AFFINE LINES ON $\mathbb{Q}$-HOMOLOGY PLANES AND GROUP ACTIONS

MIKHAIL ZAIDENBERG

Abstract. This note is a supplement to the papers [KiKo] and [GMMR]. We show the role of group actions in classification of affine lines on $\mathbb{Q}$-homology planes.

Introduction

This note is a supplement to the papers [KiKo] and [GMMR]. Our aim is to shed a light on the role of group actions in classification of affine lines on $\mathbb{Q}$-homology planes with logarithmic Kodaira dimension $-\infty$. This enables us to strengthen certain results in *loc. sit.* (see Section 1).

Let us fix terminology. It is usual [Mi, Ch. 3, §4] to call a smooth $\mathbb{Q}$-acyclic ($\mathbb{Z}$-acyclic, respectively) surface over $\mathbb{C}$ a $\mathbb{Q}$-*homology plane* (a *homology plane*, respectively). By Fujita's Lemma [Fu, 2.5] such a surface is necessarily affine. Likewise we call a *homology line* an irreducible affine curve Γ with Euler characteristic $e(\Gamma) = 1$. So Γ is homeomorphic to $\mathbb{R}^2$ and its normalization is isomorphic to $\mathbb{A}^1 = \mathbb{A}^1_{\mathbb{C}}$. A smooth curve isomorphic to $\mathbb{A}^1$ will be called an *affine line*. Following [Mi] we let $\mathbb{A}^1_* = \mathbb{A}^1 \setminus \{0\}$. As usual $\bar{k}$ stands for logarithmic Kodaira dimension.

Acknowledgements: This research was done during a visit of the author to the Max Planck Institute of Mathematics in Bonn. He thanks this institution for a generous support and excellent working conditions. Our thanks also to Shulim Kaliman and to a referee of the 'Transformation groups" for useful editorial comments.

1. Main results

Theorem 1. *Let X be a $\mathbb{Q}$-homology plane and Γ a homology line on X. Then the following hold.*

(a) Suppose that $\bar{k}(X \setminus \Gamma) = -\infty$. Then Γ is either an orbit of an effective $\mathbb{C}_+$-action on X or a connected component of the fixed

---

1991 Mathematics Subject Classification: 14R05, 14R20, 14J50.
Key words: $\mathbb{Q}$-acyclic surface, $\mathbb{Q}$-homology plane.

point set of such an action. Anyhow $\Gamma \simeq \mathbb{A}^1$ *is a fiber component of the corresponding orbit map (an* $\mathbb{A}^1$*-ruling)* $\pi : X \to \mathbb{A}^1$.

(b) *Suppose that* $\bar{k}(X \setminus \Gamma) \geq 0$. *Suppose further that* $\Gamma \simeq \mathbb{A}^1$ *and* $\bar{k}(X) = -\infty$. *Then* Γ *is an orbit closure of an effective hyperbolic* $\mathbb{C}^*$*-action on* X. *Moreover* X *admits an effective action of a semidirect product* $G = \mathbb{C}^* \ltimes \mathbb{C}_+$ *with an open orbit* U. *The orbit map* $X \to \mathbb{A}^1$ *of the induced* $\mathbb{C}_+$*-action defines an* $\mathbb{A}^1$*-ruling on* X *with a unique multiple fiber say* $\Gamma' \simeq \mathbb{A}^1$ *such that* Γ *and* Γ' *meet at one point transversally and* $U = X \setminus \Gamma' \simeq \mathbb{A}^1 \times \mathbb{A}^1_*$. *Furthermore this* $\mathbb{C}_+$*-action moves* Γ. *Consequently there exists a continuous family of affine lines* Γ_t *on* X *with the same properties as* Γ.

(c) *Suppose that* Γ *is singular. Then* $X \simeq \mathbb{A}^2$ *and* $\bar{k}(X \setminus \Gamma) = 1$. *Moreover*[1] *there is an isomorphism* $X \simeq \mathbb{A}^2$ *sending* Γ *to a curve* $V(x^k - y^l)$ *with coprime* $k, l \geq 2$. *Consequently* Γ *is an orbit closure of an elliptic* $\mathbb{C}^*$*-action on* X.

