HIGHER SYMMETRIES OF THE CONFORMAL POWERS OF THE LAPLACIAN ON CONFORMALLY FLAT MANIFOLDS

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Abstract. On locally conformally flat manifolds we describe a construction which maps generalised conformal Killing tensors to differential operators which may act on any conformally weighted tensor bundle; the operators in the range have the property that they are symmetries of any natural conformally invariant differential operator between such bundles. These are used to construct all symmetries of the conformally invariant powers of the Laplacian (often called the GJMS operators) on manifolds of dimension at least 3. In particular this yields all symmetries of the powers of the Laplacian Δ^k, $k \in \mathbb{Z} > 0$, on Euclidean space $\mathbb{E}^n$. The algebra formed by the symmetry operators is described explicitly.

1. Introduction

Given a differential operator P, say on functions, it is natural to consider smooth differential operators which locally preserve the solution space of P. A refinement is to seek differential operators S with the property that $P \circ S = S' \circ P$, for some other differential operator S'. In this case we shall say that S is a symmetry of P. On Euclidean n-space $\mathbb{E}^n$ with $n \geq 3$ the space of first order symmetries of the Laplacian Δ is finite dimensional with commutator subalgebra isomorphic to $\mathfrak{so}(n+1,1)$, the Lie algebra of conformal motions of $\mathbb{E}^n$. Second order symmetries have applications in the problem of separation of variables for the Laplacian, see [43] and references therein; on $\mathbb{E}^3$ the second order symmetries were classified by Boyer et al. [4].

Symmetries are closely related to conformal Killing tensors and their generalisations, see Theorem 2.1 below. Such operators also play a role in physics [42, 46]. Partly motivated by these links, Eastwood has recently given a complete algebraic description of the symmetry algebra for the Laplacian on $\mathbb{E}^{n\geq 3}$ [20]. His treatment uses conformal geometry and in particular a treatment of the conformal Laplacian due to Hughston and Hurd [26] based on the classical model of the conformal n-sphere as the projective image of an indefinite quadratic variety in $\mathbb{R}^{n+2}$. There are close links to the Fefferman-Graham ambient metric [24, 25], which provides a curved version of this model, and the ideas of Maldacena's AdS/CFT correspondence [40, 36, 47]. Eastwood's work was extended in [22], via similar techniques, where the authors found the symmetry algebra for Δ^2 on $\mathbb{E}^{n\geq 3}$.

Here the main result of the article is a simultaneous treatment of all powers of the Laplacian on pseudo-Euclidean space $\mathbb{E}^{s,s'}$ (i.e. $\mathbb{R}^{s+s'}$ equipped with a constant signature (s,s') metric) with $s+s' \geq 3$; we obtain an explicit construction of all

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symmetries and a description of the algebra these generate. See Theorem 2.1, and Theorem 2.5. (In lower dimensions a corresponding result is not to be expected as, in that case, the space of conformal Killing vectors is infinite dimensional). As will shortly be clear, the problem is fundamentally linked to conformal geometry. Thus it is natural to also formulate and treat analogous questions for the conformally invariant generalisations P_k of the powers Δ^k ($k \in \mathbb{Z}_{>0}$) on conformally flat manifolds, and we do this; via Theorem 2.4 and surrounding discussion we see that the algebra is again described by Theorem 2.5. In dimension 4 the operators P_k were discussed in [37]. Conformally curved versions in general dimensions ($n \geq 2k$ if even) are due to Paneitz ($k = 2$) [44] and Graham-Jenne-Mason-Sparling [35], and have been the subject of tremendous recent interest in both the mathematics and physics community [16, 18, 38]. For convenience we shall refer to these operators as the GJMS operators.

Although the current work is inspired by [20, 22], we follow a rather different approach that is designed to be easily adapted to study the symmetries of other classes of differential operators. Indeed with minor adaption our techniques also apply to the entire class of parabolic geometries. Firstly, rather than work on a higher dimensional "ambient" manifold, we calculate directly on the n dimensional space and use the tractor calculus of [1, 29, 11]. Using this machinery we construct a map which takes solutions of certain overdetermined PDE (solutions called generalised conformal Killing tensors) to differential operators which have the universality property that they are symmetries for *any* conformally invariant operator between irreducible bundles. This is Theorem 5.2. These universal symmetry operators form an algebra under formal composition; by construction this is a quotient of the tensor algebra $\bigotimes \mathfrak{so}(s+1, s'+1)$. On the other hand for the case of GJMS operators, Theorem 2.4 states that, conversely, all symmetries arise from the operators in this algebra. Determining the algebra of symmetries of a given order $2k$ GJMS operator P_k then proceeds in two steps. The order $2k$ determines the domain (density) bundle (for P_k and hence) on which the universal symmetry operators should act. From the latter we obtain an ideal of identities satisfied by the universal symmetries; the ideal is specific to the domain. This is the subject of Theorem 7.1. A further ideal is generated by symmetries that are trivial in a sense to be made precise below, see Theroem 7.2. The result is an explicit description in Theorem 2.5 of the ideal, the quotient of $\bigotimes \mathfrak{so}(s+1, s'+1)$ by which yields the symmetry algebra of P_k.

2. The main theorems

2.1. **Symmetries and triviality.** Throughout we shall retrict to conformally flat pseudo-Riemannian manifolds (M, g) of dimension $n \geq 3$ and signature (s, s'), or the conformal structures $(M, [g])$ that these determine. In the spirit of Penrose's *abstract index notation* [45], we shall denote write $\mathcal{E}^a$ as an alternative notation for TM and $\mathcal{E}_a$ for the dual bundle T^*M. Thus for example $\mathcal{E}_{ab} = \otimes^2 T^*M$. According to context we may also use *concrete* indices from time to time. That is indices refering to a frame. All manifolds, structures, functions and tensor fields will be taken to be smooth (i.e. to infinite order) and all differential operators will be linear with smooth coefficients. Since our later treatment generalises easily, we define here the notion of symmetry in greater generality than is strictly needed for our main results. This also serves to indicate the general context for the devolopments.

Suppose that $P : \mathcal{V} \to \mathcal{W}$ is a smooth differential operator between (section spaces of) irreducible bundles. (In our notation we shall not distinguish bundles from their smooth section spaces.) We shall say that linear differential operators $S : \mathcal{V} \to \mathcal{V}$ and $S' : \mathcal{W} \to \mathcal{W}$ form a (S, S') a *symmetry (pair)* of P if the operator compositions PS and $S'P$ satisfy

$$PS = S'P.$$

An example is the pair (TP, PT), where T is a differential operator $T : \mathcal{W} \to \mathcal{V}$. However for obvious reasons such symmetries shall be termed *trivial*.

Following the treatment of Δ and Δ^2 of [20, 22], we note that there is an algebraic structure on the symmetries modulo trivial symmetries as follows. First the symmetries of P form a vector space under the obvious operations. Then if (S_1, S_1') and (S_2, S_2') are symmetries then so too is the composition $(S_1S_2, S_1'S_2')$. So the symmetries of P form an algbera $\tilde{\mathcal{S}}$. Next we say that two symmetries (S_1, S_1') and (S_2, S_2') are equivalent, $(S_1, S_1') \sim (S_2, S_2')$, if and only if $(S_1 - S_2, S_1' - S_2')$ is a trivial symmetry. It is easily verified that trivial symmetries form a two-sided ideal in the algebra $\tilde{\mathcal{S}}$ and the quotient by this yields an algebra $\mathcal{S}$. For the case that P is a GJMS operator it is this algebra that we shall study in detail.

To simplify our discussion we shall often work with just the first operator $S : \mathcal{V} \to \mathcal{V}$ in a symmetry pair. That is an operator $S : \mathcal{V} \to \mathcal{V}$ shall be called a symmetry if there exists some $S' : \mathcal{W} \to \mathcal{W}$ that makes (S, S') a symmetry as above. (In fact for the main class of operators we treat it is easily verified that S' is uniquely determined by S.) Note that with this language, and in the class of cases satisfying $\mathcal{V} = \mathcal{W}$, the composition PS is a trivial symmetry if and only if S is a symmetry.

2.2. **Symmetries of Δ^k on $\mathbb{E}^{s,s'}$.** We shall write $\mathbb{E}^{s,s'}$ to mean $\mathbb{R}^n$, $n = s + s'$, equipped with the standard flat diagonal signature (s, s') metric g; in the $s = n$, $s' = 0$ case this is n-dimensional Euclidean space. Here and throughout we shall make the restriction $n \geq 3$. In this setting the Levi-Civita connection ∇ is flat and, with tensors expressed in terms of the standard $\mathbb{R}^n$ coordinates x^i, the action of ∇_i on these agrees with $\partial/\partial x^i$. We shall use the metric g_{ij} and its inverse g^{ij} to lower and raise indices in the usual way. For example, and capturing also our sign convention for the Laplacian, $\Delta = g^{ij}\nabla_i\nabla_j = \nabla^i\nabla_i$. (We use the summation convention here and below without further mention.)

Recall that a vector field v is a conformal Killing field (or infinitesimal conformal isometry) if $\mathcal{L}_v g = \rho g$ for some function ρ. Otherwise written, this equation is

$$\nabla^i v^j + \nabla^j v^i = \rho g^{ij},$$

and so, for solutions, $\rho = 2\mathrm{div}\ v/n$. Suppose now that φ is a symmetric trace-free covariant tensor satisfying

$$\nabla^{(i} \cdots \nabla^l \varphi^{m \cdots n)} = g^{(ij} \rho^{k \cdots n)}, \quad \text{with} \quad |\{i, \cdots, l\}| \text{ an odd integer} \tag{1}$$

for some tensor $\rho^{k\cdots n}$, and where $\phi^{(i\cdots n)}$ indicates the symmetric part of the tensor $\phi^{i\cdots n}$. Then, following [20], we shall term φ a *generalised conformal Killing tensor*.

In Sections 5 below we shall construct a canonical 1-1 map

$$\varphi \mapsto (S_\varphi, S'_\varphi) \tag{2}$$

which takes solutions of (1) to symmetries of Δ^k, see Definition 5.1 and Theorem 5.2 (which, in fact, deal with a far more general setting). Although we defer the construction of (2), let us already term (S_φ, S'_φ) the *canonical symmetry* corresponding to φ. Our main classification result is that all symmetries of Δ^k arise this way, and this is established in Theorem 6.4. Putting these results together, on $\mathbb{E}^{s,s'}$ we have the following.

Theorem 2.1. *Let us fix $k \in \mathbb{Z}_+$. For the Laplacian power Δ^k on $\mathbb{E}^{s,s'}$ we have the following. For each φ, a solution of (1), there is canonically associated a symmetry (S_φ, S'_φ) for Δ^k with S_φ and S'_φ each having leading term*

$$\varphi^{a_1\ldots a_p}(\nabla_{a_1}\cdots\nabla_{a_p})\Delta^r.$$

$p \in \mathbb{Z}_{\geq 0}$, $r \in \{0, 1, \cdots, k-1\}$.

Modulo trivial symmetries, any symmetry of Δ^k is a linear combination of such pairs (S_φ, S'_φ), with various solutions φ of (1) as above.

2.3. **Conformal geometry and the GJMS operators.** Although the question of symmetries of Δ^k is not phrased in terms of conformal geometry, it turns out that there is a deep connection. According to the Theorem 2.1 above, all symmetries of Δ^k arise from the solutions of the equations (1). As we shall explain, these equations are each conformally covariant, and in fact this class of equations can only be fully understood via consideration of their conformal properties. First note that we may alternatively write the equation (1) as

$$\nabla_{(b_0}\cdots\nabla_{b_{2r}}\varphi_{a_1\ldots a_p)_0} = 0$$

where we have lowered the indices for convenience and $(\cdots)_0$ indicates the trace-free part over the enclosed indices. For a given (say symmetric) tensor taking the trace-free part is a conformally invariant notion. Then for example in the case of $r = 0$ this is the well known conformal Killing tensor operator. In that case, if (on any pseudo-Riemannian manifold (M, g)) we replace the metric g with the conformally related $\widehat{g} := e^{2\Upsilon}g$, where $\Upsilon \in C^\infty(M)$, and replace φ with $\widehat{\varphi} := e^{2p\Upsilon}\varphi$ then

$$\nabla^{\widehat{g}}_{(b_0}\widehat{\varphi}_{a_1\ldots a_p)_0} = e^{2p\Upsilon}\nabla_{(b_0}\varphi_{a_1\ldots a_p)_0}.$$

One may think of φ here as representing a *density* valued tensor of weight $2p$. Recall that on a smooth manifold the density bundles $\mathcal{E}[w]$ are the bundles associated to the frame bundle by 1-dimensional (real) representations arising as the roots (or powers) of the square of the determinant representation. These representations and the associated bundles are thus naturally parametrised by *weights* w from $\mathbb{R}$. These weights are normalised so that $\mathcal{E}[-2n] \cong \Lambda^{2n}T^*M$, and with this normalisation the weights are often called *conformal weights*. Note that $\Lambda^{2n}T^*M$ is trivialised by a choice of metric and hence so are all the line bundles $\mathcal{E}[w]$. There is a section $\tilde{\varphi}$ of $\mathcal{E}_{(a_1\cdots a_p)_0}[2p] = \mathcal{E}_{(a_1\cdots a_p)_0} \otimes \mathcal{E}[2p]$ which, in the trivialisation of $\mathcal{E}[2p]$ afforded by g, has the component φ, while $\tilde{\varphi}$ has the component $\widehat{\varphi} = e^{2p}\varphi$ with respect to the trivialisation from $\widehat{g}$. As an associated connection, it is clear the Levi Civita connection, determined by a metric g, yields a connection on density weighted tensor bundles. Thus dropping the tilde, for $\varphi \in \mathcal{E}_{(a_1\cdots a_p)_0}[2p]$ we have $\nabla^{\widehat{g}}_{(b_0}\varphi_{a_1\ldots a_p)_0} = \nabla_{(b_0}\varphi_{a_1\ldots a_p)_0}$. This means that the operator descends to a well defined differential operator on a conformal manifold (M, c). Here (M, c) means a manifold equipped with just an an equivalence class of conformally related metrics: if $g, \widehat{g} \in c$ then $\widehat{g} = e^{2\Upsilon}g$ for some $\Upsilon \in C^\infty(M)$.

Henceforth, it will be convenient to use the notation and language of conformal densities, for further details and conventions see e.g. [10] or [32]. In particular below we shall use the *conformal metric* $\boldsymbol{g}_{ab}$ to raise and lower indices. On a conformal manifold this is a tautological section of $\mathcal{E}_{(ab)}[2] = \mathcal{E}_{(ab)} \otimes [2]$ which gives an isomorphism $\mathcal{E}^a = \mathcal{E}^a[0] \cong \mathcal{E}_b[2]$. In particular, via the conformal metric, we shall identify $\mathcal{E}_{(a_1\ldots a_p)_0}[2p+2r]$ and $\mathcal{E}^{(a_1\ldots a_p)_0}[2r]$. Note also that with these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$ and so this carries a conformal weight of -2.

>From [3] (interpreted using the ideas of [23]) we have the following.

Proposition 2.2. *For each pair (p, r), of non-negative integers, there is a conformally invariant operator*

$$\begin{aligned} &\mathcal{E}_{(a_1\ldots a_p)_0}[2p+2r] \to \mathcal{E}_{(b_0\ldots b_{2r}a_1\ldots a_p)_0}[2p+2r] \\ &\varphi_{a_1\ldots a_p} \mapsto \nabla_{(b_0}\cdots\nabla_{b_{2r}}\varphi_{a_1\ldots a_p)_0} + lot \end{aligned} \tag{3}$$

where "lot" denotes lower order terms.

In fact there is a larger class of similar operators, but we shall not need the even order analogues of the operators above for our current discusssion. An algorithm for generating explicit formulae for these operators is given in [27] (in dimension four but same formulae hold in all dimensions [28], see also [12, 9]). The lower order terms are given by Ricci curvature and its derivatives; in particular on $\mathbb{E}^{s,s'}$ we recover the operator of (1). On any manifold we shall term φ in the kernel of (3) a *(generalised) conformal (Killing) tensor.*

By construction the GJMS operator P_k is conformally invariant [35]. This means that it is a natural operator on pseudo-Riemannian manifolds M that descends to a well defined differential operator on densities

$$P_k : \mathcal{E}[k - \frac{n}{2}] \to \mathcal{E}[-k - \frac{n}{2}],$$

on conformal manifolds. Recall that we say (M, g) is *locally conformally flat*, if locally there is a metric $\widehat{g}$, conformally related to g, so that on this neighbourhood $(M, \widehat{g})$ is isometric to $\mathbb{E}^{s,s'}$. If (M, g) is locally conformally flat then in all dimensions $n \geq 3$ the operators P_k exist for every $k \geq 1$.

