooth Y . Using Künneth formula in algebraic De Rham cohomology, it is enough to prove this statement for $\nu = 1$. If $b \in \mathbb{N}^{x}$, then ${}_2F_1(a,b,x)$ is algebraic and the statement holds with $\mu = 0$.

If $b \in \mathbb{N}^{\times}$ (so that $b \notin \mathbb{Q}$ by our hypergeometric series

$$(b-a-1)\,{}_2F_1(a,1,b,x)+a\,{}_2F_1(a+1,1,b,x)-(b-1)\,{}_2F_1(a,1,b-1,x)=0$$

$$b[a-(b-1)x]\,{}_2F_1(a,1,b,x)+ab(1-x)\,{}_2F_1(a+1,1,b,x)+(b-1)(b-a)x\,{}_2F_1(a,1,b+1,x)=0$$

in order to reduce ourselves to the case $a > 0$, $1 < b < 2$. In this case, Euler's integral representation $_2F_1(a,1,b,x) = (b-1)\int_0^1 (1-t)^{b-2}(1-tx)^{-a}dt$ shows that $_2F_1(a,1,b,x)$ satisfies the Picard-Fuchs equation associated to the differential $\frac{dt}{u}$ over the smooth completion of the curve

$$u^N = (1-t)^{(2-b)N}(1-tx)^{aN}, \quad N = \mathrm{den}(a,b) .$$

□

§ 5. COUNTEREXAMPLES

In this paragraph, we gather some "pathological" examples to show that there is no link in general between ρ and σ (we shall show elsewhere that for solutions of linear homogeneous differential equations with coefficients in $\bar{\mathbb{Q}}(x)$, ρ and σ are in contrast strongly related). We also state that ρ and σ are bad-behaved under inversion of functions.

5.1 A G-function whose inverse is not a G-function

Recall that $\rho(L_1/x) = 0$, $\sigma(L_1/x) = 1$. Let $y = x/L_1$, so that $y_0 = 1$ and $y_n = \sum_{m=1}^{n} \frac{y_{n-m}}{m+1}$. For each p^{th} root of unity

$\zeta \in \mathbb{C}_p \setminus \{1\}$, L_1/x vanishes at $1-\zeta$, and $|1-\zeta|_v = |p|_v^{1/p-1}$. Therefore $R_v(y) \leq |p|_v^{1/p-1}$ and $\rho(y) = \infty$. It will be shown elsèwhere that $\sigma(y) = \infty$; it will follow that the composite series $L_1 \circ L_1 \in \mathbb{Q}[[x]]$ is not a G-function, since

$$x(1-x)\, d/dx\, (L_1 \circ L_1) = y \; .$$

5.2 An example with $\rho = 0$ and $\sigma = \infty$

We set $y = \sum_{k \geq 1} k^{-[k/\log^2 k]} x^k$. We readily compute

$$h_{p,n}(y) = \begin{cases} 0 \quad \text{for} \quad p = \infty \\ \frac{1}{n} \operatorname{Max}_{k \leq n} [{}^k/\log^2 k][\log k/_{\log p}] \log p = o_n(1) , \\ \qquad \text{for} \quad p \text{ a finite prime.} \end{cases}$$

Thus $\overline{\lim_{n\to\infty}}\, h_{p,n}(y) = 0$ and $\rho(y) = 0$. On the other side

$$\sum_{p \text{ prime} \neq \infty} h_{p,n}(y) = \frac{1}{n} \log \underset{k \leq n}{g\cdot c\cdot m} \left(k^{[k/\log^2 k]}\right)$$

$$\geq \frac{1}{n} \sum_{p \leq n} (p/\log p - \log p) \longrightarrow \infty \quad \text{when} \quad n \to \infty \; .$$

This shows that $\sigma(y) = \infty$.

5.3 An example with $\rho = \infty$ and σ arbitrarily small.

Let $N \geq 0$ and let us set

$$y = \sum_{\substack{p \text{ prime} \\ \neq \infty}} \sum_{k \geq 0} p^{-[2^{p^k - N}/\log p]} \cdot x^{p \cdot 2^{p^k}} \; .$$

$$\text{We have}\quad h_{p,n}(y) = \begin{cases} 0 & \text{for } p = \infty \\ [2^{\{n,p\}-N}/\log p]\,\dfrac{\log p}{n} & \text{for any finite prime } p, \end{cases}$$

denoting by $\{n,p\}$ the maximal power of p such that $2^{\{n,p\}} \le n/p$. Thus $\overline{\lim\limits_{n\to\infty}}\, h_{p,n}(y) = 2^{-N}/p$ in the latter case, and $\rho(y) = \infty$. Now we have

$$\Sigma h_{p,n}(y) = \frac{1}{n}\sum_{p\le n}[2^{\{n,p\}-N}/\log p]\log p$$

$$\le \frac{1}{n}\sum_{p\le n} 2^{\{n,p\}-N} .$$

We note that for $p \neq q$, then $\{n,p\} \neq \{n,q\}$, so that $\sum\limits_{p\le n} 2^{\{n,p\}} \le 2^{\{n,p_0\}}\sum\limits_{k=1}^{\infty} 2^{-k}$ for some p_0 , $2 \le p_0 \le n$.
Therefore $\sigma(y) \le 2^{-N}$.