We indicate below a proof of the theorem. The cases (a), (b) and (c) are proven in Sections 2, 3 and 4, respectively. Besides, in cases (a) and (b) we provide in Lemmas 3 and 7, respectively, a description of the pairs (X, Γ) satisfying their assumptions. The assertion of (b) follows from Theorem 1.1 in [KiKo], cf. also Theorem 3.10 in [GMMR]. In the case of a $\mathbb{Z}$-homology plane (c) was established in [Za]; the proof for a $\mathbb{Q}$-homology plane is similar. This gives a strengthening of Theorem 1.3 in [KiKo].

The cases (a)-(c) of Theorem 1 do not exhaust all the possibilities for the pair (X, Γ) as above. To complete the picture let us summarize some known facts, see e.g. [Za, GuPa, Mi, Ch. 3, §4] and the references therein.

Theorem 2. *We let as before* X *be a* $\mathbb{Q}$*-homology plane and* $\Gamma \subseteq X$ *a homology line. If* Γ *is singular then* (X, Γ) *is as in Theorem 1(c). Suppose further that* Γ *is smooth i.e. is an affine line. Then* $\bar{k}(X) \leq \bar{k}(X \setminus \Gamma) \leq 1$[2] *and one of the following cases occurs.*

(a) $\big(\bar{k}(X), \bar{k}(X \setminus \Gamma)\big) = (-\infty, -\infty)$ *and* (X, Γ) *is as in Theorem 1(a) that is,* Γ *is of fiber type and* $X \setminus \Gamma$ *carries a family of disjoint affine lines*[3].

(b) $\big(\bar{k}(X), \bar{k}(X \setminus \Gamma)\big) = (-\infty, 0)$ *or* $(-\infty, 1)$ *and* (X, Γ) *is as in Theorem 1(b)*[4].

[1] This is due to the Lin-Zaidenberg Theorem [LiZa, Mi, Ch. 3, §3].

[2] See [Mi, Ch.2, Theorem 6.7.1].

[3] See also Lemma 3 below.

[4] The both possibilities actually occur, see the Construction in Section 3 and also Lemma 7.

(c) $\big(\bar{k}(X), \bar{k}(X \setminus \Gamma)\big) = (0,0)$ *and either X is not NC minimal or X is one of the Fujita's surfaces $H[-k,k]$ $(k \geq 1)$*[5]*. Anyhow Γ is a unique affine line on X unless $X = H[-1,1]$.*

(d) $\big(\bar{k}(X), \bar{k}(X \setminus \Gamma)\big) = (0,1)$*, $X = H[-1,1]$ and there are exactly two affine lines, say, Γ_0 and $\Gamma_1 = \Gamma$ on X. These lines meet transversally in two distinct points, moreover $\bar{k}(X \setminus \Gamma_0) = 0$ and $\bar{k}(X \setminus \Gamma_1) = 1$.*

(e) $\big(\bar{k}(X), \bar{k}(X \setminus \Gamma)\big) = (1,1)$*, there is a unique $\mathbb{A}^1_*$-fibration on X and Γ is a fiber component of its degenerate fiber*[6]*. There can be at most one further affine line on X, which is then another component of this same degenerate fiber, and these components meet transversally in one point.*

Remark 1. Let X be a $\mathbb{Z}$-homology plane. By [Fu] then $\bar{k}(X) \neq 0$. By [Za] (supplement) $\bar{k}(X) = 1$ if and only if there exists a unique homology (in fact, affine) line on X.

2. $\mathbb{Q}$-homology planes with an $\mathbb{A}^1$-ruling

These occur to be smooth affine surfaces with $\mathbb{A}^1$-rulings $X \to \mathbb{A}^1$ which possess only irreducible degenerate fibers. They were studied in details e.g. in [Fu, 4.14], [Be], [Fi], [FlZa$_1$, §4]. See also [Mi, Ch. 3, 4.3.1] for a brief summary[7]. In Lemma 3 below we show that every $\mathbb{A}^1$-ruling $\pi : X \to \mathbb{A}^1$ on a $\mathbb{Q}$-homology plane X can be obtained starting from a standard linear $\mathbb{A}^1$-ruling $\mathbb{A}^2 \to \mathbb{A}^1$ and replacing several fibers by multiple fibers via a procedure called in [FlZa$_1$] a *comb attachment*. More precisely, this replacement goes as follows.