Definition 2.3. Let us fix a conformal manifold (M, c). Suppose that (S, S') is a pair of differential operators

$$S : \mathcal{E}[k - \frac{n}{2}] \to \mathcal{E}[k - \frac{n}{2}], \quad \text{and} \quad S' : \mathcal{E}[-k - \frac{n}{2}] \to \mathcal{E}[-k - \frac{n}{2}]$$

on the given conformal manifold (M, c). If locally (i.e. in contractable neighbourhoods) on (M, c) we have agreement of the compositions as follows

$$P_k S = S' P_k,$$

as operators on $\mathcal{E}[k - \frac{n}{2}]$, then we shall say that (S, S') is a conformal symmetry (pair) of P_k on (M, c).

Note that the definition does not require/impose naturality properties of the pair (S, S'). They are simply required to be well defined differential operators on the given (M, c).

For a given conformal manifold, and suitable natural number k, we may ask for some description of all conformal conformal symmetries of P_k. From Theorem 2.1 we have the following Theorem. Here and below we use $\mathcal{E}_r^{(p)_0}$ as shorthand for the bundle $\mathcal{E}^{(a_1 \cdots a_p)_0}[2r]$ (and its section space). We will often write φ_r^p to denote some section of this bundle.

Theorem 2.4. *Let (M, c) be a (locally) conformally flat manifold of signature (s, s'). For each non-zero $\varphi \in \mathcal{E}_r^{(p)_0}$, $p \in \mathbb{Z}_{\geq 0}$, $r \in \{0, 1, \cdots, k-1\}$, a solution of (3), there is canonically associated a non-trivial conformal symmetry (S_φ, S'_φ) for P_k, with S_φ and S'_φ each having leading term*

$$\varphi_r^{a_1 \cdots a_p} (\nabla_{a_1} \cdots \nabla_{a_p}) \Delta^r.$$

Modulo trivial symmetries, locally any conformal symmetry of P_k is a linear combination of such pairs (S_φ, S'_φ), for various solutions φ of (3), with p and r in the range assumed here.

The question of conformal symmetries is not a priori the same question as that addressed in Theorem 2.1. However using that S, S' and P_k are well defined on (M, c), we may use any metric $g \in c$ to calculate. This is a choice similar to choosing coordinates in order to calculate; indeed g gives a trivialisation of the density bundles. Now, by working locally and choosing a flat metric, the result here follows immediately from Theorem 2.1, since by the definition of the canonical symmetries in Definition 5.1 and Theorem 5.2, they are well defined on locally conformally flat conformal manifolds.

2.4. **Algebraic structure.** Let us denote by $\mathcal{A}_k$ the algebra of symmetries of Δ^k on $\mathbb{E}^{s,s'}$ modulo trivial symmetries. As usual we write $n = s + s'$. It follows from the theorem 2.1 we have the vector space isomorphism

$$\mathcal{A}_k \cong \bigoplus_{j=0}^{\infty} \bigoplus_{i=0}^{k-1} \mathcal{K}_i^j \tag{4}$$

where $\mathcal{K}_i^j \subseteq \mathcal{E}_i^{(j)_0}$ is the space of solutions of (3) with $r = j$ and $p = i$.

Now we turn to the algebra structure of $\mathcal{A}_k$. It is well known [39, 13], and given explicitly by (23) below, that that the (finite dimensional) spaces $\mathcal{K}_i^j$ are isomorphic to irreducible $\mathfrak{g} := \mathfrak{so}_{s+1,s'+1}$–modules

$$\mathcal{K}_i^j \cong \overbrace{\ydiagram{2,2}\cdots\ydiagram{1,1}}^{j}\overbrace{\ydiagram{2}\cdots\ydiagram{1}}^{2i}{}_0 \tag{5}$$

in the notation of Young diagrams. (Using the highest weights, expressed as a vector of coefficients over the Dynkin diagram as in [2], $\mathcal{K}_i^j$ corresponds to the coefficient $2i$ over the first node, the coefficient j over the second one and with remaining coefficients equal to zero. At least this applies in dimensions at least 5, but there is an obvious adjustment in lower dimensions.)

We follow [22] in the discussion of the algebraic structure of $\mathcal{A}_k$. Decomposing the tensor product of two copies of $\mathfrak{g} = \ydiagram{1,1}$ we obtain

$$\mathfrak{g} \otimes \mathfrak{g} = \underbrace{\ydiagram{2,2}_0 \oplus \ydiagram{2}_0 \oplus \mathbb{R} \oplus \ydiagram{1,1,1,1}}_{\mathfrak{g}\odot\mathfrak{g}} \oplus \underbrace{\ydiagram{2,1,1}_0 \oplus \ydiagram{1,1}}_{\mathfrak{g}\wedge\mathfrak{g}} \tag{6}$$

where $\odot$ is the symmetric tensor product. All these components occur with multiplicity one. We shall need notation for the projections of $V_1 \otimes V_2 \in \mathfrak{g}\otimes\mathfrak{g}$ to some of the irreducible components on the right hand side of the previous display. In particular, we put

$$V_1 \boxtimes V_2 \in \ydiagram{2,2}_0, \quad V_1 \bullet V_2 \in \ydiagram{2}_0, \quad \langle V_1, V_2\rangle \in \mathbb{R} \quad \text{and} \quad [V_1, V_2] \in \ydiagram{1,1}, \tag{7}$$

and we write the same notation for the projections. Here the $\boxtimes$ denotes the Cartan product, $\langle , \rangle$ the Killing form on $\mathfrak{g}$ (normalized as in [22]) and $[,]$ is the Lie bracket. These projections are described explicitly in (41) below. There is also the inclusion

$$\boxtimes^{2k}\Box = \underbrace{\ydiagram{2k}}_{2k}{}_0 \hookrightarrow \underbrace{\ydiagram{1,1}\odot\ydiagram{1,1}\odot\cdots\odot\ydiagram{1,1}}_{2k} \subset \underbrace{\ydiagram{1,1}\otimes\ydiagram{1,1}\otimes\cdots\otimes\ydiagram{1,1}}_{2k},$$

see (44) for the explicit form. That is, there is an (obviously unique) irreducible component in $\bigodot^{2k}\mathfrak{g}$ of the type specified on the left hand side.

With this notation, we obtain the following generalization of [22, Theorem 3]:

Theorem 2.5. *The algebra $\mathcal{A}_k$ is isomorphic to the tensor algebra $\bigotimes\mathfrak{g}$ modulo the two sided ideal generated by*

$$V_1\otimes V_2 - V_1\boxtimes V_2 - V_1\bullet V_2 - \frac{1}{2}[V_1,V_2] + \frac{(n-2k)(n+2k)}{4n(n+1)(n+2)}\langle V_1,V_2\rangle, \quad V_1,V_2\in\mathfrak{g} \tag{8}$$

and the image of $\boxtimes^{2k}\Box$ in $\otimes^{2k}\mathfrak{g}$.

Note that, from Theorem 2.4, $\mathcal{A}_k$ is also the algebra of local symmetries of P_k on any conformally flat conformal manifold of dimension n.

3. Conformal tractor calculus

We first recall the basic elements of tractor calculus following [10, 32].

3.1. **Tractor bundles.** Let M be a smooth manifold of dimension $n \geq 3$ equipped with a conformal structure (M, c) of signature (s, s'). Since the Levi-Civita connection is torsion-free, the (Riemannian) curvature $R_{ab}{}^c{}_d$ is given by $[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^d$ where $[\cdot, \cdot]$ indicates the commutator bracket. The Riemannian curvature can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Schouten tensor* P_{ab}, according to $R_{abcd} = C_{abcd} + 2\boldsymbol{g}_{c[a}P_{b]d} + 2\boldsymbol{g}_{d[b}P_{a]c}$, where $[\cdots]$ indicates antisymmetrisation over the enclosed indices. We shall write $J := P^a{}_a$. The *Cotton tensor* is defined by

$$A_{abc} := 2\nabla_{[b}P_{c]a}.$$

The standard tractor bundle over $(M, [g])$ is a vector bundle of rank $n+2$ defined, for each $g \in c$, by $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. If $\widehat{g} = e^{2\Upsilon}g$ ($\Upsilon \in C^\infty(M)$), we identify $(\alpha, \mu_a, \tau) \in [\mathcal{E}^A]_g$ with $(\widehat{\alpha}, \widehat{\mu}_a, \widehat{\tau}) \in [\mathcal{E}^A]_{\widehat{g}}$ by the transformation

$$\begin{pmatrix} \widehat{\alpha} \\ \widehat{\mu}_a \\ \widehat{\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a{}^b & 0 \\ -\frac{1}{2}\Upsilon_c\Upsilon^c & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix}, \tag{9}$$

where $\Upsilon_a := \nabla_a \Upsilon$. These identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the *standard tractor bundle* $\mathcal{T}$, or $\mathcal{E}^A$ in an abstract index notation, over the conformal manifold. The bundle $\mathcal{E}^A$ admits an invariant metric h_{AB} of signature $(s+1, s'+1)$ and an invariant connection, which we shall also denote by ∇_a, preserving h_{AB}. In a conformal scale g, these are given by

$$h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \nabla_a \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \alpha - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab}\tau + P_{ab}\alpha \\ \nabla_a \tau - P_{ab}\mu^b \end{pmatrix}. \tag{10}$$

It is readily verified that both of these are conformally well-defined, i.e., independent of the choice of a metric $g \in [g]$. Note that h_{AB} defines a section of $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$, where $\mathcal{E}_A$ is the dual bundle of $\mathcal{E}^A$. Hence we may use h_{AB} and its inverse h^{AB} to raise or lower indices of $\mathcal{E}_A$, $\mathcal{E}^A$ and their tensor products.

In computations, it is often useful to introduce the 'projectors' from $\mathcal{E}^A$ to the components $\mathcal{E}[1]$, $\mathcal{E}_a[1]$ and $\mathcal{E}[-1]$ which are determined by a choice of scale. They are respectively denoted by $X_A \in \mathcal{E}_A[1]$, $Z_{Aa} \in \mathcal{E}_{Aa}[1]$ and $Y_A \in \mathcal{E}_A[-1]$, where $\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$, etc. Using the metrics h_{AB} and $\boldsymbol{g}_{ab}$ to raise indices, we define X^A, Z^{Aa}, Y^A. Then we immediately see that

$$Y_A X^A = 1, \quad Z_{Ab} Z^A{}_c = \boldsymbol{g}_{bc}, \tag{11}$$

and that all other quadratic combinations that contract the tractor index vanish. In (9) note that $\widehat{\alpha} = \alpha$ and hence X^A is conformally invariant. Using this notation the tractor V^A given by

$$[V^A]_g = \begin{pmatrix} \alpha \\ \mu_a \\ \tau \end{pmatrix}$$

may be written

$$V^A = \alpha Y^A + \mu^a Z^A{}_a + \tau X^A. \tag{12}$$

The curvature Ω of the tractor connection is defined by

$$[\nabla_a, \nabla_b]V^C = \Omega_{ab}{}^C{}_E V^E$$

for $V^C \in \mathcal{E}^C$. Using (10) and the formulae for the Riemannian curvature yields

$$\Omega_{abCE} = Z_C{}^c Z_E{}^e C_{abce} - 2X_{[C} Z_{E]}{}^e A_{eab}\ . \tag{13}$$

In the following we shall also need 2-form tractors, that is $\Lambda^2\mathcal{T}$, or in abstract indices $\mathcal{E}_{[AB]}$. To simplify notation we shall set the rule that indices labelled sequentially by a superscript are implicitly skewed over and then denote skew pairs with a bold multi-index. Here we shall need this only for valence 2 forms. This convention does not apply to subscripts. That is, A^0A^1 means $[A^0A^1] = \mathbf{A}$ but e.g. the notation $A_1A_2A_3$ does not assume any implicit projection to a tensor part. The same convention will be used for tensor indices, i.e. $[a^0a^1]$ means $a^0a^1 = \mathbf{a}$.

With $\mathcal{E}^k[w]$ denoting the space of k-forms of weight w, the structure of $\mathcal{E}_{\mathbf{A}} = \mathcal{E}_{A^0A^1}$ is [6, 33]

$$\mathcal{E}_{\mathbf{A}} = \mathcal{E}^1[2] \niplus \big(\mathcal{E}^2[2] \oplus \mathcal{E}[0]\big) \niplus \mathcal{E}^1[0]; \tag{14}$$

this means that in a choice of scale the *semidirect sums* $\niplus$ may be replaced by direct sums and otherwise they indicate the composition series structure arising from the tensor powers of (9).

In a choice of metric g from the conformal class, the projectors (or splitting operators) X, Y, Z for $\mathcal{E}_A$ determine corresponding projectors $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}$ for $\mathcal{E}_{\mathbf{A}}$, These execute the splitting of this space into four components and are given as follows.

$$\begin{aligned}
\mathbb{Y} &= \mathbb{Y}_{A^0A^1}{}^{a^1} &&= Y_{A^0} Z_{A^1}^{a^1} &&\in \mathcal{E}_{\mathbf{A}}^{a^1}[-2]\\
\mathbb{Z} &= \mathbb{Z}_{A^1A^2}^{a^1a^2} &&= Z_{A^1}^{a^1} Z_{A^2}^{a^2} &&\in \mathcal{E}_{\mathbf{A}}^{\mathbf{a}}[-2]\\
\mathbb{W} &= \mathbb{W}_{A^0A^1} &&= X_{A^0} Y_{A^1} &&\in \mathcal{E}_{\mathbf{A}}[0]\\
\mathbb{X} &= \mathbb{X}_{A^0A^1}{}^{a^1} &&= X_{A^0} Z_{A^1}^{a^1} &&\in \mathcal{E}_{\mathbf{A}}^{a^1}[0].
\end{aligned}$$

Further they satisfy $\mathbb{X}^{\mathbf{A}}_a \mathbb{Y}_{\mathbf{A}}^c = \frac{1}{2}\delta_a^c$, $\mathbb{Z}^{\mathbf{A}}_{\mathbf{a}} \mathbb{Z}^{\mathbf{c}}_{\mathbf{A}} = \delta^{c^1}_{a^1}\delta^{c^2}_{a^2}$ and $\mathbb{W}^{\mathbf{A}}\mathbb{W}_{\mathbf{A}} = -\frac{1}{2}\,\mathrm{id}$, the remaining contractions are zero. The explicit formula for the tractor connection is then determined by how it acts on these (cf. [33, 6]):

$$\begin{aligned}
\nabla_p \mathbb{Y}_{A^0A^1}{}^{a^1} &= P_{pa_0} \mathbb{Z}_{A^0A^1}^{a^0a^1} + P_p{}^{a^1} \mathbb{W}_{A^0A^1}\\
\nabla_p \mathbb{Z}_{A^0A^1}^{a^0a^1} &= -2\delta_p^{a^0} \mathbb{Y}_{A^0A^1}{}^{a^1} - 2P_p{}^{a^0} \mathbb{X}_{A^0A^1}{}^{a^1}\\
\nabla_p \mathbb{W}_{A^0A^1} &= -\boldsymbol{g}_{pa^1} \mathbb{Y}_{A^0A^1}{}^{a^1} + P_{pa^1} \mathbb{X}_{A^0A^1}{}^{a^1}\\
\nabla_p \mathbb{X}_{A^0A^1} &= \boldsymbol{g}_{pa^0} \mathbb{Z}_{A^0A^1}^{a^0a^1} - \delta_p^{a^1} \mathbb{W}_{A^0A^1},
\end{aligned} \tag{15}$$

3.2. **Key differential operators.** Given a choice of conformal scale, Thomas' *tractor-D operator* [1] $D_A \colon \mathcal{E}_{B\cdots E}[w] \to \mathcal{E}_{AB\cdots E}[w-1]$ is defined by

$$D_A V := (n + 2w - 2)wY_A V + (n + 2w - 2)Z_{Aa}\nabla^a V - X_A(\Delta V + wJ)V. \tag{16}$$

This is conformally invariant, as can be checked directly using the formulae above (or alternatively there are conformally invariant constructions of D, see e.g. [29]). Acting on sections of weight $w \neq 1-n/2$ (16) is a *differential splitting operator* since there is a bundle homomorphism which inverts D. In this case it is a multiple of $X^A : \mathcal{E}_{AB\cdots E}[w-1] \to \mathcal{E}_{B\cdots E}[w]$; $X^A D_A$ is a multiple of the identity on the domain space. This splitting operator is particularly important on $\mathcal{E}[1]$, the densities of weight 1: for non-vanishing $\sigma \in \mathcal{E}[1]$, $g := \sigma^{-2}\boldsymbol{g}$ is Einstein if and only if $D_A\sigma$ is parallel for the tractor connection. The point is that the tractor connection (10) gives a prolonged system essentially equivalent to the equation $\nabla_{(a}\nabla_{b)_0}\sigma + P_{(ab)_0}\sigma = 0$ which controls whether the metric $g \in c$ is Einstein [1].

