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by

Tatiana Gateva-Ivanova

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Tatiana Gateva-Ivanova

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
1113 Sofia
Bulgaria

MPIM 12-05

QUADRATIC ALGEBRAS, YANG-BAXTER EQUATION, AND ARTIN-SCHELTER REGULARITY

TATIANA GATEVA-IVANOVA

Abstract. We study two classes of quadratic algebras over a field $\mathbf{k}$: the class $\mathfrak{C}_n$ of all n-generated PBW algebras with polynomial growth and finite global dimension and the class of quantum binomial algebras. We show that a PBW algebra A is in $\mathfrak{C}_n$ *iff* its Hilbert series is $H_A(z) = 1/(1-z)^n$. Furthermore the class $\mathfrak{C}_n$ contains a unique (up to isomorphism) monomial algebra, $A = \mathbf{k}\langle x_1, \cdots, x_n\rangle/(x_j x_i \mid 1 \leq i < j \leq n)$. Surprising amount can be said when A is a quantum binomial algebra, that is its defining relations are nondegenerate square-free binomials $xy - c_{xy}zt$, $c_{xy} \in \mathbf{k}^{\times}$. Our main result shows that for an n-generated quantum binomial algebra A the following conditions are equivalent: (i) A is an Artin-Schelter regular PBW algebra. (ii) A is a Yang-Baxter algebra, that is the set of relations $\Re$ defines canonically a solution of the Yang-Baxter equation. (iii) A is a binomial skew polynomial ring, with respect to an enumeration of X. (iv) The Koszul dual $A^!$ is a quantum Grassmann algebra.

1. Introduction

A quadratic algebra is an associative graded algebra $A = \bigoplus_{i \geq 0} A_i$ over a ground field $\mathbf{k}$ determined by a vector space of generators $V = A_1$ and a subspace of homogeneous quadratic relations $R = R(A) \subset V \otimes V$. We assume that A is finitely generated, so $\dim A_1 < \infty$. Thus $A = T(V)/(R)$ inherits its grading from the tensor algebra $T(V)$. The Koszul dual algebra of A, denoted by $A^!$ is the quadratic algebra $T(V^*)/(R^{\perp})$, see [29, 30].

Following a classical tradition (and a recent trend), we take a combinatorial approach to study A. The properties of A will be read off a presentation $A = \mathbf{k}\langle X\rangle/(\Re)$, where by convention X is a fixed finite set of generators of degree 1, $|X| = n$, $\mathbf{k}\langle X\rangle$ is the unital free associative algebra generated by X, and $(\Re)$ is the two-sided ideal of relations, generated by a *finite* set $\Re$ of homogeneous polynomials of degree two. Clearly A is a connected graded $\mathbf{k}$-algebra (naturally graded by length) $A = \bigoplus_{i \geq 0} A_i$, where $A_0 = \mathbf{k}$, A is generated by $A_1 = Span_{\mathbf{k}} X$, so each A_i is finite dimensional. A quadratic algebra A is *a PBW algebra* if there exists an enumeration of X, $X = \{x_1, \cdots x_n\}$ such that the quadratic relations $\Re$ form a (noncommutative) Gröbner basis with respect to the degree-lexicographic ordering $<$ on $\langle X\rangle$ induced from $x_1 < x_2 < \cdots < x_n$. In this case the set of normal monomials (mod $\Re$) forms a $\mathbf{k}$-basis of A called a *PBW basis* and $x_1, \cdots, x_n$ (taken

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exactly with this enumeration) are called *PBW-generators of A*. The notion of a PBW algebra was introduced by Priddy, [33], his *PBW basis* is a generalization of the classical Poincaré-Birkhoff-Witt basis for the universal enveloping of a finite dimensional Lie algebra. PBW algebras form an important class of Koszul algebras. The interested reader can find information on PBW algebras and more references in [32]. One of the central problems that we consider is the classification of Artin-Schelter regular PBW algebras. (We shall often use the abbreviation "AS" instead of "Artin-Shelter"). The first question to be asked is: what can be said about PBW algebras with polynomial growth and finite global dimension?

Theorem 1.1. *Let $A = \boldsymbol{k}\langle X\rangle/(\Re)$ be a quadratic PBW algebra, where $X = \{x_1, x_2, \cdots, x_n\}$ is a set of PBW generators. The following are equivalent:*

(1) *A has polynomial growth and finite global dimension.*
(2) *A has exactly $\binom{n}{2}$ relations and finite global dimension.*
(3) *A has polynomial growth, exactly $\binom{n}{2}$ relations, and the leading monomial of each relation (w.r.t. deg-lex ordering) has the shape $xy, x \neq y$.*
(4) *The Hilbert series of A is*

$$H_A(z) = \frac{1}{(1-z)^n}.$$

(5) *There exists a permutation $y_1, \cdots, y_n$ of $x_1 \cdots x_n$, such that the set*

$$\mathcal{N} = \{y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \mid \alpha_i \geq 0 \text{ for } 1 \leq i \leq n\} \tag{1.1}$$

is a $\boldsymbol{k}$-basis of A.

Furthermore the class $\mathfrak{C}_n$ of all n-generated PBW algebras with polynomial growth and finite global dimension contains a unique (up to isomorphism) monomial algebra: $A^0 = \langle x_1, \cdots, x_n\rangle/(x_j x_i \mid 1 \leq i < j \leq n)$.

Note that $y_1, y_2, \cdots, y_n$ is a possibly "new" enumeration of X, which induces a degree-lexicographic ordering $\prec$ on $\langle X\rangle$ (with $y_1 \prec y_2 \prec \cdots \prec y_n$) different from the original ordering. The defining relations remain the same, but their leading terms w.r.t. $\prec$ may be different from the original ones, and $y_1, y_2, \cdots, y_n$ are not necessarily PBW generators of A. In the terminology of Gröbner bases, $\mathcal{N}$ is not necessarily *a normal basis* of A w.r.t. $\prec$. A class of AS regular PBW algebras of arbitrarily high global dimension n were introduced and studied in [16, 23, 18, 17]. These are *the binomial skew-polynomial rings*. It was shown in [23] that they are also closely related to the set-theoretic solutions of the Yang-Baxter equation (YBE).

More generally, we consider the so-called *quantum binomial algebras* introduced and studied in [18, 20]. These are quadratic algebras (not necessarily PBW) with square-free binomial relations. Each binomial skew-polynomial ring is an example of a quantum binomial PBW algebra. So it is natural to ask: which are the Artin-Schelter regular PBW algebras in the class of quantum binomial algebras? We prove that each quantum binomial PBW algebra with finite global dimension is a Yang-Baxter algebra and therefore a binomial skew-polynomial ring. This implies that in the class of quantum binomial algebras the three notions: an Artin-Schelter regular PBW algebra, a binomial skew-polynomial ring, and a Yang-Baxter algebra (in the sense of Manin) are equivalent. The main result of the paper is the following.

Theorem 1.2. *Let $A = \boldsymbol{k}\langle X\rangle/(\Re)$ be a quantum binomial algebra, $|X| = n$, let $A^!$ be its Koszul dual algebra. The following conditions are equivalent:*

(1) *A is a PBW algebra with finite global dimension.*
(2) *A is a PBW algebra with polynomial growth.*
(3) *A is an Artin-Schelter regular PBW algebra.*
(4) *A is a Yang-Baxter algebra, that is the set of relations $\Re$ defines canonically a solution of the Yang-Baxter equation.*
(5) *A is a binomial skew polynomial ring, with respect to an enumeration of X.*
(6)
$$\dim_{\boldsymbol{k}} A_3 = \binom{n+2}{3}.$$
(7)
$$\dim_{\boldsymbol{k}} A^!_3 = \binom{n}{3}.$$
(8) *The Hilbert series of A is*
$$H_A(z) = \frac{1}{(1-z)^n}.$$
(9) *The Koszul dual $A^!$ is a quantum Grassmann algebra of dimension n.*

Each of these conditions implies that A is Koszul and a Noetherian domain.

Consider now the intersection $\mathfrak{I}$ of the two classes quadratic algebras generated by the same set of generators X, $|X| = n$: the class $\mathfrak{C}_n$ of PBW algebras with polynomial growth and finite global dimension, and the class of quantum binomial algebras. Theorem 1.2 shows that $\mathfrak{I}$ consists of Artin-Shelter regular algebras of global dimension n or equivalently of Yang-Baxter algebras. Moreover, $\mathfrak{I}$ coincides with the class of binomial skew-polynomial rings (w.r.t. an enumeration of X). It follows from Theorem 1.2 that the problem of classifying Artin-Schelter regular PBW algebras with quantum binomial relations and global dimension n is equivalent to finding the classification of square-free set-theoretic solutions of YBE, (X, r), on sets X of order n. Even under these strong restrictions on the shape of the relations, the problem remains highly nontrivial. However, for reasonably small n (say $n \leq 10$) the square-free solutions of YBE (X, r) are known. A possible classification for general n can be based on the so called *multipermutation level* of the solutions, see [22].

The paper is organized as follows. In section 2 we recall some basic definitions and results used throughout the paper. In section 3 we study the general case of PBW algebras with finite global dimension and polynomial growth and prove Theorem 1.1. The approach is combinatorial. To each PBW algebra A we associate two finite oriented graphs. The first one is *the graph of normal words* $\Gamma_{\mathbf{N}}$ which determines the growth and the Hilbert series of A (this is a particular case of the Ufnarovski graph [38]). The second graph, $\Gamma_{\mathbf{W}}$, dual to $\Gamma_{\mathbf{N}}$, gives a precise information about *the global dimension* of the algebra A. We prove that all algebras in the class $\mathfrak{C}_n$ of n-generated PBW algebras with polynomial growth and finite global dimension determine a unique (up to isomorphism) graph of obstructions Γ_W, which is the complete oriented graph K_n without cycles. In section 4 we find some interesting combinatorial results on quantum binomial sets (X, r) and the corresponding quadratic algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$. We show that a quantum binomial set (X, r) of order n is a set-theoretic solution of the Yang-Baxter equation *iff* $\dim_{\mathbf{k}} \mathcal{A}_3 = \binom{n+2}{3}$. In section 5 we prove Theorem 1.2.