5.4 <u>A globally bounded function with</u> $\sigma < \rho$

Let us consider

$$y = \sum_{k\ge 0} 2^{(-2)^k} x^{2^k} .$$

$$\text{We have}\qquad h_{p,n}(y) = \begin{cases} 0 & \text{for } p \neq 2,\ p \neq \infty \\ 2^{2\left[\frac{1}{2}\left[\frac{\log n}{\log 2}\right]\right]}\log 2 , & \text{for } p = 2 \\ 2^{2\left[\frac{-1}{2}\left[\frac{\log n}{\log 2}\right]\right]}\log 2 , & \text{for } p = \infty . \end{cases}$$

Thus $\sum_p \overline{\lim_{n\to\infty}} h_{p,n}(y) = 2 \log 2 = \rho(y) = \sigma_f(y) + \sigma_\infty(y)$, and

$\overline{\lim_{n\to\infty}} \sum_p h_{p,n}(y) = 3/2 \log 2 = \sigma(y)$.

EXERCISES. 1) Show that $\sigma(y) = 0 \Rightarrow \rho(y) = 0$.

2) Assume that for all $v \in \Sigma(K)$, $\lim_{n\to\infty} h_{v,n}(y)$ exists. Show that $\rho(y) \leq \sigma(y)$.

3) Let $y \in K[[x]]$ and <u>assume</u> that $\rho(y,1/y) < \infty$,

a) Show that this condition is equivalent to

$\sum_{v\in\Sigma_f} \operatorname{Sup}_{n\geq 1} \frac{1}{n} \log^+|y_n|_v < \infty$ (use the fact that for any $v \in \Sigma_f$, $y^{(v)}$ has no zero $\xi \in \mathbb{C}_v$ satisfying $0 < |\xi|_v < R$, if and only if $r \longmapsto \operatorname{Sup}_n |y_n|r^n$ is a constant function on $]0,R[$),

b) deduce that this condition is satisfied in particular if y is globally bounded,

c) show that $\sigma(y) \leq \rho(y) < \infty$,

d) deduce that $\tau(y) = 0$,

e) show that if $y(0) \neq 0$, $1/y$ is a G-function; give upper bounds for $\rho(1/y)$, $\sigma(1/y)$,

f) show that if $y(0) = 0$, then for every G-function z , the composed series $z \circ y$ is again a G-function.

4) Consider the G-function y of § 5.3: assume the finiteness of the set of solutions of the equation $p^k - q^l = m$ (m fixed but arbitrary), and show that, in point of fact, $\sigma(y) = 2^{-N-1}$.

Appendix

Calculus of factorials

Following [6]3, we give estimates for the p-adic valuation $v_p((a)_n)$ of the rational number $(a)_n = \prod_{i=0}^{n-1}(a+i)$, for $a \in \mathbb{Q} - (-\mathbb{N})$. We first introduce general notations:

let p be a fixed prime, and let $a \in \mathbb{Q} \cap \mathbb{Z}_p$, i.e. the denominator of a is prime to p.

We define R, Q, and f by the formulae:

$$a = -R(a,p^k) + p^k Q(a,p^k)$$

$$\text{with } R(a,p^k) \in \mathbb{N} , \quad R(a,p^k) < p^k ,$$

$$f(a,p^k,n) = \left[\frac{n + p^k - 1 - R(a,p^k)}{p^k}\right]$$

For instance, when $a = 1$, we have $R(1,p^k) = p^k - 1$ and $f(1,p^k,n) = [n/p^k]$. Let us remark that $f(a,p^k,n) - f(1,p^k,n)$ is periodic, with period p^k in n ; this leads to the equality

$$(1) \quad f(a,p^k,n) - f(1,p^k,n) = y(< n/p^k > - R(a,p^k)/p^k)$$

$$\text{where} \quad y(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

and $<x> = x - [x]$; we shall also use the notation

$$\{ x \} = - x - [-x] .$$

We extract from [6][14] a formula for $R(a,p^k)$:

(2) $R(a,p^k)/p^k = \{a\Delta^k\} - a/p^k$ where the integer Δ satisfies the condition:

for some $N \in N^*$, such that $N|a| < p$ and $Na \in \mathbb{Z}$,

$\Delta p \equiv 1 \bmod N$ (in fact $N|a| < p^k$ is enough) .

At last we recall the generalization for $(a)_n$ (see [6]) of the classical equality $v_p((1)_n) = \sum_{k=1}^{\infty} [n/p^k]$:

$$v_p((a)_n) = \sum_{k=1}^{\infty} f(a,p^k,n) . \tag{3}$$

Putting together (1), (2), (3), we find:

LEMMA, <u>The following equality holds</u>:

$$v_p((a)_n) = \sum_{k=1}^{\infty} [n/p^k] + \#\{ k \text{ such that } \{\Delta^k a\} \leq (a/p^k + < n/p^k >)\} . \tag{4}$$

REMARK: For $p^k > (a+n)N$, we have $\{\Delta^k a\} \geq 1/N \geq a/p^k + <n/p^k>$, so that the second term of the right-hand side of (4) is bounded by $\dfrac{\log \text{Max } ((a+n)N, 0)}{\log p}$.

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