The GJMS operators on conformally flat manifolds can easily be constructed using the tractor D-operator. It turns out

$$(-1)^k X_{A_1} \ldots X_{A_k} P_k = D_{A_1} \ldots D_{A_k} \quad \text{on} \quad \mathcal{E}_\bullet[-n/2+k],$$

see [29] for details. Here $\bullet$, in $\mathcal{E}_\bullet$, denotes any system of tractor indices (or $\mathfrak{so}(h)$ tensor part thereof).

In addition to the tractor-D operator D_A, one has also the conformally invariant *double-D* operator $\mathbb{D}_{\mathbf{A}}$ and its "square" $\mathbb{D}^2_{AB} = -\mathbb{D}_{(A}{}^P\mathbb{D}_{|P|B)}$ defined as

$$\begin{aligned} \mathbb{D}_{\mathbf{A}} &= 2(w\mathbb{W}_{\mathbf{A}} + \mathbb{X}^a_{\mathbf{A}}\nabla_a) : \mathcal{E}_\bullet[w] \longrightarrow \mathcal{E}_{\mathbf{A}} \otimes \mathcal{E}_\bullet[w], \quad w \in \mathbb{R}, \\ \mathbb{D}^2_{AB} &= -(wh_{AB} + X_{(A}D_{B)}) : \mathcal{E}_\bullet[w] \longrightarrow \mathcal{E}_{(AB)} \otimes \mathcal{E}_\bullet[w], \quad w \in \mathbb{R}. \end{aligned} \tag{17}$$

The operator $\mathbb{D}_{\mathbf{A}}$ (but with the opposite sign) was originally defined in [30]. Note that, $2X_{[A^0}D_{A^1]} = (n+2w-2)\mathbb{D}_{\mathbf{A}}$ on $\mathcal{E}_\bullet[w]$. We shall also need the commutation relation on $\mathcal{E}_\bullet[w]$

$$[D_A, X_B] = -2\mathbb{D}_{AB} + (n+2w)h_{AB} \tag{18}$$

from [29]; alternatively this may be viewed as defining $\mathbb{D}$ as (one half of) the skew part of the left hand side.

Finally some points of notation: In the following we shall sometimes write ∇^q to denote the composition of q applications of ∇. By context it will be clear that q is not to be interpreted as an abstract index. Next if $\mathcal{V}$ is a tensor bundle, or a tensor product of the standard tractor bundle then for $F \in \mathcal{V}$ we shall write $F|_\boxtimes$ to denote the projection of the section F to the Cartan component (with respect to the $\mathfrak{co}(g)$ structure, or $\mathfrak{so}(h)$ tensor structure, respectively) of the bundle $\mathcal{V}$. For example on $\mathbb{E}^{s,s'}$ equipped with the standard flat diagonal signature (s,s') metric the equation (3) may be expressed as $[\nabla^{2r+1}\varphi]|_\boxtimes = 0$.

4. The double-D and conformally invariant operators

We work on $(M,[g])$, assumed to be locally conformally flat. We outline a rather general picture here. The theorem below provides a general technique for the construction of symmetries of *any* conformally invariant operator between irreducibles. Moreover, since the tools used are general in nature, this result indicates how to deal with symmetries of invariant operators on a bigger class of structures, the so-called parabolic geometries [14]. This will be taken up elsewhere.

Consider a conformally invariant differential operator $P : \mathcal{V} \to \mathcal{W}$ between irreducible (or completely reducible will suffice) conformal bundles $\mathcal{V}$ and $\mathcal{W}$. More specifically, we restrict only to subbundles of $(\bigotimes \mathcal{E}_a)\otimes(\bigotimes \mathcal{E}^b)\otimes\mathcal{E}[w]$ which we shall term tensor bundles. The case of spinor bundles is however completely analogous.

Assume for a moment the general (i.e. possibly curved) conformal setting. Following [11], the double-D operator $\mathbb{D}_{\mathbf{A}}$ can be extended to all irreducible bundles (see the discussion on the fundamental derivative below for details). This extension obeys the Leibniz rule, and since (17) describes $\mathbb{D}_{\mathbf{A}}$ on $\mathcal{E}_\bullet[w]$, it remains to understand the action of $\mathbb{D}_{\mathbf{A}}$ on $\mathcal{E}_a \cong \mathcal{E}^b[-2]$. In this case we obtain

$$\mathbb{D}_{\mathbf{B}} f_a = -2\mathbb{W}_{\mathbf{B}} f_a + 2\mathbb{Z}_{\mathbf{B}}^{\mathbf{b}} \boldsymbol{g}_{b^0 a} f_{b^1} + 2\mathbb{X}_{\mathbf{B}}{}^b \nabla_b f_a \quad \text{for} \quad f_a \in \mathcal{E}_a \tag{19}$$

where $\mathbf{B} = \mathbf{B}^2$.

Our use of $\mathbb{D}$ is linked to the following proposition. For a tangent vector $\varphi^a \in \mathcal{E}^a$ we denote by L_φ the Lie derivative on sections of natural bundles. Recall $\mathcal{E}[w]$ is such a natural bundle, cf. the definition of $\mathcal{E}[w]$ in Section 2, as well as $\mathcal{E}_a$ and $\mathcal{E}^b$.

Proposition 4.1. *Let M be any conformally flat manifold and assume $\varphi^a \in \mathcal{E}^a$ is a conformal Killing vector (i.e. a solution of (3)). Then there is a unique parallel tractor $I_\varphi^{\mathbf{A}} \in \mathcal{E}^{\mathbf{A}}$, $\mathbf{A} = \mathbf{A}^2$ such that $\varphi^a = 2\mathbb{X}_{\mathbf{A}}^{\ a} I_\varphi^{\mathbf{A}}$* [33], *cf. (43). Then*

$$I_\varphi^{\mathbf{A}} \mathbb{D}_{\mathbf{A}} = L_\varphi \quad \textit{on} \quad (\bigotimes \mathcal{E}_b) \otimes (\bigotimes \mathcal{E}^c) \otimes \mathcal{E}[w].$$

Proof. It is sufficient to verify the theorem on $\mathcal{E}[w]$ and $\mathcal{E}^a$ since both operators L_φ and $I_\varphi^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}$ obey the Leibniz rule and $\mathcal{E}_b \cong \mathcal{E}^a[-2]$. Using using (17) and (43) we have $I_\varphi^{\mathbf{A}}\mathbb{D}_{\mathbf{A}} = \varphi^a\nabla_a - \frac{2}{n}(\nabla_a\varphi^a)$ on $\mathcal{E}[2]$. Thus using (19) (and (43) below) we obtain

$$\begin{aligned} I_\varphi^{\mathbf{B}}\mathbb{D}_{\mathbf{B}} f_a &= \varphi^b\nabla_b f_a + (\nabla_{[a}\varphi_{b]})f^b - \frac{1}{n}(\nabla_b\varphi^b) f_a \\ &= \varphi^b\nabla_b f_a - f^b\nabla_b\varphi_a + f^b\big[\frac{1}{2}(\nabla_b\varphi_a + \nabla_a\varphi_b) - \frac{1}{n}\boldsymbol{g}_{ab}\nabla^c\varphi_c\big] \end{aligned}$$

on $f_a \in \mathcal{E}_a[2] \cong \mathcal{E}^b$. The square bracket in the display is the conformal Killing operator, and thus vanishes. The equality of L_φ and $I_\varphi^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}$ on $\mathcal{E}[w]$ is even simpler, and hence the general case follows. ☐

Note it obvious from the proof that the proposition does not hold without the assumption that $\varphi^a \in \mathcal{E}^a$ is a conformal Killing vector.

The conformal invariance of the operator $P : \mathcal{V} \to \mathcal{W}$ is given by the property $L_\varphi P = PL_\varphi$ for every conformal Killing field $\varphi^a \in \mathcal{E}^a$. That is, every conformal Killing vector φ^a provides a symmetry of the operator P.

As is well known, conformal invariance can equivalently be verified from a formula for the operator P. In particular for each conformally invariant operator, and a choice of metric from the conformal class, there is a formula in terms of the Levi-Civita connection ∇, its curvature, and various algebraic projections which express the operator as a natural (pseudo-)Riemmanian differential operator. The hallmark of conformal invariance is then that this operator is unchanged if we use the same formula when starting with a different metric form the conformal class. Now, given such a formula for $P : \mathcal{V} \to \mathcal{W}$, we have also the (tractor coupled) operator $P^\nabla : \mathcal{V} \otimes \mathcal{E}_\bullet \to \mathcal{W} \otimes \mathcal{E}_\bullet$ given by the same formula where ∇ is now assumed to be coupled Levi-Civita-tractor connection. Then P^∇ is also conformally invariant. We shall often write P instead of P^∇ to simplify the notation.

Theorem 4.2. *On a conformally flat manifold, let $P : \mathcal{V} \to \mathcal{W}$ be a conformally invariant operator between completely reducible tensor bundles $\mathcal{V}$ and $\mathcal{W}$. Then*

$$P^{\nabla}\mathbb{D}_{\mathbf{A}_1}\cdots\mathbb{D}_{\mathbf{A}_p} = \mathbb{D}_{\mathbf{A}_1}\cdots\mathbb{D}_{\mathbf{A}_p}P : \mathcal{V} \to \mathcal{E}_{\mathbf{A}_1\ldots\mathbf{A}_p}\otimes\mathcal{W}.$$

Proof. It is sufficient to prove the theorem in the (globally) flat case. First assume $p = 1$ and consider a conformal Killing field $\varphi^a \in \mathcal{E}^a$. Then I_φ is parallel (see e.g. [31], but this follows here easily from the fact the standard tractor connection is flat). Then $[P^\nabla, I_\varphi^{\mathbf{A}}] = 0$ and using Proposition 4.1 plus the fact that $L_\varphi P = PL_\varphi$, from conformal the invariance of P, means that $I_\varphi^{\mathbf{A}}[\mathbb{D}_{\mathbf{A}}, P] = 0$ for every conformal Killing vector φ^a. The space of conformal Killing fields on the conformally flat manifolds has the maximal dimension, i.e. the dimension of the bundle $\mathcal{E}_{\mathbf{A}}$. Therefore $[\mathbb{D}_{\mathbf{A}}, P] = 0$ on $\mathcal{V}$. Now it follows from the definition of $\mathbb{D}$ that the formulae for $[\mathbb{D}_{\mathbf{A}}, P]$ on $\mathcal{V}$ and $\mathcal{E}_\bullet \otimes \mathcal{V}$ formally coincide. Since $[\mathbb{D}_{\mathbf{A}}, P] = 0$, this formula yields a zero operator on every bundle $\mathcal{E}_\bullet \otimes \mathcal{V}$. Using an obvious induction, the theorem follows. □

Below we shall identify 2-form tractor fields $F_{\mathbf{A}} = F_{A^1A^2}$ with endomorphism fields of the standard tractor bundle according to the rule $(F\sharp f)_B := F_B{}^P f_P$ for $f_B \in \mathcal{E}_B$. This also defines the notation $\sharp$. Moreover, we shall define $\sharp$ to be trivial on the bundles $\mathcal{E}_a$ and $\mathcal{E}[w]$, and then extend this action to tensor products of $\mathcal{E}_A$, $\mathcal{E}_a$ and $\mathcal{E}[w]$ by the Leibniz rule. Note that since F is skew it yields an (pseudo-)orthogonal action pointwise and hence preserves the $SO(p+1, q+1)$ decompositions of tractor bundles.

Theorem 4.2 above is one of the primarily tools for our subsequent construction of symmetries. However there are some conceptual gains in linking this to some related results and so we complete this section with these observations.

The double-D operator discussed above reflects a more general operator called *fundamental derivative* from [11] (where it is called the fundamental-D operator). The specialisation of this to conformal geometry provides, for any natural bundle $\mathcal{V}$, a conformally invariant differential operator $\mathcal{D} : \mathcal{V} \to \mathcal{A}\otimes\mathcal{V}$, where $\mathcal{A} = \Lambda^2\mathcal{T}$ is often called the *adjoint tractor bundle* (because it is modelled on $\mathfrak{g} = \mathfrak{so}_{s+1,s'+1}$). Since there is a natural inclusion $\mathcal{A} \hookrightarrow \operatorname{End}\mathcal{E}_\bullet$ via $\sharp$, we may form ((−1)–times) the symmetrisation of the contracted composition, to be denoted by

$$\mathcal{D}^2 : \mathcal{V} \to (\operatorname{End}\mathcal{V}) \otimes \mathcal{V}.$$

In the abstract index notation we write $\mathcal{D}_A{}^B$ (or $\mathcal{D}_{\mathbf{A}}$, using the identification $\mathcal{A} \cong \mathcal{E}_{A^1A^2}$) for the fundamental derivative and so $\mathcal{D}^2_{AB} = -\mathcal{D}^C{}_{(A}\mathcal{D}_{B)C}$.

We shall use $\mathcal{D}$ only on weighted tensor bundles $\mathcal{V} \subseteq (\bigotimes \mathcal{E}_a)\otimes(\bigotimes \mathcal{E}^b)\otimes\mathcal{E}_\bullet[w]$. Recall the fundamental derivative obeys the Leibniz rule and actually $\mathcal{D}_{\mathbf{A}} = \mathbb{D}_{\mathbf{A}}$ on irreducible bundles. (In fact, the double-D was defined in such way in [11].) To show the difference between $\mathbb{D}$ and $\mathcal{D}$ and, more generally, the analogue of (17) we shall need certain special tractor sections and their corresponding algebraic actions on tractor bundles as follows:

$$\begin{aligned} &\mathbb{H}_{\mathbf{AB}} = h_{A^0B^0}h_{A^1B^1}, && \mathbb{H}_{\mathbf{A}}\sharp = h_{A^0B^0}h_{A^1B^1}\,\sharp_{\mathbf{B}} \\ &\widetilde{\mathbb{H}}_{AD\mathbf{BC}} = h_{(A|B^0|}h_{D)C^0}h_{B^1C^1}, && \widetilde{\mathbb{H}}_{AD}\sharp\sharp = h_{(A|B^0|}h_{D)C^0}h_{B^1C^1}\,\sharp_{\mathbf{B}}\,\sharp_{\mathbf{C}} \end{aligned} \tag{20}$$

where, as usual, we skew over the index pairs A^0A^1, B^0B^1 and C^0C^1. Here the subscript of $\sharp$ indicates which skew symmetric component is considered as an endomorphism. That is, for example, $(\mathbb{H}_{\mathbf{A}}\sharp f)_C = h_{A^0C}f_{A^1}$ for $f_C \in \mathcal{E}_C$, and this extends to tensor powers of the tractor bundle by the Leibniz rule. It also indicates the order of applications of these endomorphisms in the case of $\widetilde{\mathbb{H}}$.