2. Preliminaries - some definitions and facts

The paper is a natural continuation of [17]. We shall use the terminology, notation and results from our previous works [16, 23, 17, 18, 20]. The reader acquainted with these may proceed to the next section.

2.1. **Artin-Schelter regular algebras.** A connected graded algebra A is called *Artin-Schelter regular* (or *AS regular*) if

(i) A has *finite global dimension* d, that is, each graded A-module has a free resolution of length at most d.
(ii) A has *finite Gelfand-Kirillov dimension*, meaning that the integer-valued function $i \mapsto \dim_{\mathbf{k}} A_i$ is bounded by a polynomial in i.
(iii) A is *Gorenstein*, that is, $Ext^i_A(\mathbf{k}, A) = 0$ for $i \neq d$ and $Ext^d_A(\mathbf{k}, A) \cong \mathbf{k}$.

AS regular algebras were introduced and studied first in [3, 4, 5]. Since then AS regular algebras and their geometry have intensively been studied. When $d \leq 3$ all regular algebras are classified. The problem of classification of regular algebras seems to be difficult and remains open even for regular algebras of global dimension 4. The study of Artin-Schelter regular algebras, their classification, and finding new classes of such algebras is one of the basic problems in noncommutative geometry. Numerous works on this topic appeared during the last two decades, see for example [4, 5], [7, 26, 37, 28, **?**], et all.

Definition 2.1. [16] A *binomial skew polynomial ring* is a quadratic algebra $A = \mathbf{k}\langle x_1, \cdots, x_n\rangle/(\Re)$ with precisely $\binom{n}{2}$ defining relations

$$\Re = \{x_j x_i - c_{ij} x_{i'} x_{j'}\}_{1\leq i<j\leq n}$$

such that

(a) For every pair $i, j,\ 1 \leq i < j \leq n$, the relation $x_j x_i - c_{ij} x_{i'} x_{j'} \in \Re$, satisfies $c_{ij} \in \mathbf{k}^{\times}$, $j > i'$, $i' < j'$.
(b) Every ordered monomial $x_i x_j$, with $1 \leq i < j \leq n$ occurs in the right hand side of some relation in $\Re$.
(c) $\Re$ is the *reduced Gröbner basis* of the two-sided ideal $(\Re)$, with respect to the order $<$ on $\langle X\rangle$, or equivalently the ambiguities $x_k x_j x_i$, with $k > j > i$ do not give rise to new relations in A.

We say that $\Re$ are *relations of skew-polynomial type* if conditions 2.1 (a) and (b) are satisfied (we do not assume (c)).

By [6] condition 2.1 (c) may be rephrased by saying that *the set of ordered monomials*

$$\mathcal{N}_0 = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_n \geq 0 \text{ for } 1 \leq i \leq n\}$$

is a $\mathbf{k}$-basis of A.

The binomial skew polynomial rings were introduced and studied first in [15, 16], where by definition they satisfy conditions (a) and (c) of Definition 2.1 but not necessarily (b) (see [16], Definition 1.11). It was proven that condition (b) is essential for the "good" algebraic and homological properties of the skew polynomial rings, so in our later works (b) became a part of the definition. We recall first the following result which can be extracted from [16], Theorem II.

Fact 2.2. *Assume that $A = \boldsymbol{k}\langle x_1, \cdots, x_n\rangle/(\Re)$ is a quadratic algebra, with relations $\Re$ satisfying conditions (a) and (c) of Definition 2.1. Then the following three* *conditions are equivalent: (i) A is left Noetherian; (ii) A is right Noetherian; (iii) Condition (b) of Definition 2.1 holds.*

Remark 2.3. In the terminology of this paper *a binomial skew polynomial ring* is a PBW algebra A with PBW generators $x_1 \cdots , x_n$ and relations of skew-polynomial type.

More generally, we will consider a class of quadratic algebras with binomial relations, so-called *quantum binomial algebras*, these are not necessarily PBW algebras.

2.2. **Quadratic sets. Quantum binomial algebras.** Quadratic sets were introduced and studied in the context of set-theoretic solutions of YBE, see [20]. They are also closely related to quadratic algebras with binomial relations.

Definition 2.4. Let X be a nonempty set and let $r : X \times X \longrightarrow X \times X$ be a bijective map. In this case we shall use notation (X, r) and refer to it as a *quadratic set*. We present the image of (x, y) under r as

$$r(x, y) = ({}^{x}y, x^{y}). \tag{2.1}$$

Formula (2.1) defines a "left action" $\mathcal{L} : X \times X \longrightarrow X$, and a "right action" $\mathcal{R} : X \times X \longrightarrow X$, on X as:

$$\mathcal{L}_x(y) = {}^{x}y, \quad \mathcal{R}_y(x) = x^{y}, \quad \text{for all } x, y \in X.$$

(1) r is *nondegenerate* if the maps $\mathcal{L}_x$ and $\mathcal{R}_x$ are bijective for each $x \in X$.
(2) r is *involutive* if $r^2 = id_{X\times X}$.
(3) (X, r) is said to be *square-free* if $r(x, x) = (x, x)$ for all $x \in X$.
(4) (X, r) is called a *quantum binomial set* if it is nondegenerate, involutive and square-free.
(5) (X, r) is a *set-theoretic solution of the Yang-Baxter equation* (YBE) if the braid relation
$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$
holds in $X \times X \times X$, where $r^{12} = r \times \mathrm{id}_X$ and $r^{23} = \mathrm{id}_X \times r$. In this case we refer to (X, r) also as *a braided set*.
(6) A braided set (X, r) with r involutive is called *a symmetric set*.

More generally, let V be a $\mathbf{k}$-vector space. Recall that a linear automorphism R of $V \otimes V$ is *a solution of the Yang-Baxter equation*, (YBE) if the equality

$$R^{12}R^{23}R^{12} = R^{23}R^{12}R^{23} \tag{2.2}$$

holds in the automorphism group of $V \otimes V \otimes V$, where R^{ij} means R acting on the i-th and j-th component. Each braided set (X, r) extends canonically to a linear solution R of YBE defined on $V \otimes V$, where $V = Span_{\mathbf{k}}X$.

Set-theoretic solutions were introduced in [11, 39] and have been under intensive study during the last decade. There are numerous works on set-theoretic solutions and related structures, of which a relevant selection for the interested reader is [39, 23, 34, 12, 27, 18, 17, 21, 24, 35, 36, 20, 21, 22, 8].

As a notational tool, we shall often identify the sets $X^{\times m}$ of ordered m-tuples, $m \geq 2$, and X^m, the set of all monomials of length m in the free monoid $\langle X\rangle$.

To each quadratic set (X, r) we associate canonically algebraic objects generated by X and with quadratic relations $\Re_0 = \Re_0(r)$ naturally determined as

$$xy = y'x' \in \Re_0(r) \text{ iff } r(x, y) = (y', x') \text{ and } (x, y) \neq (y', x') \text{ hold in } X \times X.$$

The monoid $S = S(X,r) = \langle X;\ \Re_0(r)\rangle$ with a set of generators X and a set of defining relations $\Re_0(r)$ is called *the monoid associated with* (X,r). The *group* $G = G(X,r)$ *associated with* (X,r) is defined analogously. For an arbitrary fixed field $\mathbf{k}$, *the* $\mathbf{k}$*-algebra associated with* (X,r) is defined as

$$\begin{gathered}\mathcal{A} = \mathcal{A}(\mathbf{k},X,r) = \mathbf{k}\langle X\rangle/(\Re) \simeq \mathbf{k}\langle X;\ \Re_0(r)\rangle, \text{ where}\\ \Re = \Re(r) = \{xy - y'x' \mid xy = y'x' \in \Re_0(r)\}.\end{gathered}$$

Clearly, the quadratic algebra $\mathcal{A}$ generated by X and with defining relations $\Re(r)$ is isomorphic to the monoid algebra $\mathbf{k}S(X,r)$.

Remark 2.5. The number of relations in $\Re(r)$ depends on r and its properties. For example, if $r = id_{X^2}$ then $\Re(r) = \emptyset$ and $\mathcal{A}$ is the free associative algebra. It follows from [19], Proposition 2.3, that the set $\Re(r)$ consists of precisely $\binom{n}{2}$ quadratic relations whenever (X,r) is nondegenerate and involutive, with $|X| = n$.

In many cases the associated algebra will be *standard finitely presented* with respect to the degree-lexicographic ordering induced by an appropriate enumeration of X, that is, it will be a PBW algebra.

Definition 2.6. Let $V = Span_{\mathbf{k}}X$. Let $\Re \subset \mathbf{k}\langle X\rangle$ be a set of quadratic binomials, satisfying the following conditions:

- **B1** Each $f \in \Re$ has the shape $f = xy - c_{yx}y'x'$, where $c_{xy} \in \mathbf{k}^{\times}$ and $x,y,x',y' \in X$.
- **B2** Each monomial xy of length 2 occurs at most once in $\Re$.

The canonically *associated quadratic set* $(X, r(\Re))$ with an involutive bijection $r = r(\Re) : X \times X \longrightarrow X \times X$ is defined as

$$\begin{aligned}&r(x,y) = (y',x'), \text{ and } r(y',x') = (x,y) \text{ iff } xy - c_{xy}y'x' \in \Re.\\ &r(x,y) = (x,y) \text{ iff } xy \text{ does not occur in } \Re.\end{aligned}$$

The (involutive) automorphism $R = R(\Re) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$ *associated with* $\Re$ is defined analogously.

$$\begin{aligned}&R(x\otimes y) = c_{xy}y'\otimes x', \text{ and } R(y'\otimes x') = (c_{xy})^{-1}x\otimes y \text{ iff } xy - c_{xy}y'x' \in \Re.\\ &R(x\otimes y) = x\otimes y \text{ iff } xy \text{ does not occur in } \Re.\end{aligned}$$

R is called *nondegenerate* if r is nondegenerate.