We need $\mathcal{D}$ only up to a (nonzero) scalar multiple and our choice will differ from [7] by -1. Explicit formulae of $\mathcal{D}$ and $\mathcal{D}^2$ on weighted tractor bundles $\mathcal{E}_\bullet[w]$ are given by

$$\begin{aligned}\mathcal{D}_{\mathbf{A}} &= 2(w\mathbb{W}_{\mathbf{A}} + \mathbb{X}^a_{\mathbf{A}}\nabla_a + \mathbb{H}_{\mathbf{A}}\sharp) \\ \mathcal{D}^2_{AD} &= -(wh_{AD} + X_{(A}D_{D)} + 4h_{(A|B^0|}\mathbb{D}_{D)B^1}\sharp_{\mathbf{B}} - 4\widetilde{\mathbb{H}}_{AD}\sharp\sharp)\end{aligned} \tag{21}$$

where we skew over $[B^0B^1]$ and $\sharp_{\mathbf{B}}$ indicates the skewed symmetric component which is considered as an endomorphism. That is, $\mathcal{D}_{\mathbf{A}} = \mathbb{D}_{\mathbf{A}} + 2\mathbb{H}_{\mathbf{A}}\sharp$.

Corollary 4.3. *Assume the locally conformally flat setting. Let $P : \mathcal{V} \to \mathcal{W}$ be a conformally invariant operator between irreducible weighted tensor bundles $\mathcal{V}$ and $\mathcal{W}$. Then*

$$P^{\nabla}\mathcal{D}_{\mathbf{A}_1}\cdots\mathcal{D}_{\mathbf{A}_p} = \mathcal{D}_{\mathbf{A}_1}\cdots\mathcal{D}_{\mathbf{A}_p}P : \mathcal{V} \to \mathcal{E}_{\mathbf{A}_1\ldots\mathbf{A}_p}\otimes\mathcal{W}.$$

Proof. We shall use an induction. The case $p = 1$ is obvious as $\mathcal{D}_{\mathbf{A}} = \mathbb{D}_{\mathbf{A}}$ on $\mathcal{V}$ and $\mathcal{W}$. Assume the corollary holds for a fixed integer p. Then

$$\mathcal{D}_{\mathbf{A_0}}\mathcal{D}_{\mathbf{A}_1}\cdots\mathcal{D}_{\mathbf{A}_p} = \mathbb{D}_{\mathbf{A}_0}\mathcal{D}_{\mathbf{A}_1}\cdots\mathcal{D}_{\mathbf{A}_p} + 2\mathbb{H}_{\mathbf{A}_0}\sharp\,\mathcal{D}_{\mathbf{A}_1}\cdots\mathcal{D}_{\mathbf{A}_p}.$$

The operator P commutes with the first term on the right hand side using $[P, \mathbb{D}_{\mathbf{A}_0}] = 0$ and the inductive assumption. Since the second term involves only $\mathcal{D}_{\mathbf{A}_1}\cdots\mathcal{D}_{\mathbf{A}_p}$ with some additional trace factors, P commutes with the second term (using the induction) as well. ☐

Lemma 4.4. *Assume the locally conformally flat setting. Then $[\mathcal{D}_{\mathbf{A}}, \mathbb{D}_{\mathbf{B}}] = 0$ on $\mathcal{V}\otimes\mathcal{E}_\bullet$ for $\mathcal{V}$ irreducible.*

Proof. >From (17) and (21) we obtain

$$[\mathcal{D}_{\mathbf{A}}, \mathbb{D}_{\mathbf{B}}] = [\mathcal{D}_{\mathbf{A}}, \mathcal{D}_{\mathbf{B}}] - 2\mathcal{D}_{\mathbf{A}}\mathbb{H}_{\mathbf{B}}\sharp + 2\mathbb{H}_{\mathbf{B}}\sharp\mathcal{D}_{\mathbf{A}} = [\mathcal{D}_{\mathbf{A}}, \mathcal{D}_{\mathbf{B}}] + 4h_{B^0A^0}\mathcal{D}_{B^1A^1}.$$

Thus contracting arbitrary sections $I^{\mathbf{A}} \in \mathcal{E}^{\mathbf{A}}$, $\bar{I}^{\mathbf{B}} \in \mathcal{E}^{\mathbf{B}}$ into the previous display we get

$$I^{\mathbf{A}}\bar{I}^{\mathbf{B}}[\mathcal{D}_{\mathbf{A}}, \mathbb{D}_{\mathbf{B}}] = I^{\mathbf{A}}\bar{I}^{\mathbf{B}}[\mathcal{D}_{\mathbf{A}}, \mathcal{D}_{\mathbf{B}}] + 4I^{A^1P}\bar{I}_P{}^{B^1}\mathcal{D}_{A^1B^1}.$$

We put $[I, \bar{I}]^{\mathbf{C}} := 4I^{C^0P}\bar{I}_P{}^{C^1}$. On the one hand, $I^{\mathbf{A}}\bar{I}^{\mathbf{B}}[\mathcal{D}_{\mathbf{A}}, \mathcal{D}_{\mathbf{B}}]$ is given [11, Proposition, p. 21]. On the other hand, a direct computation verifies the statement on $\mathcal{E}_\bullet[w]$, cf. (40) below. Therefore by restricting to this case (of $\mathcal{E}_\bullet[w]$), it follows that our notation $[I, \bar{I}]$ coincides precisely with $\{I, \bar{I}\}$ used in [11]. Thus using [11, Proposition, p. 21] on $\mathcal{V}\otimes\mathcal{E}_\bullet[w]$, the lemma follows. ☐

Remark 4.5. There is also a more conceptual proof of the previous corollary (thus also of Theorem 4.2). Motivated by [11, Theorem 3.3], we note that, at each point $x \in M$, the section

$$\overline{\mathbb{D}}^{(k)}\sigma := (\sigma, \mathbb{D}\sigma, \mathbb{D}^{(2)}\sigma = \mathbb{D}\mathbb{D}\sigma, \ldots, \mathbb{D}^{(k)}\sigma) \in \overline{\mathcal{A}}^{(k)}(\mathcal{V}) \subseteq \mathcal{V}\oplus\mathcal{E}_{\mathbf{A}}\otimes\mathcal{V}\oplus\ldots\oplus\bigotimes^k\mathcal{E}_{\mathbf{A}}\otimes\mathcal{V},$$

contains the data of the entire k-jet of $\sigma \in \mathcal{V}$. Note although here we assume $\mathcal{V}$ is irreducible, the operator $\overline{\mathbb{D}}^{(k)}$ is defined also on bundles of the form $\mathcal{V} \otimes \mathcal{E}_\bullet$. From the general theory, the subbundle $\overline{\mathcal{A}}^{(k)}(\mathcal{V})$ (defined in the obvious way by the display) is an induced bundle of a principle H–bundle where $H \subseteq SO(s+1, s'+1)$ is a parabolic subgroup. It is straightforward to argue that any conformally invariant k-order operator on $\mathcal{V}$ is given by $\overline{\mathbb{D}}^{(k)}$ followed by a suitable H-homomorphism Φ on this subbundle. We denote this homomorphism by Φ_P in the case of the operator P.

Our aim is to commute $P = \Phi_P \circ \overline{\mathbb{D}}^{(k)}$ and $\mathbb{D}_{\mathbf{B}}$. More precisely, we put

$$P^\nabla := (\mathrm{id}\,|_{\mathcal{E}_{\mathbf{B}}} \otimes \Phi_P) \circ \overline{\mathbb{D}}^{(k)} : \mathcal{E}_{\mathbf{B}} \otimes \mathcal{V} \to \mathcal{E}_{\mathbf{B}} \otimes \mathcal{W}.$$

Observe the formulae for $\overline{\mathbb{D}}^{(k)} : \mathcal{V} \to \overline{\mathcal{A}}^{(k)}(\mathcal{V})$ and $\overline{\mathbb{D}}^{(k)} : \mathcal{E}_{\mathbf{B}} \otimes \mathcal{V} \to \mathcal{E}_{\mathbf{B}} \otimes \overline{\mathcal{A}}^{(k)}(\mathcal{V})$ are formally the same. (Note the implicit ∇ is interpreted as the coupled Levi-Civita-tractor connection in the latter case). That means also the formulae for $P : \mathcal{V} \to \mathcal{W}$ and $P^\nabla : \mathcal{E}_{\mathbf{B}} \otimes \mathcal{V} \to \mathcal{E}_{\mathbf{B}} \otimes \mathcal{W}$ are given by the same formal expression. Hence our definition of P^∇ coincides with that given before Theorem 4.2.

Now we are ready to show that $\mathcal{D}_{\mathbf{B}} P = P^\nabla \mathcal{D}_{\mathbf{B}}$ on $\mathcal{V}$, i.e.

$$(\Phi_p \otimes \mathrm{id}\,|_{\mathcal{E}_{\mathbf{B}}}) \circ \overline{\mathbb{D}}^{(k)} \mathcal{D}_{\mathbf{B}} = \mathcal{D}_{\mathbf{B}} \big(\Phi_p \circ \overline{\mathbb{D}}^{(k)}\big) : \mathcal{V} \to \mathcal{E}_{\mathbf{B}} \otimes \mathcal{W}.$$

Clearly $\mathcal{D}_{\mathbf{B}} \Phi_P = (\Phi_P \otimes \mathrm{id}\,|_{\mathcal{E}_{\mathbf{B}}}) \mathcal{D}_{\mathbf{B}}$. Since $[\mathcal{D}_{\mathbf{B}}, \mathbb{D}_{\mathbf{A}}] = 0$ from Lemma 4.4 and $\mathcal{D}_{\mathbf{B}}$ preserves subbundles (of the space $\mathcal{D}_{\mathbf{B}}$ acts on), $(\Phi_P \otimes \mathrm{id}\,|_{\mathcal{E}_{\mathbf{B}}}) \mathbb{D}_{\mathbf{A}_1} \ldots \mathcal{D}_{\mathbf{B}} \ldots \mathbb{D}_{\mathbf{A}_i}$ is conformally invariant and the previous display follows.

Henceforth we shall write P instead of P^∇ for simplicity. Finally note although we have shown $[\mathcal{D}_{\mathbf{B}}, P] = 0$ only on an irreducible $\mathcal{V}$, the same reasoning shows $[\mathcal{D}_{\mathbf{B}}, P] = 0$ also on bundles $\mathcal{V} \otimes \mathcal{E}_\bullet$. Therefore this remark offers an alternative proof of the previous corollary (thus also of Theorem 4.2).

The previous results provide an obvious way to construct symmetries of conformally invariant operators. Assume the section

$$I^{\mathbf{A}_1 \ldots \mathbf{A}_p B_1 B'_1 \ldots B_r B'_r} \in \mathcal{E}^{\mathbf{A}_1 \ldots \mathbf{A}_p B_1 B'_1 \ldots B_r B'_r}$$

is parallel. Then from Theorem 4.2 and Corollary 4.3 the differential operators

$$\begin{aligned} S &= I^{\mathbf{A}_1 \ldots \mathbf{A}_p B_1 B'_1 \ldots B_r B'_r} \mathcal{D}_{\mathbf{A}_1} \ldots \mathcal{D}_{\mathbf{A}_p} \mathcal{D}^2_{B_1 B'_1} \ldots \mathcal{D}^2_{B_r B'_r} \quad \text{and} \\ \mathbb{S} &= I^{\mathbf{A}_1 \ldots \mathbf{A}_p B_1 B'_1 \ldots B_r B'_r} \mathbb{D}_{\mathbf{A}_1} \ldots \mathbb{D}_{\mathbf{A}_p} \mathbb{D}^2_{B_1 B'_1} \ldots \mathbb{D}^2_{B_r B'_r} \end{aligned} \tag{22}$$

commute with P. That is S and $\mathbb{S}$ are symmetries of the operator P.

Proposition 4.6. *Assume the tractor $I^{\mathbf{A}_1 \ldots \mathbf{A}_p B_1 B'_1 \ldots B_r B'_r}$ is parallel and irreducible, $I = I|_{\boxtimes}$. Then $S = \mathbb{S}$ on $\mathcal{E}[w]$.*

Proof. Consider the parallel and irreducible tractor $I^{\mathbf{A}_1 \ldots \mathbf{A}_p B_1 B'_1 \ldots B_r B'_r}$ and the symmetry S from (22). Since $\mathcal{D}_{\mathbf{A}} = \mathbb{D}_{\mathbf{A}} + 2\mathbb{H}_{\mathbf{A}}\sharp$, the difference

$$\mathcal{D}_{\mathbf{A}_1} \mathcal{D}_{\mathbf{A}_2} \ldots \mathcal{D}_{\mathbf{A}_p} \mathcal{D}^2_{B_1 B'_1} \ldots \mathcal{D}^2_{B_r B'_r} - \mathbb{D}_{\mathbf{A}_1} \mathcal{D}_{\mathbf{A}_2} \ldots \mathcal{D}_{\mathbf{A}_p} \mathcal{D}^2_{B_1 B'_1} \ldots \mathcal{D}^2_{B_r B'_r}$$

lives in the trace part of $\mathcal{E}_{\mathbf{A}_1\ldots\mathbf{A}_pB_1B_1'\ldots B_rB_r'}[w]$, cf. (20). Therefore this difference is killed after contraction with $I_\varphi^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B_1'\ldots B_rB_r'}$. Repeating this argument for $\mathcal{D}_{\mathbf{A}_2},\ldots,\mathcal{D}_{\mathbf{A}_p}$, we obtain

$$S=I^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B_1'\ldots B_rB_r'}\mathbb{D}_{\mathbf{A}_1}\ldots\mathbb{D}_{\mathbf{A}_p}\mathcal{D}^2_{B_1B_1'}\ldots\mathcal{D}^2_{B_rB_r'}:\mathcal{E}[w]\to\mathcal{E}[w].$$

Now we replace $\mathcal{D}^2_{B_1B_1'}$ in the previous display by $\mathbb{D}^2_{B_1B_1'}$. Note $I^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B_1'\ldots B_rB_r'}$ commutes with $\mathbb{D}_{\mathbf{A}_i}$ and consider $I^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B_1'\ldots B_rB_r'}$ contracted with

$$\begin{aligned}\mathcal{D}^2_{B_1B_1'}\mathcal{D}^2_{B_2B_2'}\ldots\mathcal{D}^2_{B_rB_r'}-\mathbb{D}^2_{B_1B_1'}\mathcal{D}^2_{B_2B_2'}\ldots\mathcal{D}^2_{B_rB_r'}=&\\=-\big(4h_{(B_1|C^0|}\mathbb{D}_{B_1')C^1}\sharp_{\mathbf{C}}-4\widetilde{\mathbb{H}}_{B_1B_1'}\sharp\sharp\big)\mathcal{D}^2_{B_2B_2'}\ldots\mathcal{D}^2_{B_rB_r'}.&\end{aligned}$$

where we have used (21) and (17). The second term in the round brackets on the right hand side vanishes after the contraction (using trace-freeness of I again) so it remains to contract $I^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B_1'\ldots B_rB_r'}$ with

$$\begin{aligned}4h_{C^0(B_1}\mathbb{D}_{B_1')C^1}\sharp_{\mathbf{C}}\mathcal{D}^2_{(B_2B_2'}\ldots\mathcal{D}^2_{B_rB_r')}=&4(r-1)h_{(B_2B_1}\mathbb{D}_{B_1'}{}^P\mathcal{D}^2_{|P|B_2'}\ldots\mathcal{D}^2_{B_rB_r')}\\&-4(r+1)\mathbb{D}_{(B_1'B_2}\mathcal{D}^2_{B_1B_2'}\ldots\mathcal{D}^2_{B_rB_r')}\end{aligned}$$

Here we have used the fact that the indices $B_1B_1'\ldots B_rB_r'$ of I are symmetric (because I is irreducible). Now the second term on the right hand side is zero due to skew symmetry of indices of $\mathbb{D}_{B_1'B_2}$ and the first term vanishes after contraction with I which is trace-free. Repeating the same argument for $\mathcal{D}^2_{B_1B_2'},\ldots,\mathcal{D}^2_{B_rB_r'}$, the proposition follows. $\square$

Note an analogous statement to the Proposition above holds where $\mathcal{E}[w]$ is replaced by any irreducible bundle $\mathcal{V}$. This may be proved along the same lines as in the treatment above. However since the details are technical and not required here, this proof is omitted.

Finally note the operators given by (22) are well defined also on bundles $\mathcal{E}_\bullet[w]$. In this setting, however, they yield generally different operators $\mathcal{E}_\bullet[w]\to\mathcal{E}_\bullet[w]$.

5. A Construction of symmetries

We are now ready to construct canonical symmetries. For a section $\varphi_r^{a_1\ldots a_p}\in\mathcal{E}^{(a_1\ldots a_p)_0}[2r]$ we shall define the operators (S_φ,S_φ') where S_φ and S_φ' have leading term $\varphi_r^{a_1\ldots a_p}\nabla_{a_1}\cdots\nabla_{a_p}\Delta^r$. To do this we use the bijective correspondence between the linear space of solutions of (3) and certain finite dimensional $\mathfrak{g}$–modules, cf. the discussion around (4). Explicitly, this is given by differential prolongation in the form of a differential splitting operator $\mathcal{E}^{(a_1\ldots a_p)_0}[2r]\to\mathcal{E}^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B_1'\ldots B_rB_r'}|_\boxtimes$. There are many ways of constructing this, but for our current purposes the splitting operator can be conveniently expressed using the fundamental derivative. There is a certain operator $\mathcal{C}$ known as the *curved Casimir* [15] which is given by $h^{AB}\mathcal{D}^2_{AB}$. This acts on any natural bundle and, in particular, on weighted tractor bundles. It can thus be iterated. One gets the splitting operator as

$$\varphi_r^{a_1\ldots a_p}\mapsto\mathbb{Y}^{\mathbf{A}_1}{}_{a_1}\cdots\mathbb{Y}^{\mathbf{A}_p}{}_{a_p}Y^{B_1}Y^{B_1'}\cdots Y^{B_r}Y^{B_r'}\varphi_r^{a_1\ldots a_p}\xrightarrow{Q}\mathcal{E}^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B_1'\ldots B_rB_r'}\tag{23}$$

where Q is an operator polynomial in $\mathcal{C}$, and hence is polynomial in $\mathcal{D}$, see [15, 34]. We shall denote the image by $I_\varphi^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B_1'\ldots B_rB_r'} \in \mathcal{E}^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B_1'\ldots B_rB_r'}|_{\boxtimes}$. The main point we need is that the tractor I_φ is parallel if and only if φ is a solution of the operator (3).

Definition 5.1. Let $\varphi = \varphi_r^{(a_1\ldots a_p)_0} \in \mathcal{E}^{(a_1\ldots a_p)_0}[2r]$, $r,p \geq 0$ be a solution of (3). Given such a solution φ we shall associate a differential operator S_φ as follows. Let I_φ denote the parallel tractor corresponding to φ, in the sense of the discussion surrounding (23) above. Then via (22),

$$S_\varphi := I_\varphi^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B_1'\ldots B_rB_r'}\mathbb{D}_{\mathbf{A}_1}\ldots\mathbb{D}_{\mathbf{A}_p}\mathbb{D}^2_{B_1B_1'}\ldots\mathbb{D}^2_{B_rB_r'}\ , \tag{24}$$

is a well defined differential operator $S_\varphi : \mathcal{V} \to \mathcal{V}$, for any weighted tensor-tractor bundle $\mathcal{V}$.

It follows immediately from Theorem 4.2, and the fact that I_φ is parallel, that S_φ is a universal symmetry operator. That is, using also that $\varphi \mapsto I_\varphi$ is a splitting operator, we have the following.

Theorem 5.2. *On a conformally flat manifold, let $P : \mathcal{V} \to \mathcal{W}$ be a conformally invariant operator between irreducible tensor bundles $\mathcal{V}$ and $\mathcal{W}$, and suppose that $\varphi = \varphi_r^{(a_1\ldots a_p)_0} \in \mathcal{E}^{(a_1\ldots a_p)_0}[2r]$, $r,p \geq 0$ is a solution of (3). Then with $S_\varphi : \mathcal{V} \to \mathcal{V}$ and $S'_\varphi : \mathcal{W} \to \mathcal{W}$ given by (24), the pair (S_φ, S'_φ) is symmetry of P. For $\varphi \neq 0$ and $r < k$ this is a non-trivial symmetry.*

Note that S_φ and S'_φ are not the same differential operators. The point is that (24) really defines a family of differential operators parametrised by the space of domain bundles.

We shall henceforth only pursue the case that P is a GJMS operator. Proposition 6.1 (which we will come to later) shows that, acting on any density bundle, φ is the leading symbol of the operator (24). Also note that in this case the use of $\mathcal{D}$ and $\mathcal{D}^2$ instead of $\mathbb{D}$ and $\mathbb{D}^2$ in (24) yields the same symmetries, cf. Proposition 4.6.

Remark. Consider an operator $F : \mathcal{E}[w] \to \mathcal{E}[w]$, of order $\tilde{p} \geq 0$, on a smooth conformal manifold manifold $(M,[g])$ and its symbol $\tilde{\varphi}^{(a_1\ldots a_{\tilde{p}})} \in \mathcal{E}^{(a_1\ldots a_{\tilde{p}})}$. Then, via the conformal structure $[g]$ we may decompose $\tilde{\varphi}$ into irreducibles. Each irreducible component φ of $\tilde{\varphi}$ can be realised as $\varphi^{(a_1\ldots a_p)_0} \in \mathcal{E}^{(a_1\ldots a_p)_0}[2r]$ where $p = \tilde{p} - 2r$. Thus we have also the operator S_φ, constructed as above except that we here do not require φ to solve (3). We may then take the difference $F - S_\varphi : \mathcal{E}[w] \to \mathcal{E}[w]$. Now the whole procedure can be repeated for the operator $F - S_\varphi$. It is clear that after a finite number of steps we obtain the form $F = \sum_{\varphi \in U} S_\varphi$ for a (finite) index set $U \subseteq \mathbb{N}$. That is, given an operator $F : \mathcal{E}[w] \to \mathcal{E}[w]$ on a smooth manifold M, any conformal structure on M yields a decomposition of F as a sum of canonical operators S_φ.

In the other direction, the operators S_φ provide the conformally invariant quantization introduced in [19], in particular the special case [19, 3.1]. Also note the Section 4 shows how to rewrite the general construction [8] using an affine connection.

6. Classification of leading terms of symmetries

According to the discussion following Theorem 2.4, the problem of conformal symmetries for the GJMS operators (on locally conformally flat manifolds) is reduced to the setting of Theorem 2.1. So throughout this section we work on $\mathbb{E}^{s,s'}$ equipped with the standard flat diagonal signature (s, s') metric g with $s + s' =: n \geq 3$.

All linear differential operators $L : \mathcal{E}[w] \to \mathcal{E}[w]$ may be expressed as sums of the form

$$P = \sum_{p,r\geq 0} \varphi_r^{a_1\dots a_p}(\nabla_{a_1}\cdots\nabla_{a_p})\Delta^r, \quad \varphi_r^{a_1\dots a_p} \in \mathcal{E}^{(a_1\dots a_p)_0}[2r] = \mathcal{E}_r^{(p)_0}; \tag{25}$$

We shall term such a *standard expression* for L. For the operator in the previous display, we shall use the shorthand notation $\varphi_r^p(\odot^p\nabla)\Delta^r$ instead of $\varphi_r^{a_1\dots a_p}(\nabla_{a_1}\cdots\nabla_{a_p})\Delta^r$.

We use the standard expressions as above to analyze the structure of potential symmetries and their compositions with Δ^k. In particular we shall use the following properties/descriptions of a given coefficient φ_r^p. We shall write $o(\varphi_r^p) = p+2r$ and term this the *formal order* of φ_r^p and $\ell(\varphi_r^p) = p+r$ which will be termed *level* of φ_r^p. (These reflect properties of terms $\varphi_r^{a_1\dots a_p}(\nabla_{a_1}\cdots\nabla_{a_p})\Delta^r$ and how they appear naturally in appropriate tractor formulae. However these quantities are fully determined by the coefficients φ_r^p, so it is sufficient to consider formal order and level of coefficients.) We also say $\left[\begin{smallmatrix}p\\r\end{smallmatrix}\right]$ is the *type* of φ_r^p. We shall write $o(R) = a$ and $\ell(R) = b$ if all terms of a differential operator $R : \mathcal{E}[w] \to \mathcal{E}[w]$ are of the formal order at most a, respectively level at most b. Finally if L is a symmetry of Δ^k, then we shall say L is a *normal* symmetry (of Δ^k) if $r < k$ for all terms in the standard expression (25). Modulo trivial symmetries, any symmetry Δ^k may be represented by a normal symmetry.

Further we shall need a suitable ordering of the terms in a standard expression. This will be defined via the coefficients as follows:

$$\varphi_r^p \lhd \psi_{r'}^{p'} \quad \text{iff} \quad \ell(\varphi_r^p) < \ell(\psi_{r'}^{p'}) \;\text{ or }\; \big(\ell(\varphi_r^p) = \ell(\psi_{r'}^{p'})\big) \wedge \big(o(\varphi_r^p) < o(\psi_{r'}^{p'})\big). \tag{26}$$

Since the coefficient φ_r^p determines a corresponding term in the standard expression completely, we shall use the ordering $\lhd$ for both coefficients and terms of an operator (25).

First we shall study the canonical symmetries. Since these are constructed using tractor operators we need a further weight type measure as follows. In the tractor formulae, we use strings of the symbols X, Y, Z and $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ and $\mathbb{W}$ from Section 3.1. We define the *homogenity* $\mathrm{h}(\omega)$ of a string $\omega \in \{X, Y, Z, \mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}\}$ by

$$\begin{gathered}\mathrm{h}(Y) = 1, \mathrm{h}(Z) = 0, \mathrm{h}(X) = -1, \mathrm{h}(\mathbb{Y}) = 1, \mathrm{h}(\mathbb{Z}) = h(\mathbb{W}) = 0, \mathrm{h}(\mathbb{X}) = -1 \\ \text{and} \quad \mathrm{h}(\omega_1\omega_2) := \mathrm{h}(\omega_1) + \mathrm{h}(\omega_2)\end{gathered} \tag{27}$$

where $\omega_1\omega_2$ means a concatenation of the strings ω_1 and ω_2.

Now we are set to describe some properties the canonical symmetries, as follows.

Proposition 6.1. *Consider $\varphi = \varphi_r^p \in (\odot^p TM) \otimes \mathcal{E}[2r]$ such that $[\nabla^{2r+1}\varphi]|_{\boxtimes} = 0$ and the corresponding canonical symmetry (S_φ, S'_φ) given by given by (24). Then, in the standard expressions for S_φ and S'_φ, the following properties hold:*

(i) S_φ and S'_φ have the same leading term φ.

(ii) $\ell(S'_\varphi) = \ell(S_\varphi) = r + p = \ell(\varphi_r^p)$, that is every term ψ of S_φ or S'_φ satisfies $\ell(\psi) \leq p + r$. Moreover, the greatest terms of S_φ and S'_φ have the coefficient φ.

(iii) $o(S'_\varphi) = o(S_\varphi) = p + 2r = o(\varphi_r^p)$, that is every term ψ of S_φ or S'_φ satisfies $o(\psi) \leq p + 2r$. Moreover, the equality happens only for $\psi = \varphi$.

(iv) Every term ψ of type $\left[\begin{smallmatrix}\bar{p}\\ \bar{r}\end{smallmatrix}\right]$ of S_φ or S'_φ satisfies $r \geq \bar{r}$.

Proof. First note that because S_φ and S'_φ are given by the same operator (24) acting on different density bundles, it turns out to be sufficient to establish facts only for S_φ. >From (24) S_φ is defined as the contraction of the parallel tractor $I_\varphi^{\mathbf{A}_1\ldots\mathbf{A}_p B_1 B'_1 \ldots B_r B'_r}$, corresponding to φ, with the operator

$$\widetilde{\mathbb{D}}_{\mathbf{A}_1\ldots\mathbf{A}_p B_1 B'_1 \ldots B_r B'_r} := \mathbb{D}_{\mathbf{A}_1} \ldots \mathbb{D}_{\mathbf{A}_p} \mathbb{D}^2_{B_1 B'_1} \ldots \mathbb{D}^2_{B_r B'_r} : \mathcal{E}_\bullet[w] \longrightarrow \mathcal{E}_{\bullet\, \mathbf{A}_1\ldots\mathbf{A}_p B_1 B'_1 \ldots B_r B'_r}[w].$$

We need some broad facts about the structure of the tractor formulae for I_φ and $\widetilde{\mathbb{D}}$. When working in a metric scale and using (12), (10), (15), and (17) it follows that terms of these are built respectively from tensor fields and tensor valued differential operators contracted into 'projectors'

$$\omega \in \mathcal{B}.$$

Here $\mathcal{B}$ is a set of fields taking values in the appropriate tractor bundle tensor product with an irreducible weighted trace-free tensor bundle. Each element $\omega \in \mathcal{B}$ is an appropriate projection (onto the irreducible part with respect to the tensor indices) of a p-fold tensor product of elements from $\{\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}\}$ with a $2r$-fold tensor product of elements from $\{X, Y, Z\}$, and we may take $\mathcal{B}$ to be all such. Similarly, the elements of $\mathcal{B}$ can be considered as 'injectors', i.e. a mapping going in the opposite direction. For example, since I_φ is obtained from φ by a splitting operator, it has the form

$$I_\varphi^{\mathbf{A}_1\ldots\mathbf{A}_p B_1 B'_1 \ldots B_r B'_r} = \sum_{\omega \in \mathcal{B}} \omega^{\mathbf{A}_1\ldots\mathbf{A}_p B_1 B'_1 \ldots B_r B'_r} \cdot F_\omega(\varphi) \tag{28}$$

where, for each $\omega \in \mathcal{B}$, $F_\omega(\varphi)$ is the result of a (weighted tensor valued) differential operator F_ω acting on φ (a section of $(\odot^p TM) \otimes \mathcal{E}[2r]$) and '$\cdot$' indicates a contraction of tensor indices (which are suppressed); cf. (43) below which shows I_φ for $\varphi^a \in \mathcal{E}^a$ explicitly. Note also that we sum over all strings in $\mathcal{B}$ in the previous display, so many of the F_ω will be zero. Similarly, it follows from the definition of $\widetilde{\mathbb{D}}$ that

$$\widetilde{\mathbb{D}}_{\mathbf{A}_1\ldots\mathbf{A}_p B_1 B'_1 \ldots B_r B'_r} = \sum_{\omega \in \mathcal{B}} \omega_{\mathbf{A}_1\ldots\mathbf{A}_p B_1 B'_1 \ldots B_r B'_r} \cdot G_\omega \tag{29}$$

where G_ω is a (weighted tensor valued) differential operator acting on densities and, again, '$\cdot$' denotes contraction of (suppressed) tensor indices. See (17) and (39) for explicit examples. Contracting the last two displays we obtain the canonical symmetry S_φ as in (24). Thus, using (11) and the surrounding observations, we have

$$S_\varphi = \sum_{\substack{\omega,\omega'\in\mathcal{B},\\ \mathrm{h}(\omega)+h(\omega')=0}} (F_\omega(\varphi))\cdot G_{\omega'}$$

where '$\cdot$' indicates the contraction of suppressed tensor indices. Note pairs (ω,ω') not satisfying $\mathrm{h}(\omega)+\mathrm{h}(\omega')=0$ have dropped out of the sum by properties of the tractor metric. (Also note that the same property implies that if the tensor indices of F_ω and $G_{\omega'}$ are not compatible for complete contraction then the term $(F_\omega(\varphi))\cdot G_{\omega'}$ is necessarily zero.)