Definition 2.7. Suppose $A = \mathbf{k}\langle X\rangle/(\Re)$ is a quadratic algebra with binomial relations $\Re$ satisfying conditions **B1** and **B2**. A is said to be *a quantum binomial algebra* if the associated quadratic set $(X, r(\Re))$ is quantum binomial. In this case we say that $\Re$ is a set of *quantum binomial relations.* A is *a Yang-Baxter algebra* (in the sense of Manin [30]), if the associated map $R = R(\Re) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$, is a solution of the Yang-Baxter equation.

Note that (although this is not a part of the definition) every n-generated quantum binomial algebra has exactly $\binom{n}{2}$ relations, see Remark 2.5.

Remark 2.8. Each binomial skew-polynomial ring A is a quantum binomial PBW algebra. Indeed A has square-free defining relations satisfying conditions **B1** and **B2**, so the associated (X,r) is square-free and involutive. Furthermore, A satisfies the Ore conditions, see [23], which imply non-degeneracy of r.

The results below can be extracted from [23], [16], and [17], Theorem B.

Fact 2.9. *Let $A = \boldsymbol{k}\langle X\rangle/(\Re)$ be a quantum binomial algebra. Then the following two conditions are equivalent:*

(1) *A is a binomial skew polynomial ring, with respect to some appropriate enumeration of X.*
(2) *The automorphism $R = R(\Re) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$ is a solution of the Yang-Baxter equation, so A is a Yang-Baxter algebra.*

Each of these conditions implies that A is an Artin-Schelter regular PBW algebra. Furthermore A is a left and right Noetherian domain.

2.3. **Examples.** We end up the section with three concrete examples of quadratic algebras.

Example 2.10. Let $A = \mathbf{k}\langle X\rangle/(\Re)$, where $X = \{x, y, z, t\}$ and

$$\Re \quad = \quad \{xy - zt, \quad ty - zx, \quad xz - yx, \quad tz - yt, \quad xt - tx, \quad yz - zy\}.$$

A is a quantum binomial algebra. Indeed, the relations are square-free and satisfy conditions **B1** and **B2**, so the associated quadratic set (X, r) is involutive and square-free. A direct verification shows that r is nondegenerate. More sophisticated proof shows that the set of relations $\Re$ is not a Gröbner basis w.r.t. deg-lex ordering coming from any order (enumeration) of the set X. This example is studied with details in Section 4 and illustrates various statements there.

Example 2.11. Let $A = \mathbf{k}\langle X\rangle/(\Re)$, where $X = \{x, y, z, t\}$ and

$$\Re \quad = \quad \{xy - zt, \quad ty - zx, \quad xz - yt, \quad tz - yx, \quad xt - tx, \quad yz - zy\}.$$

We fix $t > x > z > y$ and take the corresponding deg-lex ordering on $\langle X\rangle$. A direct verification shows that $\Re$ is a Gröbner basis. To do this one has to show that the ambiguities txz, txy, tzy, xzy are solvable. In this case the set

$$\mathcal{N} = \{y^\alpha z^\beta x^\gamma t^\delta \mid \alpha, \beta, \gamma, \delta \geq 0\}$$

is the normal basis of A modulo $\Re$. Hence A is a binomial skew-polynomial ring and therefore A is a Yang-Baxter algebra and an AS-regular domain of global dimension 4, see Fact 2.9. Note that any order in which $\{t, x,\} > \{z, y\}$, or $\{z, y\} > \{t, x,\}$, makes A a PBW algebra (a skew polynomial ring), there are exactly eight such enumerations of X.

Example 2.12. [3] Let $V = Span_{\mathbf{k}} X, A = \mathbf{k}\langle X\rangle/(\Re)$, where $X = \{x, y, z\}$ and

$$\Re \quad = \quad \{[x, y] - z^2, \quad [y, z] - x^2, \quad [z, x] - y^2\}.$$

This is a classical example of a quadratic AS-regular algebra of global dimension 3 (type A). The algebra A has Hilbert function $\dim A_l = \binom{l+2}{2}$, [3], hence it has "the correct" Hilbert series. The algebra A is Koszul and a Noetherian domain, [4, 5]. Note that the Koszul dual $A^!$ is a quantum Grassman algebra. This follows from Fact 5.1 and the "good" shape of the Hilbert function of A. However, a direct computation shows that the set $\Re$ is not a Gröbner basis w.r.t. deg-lex ordering induced by any order (enumeration) of X. The relations canonically define an involutive automorphism $R = R(\Re) : V^{\bigotimes 2} \longrightarrow V^{\bigotimes 2}$, where:

$$R(z \otimes z) = x \otimes y - y \otimes x, \quad R(x \otimes x) = y \otimes z - z \otimes y, \quad R(y \otimes y) = z \otimes x - x \otimes z.$$

We have verified via direct computation again, that R does not satisfy YBE (2.2).

3. PBW algebras with polynomial growth and finite global dimension

Let $X = \{x_1, \cdots x_n\}$. As usual, we fix the deg-lex ordering $<$ on $\langle X\rangle$. Each element $g \in \mathbf{k}\langle X\rangle$ has the shape $g = cu + h$, where $u \in \langle X\rangle$, $c \in \mathbf{k}^{\times}, h \in \mathbf{k}\langle X\rangle$, and either $h = 0$, or $h = \sum_{\alpha} c_{\alpha}u_{\alpha}$, is a linear combination of monomials $u_{\alpha} < u$. u is called the leading monomial of g (w.r.t. $<$) and is denoted by $LM(g)$. Let $A = \mathbf{k}\langle X\rangle/I$ be a finitely presented algebra, where I is a finitely generated ideal of $\mathbf{k}\langle X\rangle$. It is known that the ideal I has a uniquely determined (w.r.t. $<$) reduced Gröbner basis $\mathbf{G}$. In general $\mathbf{G}$ may be infinite. Consider the set $\mathbf{W} = \{LM(f) \mid f \in \mathbf{G}\}$ of all leading monomials of the elements of $\mathbf{G}$.

It is straightforward that $\mathbf{W}$ coincides with a certain set of "*obstructions*" introduced by Anick, [2]. We shall follow Anick's terminology and use the term "obstructions" instead of "leading monomials of the elements of G". The set of obstructions and the elements of $\mathbf{G}$ are involved in the construction of the now famous Anick's resolution, [1, 2]. It is a free resolution of the field $\mathbf{k}$ considered as an A-module.

Consider now a PBW algebra $A = \mathbf{k}\langle X\rangle/(\Re)$ with PBW generators $X = \{x_1, \cdots x_n\}$. Then the set of defining relations $\Re$ coincides with the reduced Gröbner basis of the ideal $(\Re)$, so

$$\mathbf{W} = \{LM(f) \mid f \in \Re\},$$

and $\mathbf{N} = X^2\backslash\mathbf{W}$ is the set of *normal monomials* (mod $\mathbf{W}$) of length 2.

Notation 3.1. We set

$$\mathbf{N}^{(0)} = \{1\}, \quad \mathbf{N}^{(1)} = X,$$

$$\mathbf{N}^{(m)} = \{x_{i_1}x_{i_2}\cdots x_{i_m} \mid x_{i_k}x_{i_{k+1}} \in \mathbf{N}, \ \ 1 \leq k \leq m-1\}, \ \ m = 2, 3, \cdots$$

$$\mathbf{N}^{\infty} = \textstyle\bigcup_{m\geq 0}\mathbf{N}^{(m)}.$$

Note that $\mathbf{N}^{\infty}$ is exactly the set of all normal (mod $\mathbf{W}$) words in $\langle X\rangle$. It is well-known that $\mathbf{N}^{\infty}$ projects to a basis of A, [6]. More precisely, the free associative algebra $\mathbf{k}\langle X\rangle$ splits into a direct sum of subspaces

$$\mathbf{k}\langle X\rangle \simeq Span_{\mathbf{k}}\mathbf{N}^{\infty} \bigoplus I.$$

So there are isomorphisms of vector spaces

$$A \simeq Span_{\mathbf{k}}\mathbf{N}^{\infty}$$

$$A_m \simeq Span_{\mathbf{k}}\mathbf{N}^{(m)}, \ \dim A_m = |\mathbf{N}^{(m)}|, \ m = 0, 1, 2, 3, \cdots$$

For a PBW algebra A there is a canonically associated monomial algebra $A^0 = \mathbf{k}\langle X\rangle/(\mathbf{W})$. As a monomial algebra, A^0 is also PBW. Both algebras A and A^0 have the same set of obstructions $\mathbf{W}$ and therefore they have the same *normal basis* $\mathbf{N}^{\infty}$, the same Hilbert series and the same growth. It follows from results of Anick that $\mathrm{gl.dim}\, A = \mathrm{gl.dim}\, A^0$. More generally, the set of obstructions $\mathbf{W}$ determines uniquely the Hilbert series, the growth and the global dimension for the whole family of PBW algebras A sharing the same $\mathbf{W}$. The binomial skew polynomial rings are a well-known example of PBW algebras with polynomial growth and finite global dimension, moreover, they are AS regular Noetherian domains, see [23].

Definition 3.2. Let $M \subset X^2$ be a set of monomials of length 2. We define *the graph* Γ_M *corresponding to* M as *a directed graph* with a set of vertices $V(\Gamma_M) = X$ and a set of directed edges (arrows) $E = E(\Gamma_M)$ defined as

$$x \longrightarrow y \in E \text{ iff } x, y \in X, \text{ and } xy \in M.$$

Denote by $\overline{M}$ the complement $X^2 \backslash M$. Then the graph $\Gamma_{\overline{M}}$ is *dual to* Γ_M in the sense that

$$x \longrightarrow y \in E(\Gamma_{\overline{M}}) \text{ iff } x \longrightarrow y \text{ is not an edge of } \Gamma_M.$$

We recall that *the order* of a graph Γ is the number of its vertices, $|V(\Gamma)|$, so Γ_M is a graph of order $|X|$. A path of length $k-1$ in Γ_M is a sequence of edges $v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_k$, where $v_i \longrightarrow v_{i+1} \in E$. *A cycle (of length k)* in Γ is a path of the shape $v_1 \longrightarrow v_2 \longrightarrow \cdots v_k \longrightarrow v_1$ where $v_1, \cdots, v_k$ are distinct vertices. *A loop* is a cycle of length 0, $x \longrightarrow x$. So the graph Γ_M contains a loop $x \longrightarrow x$ whenever $xx \in M$ and a cycle of length two $x \longrightarrow y \longrightarrow x$, whenever $xy, yx \in M$. In this case $x \longrightarrow y, y \longleftarrow x$ are called bidirected edges. Note that following the terminology in graph theory *we make difference between directed and oriented graphs.* A directed graph having no bidirected edges is known *as an oriented graph.* An oriented graph without cycles is called *an acyclic oriented* graph. In particular, such a graph has no loops.