The differential order of F_ω (and similarly $G_{\omega'}$) is exactly the maximal number of ∇'s in the corresponding expression in the splitting operator. (We consider formulae for splitting operators obtained using the curved Casimir $\mathcal{C}=h^{AB}\mathcal{D}^2_{AB}$ here.) Denoting such order of F_ω and $G_{\omega'}$ (in (28) and (29)) by, respectively, $o(F_\omega)$ and $o(G_{\omega'})$, we have

$$\mathrm{h}(\omega)+o(F_\omega)=p+2r \quad \text{and} \quad \mathrm{h}(\omega')+o(G_{\omega'})=0, \quad \omega,\omega'\in\mathcal{B}.$$

Here the first equality follows from (23) and the properties of splitting operators. The second follows from the definition of $\widetilde{\mathbb{D}}$ (in particular from the tractor expressions for $\mathbb{D}$ and $\mathbb{D}^2$ in (17)), (10), and (15). Summing up the equalities in the previous display we see that

$$S_\varphi = \sum_{\substack{\omega,\omega'\in\mathcal{B},\\ o(F_\omega)+o(G_{\omega'})=p+2r}} (F_\omega(\varphi))\cdot G_{\omega'}. \tag{30}$$

Note that all tractor indices have been eliminated, the formula (30) for S_φ is expressed using tensor operators and contractions only. Now consider a summand $(F_\omega(\varphi))\cdot G_{\omega'}$ of S_φ as in (30). First $o(F_\omega)+o(G_{\omega'})=p+2r$ implies $o(G_{\omega'})\le p+2r$; moreover the equality can happen only if $F_\omega=\mathrm{id}$ (up to a non-zero scalar multiple), since (23) is a differential splitting operator. For the same reason this term does occur. In the previous display the term with $F_\omega=\mathrm{id}$ clearly recovers the highest order term. Therefore (i) follows.

Now by assumption $F_\omega(\varphi)$ is irreducible. Since $S_\varphi:\mathcal{E}[w]\to\mathcal{E}[w]$, it follows from (25) that in the standard expression $(F_\omega(\varphi))\cdot G_{\omega'}=\gamma^{a_1\ldots a_{\bar p}}\nabla_{a_1}\ldots\nabla_{a_{\bar p}}\Delta^{\bar r}$, $\bar p,\bar r\ge 0$ where γ is symmetric and trace–free. In fact, it follows from the form of I_φ and $\widetilde{\mathbb{D}}$ that $F_\omega(\varphi)=\gamma^{a_1\ldots a_{\bar p}}$ and $G_{\omega'}=\nabla_{a_1}\ldots\nabla_{a_{\bar p}}\Delta^{\bar r}$. We denote the type of $F_\omega(\varphi)$ by $\left[\begin{smallmatrix}\bar p\\ \bar r\end{smallmatrix}\right]$. From this we get $o(G_{\omega'})=\bar p+2\bar r$ and, since F_ω takes φ of the type $\left[\begin{smallmatrix}p\\ r\end{smallmatrix}\right]$ to a section of the type $\left[\begin{smallmatrix}\bar p\\ \bar r\end{smallmatrix}\right]$, we get $o(F_\omega)\ge|p-\bar p|$. (The point is that each application of the Levi-Civita connection may increase of decrease the rank by 1, and this is the only way the rank may change.) These properties hold for every (irreducible) term $(F_\omega(\varphi))\cdot G_{\omega'}$ in (30). Therefore

$$p+2r=o(F_\omega)+o(G_{\omega'})\ge|p-\bar p|+\bar p+2\bar r$$

using (30). We prove (ii), (iii) and (iv) separately in cases $p\ge\bar p$ and $p\le\bar p$. If $p\ge\bar p$ then the previous display says $p+2r\ge p+2\bar r$ hence $r\ge\bar r$. This implies $p+r\ge\bar p+\bar r$ and $p+2r\ge\bar p+2\bar r$. If $p\le\bar p$ then the previous display means

$2p+2r \geq 2\bar{p}+2\bar{r}$ hence $p+r \geq \bar{p}+\bar{r}$. The latter inequality with $p \leq \bar{p}$ yields $r \geq \bar{r}$ and so $p+2r \geq \bar{p}+2\bar{r}$. This show (iv) and the inequalities in (ii) and (iii). Now when equality holds in (iii) then $p+2r = \bar{p}+2\bar{r}$. But then $p=\bar{p}$ from the previous display thus also $r=\bar{r}$. This means $o(G_{\omega'}) = p+2r$ and $o(F_\omega)=0$. Hence $F_\omega = id$, up to a multiple, and so if the term is non-trivial we recover the leading term. It remains to discuss the greatest term of S_φ. But since we have already proved the inequality in (ii), according to the ordering of (26) we need to consider the order of terms of level $p+r$. The maximal order is then characterized by (iii). □

Note the part (iii) of the previous proposition means that the canonical symmetry (S_φ, S'_φ), $\varphi^p_r \in (\otimes^p TM)\otimes \mathcal{E}[2r]$ is nontrivial for P_k, $k>r$. (The statement (iii) is actually stronger: no term in S_φ has Δ^k, $k>r$ as the right factor.)

Our strategy for classifying the leading terms of symmetries uses the ordering (26). We shall start with the greatest term and study what the symmetry condition imposes on its coefficient. We obtain the following

Claim: Let $\varphi^j_i \in \mathcal{E}^{(j)_0}_i$ is the greatest coefficient of a symmetry T. Then $[\nabla^{2i+1}\varphi^j_i]_\boxtimes = 0$.

The claim forms the basis for an inductive procedure, as if $[\nabla^{2i+1}\varphi^j_i]_\boxtimes = 0$ then the greatest term of $T - S_{\varphi^j_i}$ is strictly smaller (w.r.t. $\rhd$) than φ^j_i, and using Proposition 6.1, we can replace T by $T - S_{\varphi^p_0}$ and apply the previous claim again.

The Claim is proved as Proposition 6.3, and then the detailed inductive procedure is in the proof of Theorem 6.4. The proof of Proposition 6.3 requires a detailed analysis of certain terms. To demonstrate the technique, let us discuss an example first. Assume that (T,T') is a symmetry of $P_4=\Delta^4$ of order p, i.e.

$$\Delta^4 T = T'\Delta^4, \quad T = \sum_{2i+j\leq p, i<4} \varphi^j_i (\odot^j\nabla)\Delta^i$$

where we have displayed the standard expression of T. Note we have not included terms with $i \geq 4$ as they may be eliminated by the addition of trivial symmetries of Δ^4. It is useful to write the terms of T in a table as follows:

$$\begin{array}{rlllll}
\text{order } p: & \varphi^p_0(\odot^p\nabla) & +\varphi^{p-2}_1(\odot^{p-2}\nabla)\Delta^1 & +\varphi^{p-4}_2(\odot^{p-4}\nabla)\Delta^2 & +\ \varphi^{p-6}_3(\odot^{p-6}\nabla)\Delta^3 & +\\
\text{order } p-1: & \varphi^{p-1}_0(\odot^{p-1}\nabla) & +\varphi^{p-3}_1(\odot^{p-3}\nabla)\Delta^1 & +\varphi^{p-5}_2(\odot^{p-5}\nabla)\Delta^2 & +\ \varphi^{p-7}_3(\odot^{p-7}\nabla)\Delta^3 & +\\
\text{order } p-2: & \varphi^{p-2}_0(\odot^{p-2}\nabla) & +\varphi^{p-4}_1(\odot^{p-4}\nabla)\Delta^1 & +\varphi^{p-6}_2(\odot^{p-6}\nabla)\Delta^2 & +\ \varphi^{p-8}_3(\odot^{p-8}\nabla)\Delta^3 & +\\
\text{order } p-3: & \varphi^{p-3}_0(\odot^{p-3}\nabla) & +\varphi^{p-5}_1(\odot^{p-5}\nabla)\Delta^1 & +\varphi^{p-7}_2(\odot^{p-7}\nabla)\Delta^2 & +\ \varphi^{p-9}_3(\odot^{p-9}\nabla)\Delta^3 & +\\
\text{order } p-4: & \varphi^{p-4}_0(\odot^{p-4}\nabla) & +\varphi^{p-6}_1(\odot^{p-6}\nabla)\Delta^1 & +\varphi^{p-8}_2(\odot^{p-8}\nabla)\Delta^2 & +\varphi^{p-10}_3(\odot^{p-10}\nabla)\Delta^3 & +\\
\vdots & \quad\vdots & + \quad\vdots & + \quad\vdots & + \quad\vdots & +
\end{array}$$

Every line shows terms of the same formal order and moreover every antidiagonal shows terms of the same level. So the ordering (26) in this case means

$$\varphi^p_0 \rhd \varphi^{p-2}_1 \rhd \varphi^{p-1}_0 \rhd \varphi^{p-4}_2 \rhd \varphi^{p-3}_1 \rhd \varphi^{p-2}_0 \rhd \varphi^{p-6}_3 \rhd \cdots$$

Observe the level $\ell(R)$ of an operator R is increased by k under composition with Δ^k:

$$\ell(\Delta^k R) = \ell(R) + k.$$

Moreover, only terms of the highest level in R can contribute to terms of the highest level in $\Delta^k R$.

The greatest coefficient (w.r.t. $\rhd$) is φ_0^p. Recall $o(T) = p$ so we can assume $\ell(T) = p$ which means $\ell(\Delta^4 T) = p + 4$. Now we consider terms of the level $p + 4$ of $\Delta^4 T$. First we commute all covariant derivatives ∇ to the right. In fact, it is sufficient for our purpose to consider only certain terms. First we restrict to terms of the level $p + 4$ without a right factor Δ^4 and then take the candidate for the greatest among these. This is $(\nabla^1\varphi_0^p)(\odot^{p+1}\nabla)\Delta^3$. Since this does not have a right factor Δ^4, it has to vanish since T is a symmetry. Hence $(\nabla^1\varphi_0^p)_{\boxtimes} = 0$, which means that φ_0^p is a conformal Killing tensor. Now we replace the symmetry T by $T - S_{\varphi_0^p}$; this is also a symmetry. The greatest coefficient of $T - S_{\varphi_0^p}$ is now strictly smaller (w.r.t. $\rhd$) than the greatest coefficient of T. (Here we have adjusted $S_{\varphi_0^p}$ so the leading term is precisely $\varphi_0^p(\odot^p\nabla)$ rather than some non-zero multiple. We will not comment further when this sort of maneuver is used below.) It is φ_1^{p-2} according to (26). So now we may rename $T - S_{\varphi_0^p}$ as T and continue with the argument.

The next step is to assume $\varphi_0^p = 0$ and study differential conditions imposed on φ_1^{p-2}. Here we skip this and several other steps and we assume the greatest coefficient of T is φ_3^{p-6}. So suppose that $\varphi_i^j = 0$ for $\ell(\varphi_i^j) > p - 3 = \ell(\varphi_3^{p-6})$. Then $\ell(T) = p - 3$ and so $\ell(\Delta^4 T) = p + 1$. We shall examine those terms of the operator $\Delta^4 T$ of the (highest) level $p + 1$ and such that they are without a right factor Δ^4. To find these it is sufficient to consider

$$\Delta^4\left[\varphi_3^{p-6}(\odot^{p-6}\nabla)\Delta^3 + \varphi_2^{p-5}(\odot^{p-5}\nabla)\Delta^2 + \varphi_1^{p-4}(\odot^{p-4}\nabla)\Delta^1 + \varphi_0^{p-3}(\odot^{p-3}\nabla)\right].$$

We use the Leibniz rule to move Δ^4 to the right in the previous display. We need to know the form of (level $p + 1$) terms of types $\left[{p-2 \atop 3}\right]$, $\left[{p-1 \atop 2}\right]$, $\left[{p \atop 1}\right]$ and $\left[{p+1 \atop 0}\right]$. The simplest case is the type $\left[{p+1 \atop 0}\right]$, we obtain only the term $2^4(\nabla^4\varphi_0^{p-3}) \odot^{p+1} \nabla$. The operator $\odot^{p+1}\nabla$ does not arise in any other way, so the given term must vanish through φ_0^{p-3} satisfying the obvious equation. In the case of the type $\left[{p \atop 1}\right]$ we similarly get the equation

$$2^4(\nabla^4\varphi_1^{p-4})(\odot^p\nabla)\Delta + 2^3 \cdot 4(\nabla^3\varphi_0^{p-3})(\odot^p\nabla)\Delta = 0.$$

Here $2^3 \cdot 4 = 2^3\binom{4}{1} = 2^3\binom{4}{3}$; generally we put $C^s(4) = 2^s\binom{4}{s}$. The types $\left[{p-2 \atop 3}\right]$ and $\left[{p-1 \atop 2}\right]$ yield two more equations which give conditions for the coefficients φ_0^{p-3}, φ_1^{p-4}, φ_2^{p-5} and φ_3^{p-6}. Together these four equations yield the following differential equations for the coefficients φ_i^j:

$$\begin{array}{lccccccccl}
\text{type } \left[{p-2 \atop 3}\right]: & C^4(4)\nabla^4\varphi_3^{p-6} & + & C^3(4)\nabla^3\varphi_2^{p-5} & + & C^2(4)\nabla^2\varphi_1^{p-4} & + & C^1(4)\nabla^1\varphi_0^{p-3} & = & 0 \\
\text{type } \left[{p-1 \atop 2}\right]: & 0 & + & C^4(4)\nabla^4\varphi_2^{p-5} & + & C^3(4)\nabla^3\varphi_1^{p-4} & + & C^2(4)\nabla^2\varphi_0^{p-3} & = & 0 \\
\text{type } \left[{p \atop 1}\right]: & 0 & + & 0 & + & C^4(4)\nabla^4\varphi_1^{p-4} & + & C^3(4)\nabla^3\varphi_0^{p-3} & = & 0 \\
\text{type } \left[{p+1 \atop 0}\right]: & 0 & + & 0 & + & 0 & + & C^4(4)\nabla^4\varphi_0^{p-3} & = & 0
\end{array}$$

Here we implicitly consider the symmetric trace–free parts in every equation. Now applying ∇^3 to the first equation, ∇^2 to the second and ∇ to the third, and then taking the trace–free symmetric part in all cases, we obtain a linear system in variables $[\nabla^7\varphi_3^{p-6}]_{\boxtimes}$, $[\nabla^6\varphi_2^{p-5}]_{\boxtimes}$, $[\nabla^5\varphi_1^{p-4}]_{\boxtimes}$ and $[\nabla^4\varphi_0^{p-3}]_{\boxtimes}$. The matrix of (integer) coefficient is

$$\begin{pmatrix} C^4(4) & C^3(4) & C^2(4) & C^1(4) \\ 0 & C^4(4) & C^3(4) & C^2(4) \\ 0 & 0 & C^4(4) & C^3(4) \\ 0 & 0 & 0 & C^4(4) \end{pmatrix}.$$

This is non-singular. So all the variables must vanish, and in particular $[\nabla^7\varphi_3^{p-6}]_{\boxtimes} = 0$, which is what we wanted to prove.

This was the case with greatest coefficient φ_3^{p-6}. It suggests a route to solving the remaining cases, as they yield linear systems in the same way. Actually it turns out that in each of the cases with the greatest terms between φ_0^p and φ_3^{p-6} (which were skipped above), the matrix of coefficients includes a square "upper right" submatrix of the matrix above, i.e. a matrix obtained by removing the first q columns and the last q rows for some choice of q, that is sufficient if non-degenerate. That is it suffices to prove that determinants of these matrices are nonzero. This necessitates analysing the combinatorial coefficients $C^s(4)$ in more detail.

The general case is analogous; in the case of Δ^k, $k \in \mathbb{N}$ we shall need the scalars

$$C^s(k) := 2^s\binom{k}{s}, \quad C^s(k) := 0 \text{ for } s > k$$

and matrices

$$\begin{aligned} &\boldsymbol{C}(k;d) \in \mathrm{Mat}_{k-d},\ 0 \le d \le k-1 \quad \text{where} \\ &\boldsymbol{C}(k;d)_{s,t} = C^{k-d+s-t}(k),\ 1 \le s,t \le k-d. \end{aligned} \tag{31}$$

The matrices $\boldsymbol{C}(k,0)$ are upper diagonal with $C^k(k)$ on the diagonal; the matrix $\boldsymbol{C}(4,0)$ appeared in the previous example. In fact, $\boldsymbol{C}(k,d)$ is obtained from $\boldsymbol{C}(k,0)$ by removing d first columns and d last rows. Note also that considering (any) diagonal of $\boldsymbol{C}(k,d)$, all the coefficients are the same.

Clearly the $\boldsymbol{C}(k,0)$ are regular.

Theorem 6.2. *The matrices $\boldsymbol{C}(k,d)$, $k \in \mathbb{N}$, $0 \le d \le k-1$ are regular.*

By J. Kadourek, Masaryk University. First observe that for $d = 0$ the matrix $\boldsymbol{C}$ is upper triangular with nonzero entries on the diagonal. Thus it is regular so it is sufficient to assume $1 \le d \le k-1$. Also to simplify the notation we put $k_d := k - d$. Clearly $1 \le k_d \le k-1$.