Let A be a PBW algebra, let $\mathbf{W}$ and $\mathbf{N}$ be the set of obstructions and the set of normal monomials of length 2, respectively. The graph $\Gamma_{\mathbf{N}}$ called *the graph of normal words of A*, was introduced in more general context by Ufnarovski, [38]. It gives a complete information about the growth of A. The global dimension of A, can be read off its *graph of obstructions* $\Gamma_{\mathbf{W}}$. Note that, in general, $\Gamma_{\mathbf{N}}$ is a directed graph which may contain bidirected edges, so $\Gamma_{\mathbf{N}}$ is *not necessarily an oriented graph.* Similarly, its dual graph $\Gamma_{\mathbf{W}}$ is a directed graph, which, in general, may contain pairs of bidirected edges, or loops.

The following is a particular case of a more general result of Ufnarovski.

Fact 3.3. [38] *Let A be a PBW algebra.*

(1) *For every $m \geq 1$ there is a one-to-one correspondence between the set $\boldsymbol{N}^{(m)}$ of normal words of length m and the set of paths of length $m-1$ in the graph $\Gamma_{\boldsymbol{N}}$. The path $a_1 \longrightarrow a_2 \longrightarrow a_2 \longrightarrow \cdots \longrightarrow a_m$ (these are not necessarily distinct vertices) corresponds to the word $a_1a_2\cdots a_m \in \boldsymbol{N}^{(m)}$.*
(2) *A has exponential growth* iff *the graph $\Gamma_{\boldsymbol{N}}$ has two intersecting cycles.*
(3) *A has polynomial growth of degree m* iff *$\Gamma_{\boldsymbol{N}}$ has no intersecting cycles and m is the largest number of (oriented) cycles occurring in a path of $\Gamma_{\boldsymbol{N}}$.*

Example 3.4. All binomial skew-polynomial algebras A with five PBW generators $x_1, x_2, \cdots, x_5$ have the same graph $\Gamma_{\mathbf{N}}$ as in Figure 1. The graph of obstruction $\Gamma_{\mathbf{W}}$ for A can be seen in Figure 2. The graph of normal words for the Koszul dual $A^!$ denoted by $\Gamma_{\mathbf{N}^!}$ is represented in Figure 3. The graphs in Figure 2 and Figure 3 are acyclic tournaments.

It is straightforward from Anick's general definition of an m-chain, see [1, 2] that in the case of PBW algebras, each m-chain is a monomial of length $m+1$, $y_{m+1}y_m\cdots y_1$, where $y_{i+1}y_i \in \mathbf{W}$, $1 \leq i \leq m$. (For completeness, the 0-*chains* are the elements of X, by definition). This implies that for every $m \geq 1$ there is a one-to-one correspondence between the set of m-chains, in the sense of Anick, and the set of paths of length m in the directed graph $\Gamma_{\mathbf{W}}$. The m-chain $y_{m+1}y_m\cdots y_1$, with $y_{i+1}y_i \in \mathbf{W}$, $1 \leq i \leq m$, corresponds to the path $y_{m+1} \longrightarrow y_m \longrightarrow \cdots \longrightarrow y_1$ of length m in $\Gamma_{\mathbf{W}}$. Note that Anick's resolution [1, 2] is minimal for PBW algebras and for monomial algebras (not necessarily quadratic), and therefore a PBW algebra A has finite global dimension $d < \infty$ *iff* there is a $(d-1)$-chain, but there are no d-chains in $\langle X \rangle$. We "translate" this in terms of the properties of $\Gamma_{\mathbf{W}}$.

Lemma 3.5. *A PBW algebra A has finite global dimension d* iff $\Gamma_{\boldsymbol{W}}$ *is an acyclic oriented graph and $d-1$ is the maximal length of a path occurring in* $\Gamma_{\boldsymbol{W}}$.

All PBW algebras with the same set of PBW generators $x_1, \cdots, x_n$ and the same sets of obstructions, $\mathbf{W}$, share the same graphs $\Gamma_{\mathbf{N}}$ and $\Gamma_{\mathbf{W}}$. In some cases it is convenient to study the corresponding monomial algebra A^0 instead of A.

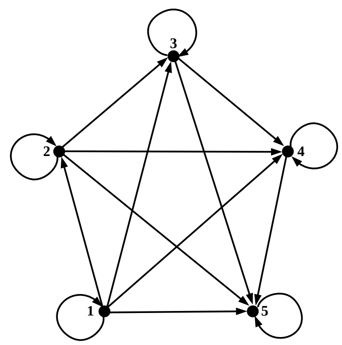


FIGURE 1. This is the graph of normal words $\Gamma_{\mathbf{N}}$ for a PBW algebra A with 5 generators, polynomial growth and finite global dimension.

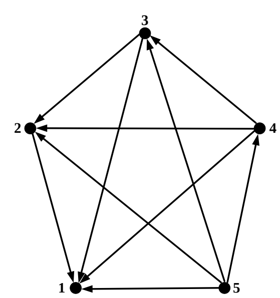


FIGURE 2. This is the graph of obstructions $\Gamma_{\mathbf{W}}$, dual to $\Gamma_{\mathbf{N}}$. It is **an acyclic tournament of order 5**, labeled "properly", as in (3.1).

Definition 3.6. A complete oriented graph Γ is called *a tournament or tour.*

In other words, a tournament is a directed graph in which each pair of vertices is joined by a single edge having a unique direction. Clearly, a complete directed graph without cycles (of any length) is *an acyclic tournament*. An acyclic oriented graph with n vertices is a tournament *iff* it has exactly $\binom{n}{2}$ (directed) edges. The following is well-known in graph theory.

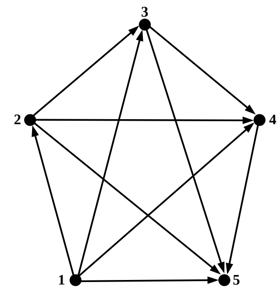


FIGURE 3. This is the graph of normal words $\Gamma^!$ for the Koszul dual $A^!$. It is **an acyclic tournament of order 5**

.

Remark 3.7. Let Γ be an acyclic tournament of order n. Then the set of its vertices $V = V(\Gamma)$ can be labeled $V = \{y_1, y_2, \cdots, y_n\}$, so that the set of edges is

$$E(\Gamma) = \{y_j \longrightarrow y_i \mid 1 \leq i < j \leq n\}. \tag{3.1}$$

Analogously, the vertices can be labeled $V = \{z_1, z_2, \cdots, z_n\}$, so that $E(\Gamma) = \{z_i \longrightarrow z_j \mid 1 \leq i < j \leq n\}$.

A^0 is *a quadratic monomial algebra* if it has a presentation $A^0 = \mathbf{k}\langle X\rangle/(W)$, where W is a set of monomials of length 2. Any quadratic monomial algebra A^0 is a PBW algebra, moreover, any enumeration $x_1, \cdots, x_n$ of X gives PBW generators of A^0.

Theorem 3.8. *Let $A^0 = \boldsymbol{k}\langle x_1 \cdots, x_n\rangle/(\boldsymbol{W})$ be a quadratic monomial algebra. The following conditions are equivalent:*

(1) *A^0 has finite global dimension and polynomial growth.*
(2) *A^0 has finite global dimension and $|\boldsymbol{W}| = \binom{n}{2}$.*
(3) *A^0 has polynomial growth, $\boldsymbol{W} \bigcap \operatorname{diag} X^2 = \emptyset$, and $|\boldsymbol{W}| = \binom{n}{2}$.*
(4) *The graph $\Gamma_{\boldsymbol{W}}$ is an acyclic tournament.*
(5)
$$H_{A^0}(z) = \frac{1}{(1-z)^n}.$$
(6) *There is a permutation $y_1, \cdots, y_n$ of $x_1 \cdots, x_n$ such that*
$$\boldsymbol{N}^\infty = \{y_1^{\alpha_1} \cdots y_n^{\alpha_n} \mid \alpha_i \geq 0,\ 1 \leq i \leq n\}. \tag{3.2}$$
(7) *There is a permutation $y_1, \cdots, y_n$ of $x_1 \cdots, x_n$, such that*
$$\boldsymbol{W} = \{y_j y_i \mid\ 1 \leq i < j \leq n\}.$$

Furthermore, in this case

$$\operatorname{gl.dim} A^0 = n = \text{ the degree of the polynomial growth of } A.$$

Proof. Condition (4) is central for our proof.

The following are straightforward

$$(4) \Longrightarrow (1); \quad (4) \Longrightarrow (3); \quad (7) \Longrightarrow (3); \quad (7) \Longrightarrow (4);$$
$$(7) \Longleftrightarrow (6) \Longrightarrow (5).$$

(4) $\Longrightarrow$ (7). Suppose (4) holds, so $\Gamma_{\mathbf{W}}$ is an acyclic tournament. By Remark 3.7 the set of its vertices $V = V(\Gamma_{\mathbf{W}})$ can be relabeled $V = \{y_1, y_2, \cdots, y_n\}$, so that the set of edges satisfies (3.1), this implies (7).