It turns out to be useful to consider also the closely related matrix

$$\begin{aligned} &\widetilde{\boldsymbol{C}}(k;d) \in \mathrm{Mat}_{k_d},\ 0 \le d \le k-1 \quad \text{where} \\ &\widetilde{\boldsymbol{C}}(k;d)_{s,t} = \binom{k}{k_d+s-t},\ 1 \le s,t \le k_d, \end{aligned} \tag{32}$$

where the latter is taken to be 0 if $s - t > d$. That is, the entries of $\boldsymbol{C}$ and $\widetilde{\boldsymbol{C}}$ differ by a power of 2. Now writing the determinant as a sum (over permutations of $\{1, \dots, k_d\}$) of products of entries of a matrix, one easily shows that determinants of $\boldsymbol{C}$ and $\widetilde{\boldsymbol{C}}$ differ by a power of 2. That is, the matrix $\boldsymbol{C}$ is regular if and only if $\widetilde{\boldsymbol{C}}$ is regular. We shall prove regularity for the latter.

First recall the well-know relation

$$\binom{q}{m} + \binom{q}{m+1} = \binom{q+1}{m+1}, \quad q, m \geq 0. \tag{33}$$

Henceforth we fix the values k, d from the allowed range. The proof now consists of several series of row or column elementary operations which change the determinant by a nonzero multiple. During certain stages of this process we shall obtain matrices $D_1, D_2, D_3, D_4 \in \mathrm{Mat}_{k_d}$ whose determinants differ from each other only by nonzero multiples. The last of these, D_4 is upper triangular with nonzero entries on the diagonal, and so this concludes the proof.

The construction of D_1 from $\widetilde{\boldsymbol{C}}$ consists of $k_d - 1$ steps; in each of these we undertake a series of elementary column operations, as follows. In the first step, we add the second column to the first one, then the third column to the second and so on; finally we add the last column to the last but one. In the second step, we add the second column to the first one, then the third column to the second and so on but finish by adding the $(k_d - 1)$th column to the $(k_d - 2)$th column. Continuing in this way, in the last step (i.e. the step number $k_d - 1$) we add only the second column to the first one. Note the determinants of D_1 and $\widetilde{\boldsymbol{C}}$ differ by a nonzero multiple.

Overall we obtain the matrix

$$D_1(s,t) = \binom{k + k_d - t}{k_d + s - t} = \frac{(k + k_d - t)!}{(k_d + s - t)!(k - s)!}; \tag{34}$$

note $1 \leq k_d + s - t \leq k + k_d - t$. The reasoning uses (33) in every addition of two binomial numbers and goes as follows. Consider how the (s,t)-entry changes during the procedure described in the previous paragraph. First observe that after the ith step of elementary column operations, this entry has the form $\binom{a_i}{k_d+s-t}$. That is, the "denominator" of the binomial number on the position (s,t) does not change during this procedure. This follows from (33). Second, the "numerator" of the binomial number on the (s,t)-position increases by 1 if we add the $(s, t+1)$–entry, see (33). Thus the "numerator" depends on the number of additions of the $(t+1)$st column, as stated in (34).

Now we modify the matrix D_1 as follows. First we multiple the tth column by $\frac{1}{(k+k_d-t)!}$, where we note that $k + k_d - t \geq k \geq 1$. Then we multiply the sth row by $(k - s)!$ where $k - s \geq 1$ because $s \leq k_d \leq k - 1$. We obtain the matrix D_2, the determinants of D_1 and D_2 differ by a nonzero multiple. It follows from the fractional form of entries of D_1 in (35) that

$$D_2(s,t) = \frac{1}{(k_d + s - t)!}. \tag{35}$$

We continue with the following modification of D_2. First we multiply the sth row by $(k_d+s-1)! \geq 1$. Then we multiply the tth column by $\frac{1}{(t-1)!}$, $t-1 \geq 0$ (thus $(t-1)! \geq 1$). The result is a matrix D_3, the determinants of D_3 and D_2 differ by nonzero multiple. It follows from (35) that

$$D_3(s,t) = \frac{(k_d+s-1)!}{(k_d+s-t)!(k-1)!} = \binom{k_d+s-1}{k_d+s-t}. \tag{36}$$

In the last stage we apply the following k_d-1 steps of elementary row transformations to the matrix D_3. Observe that the first column of D_3 has all its entries equal to 1. In the first step, we subtract the (k_d-1)-st row from the k_d-th row, then we subtract (k_d-2)-nd row from the (k_d-1)-st row and so on; finally we subtract the first row from the second one. Thus the first column has now 1 as its top entry and 0's below this. In the second step, we subtract the (k_d-1)-st row from the k_d-th row, then we subtract (k_d-2).row from the (k_d-1)-st row and so on, as before except in this step we finish at the point of subtracting the 2nd row from the 3rd row. Continuing in this way, in the last step we subtract only (k_d-1)-st row from the k_d-th row. We shall denote the resulting matrix by D_4.

It turns out D_4 is upper triangular with all entries on the diagonal equal to 1. To show this note we use (33) at every step of the above procedure. In fact, the final form of D_4 can be foreseen already from the first step, after which we obtain a matrix that we shall denote $O \in \mathrm{Mat}_{k_d}$. We already know the first column of O is $(1,0,\ldots,0)^T$. From this it follows that in the second step we effectively work only with submatrix of O with entries (s,t), $2 \leq s,t \leq k_d$. Since

$$O(s,t) = \binom{k_d+s-2}{k_d+s-t} = D_3(s-1,t-1), \quad 2 \leq s,t \leq k_d$$

using (33), we see this submatrix of O is exactly the submatrix of D_3 without the last row and the last column. Applying the second step to the displayed submatrix corresponds to applying the first step to the corresponding submatrix of D_3 (the last row and column clearly have no influence on the previous ones). These observations yield an inductive procedure which demonstrates the claimed form of D_4. □

Proposition 6.3. *Let (T,T') be a normal symmetry of Δ^k and suppose that, in a standard expression for T, $\varphi^p_r(\odot^p\nabla)\Delta^r$ is the greatest non-zero term of T with respect to $\rhd$. Then $[\nabla^{2r+1}\varphi^p_r]|_{\boxtimes} = 0$.*

Proof. The ordering $\lhd$ can be equivalently described as $\varphi^j_i \lhd \varphi^{j'}_{i'}$ if and only if either $i+j < i'+j'$ or $i+j = i'+j'$ and $i < i'$. Thus

$$\Delta^k T = T'\Delta^k, \quad T = \varphi^p_r(\odot^p\nabla)\Delta^r + \sum_{\substack{i<k \\ i+j<r+p \text{ or} \\ (i+j=r+p)\ \wedge\ (i<r)}} \varphi^j_i(\odot^j\nabla)\Delta^i.$$

Note φ^p_r might not be a leading term of T.

Note, $\ell(T) = p+r$ and $\ell(\Delta^k T) = p+r+k$. We shall discuss the terms of the highest level in $\Delta^k T$. For this it is sufficient to apply Δ^k only to level $p+r$ terms of T. That is, we need to understand the right hand side of

$$\Delta^k \left[\varphi^p_r(\odot^p\nabla)\Delta^r + \varphi^{p+1}_{r-1}(\odot^{p+1}\nabla)\Delta^{r-1} + \ldots + \varphi^{p+r}_0(\odot^{p+r}\nabla)\right] - F\Delta^k$$
$$= \psi^{p+r+1}_{k-1}(\odot^{p+r+1}\nabla)\Delta^{k-1} + \psi^{p+r+2}_{k-2}(\odot^{p+r+2}\nabla)\Delta^{k-2} + \ldots + \psi^{p+r+k}_0(\odot^{p+r+k}\nabla) + \mathrm{llt}$$

where F is a differential operator. Here "llt" denotes terms of the level at most $p+r+k-1$ (with powers of Δ strictly less than k) and ψ^j_i is of type $\left[\begin{smallmatrix} j \\ i \end{smallmatrix}\right]$. Since $i < k$ for every ψ^j_i on the right had side, imposing the symmetry condition, each of these terms has to vanish. This yields k differential conditions

$$\psi^{p+r+1}_{k-1}(\odot^{p+r+1}\nabla)\Delta^{k-1} = 0, \psi^{p+r+2}_{k-2}(\odot^{p+r+2}\nabla)\Delta^{k-2} = 0, \ldots, \psi^{p+r+k}_0(\odot^{p+r+k}\nabla) = 0.$$

Thus $\psi^{p+r+q+1}_{k-q-1} = 0$ for $q \in \{0, \ldots, k-1\}$. For our purposes it turns out to be sufficient to take q in the (in general smaller) range $\{0, \ldots, r\}$. So we have $r+1$ differential conditions. Now fix such a q; we have more explicitly

$$\psi^{p+r+q+1}_{k-q-1} = \left[a_{q,0}\nabla^{r+q+1}\varphi^p_r + a_{q,1}\nabla^{r+q}\varphi^{p+1}_{r-1} + \ldots + a_{q,r}\nabla^{q+1}\varphi^{p+r}_0\right]|_\boxtimes$$

for some integer coefficients $a_{q,q'}$, $q' \in \{0, \ldots, r\}$. Via the Leibniz rule and a counting argument, it is straightforward to verify that $a_{q,q'} = C^{r+q-q'+1}(k)$. Recall $\psi^{p+r+q+1}_{k-q-1} = 0$ hence the right hand side of the previous display vanishes. Finally, let us apply ∇^{r-q} to both sides of the previous display. Projecting to the Cartan component, we obtain

$$\left[C^{r+q+1}(k)(\nabla^{2r+1}\varphi^p_r) + C^{r+q}(k)(\nabla^{2r}\varphi^{p+1}_{r-1}) + \ldots + C^{q+1}(k)(\nabla^{r+1}\varphi^{p+r}_0)\right]|_\boxtimes = 0.$$

This is a linear equation in the $r+1$ variables $(\nabla^{2r+1}\varphi^p_r)|_\boxtimes$, $(\nabla^{2r}\varphi^{p+1}_{r-1})|_\boxtimes$, …, $(\nabla^{r+1}\varphi^{p+r}_0)|_\boxtimes$. These variables obviously do not depend on q. That is for every $q \in \{0, \ldots, r\}$ we obtain one equation in these variables. Overall we have a system of $r+1$ linear equations in $r+1$ variables $(\nabla^{2r+1}\varphi^p_r)|_\boxtimes$, $(\nabla^{2r}\varphi^{p+1}_{r-1})|_\boxtimes$, …, $(\nabla^{r+1}\varphi^{p+r}_0)|_\boxtimes$. The integer coefficients are $a_{q,q'} = C^{r+q-q'+1}(k) = C^{(r+1)+(q+1)-(q'+1)}(k)$, $q, q' \in \{0, \ldots, r\}$ thus the $(r+1) \times (r+1)$ matrix of integer coefficients is exactly $\boldsymbol{C}(k,d)$ for $d = k-r-1$ from (31). (Note $r < k$ hence $d \in \{0, \ldots, k-1\}$.) But matrices $\boldsymbol{C}(k,d)$ are regular according to Theorem 6.2. Therefore this linear system has only the zero solution, i.e.

$$(\nabla^{2r+1}\varphi^p_r = 0)|_\boxtimes = 0, \nabla^{2r}(\varphi^{p+1}_{r-1})|_\boxtimes = 0, \ldots, (\nabla^{r+1}\varphi^{p+r}_0)|_\boxtimes = 0.$$

In particular $(\nabla^{2r+1}\varphi^p_r)|_\boxtimes = 0$, which is what we wanted to prove. □

Finally we have the key theorem of this section. By an obvious induction this establishes the second part of Theorem 2.1.

Theorem 6.4. *Let (S, S') be a normal symmetry of Δ^k and suppose that, in a standard expression for S, $\varphi^p_r(\odot^p\nabla)\Delta^r$, $r < k$ is a leading term. Then $[\nabla^{2r+1}\varphi^p_r]|_\boxtimes = 0$.*

This establishes the second part of Theorem 2.1. Note using the conformal metric, we can view all $p+2r+1$ abstract indices of $\nabla^{2r+1}\varphi^p_r$ as contravariant. Then the projection to the Cartan component in $[\nabla^{2r+1}\varphi^p_r]|_\boxtimes = 0$ simply means taking the symmetric trace-free part.

Proof. Consider the coefficients of the maximal level $\ell(S)$ of S; among them, denote by ψ_i^j the term of the highest order. In the other words, ψ_i^j is the greatest coefficient in S w.r.t. $\lhd$. Now $[\nabla^{2i+1}\psi_i^j]|_{\boxtimes} = 0$ according to Proposition 6.3 hence ψ_i^j yields the corresponding canonical symmetry (S_ψ, S'_ψ) of Δ^k. Therefore $(S - S_\psi, S' - S'_\psi)$ is also a symmetry of Δ^k.

First observe using Proposition 6.1 (iii), the leading terms of S and $S - S_\psi$ can differ only if $\psi_i^j(\odot^j\nabla)\Delta^i$ is a leading term of S. But in that case we have proved the theorem for $\psi_i^j(\odot^j\nabla)\Delta^i$. Therefore, it is sufficient to prove the theorem for $S - S_\psi$. So we can take $S := S - S_\psi$ and continue inductively.

Proposition 6.1 (ii) guarantees that the greatest term of $S := S - S_\psi$ is smaller than the greatest term of S. Hence this is induction w.r.t. $\lhd$ which is finite. □

7. Algebra of symmetries

Here we shall prove Theorem 2.5. Recall that the finite dimensional space of solutions of (3) may be realised as a standard linear "matrix" representation of $\mathfrak{g} = \mathfrak{so}_{s+1,s'+1}$ via the map from solutions to parallel tractors $\varphi \mapsto I_\varphi$. In the case of conformal Killing vectors (i.e. (3) with $p = 1$, $r = 0$) the range space is $\mathfrak{g}$, on which $\mathfrak{g}$ acts by the adjoint representation. Then the identification of $\mathfrak{g}$ with differential symmetries is given by the mapping $\mathfrak{g} \ni I_\varphi \mapsto S_\varphi = I_\varphi^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}$, as a special case of (24). The mapping $S_\varphi = I_\varphi^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}$ extends to

$$\mathfrak{g}\otimes\mathfrak{g}\otimes\cdots\mathfrak{g} \ni I_{\varphi_1}\otimes\cdots\otimes I_{\varphi_m} \mapsto S_{\varphi_1}\cdots S_{\varphi_m}, \quad m \geq 1\ , \tag{37}$$

and hence to the full tensor algebra $\bigotimes \mathfrak{g}$ by linearity.

The first step in the proof of Theorem 2.5 is to express the composition $S_\varphi S_{\bar\varphi}$ for $I_\varphi, I_{\bar\varphi} \in \mathfrak{g}$ in terms of canonical symmetries. This is done [22, Theorem 5.1] and necessarily our results must agree with those from their construction (as uniqueness of the low order symmetries involved is easily verified). We present the details here to keep this text self–contained and also because we derive the formulae for all conformally flat manifolds.

Putting $I := I_\varphi, \bar I := I_{\bar\varphi}$ to simplify the notation, one has

$$S_\varphi S_{\bar\varphi} = I^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}\bar I^{\mathbf{B}}\mathbb{D}_{\mathbf{B}} = I^{\mathbf{A}}\bar I^{\mathbf{B}}\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}. \tag{38}$$

on $\mathcal{E}[w]$, since I is parallel. This gives an explicit and key link between the algebraic structure of symmetries $\mathcal{A}_k$ and operations on the tensor algebra $\bigotimes\mathfrak{g}$. We shall consider the displayed operator acting on $\mathcal{E}[w]$ for all $w \in \mathbb{R}$ at this stage.