(4) $\Longrightarrow$ (2). Suppose (4) holds. As an acyclic tournament $\Gamma_{\mathbf{W}}$ contains exactly $\binom{n}{2}$ edges and therefore $|\mathbf{W}| = \binom{n}{2}$. Chose a "good" labeling of $\Gamma_{\mathbf{W}}$, such that (3.1) holds. Then the set $E(\Gamma_{\mathbf{W}})$ contains the edges $y_j \longrightarrow y_{j-1}, 2 \leq j \leq n$, so the graph $\Gamma_{\mathbf{W}}$ has a path $y_n \longrightarrow y_{n-1} \longrightarrow \cdots \longrightarrow y_1$ of length $n-1$. Clearly, there are no longer paths in $\Gamma_{\mathbf{W}}$, so by Lemma 3.5 $\operatorname{gl.dim} A^0 = n$.

(2) $\Longrightarrow$ (4). Assume (2). Clearly, $\Gamma_{\mathbf{W}}$ has exactly $\binom{n}{2}$ edges. Lemma 3.5 and $\operatorname{gl.dim} A^0 < \infty$ imply that $\Gamma_{\mathbf{W}}$ is an acyclic oriented graph, so, $\Gamma_{\mathbf{W}}$ is an acyclic tournament.

(3) $\Longrightarrow$ (4). Assume (3) holds. Then $\Gamma_{\mathbf{W}}$ has exactly $\binom{n}{2}$ edges, where each edge has the shape $x \longrightarrow y$, $x \neq y$. Therefore its dual graph $\Gamma_{\mathbf{N}}$ has a loop $x \longrightarrow x$ at every vertex and exactly $\binom{n}{2}$ edges $y \longrightarrow x$, where $x \longrightarrow y \in E(\Gamma_{\mathbf{W}})$. The polynomial growth of A^0 implies that $\Gamma_{\mathbf{N}}$ has no cycles of length ≥ 2, therefore every two vertices in $\Gamma_{\mathbf{N}}$ are connected with a single directed edge, so $\Gamma_{\mathbf{N}}$ is an oriented graph. It follows then that $\Gamma_{\mathbf{W}}$ is an acyclic oriented tournament.

(5) $\Longrightarrow$ (4). Note first that

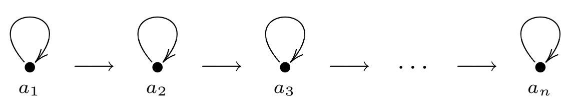
$$H_{A^0}(z) = \frac{1}{(1-z)^n} = 1 + nz + \binom{n+1}{2} z^2 + \binom{n+2}{3} z^3 + \cdots . \tag{3.3}$$

So

$$\dim A_2 = |\mathbf{N}| = \binom{n+1}{2}, \quad \text{which implies} \quad |\mathbf{W}| = \binom{n}{2}.$$

Secondly, the special shape of Hilbert series $H_{A^0}(z)$ implies that A^0 has polynomial growth of degree n. Therefore by Fact 3.3 the graph $\Gamma_{\mathbf{N}}$ contains a path with n cycles. The only possibility for such a path is to have n distinct vertices and a loop at every vertex:

(3.4)



Indeed $\Gamma_{\mathbf{N}}$ has exactly n vertices and possesses no intersecting cycles. Each loop $x \longrightarrow x$ in $\Gamma_{\mathbf{N}}$ implies $xx \in N$, so $\Delta_2 \subset \mathbf{N}$ ($\Delta_2 = \operatorname{diag}(X^2)$). Then the complement $\mathbf{N} \backslash \Delta_2$ contains exactly $\binom{n}{2}$ monomials of the shape $xy, x \neq y$, or equivalently, $\Gamma_{\mathbf{N}}$ has $\binom{n}{2}$ edges of the shape $x \longrightarrow y, x \neq y$. Clearly, $E(\Gamma_{\mathbf{N}})$ does not contain bidirected edges otherwise the graph will have two intersecting cycles $x \longrightarrow x$ and $x \longrightarrow y \longrightarrow x$ which is impossible since A^0 has polynomial growth. Therefore $\Gamma_{\mathbf{N}}$ is an oriented graph. Consider now the dual graph $\Gamma_{\mathbf{W}}$. The properties of $\Gamma_{\mathbf{N}}$ imply that: (a) $\Gamma_{\mathbf{W}}$ has no loops; (b) $\Gamma_{\mathbf{W}}$ is without cycles of length ≥ 2. Each edge $x \longrightarrow y$ in $E(\Gamma_{\mathbf{N}})$ has a corresponding edge $x \longleftarrow y \in E(\Gamma_{\mathbf{W}})$. So $\Gamma_{\mathbf{W}}$ is an acyclic oriented graph with $\binom{n}{2}$ edges, therefore it is an acyclic tournament.

(1) $\Longrightarrow$ (4). Assume that A^0 has polynomial growth and finite global dimension. We shall use once more the nice balance between the dual graphs $\Gamma_{\mathbf{W}}$ and $\Gamma_{\mathbf{N}}$. The graph $\Gamma_{\mathbf{N}}$ is without intersecting cycles (otherwise A would have exponential growth). By Lemma 3.5 $\Gamma_{\mathbf{W}}$ is also acyclic, therefore it is an acyclic oriented graph. In particular, $\Gamma_{\mathbf{W}}$ has no loops, therefore its dual graph $\Gamma_{\mathbf{N}}$ has loops $x \longrightarrow x$ at every vertex. $\Gamma_{\mathbf{W}}$ has no bidirected edges either, thus for each each pair $x \neq y$ of

vertices, its dual $\Gamma_{\mathbf{N}}$ contains at least one of the edges $x \longrightarrow y$, or $y \longrightarrow x$. The algebra A^0 has polynomial growth, hence $\Gamma_{\mathbf{N}}$ has no bidirected edges. It follows then that $\Gamma_{\mathbf{W}}$ is an acyclic oriented graph with $\binom{n}{2}$ edges, therefore it is an acyclic tournament. □

Remark 3.9. The implication (1) $\Longrightarrow$ (5) also follows straightforwardly from a result of Anick, see [1] Theorem 6.

Proof of Theorem 1.1. Assume now that $A = \mathbf{k}\langle X\rangle/(\Re)$ is a quadratic PBW algebra, with PBW generators $X = \{x_1, \cdots x_n\}$. Let $\mathbf{W}$ be the set of obstructions and let $A^0 = \mathbf{k}\langle X\rangle/(\mathbf{W})$ be the corresponding monomial algebra. As we have noticed before, the two algebras share the same set of obstruction, so the set $\mathbf{N}^\infty$ of normal (mod $\mathbf{W}$) monomials is a $\mathbf{k}$-basis for both algebras A and A^0, they have the same Hilbert series, equal degrees of growth and equal global dimension.

(2) $\Longleftrightarrow$ (3). This is straightforward from Theorem 3.8.

(1) $\Longrightarrow$ (5). Suppose A has finite global dimension and polynomial growth. Then the same is valid for A^0. By Theorem 3.8.6 there is a permutation $y_1, \cdots, y_n$ of $x_1 \cdots, x_n$, such that

$$\mathbf{N}^\infty = \{y_1^{\alpha_1} \cdots y_n^{\alpha_n} \mid \alpha_i \geq 0,\ 1 \leq i \leq n\},$$

so A has a $\mathbf{k}$-basis of the desired form. (In general, it is not true that $\mathbf{N}^\infty$ is *a normal basis* for A w.r.t. the deg-lex ordering $\prec$ defined via $y_1 \prec \cdots \prec y_n$).

(5) $\Longrightarrow$ (4) is clear.

(4) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (2). Assume (4) holds. Then obviously A has polynomial growth of degree n. The equalities

$$H_{A^0}(z) = H_A(z) = \frac{1}{(1-z)^n}$$

and Theorem 3.8 imply that the monomial algebra A^0 has $\operatorname{gl.dim} A^0 = n$, and $\mid W \mid = \binom{n}{2}$. Clearly, then A has $\binom{n}{2}$ relations and global dimension n.

(2) $\Longrightarrow$ (4). Condition (2) is satisfied simultaneously by A and A^0, so, by Theorem 3.8, the Hilbert series $H_{A^0}(z)$ has the desired form and therefore (4) is in force. □

4. Combinatorics in quantum binomial sets

In this section (X, r) is a quantum binomial set of arbitrary cardinality. When we consider only finite sets X this will be clearly indicated. As usual, $S = S(X, r)$, $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) \simeq \mathbf{k}\ [S]$, respectively, denote the monoid and the quantum binomial $\mathbf{k}$-algebra, associated with (X, r).

We shall consider the action of the group

$$\mathcal{D} = {}_{\mathrm{gr}}\langle r^{12}, r^{23}\rangle$$

on the set $X \times X \times X$, or equivalently on X^3. Two monomials $\omega, \omega' \in X^3$ are equal as elements of S *iff* they belong to the same orbit of $\mathcal{D}$ in X^3. Clearly $r^2 = 1$ implies $(r^{12})^2 = (r^{23})^2 = 1$, so $\mathcal{D}$ is a dihedral group, finite or infinite depending on the order of the element $r^{12}r^{23}$. More precisely, $\mathcal{D}$ is the dihedral group of order $2m$ *iff* $r^{12}r^{23}$ has order $m < \infty$. We shall find some counting formulae and inequalities involving the orders of the $\mathcal{D}$-orbits in X^3 and their number, see Proposition 4.6. When X is finite we use these to find necessary and sufficient conditions for (X, r)

to be a symmetric set, Proposition 4.10, and to give upper bounds for $\dim A_3$ and $\dim A_3^!$ in the general case of a quantum binomial algebra A, Corollary 4.13.

As usual, the orbit of a monomial $\omega \in X^3$ under the action of $\mathcal{D}$ will be denoted by $\mathcal{O} = \mathcal{O}(\omega)$. Denote by Δ_i the diagonal of $X^{\times i}, 2 \leq i \leq 3$. One has $\Delta_3 = (\Delta_2 \times X) \bigcap (X \times \Delta_2)$.

Definition 4.1. We call a $\mathcal{D}$-orbit $\mathcal{O}$ *square-free* if

$$\mathcal{O} \bigcap (\Delta_2 \times X \bigcup X \times \Delta_2) = \emptyset.$$

A monomial $\omega \in X^3$ is *square-free* in S if its orbit $\mathcal{O}(\omega)$ is square-free.

Remark 4.2. If (X, r) is a square-free and nondegenerate quadratic set of arbitrary cardinality then

$$\begin{array}{ccccc} {}^z t = {}^z u & \Longrightarrow & t = u & \Longleftarrow & t^z = u^z \\ {}^z t = z & \Longleftrightarrow & t = z & \Longleftrightarrow & t^z = z. \end{array} \tag{4.1}$$

Lemma 4.3. *Let (X, r) be a quantum binomial set, (not necessarily finite) and let $\mathcal{O}$ be a square-free $\mathcal{D}$-orbit in X^3. Then $|\mathcal{O}| \geq 6$.*

Proof. Suppose $\mathcal{O} = \mathcal{O}(xyz)$ is a square-free orbit. Consider the set

$$O_1 = \{v_i \mid 1 \leq i \leq 6\} \subseteq \mathcal{O}$$

consisting of the first six elements of the "Yang-Baxter" diagram

$$\begin{tikzcd} v_1 = xyz \arrow[r, "r^{12}"] \arrow[d, "r^{23}"'] & ({}^x y x^y) z = v_2 \arrow[d, "r^{23}"] \\ v_3 = x({}^y z y^z) \arrow[d, "r^{12}"'] & ({}^x y)({}^{x^y} z)(x^y)^z = v_5 \arrow[d, "r^{12}"] \\ v_4 = {}^x({}^y z)(x^{{}^y z})(y^z) & \left[{}^{x}y({}^{x^y} z)\right]\left[({}^x y)^{({}^{x^y} z)}\right]\left[(x^y)^z\right] = v_6. \end{tikzcd} \tag{4.2}$$

Clearly,

$$O_1 = U_1 \bigcup U_3 \bigcup U_5, \quad \text{where} \quad U_j = \{v_j,\ r^{12}(v_j) = v_{j+1}\}, \quad j = 1, 3, 5.$$

We claim that U_1, U_3, U_5 are pairwise disjoint sets and each of them has order 2. Note first that since v_j is a square-free monomial, for each $j = 1, 3, 5$, one has $v_j \neq r_{12}(v_j) = v_{j+1}$, therefore

$$|U_j| = 2, \ \ j = 1, 3, 5.$$

The monomials in each U_j have the same "tail". More precisely, $v_1 = (xy)z, v_2 = r(xy)z$ have a "tail" z, the tail of v_3 and v_4 is y^z, and the tail of v_5 and v_6 is $(x^y)^z$. It will be enough to show that the three elements $z, y^z, (x^y)^z \in X$ are pairwise distinct. But $\mathcal{O}(xyz)$ is square-free, so $y \neq z$ and by (4.1) $y^z \neq z$. Furthermore $v_2 = ({}^x y)(x^y) z \in \mathcal{O}(xyz)$ and therefore, $x^y \neq y$ and $x^y \neq z$. Now by (4.1) one has

$$\begin{array}{ccc} x^y \neq z & \Longrightarrow & (x^y)^z \neq z \\ x^y \neq y & \Longrightarrow & (x^y)^z \neq y^z. \end{array}$$

We have shown that the three elements $z, y^z, (x^y)^z \in X$ occurring as tails in U_1, U_3, U_5, respectively, are pairwise distinct, so the three sets are pairwise disjoint. This implies $|O_1| = 6$ and therefore $|\mathcal{O}| \geq 6$. $\square$

Corollary 4.4. *A quantum binomial set (X, r) is a symmetric set* iff *the group $\mathcal{D} = {}_{\mathrm{gr}}\langle r^{12}, r^{23}\rangle$ is isomorphic to the symmetric group S_3.*

Notation 4.5. Denote $\mathbf{E}(\mathcal{O}) = \mathcal{O}\bigcap((\Delta_2 \times X \bigcup X \times \Delta_2)\backslash\Delta_3)$.

Proposition 4.6. *Suppose (X, r) is a quantum binomial set. Let $\mathcal{O}$ be a $\mathcal{D}$-orbit in X^3.*

(1) *The following implications hold.*

$$\begin{array}{lll} \textbf{(i)} & \mathcal{O}\bigcap\Delta_3 \neq \emptyset & \Longrightarrow \quad |\mathcal{O}| = 1. \\ \textbf{(ii)} & \boldsymbol{E}(\mathcal{O}) \neq \emptyset & \Longrightarrow \quad |\mathcal{O}| \geq 3 \quad \textit{and} \quad |\boldsymbol{E}(\mathcal{O})| = 2. \end{array}$$

*In this case we say that $\mathcal{O}$ is an orbit of type **(ii)**.*

(2) *(X, r) satisfies the cyclic condition*

$$ {}^{x^y}y = {}^xy \quad \textit{and} \quad x^{{}^xy} = x^y,\ \forall x, y \in X \tag{4.3}$$

if and only if each orbit $\mathcal{O}$ of type **(ii)** *has order $|\mathcal{O}| = 3$.*

(3) *Suppose X has finite order n. Then there are exactly $n(n-1)$ orbits $\mathcal{O}$ of type* **(ii)** *in X^3.*

Proof. Clearly, the "fixed" points under the action of $\mathcal{D}$ on X^3 are exactly the monomials xxx, $x \in X$, this gives **(i)**.

Assume now that $\mathcal{O}$ is of type **(ii)**. Then it contains an element of the shape $\omega = xxy$, or $\omega = xyy$, where $x, y \in X, x \neq y$. Without loss of generality we can assume $\omega = xxy \in \mathcal{O}$. The orbit $\mathcal{O}(\omega)$ can be obtained as follows. We fix as an initial element of the orbit $\omega = xxy$. Then there is *a unique finite sequence* $r^{23}, r^{12}, r^{23}, \cdots$ which exhaust the whole orbit and produces a "new" element at every step. r is involutive and square-free, thus in order to produce new elements at every step, the sequence must start with r^{23} and at every next step we have to alternate the actions r^{23} and r^{12}.

We look at the "Yang-Baxter" diagram starting with ω and exhausting the whole orbit (without repetitions).

$$\omega = \omega_1 = xxy \overset{r^{23}}{\longleftrightarrow} \omega_2 = x({}^xy)(x^y) \overset{r^{12}}{\longleftrightarrow} \omega_3 = ({}^{x^2}y)(x^{{}^xy})(x^y) \longleftrightarrow \cdots \longleftrightarrow \omega_m. \tag{4.4}$$

Note that the first three elements $\omega_1, \omega_2, \omega_3$ are distinct monomials in X^3, hence $\mid \mathcal{O} \mid \geq 3$. Indeed, $x \neq y$ implies $r(xy) \neq xy$ in X^2, so $\omega_2 \neq \omega_1$. By assumption (X, r) is square-free, so ${}^xx = x$, but by the nondegeneracy $y \neq x$, also implies ${}^xy \neq x$. So $r(x({}^xy)) \neq x({}^xy)$, and therefore $\omega_3 \neq \omega_2$. Clearly $\omega_3 \neq \omega_1$, since $x \neq y$ implies $x^y \neq y^y = y$, see (4.1).

We claim now that the intersection $\mathbf{E} = \mathbf{E}(\mathcal{O})$ contains exactly two element. We analyze the diagram (4.4) looking from left to right. Suppose we have made $k-1$ "steps" to the right obtaining new elements, so we have obtained

$$\omega_1 = xxy \overset{r^{23}}{\longleftrightarrow} \omega_2 = x({}^xyx^y) \overset{r^{12}}{\longleftrightarrow} \omega_3 = ({}^{x^2}y)(x^{{}^xy})x^y \longleftrightarrow \cdots \longleftrightarrow \omega_{k-1} \overset{r^{ii+1}}{\longleftrightarrow} \omega_k,$$

where $\omega_1, \omega_2, \cdots, \omega_k$ are pairwise distinct. Note that all elements ω_s, $2 \leq s \leq k-1$, have the shape $\omega_s = a_sb_sc_s$, with $a_s \neq b_s$ and $b_s \neq c_s$. Two cases are possible.

(a) $\omega_k = a_kb_kc_k$, with $a_k \neq b_k$ and $b_k \neq c_k$, then applying r^{jj+1} (where $j = 2$ if $i = 1$ and $j = 1$ if $i = 2$), we obtain a new element ω_{k+1} of the orbit.

(b) $\omega_k = aac$, or $\omega_k = acc$, $a \neq c$. In this case r^{jj+1} with $j \neq i$ keeps ω_k fixed, so the process of obtaining new elements of the orbit stops at this step and the diagram is complete.

But our diagram is finite, so as a final step on the right it has to "reach" some $\omega_m = aac$, or $\omega_m = acc$, $a \neq c$ (we have already shown that $m \geq 3$). Note that $\omega_m \neq \omega_1$. Hence the intersection $\mathbf{E} = \mathbf{E}(\mathcal{O})$, contains exactly two elements, which verifies (1).

Condition (2) follows straightforwardly from (4.4).