We need to decompose $\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}$ into irreducible components. Using the definition of $\mathbb{D}_{\mathbf{A}}$, a direct computation shows that

$$\begin{aligned}\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}f =& 4w^2\mathbb{W}_{\mathbf{A}}\mathbb{W}_{\mathbf{B}}f - 4w\mathbb{X}_{\mathbf{A}}^a\mathbb{Y}_{\mathbf{B}}^b\boldsymbol{g}_{ab}f \\ &+ 4(w-1)\mathbb{X}_{\mathbf{A}}^a\mathbb{W}_{\mathbf{B}}\nabla_a f + 4w\mathbb{W}_{\mathbf{A}}\mathbb{X}_{\mathbf{B}}^b\nabla_b f + 4\mathbb{X}_{\mathbf{A}}^a\mathbb{Z}_{\mathbf{B}}^{\mathbf{b}}\boldsymbol{g}_{ab^0}\nabla_{b^1}f \\ &+ 4\mathbb{X}_{\mathbf{A}}^a\mathbb{X}_{\mathbf{B}}^b(\nabla_a\nabla_b + wP_{ab})f.\end{aligned} \tag{39}$$

>From this one easily verifies that

$$
\begin{aligned}
\frac{1}{2}(\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}+\mathbb{D}_{\mathbf{B}}\mathbb{D}_{\mathbf{A}}) =\ & \frac{1}{2}(\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}+\mathbb{D}_{\mathbf{B}}\mathbb{D}_{\mathbf{A}})|_{\boxtimes} + \frac{4}{n}h_{A^0B^0}\mathbb{D}^2_{(A^1B^1)_0} \\
& + \frac{2}{(n+1)(n+2)}h_{A^0B^0}h_{A^1B^1}\mathbb{D}^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}, \\
\frac{1}{2}(\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}-\mathbb{D}_{\mathbf{B}}\mathbb{D}_{\mathbf{A}}) =\ & 3h_{A^0[A^1}\mathbb{D}_{\mathbf{B}]} = -2h_{A^0B^0}\mathbb{D}_{A^1B^1}.
\end{aligned}
\tag{40}
$$

Hence we need the irreducible components $\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}_0$, $\square\square_0$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and $\mathbb{R}$ of $I^{\mathbf{A}}\bar{I}^{\mathbf{B}}$, cf. (6). Explicitly, we put

$$
\begin{aligned}
\langle I,\bar{I}\rangle &:= -4nI^{\mathbf{A}}\bar{I}_{\mathbf{A}} \in \mathbb{R}, \\
[I,\bar{I}]^{\mathbf{A}} &:= 4I^{A^0P}\bar{I}_p{}^{A^1} \in \begin{smallmatrix}\square\\\square\end{smallmatrix}, \\
(I\bullet\bar{I})^{BB'} &:= \frac{4}{n}I^{P(B}\bar{I}_P{}^{B')_0} \in \square\square_0
\end{aligned}
\tag{41}
$$

and we denote by $(I\boxtimes\bar{I})^{\mathbf{AB}}$ the trace–free part of the Young projection $\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}$ applied to $I^{\mathbf{A}}\bar{I}^{\mathbf{B}}$. Using this notation, the projection and decomposition of $I^{\mathbf{A}}\otimes\bar{I}^{\mathbf{B}}$ into its irreducible components in $\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}$, $\mathbb{R}$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and $\square\square_0$ is given by

$$
\begin{aligned}
\begin{smallmatrix}\square\\\square\end{smallmatrix}\otimes\begin{smallmatrix}\square\\\square\end{smallmatrix} \ni I^{\mathbf{A}}\otimes\bar{I}^{\mathbf{B}} \mapsto & (I\boxtimes\bar{I})^{\mathbf{AB}} - \frac{1}{2n(n+1)(n+2)}h^{A^0B^0}h^{A^1B^1}\langle I,\bar{I}\rangle \\
& + \frac{1}{n}h^{A^0B^0}[I,\bar{I}]^{A^1B^1} + h^{A^0B^0}(I\bullet\bar{I})^{A^1B^1}.
\end{aligned}
\tag{42}
$$

Using the computation above, we easily recover [22, Theorem 5.1]:

Theorem 7.1. *Let $\varphi^a,\bar{\varphi}^a\in\mathcal{E}^a$ be conformal Killing fields corresponding to $I^{\mathbf{A}} := I^{\mathbf{A}}_{\varphi}$ and $\bar{I}^{\mathbf{A}} := I^{\mathbf{A}}_{\bar{\varphi}}$ in $\mathfrak{g}=\mathfrak{so}_{s+1,s'+1}$. Then*

$$
S_{\varphi}S_{\bar{\varphi}}f = (I\boxtimes\bar{I})^{\mathbf{AB}}\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}f + (I\bullet\bar{I})^{BB'}\mathbb{D}^2_{BB'}f + \frac{1}{2}[I,\bar{I}]^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}f + \frac{w(n+w)}{n(n+1)(n+2)}\langle I,\bar{I}\rangle f
$$

for $f\in\mathcal{E}[w]$, cf. (7). The four summands on the right hand side are canonical symmetries, explicitly

- *$(I\boxtimes\bar{I})^{\mathbf{AB}}\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}} = S_{\Phi}$ for $\mathcal{E}^{(ab)_0}\ni\Phi^{ab}=\varphi^{(a}\bar{\varphi}^{b)_0}$,*
- *$(I\bullet\bar{I})^{BB'}\mathbb{D}^2_{BB'} = S_{\Phi}$ for $\mathcal{E}[2]\ni\Phi=\frac{1}{n}\varphi^a\bar{\varphi}_a$,*
- *$[I,\bar{I}]^{\mathbf{A}}\mathbb{D}_{\mathbf{A}} = S_{\Phi}$ for $\mathcal{E}^a\ni\Phi^a=\varphi^b\nabla_b\bar{\varphi}^a-\bar{\varphi}^b\nabla_b\varphi^a$ (the Lie bracket of vector fields),*
- *$\mathbb{R}\ni\langle I,\bar{I}\rangle = -4nI^{\mathbf{A}}\bar{I}_{\mathbf{A}} = -2[\varphi^a\nabla_a\nabla_b\bar{\varphi}^b+\bar{\varphi}^a\nabla_a\nabla_b\varphi^b]+n(\nabla_a\varphi^b)(\nabla_b\bar{\varphi}^a)-\frac{n-2}{n}(\nabla_a\varphi^a)(\nabla_b\bar{\varphi}^b)-4nP_{ab}\varphi^a\bar{\varphi}^b$.*

In all these cases, the section Φ is a solution of the corresponding equation (3).

Proof. The statement puts together the previous computations. Following (38), we need to decompose $I^{\mathbf{A}}\bar{I}^{\mathbf{B}}\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}$ into canonical symmetries. This is provided by contracting right hand sides of (42) and (40). Using in addition $\mathbb{D}^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}f =$

$-2w(n+w)f$ for $f \in \mathcal{E}[w]$ (which easily follows from (39)), the right hand side of $S_\varphi S_{\bar\varphi}$ in the display above follows.

The components $I \boxtimes \bar I$, $I \bullet \bar I$, $[I, \bar I]$ and $\langle I, \bar I \rangle$ are parallel (and irreducible) thus their projecting parts Φ are solutions of the corresponding equation from the family (3). To prove the theorem, it remains to identify how are these solutions are built from $\varphi^a, \bar\varphi^a \in \mathcal{E}^a$. Note

$$I^{\mathbf{A}} = \mathbb{Y}^{\mathbf{A}}_a \varphi^a + \frac{1}{2}\mathbb{Z}^{\mathbf{A}}_{\mathbf{a}} \nabla^{a^0}\varphi^{a^1} + \frac{1}{n}\mathbb{W}^{\mathbf{A}}\nabla_a\varphi^a + \mathbb{X}^{\mathbf{A}}_a[\frac{1}{n}\nabla_a\nabla_b\varphi^b + P_{ab}\varphi^b] \tag{43}$$

and similarly for $\bar I^{\mathbf{A}}$ [33]. Now the explicit form of such Φ for irreducible components of $I^{\mathbf{A}} \otimes \bar I^{\mathbf{B}}$ is easily obtained from (41) for $I \bullet \bar I$, $[I, \bar I]$ and $\langle I, \bar I \rangle$. Since $\frac{1}{2}(I^{\mathbf{A}}\bar I^{\mathbf{B}} + I^{\mathbf{B}}\bar I^{\mathbf{A}})$ has the projecting part $\varphi^{(a}\bar\varphi^{b)}$, the case $I \boxtimes \bar I$ follows by irreducibility. $\square$

To finish the proof of Theorem 2.5, observe the following. First we have an associative algebra morphism

$$\bigotimes \mathfrak{g} \to \mathcal{A}_k$$

determined by (37). That this is surjective is an easy consequence of Theorem 2.4 since the canonical symmetries S_ϕ of (24) clearly arise in the range of (37). We want to find all corresponding relations, that is identify the two sided ideal annihilated by this map. The ideal certainly contains (8), as follows from Theorem 7.1 with $w = -\frac{n}{2} + k$. That it also contains $\boxtimes^{2k}\square$ is due to the following result.

Lemma 7.2. *Assume $I \in \boxtimes^{2k}\square$ is parallel. Then $I = I_\varphi$ for $\varphi \in \mathcal{E}[2k]$ and $S_\varphi = \varphi P_k : \mathcal{E}[-\frac{n}{2}+k] \to \mathcal{E}[-\frac{n}{2}+k]$.*

Proof. $I \in \boxtimes^{2k}\square$ means $I^{A_1A_1'\cdots A_kA_k'} \in \mathcal{E}^{(A_1A_1'\cdots A_kA_k')_0}$ and $I = I_\varphi$ for $\varphi \in \mathcal{E}[2k]$ is due to the irreducibility of I and the fact that is parallel. Then

$$S_\varphi = I_\varphi^{A_1A_1'\cdots A_kA_k'}\mathcal{D}^2_{A_1A_1'}\cdots\mathcal{D}^2_{A_kA_k'}.$$

Now observe $\mathcal{D}^2_{(CD)_0} = -X_{(C}D_{D)_0}$ and $X_{(C}D_{D)_0} = D_{(C}X_{D)_0}$, cf. (18). On the other hand $D_{(A_1}\cdots D_{A_k)_0} = (-1)^k X_{(A_1}\cdots X_{A_k)_0}P_k$ on $\mathcal{E}[-\frac{n}{2}+k]$ [29, 32]. Thus $\mathcal{D}^2_{A_1A_1'}\cdots\mathcal{D}^2_{A_kA_k'} = X_{A_1}X_{A_1'}\cdots X_{A_k}X_{A_k'}P_k$ on $\mathcal{E}[-\frac{n}{2}+k]$. The rest follows from the relation between φ and I_φ in (23). $\square$

We have found the generators of the ideal in $\bigotimes\mathfrak{g}$ described in Theorem 2.5; it remains to show that this ideal large enough to have $\mathcal{A}_k$ as the resulting quotient. Essentially we follow [20, 22] where cases $k=1$ and $k=2$ are studied. We assume $k \geq 1$ here. Since we know $\mathcal{A}_k$, as a vector space, from (4), it is sufficient to consider the corresponding graded algebra (i.e. the symbol algebra of $\mathcal{A}_k$.) The corresponding graded ideal contains $I_1 \otimes I_2 - I_1 \boxtimes I_2 - I_1 \bullet I_2$ for $I_1, I_2 \in \mathfrak{g}$, cf. (8), hence it contains $\mathfrak{g} \wedge \mathfrak{g}$. Therefore we can pass to $\bigodot\mathfrak{g}$ and we write $\mathcal{I}$ for the ideal in $\bigodot\mathfrak{g}$ which is the image of the ideal of Theorem 2.5. We claim that as a graded structure $\mathcal{A}_k = \bigoplus \mathcal{A}_{k,t}$ where the $\mathcal{A}_{k,t}$ are defined as the submodules satisfying

$$\mathcal{A}_{k,t} = \Big\{ X \in \overbrace{\begin{array}{|c|c|c|c|c|}\hline \; & \; & \cdots & \; & \; \\ \hline \; & \; & \cdots & \; & \; \\ \hline \end{array}}^{t} \quad \text{s.t. } \underbrace{\mathrm{trace}(\ldots(\mathrm{trace}}_{k}(X)..)) = 0 \Big\} \subseteq \bigodot^t \mathfrak{g}.$$

The traces are taken via the tractor metric and note that the trace condition arises from Lemma 7.2 above. As a vector space this is the right answer as, by standard representation theory, $\mathcal{A}_{k,t} = \bigoplus_{j+2i=t} \mathcal{K}_i^j$, $t \geq 1$. To finish the proof, we need to show $\bigodot^t \mathfrak{g} = \mathcal{A}_{k,t} \oplus \mathcal{I}_t$ (as vector spaces) where $\mathcal{I}_t = \mathcal{I} \cap \bigodot^t \mathfrak{g}$, $t \geq 1$. This is based on the following

Lemma. *Assume $t \geq 3$, $k \geq 1$. Then*

$$\left(\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array} \otimes \mathcal{A}_{k,t-1} \right) \cap \left(\mathcal{A}_{k,t-1} \otimes \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array} \right) = \begin{cases} \mathcal{A}_{k,t} & t \neq 2k \\ \mathcal{A}_{k,t} \oplus \boxtimes^{2k} \square & t = 2k. \end{cases}$$

Proof. The case $t < 2k$ follows from [21, Theorem 2] or can be easily checked directly. Assume $t > 2k$. The inclusion "$\supseteq$" is obvious. To show "$\subseteq$" consider the tensor $F^{\mathbf{A_1} \ldots \mathbf{A_t}}$ in the left hand side of the display. Then

$$F^{\mathbf{A_1} \ldots \mathbf{A_i} \ldots \mathbf{A_j} \ldots \mathbf{A_t}} = F^{\mathbf{A_1} \ldots \mathbf{A_j} \ldots \mathbf{A_i} \ldots \mathbf{A_t}}$$

for any $1 \leq i < j \leq t$. From this it easily follows that the skew symmetrization over any three indices of F is zero. (This and the last display also follow from [21, Theorem 2].) Now any composition of k traces applied to F affects $2k$ indices among $2t$ indices $A_1^0, A_1^1, \ldots, A_t^0, A_t^1$, i.e. at most $2k$ form indices among $\mathbf{A}_1, \ldots, \mathbf{A}_t$. Thus there is a free form index $\mathbf{A}_i$ (as $t > 2k$) and the inclusion "$\subseteq$" follows from the symmetry given by the previous display.

Assume $t = 2k$. Following the previous case "$\subseteq$", the difference appears only if a composition of k traces affects all $2k$ form indices of F. After taking of such composition of traces we obtain a tensor in $\bigodot^t \square$ and one easily sees this tensor is trace free. On the other hand, for any symmetric trace free tensor $G^{A_1^0 \cdots A_{2k}^0} \in \boxtimes^{2k} \square$ one has

$$G^{A_1^0 \cdots A_{2k}^0} h^{A_1^1 \cdots A_{2k}^1} \in \left(\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array} \otimes \mathcal{A}_{k,t-1} \right) \cap \left(\mathcal{A}_{k,t-1} \otimes \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array} \right) \tag{44}$$

which can be easily verified by direct computation. Here $h^{A_1^1 \cdots A_{2k}^1} = h^{(A_1^1 A_2^1} \cdots h^{A_{2k-1}^1 A_{2k}^1)}$ and recall we implicitly skew over the couples $A_i^0 A_i^1$ for $1 \leq i \leq 2k$. $\square$

The final step is to use that for each s, there is (by standard theory) a projection $\odot^s \mathfrak{g} \to \mathcal{A}_{k,s}$ and that the induced projections $P_t : \bigodot^t \mathfrak{g} \to \mathfrak{g} \otimes \mathcal{A}_{k,t-1}$ and $Q_t : \bigodot^t \mathfrak{g} \to \mathcal{A}_{k,t-1} \otimes \mathfrak{g}$ have kernel in, respectively $\mathfrak{g} \otimes \mathcal{I}_{t-1}$ and $\mathcal{I}_{t-1} \otimes \mathfrak{g}$ (and hence in both cases in $\mathcal{I}_t$) where for each non-negative integer s, $\mathcal{I}_s = \mathcal{I} \cap \bigodot^s \mathfrak{g}$. Therefore, by obvious dimensional considerations,

$$\bigodot^t \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array} = (\mathrm{im}\, P_t \cap \mathrm{im}\, Q_t) \oplus (\ker P_t + \ker Q_t), \quad t \geq 3 \tag{45}$$

and the claim above and then Theorem 2.5 follow by induction.

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