Assume now that $|X| = n$. We claim that there exists exactly $n(n-1)$ orbits of type **(ii)**. Indeed, let $\mathcal{O}_1, \cdots \mathcal{O}_p$ be all orbits of type **(ii)**. The intersections $E_i = E(\mathcal{O}_i), 1 \leq i \leq p$, are disjoint sets and each of them contains two elements. Now the equalities

$$\begin{array}{lcl} \bigcup_{1\leq i\leq p} E_i & = & (\Delta_2 \times X \bigcup X \times \Delta_2)\backslash\Delta_3 \\ \mid E_i \mid = 2, & & \mid (\Delta_2 \times X \bigcup X \times \Delta_2)\backslash\Delta_3 \mid = 2n(n-1) \end{array}$$

imply $p = n(n-1)$. This verifies (3). $\square$

Example 4.7. Consider the quantum binomial algebra A given in example 2.10. Let (X, r) be the associated quadratic set, let $S = S(X, r)$ be the corresponding monoid. The relations are semigroup relations, so there is an algebra isomorphism $A = \mathcal{A}(\mathbf{k}, X, r) \simeq \mathbf{k}S$. We will find the corresponding $\mathcal{D}$-orbits in X^3. There are 12 orbits of type **(ii)**. This agrees with Proposition 4.6.

$$\begin{array}{l} \mathcal{O}_1 = xxy \xrightarrow{r^{23}} xzt \xrightarrow{r^{12}} yxt \xrightarrow{r^{23}} ytx \xrightarrow{r^{12}} tzx \xrightarrow{r^{23}} tty \\ \mathcal{O}_2 = xxz \xrightarrow{r^{23}} xyx \xrightarrow{r^{12}} ztx \xrightarrow{r^{23}} zxt \xrightarrow{r^{12}} tyt \xrightarrow{r^{23}} ttz \\ \mathcal{O}_3 = yxx \xrightarrow{r^{12}} xzx \xrightarrow{r^{23}} xty \xrightarrow{r^{12}} txy \xrightarrow{r^{23}} tzt \xrightarrow{r^{12}} ytt \\ \mathcal{O}_4 = zxx \xrightarrow{r^{12}} tyx \xrightarrow{r^{23}} txz \xrightarrow{r^{12}} xtz \xrightarrow{r^{23}} xyt \xrightarrow{r^{12}} ztt \end{array}$$

$$\begin{array}{ll} \mathcal{O}_5 = xyy \xrightarrow{r^{12}} zty \xrightarrow{r^{23}} zxz; & \mathcal{O}_6 = xzz \xrightarrow{r^{12}} yxz \xrightarrow{r^{23}} yyx \\ \mathcal{O}_7 = tyy \xrightarrow{r^{12}} zxy \xrightarrow{r^{23}} zzt; & \mathcal{O}_8 = tzz \xrightarrow{r^{12}} ytz \xrightarrow{r^{23}} yyt \\ \mathcal{O}_9 = txx \xrightarrow{r^{12}} xtx \xrightarrow{r^{23}} xxt; & \mathcal{O}_{10} = xtt \xrightarrow{r^{12}} txt \xrightarrow{r^{23}} ttx \\ \mathcal{O}_{11} = yzz \xrightarrow{r^{12}} zyz \xrightarrow{r^{23}} zzy; & \mathcal{O}_{12} = zyy \xrightarrow{r^{12}} yzy \xrightarrow{r^{23}} yyz \end{array}$$

There are only two square-free orbits, $\mathcal{O}^{(1)} = \mathcal{O}(xyz)$ and $\mathcal{O}^{(2)} = \mathcal{O}(tyz)$. Each of them has order 6.

$$\begin{tikzcd} xyz \arrow[r, "r^{12}"] \arrow[d, "r^{23}"'] & ztz \arrow[d, "r^{23}"] \\ xzy \arrow[d, "r^{12}"'] & zyt \arrow[d, "r^{12}"] \\ yxy \arrow[r, "r^{12}"] & yzt \end{tikzcd} \qquad \begin{tikzcd} tyz \arrow[r, "r^{12}"] \arrow[d, "r^{23}"'] & zxz \arrow[d, "r^{23}"] \\ tzy \arrow[d, "r^{12}"'] & zyx \arrow[d, "r^{12}"] \\ yty \arrow[r, "r^{12}"] & yzx \end{tikzcd} \tag{4.5}$$

The one element orbits are $\{xxx\}, \{yyy\}, \{zzz\}, \{ttt\}$. Clearly, r does not satisfy the braid relation, so (X, r) is not a symmetric set. A more detailed study of the orbits shows that A is not a PBW algebra w.r.t. any enumeration of X.

Lemma 4.8. *A quantum binomial set (X,r) is symmetric* iff *the orders of $\mathcal{D}$-orbits $\mathcal{O}$ in X^3 satisfy the following two conditions:*

$$\text{(a)} \qquad \boldsymbol{E(\mathcal{O})} \neq \emptyset \qquad\qquad \Longleftrightarrow |\mathcal{O}| = 3.$$

$$\text{(b)} \qquad \mathcal{O}\bigcap(\Delta_2 \times X \bigcup X \times \Delta_2) = \emptyset \quad \Longleftrightarrow |\mathcal{O}| = 6.$$

Proof. Look at the corresponding YBE diagrams. □

We fix the following notation for the $\mathcal{D}$-orbits in X^3.

Notation 4.9. We denote by $\mathcal{O}_i, 1 \leq i \leq n(n-1)$ the orbits of type **(ii)** and by $\mathcal{O}^{(j)}, 1 \leq j \leq q$ all square-free orbits in X^3. The remaining $\mathcal{D}$-orbits in X^3 are the one-element orbits $\{xxx\}, x \in X$, their union is Δ_3.

Proposition 4.10. *In notation as above, let (X,r) be a finite quantum binomial set, $|X| = n$. Then*

(1) $q \leq \binom{n}{3}$.
(2) *(X,r) is a symmetric set* iff $q = \binom{n}{3}$.

Proof. Clearly, X^3 is a disjoint union

$$X^3 = \Delta_3 \bigcup \Big(\bigcup_{1 \leq i \leq n(n-1)} \mathcal{O}_i \Big) \bigcup \Big(\bigcup_{1 \leq j \leq q} \mathcal{O}^{(j)} \Big).$$

Thus

$$| X^3 | = | \Delta_3 | + \sum_{1 \leq i \leq n(n-1)} | \mathcal{O}_i | + \sum_{1 \leq j \leq q} | \mathcal{O}^{(j)} | . \tag{4.6}$$

Denote $m_i = |\mathcal{O}_i|,\ 1 \leq i \leq n(n-1),\ \ n_j = |\mathcal{O}^{(j)}|,\ 1 \leq j \leq q$. By Proposition 4.6 one has

$$m_i \geq 3,\ 1 \leq i \leq n(n-1), \quad \text{and} \quad n_j \geq 6,\ 1 \leq j \leq q.$$

We replace these inequalities in (4.6) and obtain

$$n^3 = n + \sum_{1 \leq i \leq n(n-1)} m_i + \sum_{1 \leq j \leq q} n_j \geq\ n + 3n(n-1) + 6q. \tag{4.7}$$

So

$$q \leq \frac{n^3 - 3n^2 + 2n}{6} = \binom{n}{3},$$

which verifies (1). Assume now $q = \binom{n}{3}$. Then (4.7) implies

$$n^3 = n + \sum_{1 \leq i \leq n(n-1)} m_i + \sum_{1 \leq j \leq \binom{n}{3}} n_j \geq\ n + 3n(n-1) + 6\binom{n}{3} = n^3.$$

This is possible **iff** the following equalities hold

$$\begin{aligned} m_i &= |\mathcal{O}_i| = 3,\ 1 \leq i \leq n(n-1), \\ n_j &= |\mathcal{O}^{(j)}| = 6,\ 1 \leq j \leq q. \end{aligned} \tag{4.8}$$

By Lemma 4.8, (X,r) is a symmetric set *iff* the equalities (4.8) hold. □

Corollary 4.11. *Let (X,r) be a finite quantum binomial set. (X,r) is a symmetric set* iff *the associated quadratic algebra $\mathcal{A} = \mathcal{A}(\boldsymbol{k}, X, r)$ satisfies*

$$\dim \mathcal{A}_3 = \binom{n+2}{3}.$$

Proof. The distinct elements of the associated monoid $S = S(X, r)$, form a $\mathbf{k}$- basis of the monoid algebra $\mathbf{k}S \simeq \mathcal{A}(\mathbf{k}, X, r)$. In particular $\dim \mathcal{A}_3$ equals the number of distinct monomials of length 3 in S which is exactly the number of $\mathcal{D}$-orbits in X^3. □

There is a close relation between Yang-Baxter monoids and a special class of *Garside monoids*, see [9, 19]. Garside monoids and groups were introduced by Garside, [14]. The interested reader can find more information and references in [14, 10, 25], etall. In [19], Definition 1.10, we introduce the so-called *regular Garside monoids*. It follows from [19], Main Theorem 1.16, that a finite quantum binomial set (X, r) is a solution of YBE *iff* the associated monoid $S = S(X, r)$ is a regular Garside monoid. This together with Corollary 4.11 imply the following.

Corollary 4.12. *Let (X, r) be a finite quantum binomial set. Let $S = S(X, r)$ be the associated monoid and let S_3 be the set of distinct elements of length* 3 *in S. Suppose the cardinality of S_3 is*

$$|S_3| = \binom{n+2}{3}.$$

Then S is a Garside monoid. Moreover, S is regular in the sense of [19].

Assume now that A is an n-generated quantum binomial algebra. We want to estimate the dimension $\dim A_3$. Let (X, r) be the corresponding quantum binomial set, $S = S(X, r)$, $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$. We use Proposition 4.10 to find an upper bound for the number of distinct $\mathcal{D}$-orbits in X^3, or equivalently, the order of S_3, the set of (distinct) elements of length 3 in S. One has

$$|S_3| = n + n(n-1) + q \leq n + n(n-1) + \binom{n}{3} = \binom{n+2}{3}.$$

There is an isomorphism of vector spaces, $\mathcal{A}_3 \simeq Span S_3$, so

$$\dim \mathcal{A}_3 = |S_3| \leq \binom{n+2}{3}.$$

In the general case, a quantum binomial algebra, satisfies $\dim A_3 \leq \dim \mathcal{A}_3$, due to the coefficients c_{xy} appearing in the set of relations. Similarly, the dimension of $A^!_3$ is at most equal to the number of square-free $\mathcal{D}$-orbits in X^3. We have proven the following corollary.

Corollary 4.13. *If A is a quantum binomial algebra, then*

$$\dim A_3 \leq \binom{n+2}{3}, \quad \dim A^!_3 \leq \binom{n}{3}.$$

5. Quantum binomial algebras, Yang-Baxter equation, and Artin-Schelter regularity

A quadratic algebra $A = \bigoplus_{i \geq 0} A_i$ is called *a Frobenius quantum space of dimension d* (or *a Frobenius algebra of dimension d*) if $\dim_{\mathbf{k}}(A_d) = 1$; $A_i = 0$, for $i > d$; and the multiplication map $m : A_j \otimes A_{d-j} \to A_d$ is a perfect duality (nondegenerate pairing). A is called *a quantum Grassmann algebra* (of dimension d) if in addition $dim_{\mathbf{k}} A_i = \binom{d}{i}$, for $1 \leq i < d$, [30].

Fact 5.1. *Let A be a Koszul algebra A with finite global dimension. Then A is Gorenstein* iff *its dual $A^!$ is Frobenius.*

Remind that if $A = \mathbf{k}\langle x_1, \cdots, x_n\rangle/(\Re)$ is a quantum binomial algebra, then its Koszul dual has a presentation $A^! = \mathbf{k}\langle \xi_i, \cdots \xi_n\rangle/(\Re^\perp)$, where $\Re^\perp$ consists of $\binom{n}{2}+n$ relations: (a) $\binom{n}{2}$ binomials, $\xi_j\xi_i + c_{ij}^{-1}\xi_{i'}\xi_{j'} \in \Re^\perp$ whenever $x_jx_i - c_{ij}x_{i'}x_{j'} \in \Re$; and (b) the monomials $(\xi_i)^2$, $1 \leq i \leq n$.

Lemma 5.2. *Let $A = \boldsymbol{k}\langle X; \Re\rangle$ be a quantum binomial algebra, with $|X| = n$, let $(X, r), r = r(\Re)$ be the associated quantum binomial set (see Definition 2.7). Then each of the following three conditions implies that (X, r) is a symmetric set.*

(1)
$$\dim A_3 = \binom{n+2}{3}.$$

(2)
$$\dim A_3^! = \binom{n}{3}.$$

(3) *X can be enumerated $X = \{x_1 \cdots, x_n\}$, so that the set of ordered monomials of length 3*

$$\mathcal{N}_3 = \{x_{i_1}x_{i_2}x_{i_3} \mid 1 \leq i_1 \leq i_2 \leq i_3 \leq n\} \tag{5.1}$$

projects to a $\boldsymbol{k}$-basis of A_3.

Proof. As usual, $S = S(X, r)$ and $\mathcal{A} = \mathcal{A}(X, r)$ denote the associated monoid and quadratic algebra, respectively.

(1) $\Longrightarrow$ (3). Assume (1) holds. Then one has

$$\binom{n+2}{3} = \dim A_3 \leq \dim \mathcal{A}_3 \leq \binom{n+2}{3},$$

where the right-hand side inequality follows from Corollary 4.11. This implies $\dim \mathcal{A}_3 = \binom{n+2}{3}$, or equivalently, $|S_3| = \binom{n+2}{3}$. Therefore there are exactly $q = \binom{n}{3}$ square-free $\mathcal{D}$-orbits in X^3, so by Proposition 4.10.2 (X, r) is a symmetric set. By Fact 2.9 $\mathcal{A}$ is a skew-polynomial ring with respect to an enumeration of X, so (3) is in force. The converse implication (3) $\Longrightarrow$ (1) is straightforward.

The equivalence (1) $\Longleftrightarrow$ (2) can be proved directly using the $\mathcal{D}$-orbits in X^3. It is also straightforward from the following formula for quadratic algebras, see [32], p 85.

$$\dim A_3^! = (\dim A_1)^3 - 2(\dim A_1)(\dim A_2) + \dim A_3.$$

$\square$

Lemma 5.3. *Let A be a quadratic algebra with relations of skew-polynomal type, see Definition 2.1. Then the following conditions are equivalent:*

(1)
$$\dim A_3 = \binom{n+2}{3}.$$

(2)
$$\dim A_3^! = \binom{n}{3}.$$

(3) *The set of defining relations $\Re$ for A is a Gröbner basis, so A is a skew polynomial ring and therefore a PBW algebra with PBW generators $x_1, \cdots, x_n$.*

(4) *The set of defining relations $\Re^\perp$ for $A^!$ is a Gröbner basis, so $A^!$ is a a PBW algebra with PBW generators $\xi_1, \cdots, \xi_n$.*

Sketch of the proof. The implications (1) $\Longleftarrow$ (3) and (2) $\Longleftarrow$ (4) are clear, Lemma 5.2 gives (1) $\Longleftrightarrow$ (2).

(1) $\Longrightarrow$ (3). Assume condition (1) holds. The set $\mathbf{N}_3$ of monomials of length 3, normal mod the ideal $(\Re)$, forms a $\mathbf{k}$-basis of A_3. The relations are of skew-polynomial type (i.e. conditions (a) and (b) of Definition 2.1 hold) therefore each normal monomial is also an ordered monomial and $\mathbf{N}_3 \subseteq \mathcal{N}_3$. Now the equalities

$$\mid \mathcal{N}_3 \mid = \binom{n+2}{3} = \dim A_3 = \mid \mathbf{N}_3 \mid$$

imply $\mathbf{N}_3 = \mathcal{N}_3$. It follows then that all ambiguities $x_k x_j x_i, 1 \leq i < j < k \leq n$, are resolvable and by Bergman's Diamond lemma [6], $\Re$ is a Gröbner basis of the ideal $(\Re)$. Thus A is a PBW algebra, more precisely, A is a binomial skew-polynomial ring. An analogous argument proves (2) $\Longrightarrow$ (4). $\square$

Theorem 5.4. *Let $A = \boldsymbol{k}\langle X\rangle/(\Re)$ be a quantum binomial algebra, $|X| = n$, and let $R(\Re)$ be the associated automorphism $R = R(\Re) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$. Then the following three conditions are equivalent:*

(1) *$R = R(\Re)$ is a solution of the Yang-Baxter equation.*
(2) *A is a binomial skew-polynomal ring with respect to an enumeration $X = \{x_1, \cdots x_n\}$.*
(3)
$$\dim_{\boldsymbol{k}} A_3 = \binom{n+2}{3}.$$

Proof. The equivalence (1) $\Longleftrightarrow$ (2) is proven in [17], Theorem B.

(2) $\Longrightarrow$ (3). Assume that A is a skew polynomial ring w.r.t. an enumeration $X = \{x_1, \cdots x_n\}$. It follows from Definition 2.1 that the set $\mathcal{N}_3$ is a $\mathbf{k}$-basis of A_3, so $\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}$.

(3) $\Longrightarrow$ (2). Assume that $\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}$. Consider the corresponding quadratic set (X, r) and the monoid algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$. As before we conclude that $\dim_{\mathbf{k}} \mathcal{A}_3 = \binom{n+2}{3}$ and therefore, by Corollary 4.11 (X, r) is a symmetric set. It follows then from [18], Theorem 2.26, that there exists an ordering on X, $X = \{x_1, x_2, \cdots, x_n\}$ so that the algebra $\mathcal{A}(\mathbf{k}, X, r)$ is a skew-polynomial ring and therefore a PBW algebra, with PBW generators $x_1, x_2, \cdots, x_n$. The relations $\Re$ of A and the relations of $\mathcal{A}$ may differ only with coefficients, so $\Re$ are also relations of skew-polynomial type. It remains to show that $\Re$ is a Gröbner basis. This follows from the assumption $\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}$ and Lemma 5.3. Hence A is a binomial skew polynomial ring. $\square$

Proof of Theorem 1.2. Note that A has exactly $\binom{n}{2}$ relations, see Remark 2.5. The following implications are in force:

$$\begin{array}{ccl}
(1) \Longleftrightarrow (2) & : & \text{by Theorem 1.1} \\
(3) \Longrightarrow (1); \quad (3) \Longrightarrow (2) & : & \text{clear} \\
(5) \Longrightarrow (2); \quad (9) \Longrightarrow (7) & : & \text{clear} \\
(5) \Longrightarrow (8) \Longrightarrow (6) & : & \text{clear} \\
(4) \Longleftrightarrow (5) \Longleftrightarrow (6) & : & \text{by Theorem 5.4} \\
(6) \Longleftrightarrow (7) & : & \text{is well-known, see [32]}
\end{array}$$

(5) $\Longrightarrow$ (9). Theorem 3.1, [17] verifies that the Koszul dual $A^!$ of a binomial skew polynomial ring A is a quantum Grassman algebra.

(5) $\Longrightarrow$ (1). Assume that A is a binomial skew polynomial ring w.r.t. an enumeration $x_1, \cdots, x_n$ of X. Then A satisfies the hypothesis and condition (4) of Theorem 1.1 hence, by the same theorem, A has finite global dimension.

(5) $\Longrightarrow$ (3). Assume that A is a binomial skew polynomial ring, so A has polynomial growth, finite global dimension, and its Koszul dual $A^!$ is a quantum Grassman algebra, see [17], Theorem 3.1. Furthermore, as a PBW algebra, A is Koszul, hence by Fact 5.1, A is Gorenstein. The result that every binomial skew polynomial ring is an AS regular domain follows also from the earlier work [23].

(1) $\Longrightarrow$ (8). Assume that A satisfies (1). Let $A^0 = \mathbf{k}\langle X\rangle/(\mathbf{W})$ be the corresponding monomial algebra, where $\mathbf{W}$ is the set of obstructions. We know that a quantum binomial algebra A has exactly $\binom{n}{2}$ relations, which in this case form a reduced Gröbner basis, so $|\mathbf{W}| = |\Re| = \binom{n}{2}$. Furthermore, A and A^0 share the same global dimension, hence A^0 satisfies the equivalent conditions (3) and (5) of Theorem 3.8. But the algebras A and A^0 have the same Hilbert series, so (8) is in force.

The equivalence of conditions (1) $\cdots$ (9) has been verified. Each of these conditions imply that A is a binomial skew-polynomial ring and therefore it is a Noetherian domain, see [23], or Fact 2.9. $\square$

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Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria, AUBG, and, Max-Plank Institute for Mathematics, Vivatsgasse 7, 5311 Bonn, Germany

E-mail address: tatianagateva@yahoo.com, tatyana@aubg.bg