Attaching combs. On the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ with a $\mathbb{P}^1$-ruling $\pi_0 = \mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ we fix a finite set of points $\{A_j\}$, $j = 1, \ldots, n$ $(n \geq 0)$ in different fibers $F_j = \{t_j\} \times \mathbb{P}^1$ of π_0. We fix further a sequence $\sigma : V \to \mathbb{P}^1 \times \mathbb{P}^1$ of blowups with centers at the points A_j and infinitesimally near points. Letting $\bar{\pi} : V \to \mathbb{P}^1$ be the induced $\mathbb{P}^1$-ruling, we suppose that $\bar{\pi}$ enjoys the following properties:

- the center of every blowup over A_j except for the first one belongs to the exceptional (-1)-curve of the previous blowup;
- $D_\infty \cdot E_j = 0$ $\forall j = 1, \ldots, n$, where D_∞ is the proper transform in V of the section $\mathbb{P}^1 \times \{\infty\}$ of pr_1 and E_j is the last (-1)-curve in the fiber $\bar{\pi}^{-1}(t_j)$.

[5] We refer e.g. to [Fu, GuPa, Mi, Ch. 3, 4.4.1-4.4.2] for definitions.

[6] The same conclusions hold also in case (c) if X is not NC-minimal [GuPa].

[7] We note [Be] that $\pi_1(X)$ is a free product of cyclic groups, namely, $\pi_1(X) \cong *_j \mathbb{Z}/m_j\mathbb{Z}$, where $(m_j)_j$ is the sequence of multiplicities of degenerate fibers, and so $H_1(X;\mathbb{Z}) \cong \bigoplus_j \mathbb{Z}/m_j\mathbb{Z}$.

• E_j is a tip of the dual graph of the fiber $\bar{\pi}^{-1}(t_j)$.

Under these assumptions the dual graph as above is a comb, with all vertices of degree ≤ 3. Let $F_\infty = \bar{\pi}^{-1}(t_\infty) \subset V$ be a fiber over an extra point $t_\infty \in \mathbb{P}^1 \setminus \{t_1, \ldots, t_n\}$ and $E \subseteq V$ be the reduced exceptional divisor of $\sigma : V \to \mathbb{P}^1 \times \mathbb{P}^1$. We consider the open surface $X = V \setminus D$, where $D = F_\infty + D_\infty + E + \sum_{j=1}^n (F'_j - E_j)$ and F'_j is the proper transform of F_j in V. Then $\bar{\pi} : V \to \mathbb{P}^1$ restricts to an $\mathbb{A}^1$-ruling $\pi : X \to \mathbb{A}^1$ with only irreducible fibers; all fibers of π are reduced except possibly the fibers $\pi^{-1}(t_j) = E_j \cap X$.

The following lemma is well known, see e.g. [FlZa$_1$, Proposition 4.9].

Lemma 3. *Under the notation as above the surface X is a $\mathbb{Q}$-homology plane. Moreover, every $\mathbb{Q}$-homology plane X with an $\mathbb{A}^1$-ruling $\pi : X \to \mathbb{A}^1$ arises in this way.*

Proof. Let X be constructed as above. By the Suzuki formula [Suz, Za, Gu], $e(X) = 1$ and so the equality $b_2 = b_1 + b_3$ for the Betti numbers of X holds. Thus X is $\mathbb{Q}$-acyclic if and only if $b_2 = 0$ or equivalently, if $\mathrm{Pic}(D) \otimes \mathbb{Q}$ generates $\mathrm{Pic}(V) \otimes \mathbb{Q}$, see [Mi, Ch. 3, 4.2.1]. The latter is easily seen to be the case in our construction. The first assertion follows now by Fujita's Lemma [Fu, 2.5].

As for the second one, given an $\mathbb{A}^1$-ruling $\pi : X \to \mathbb{A}^1$ on a $\mathbb{Q}$-homology plane X it extends to a pseudominimal $\mathbb{P}^1$-ruling $\pi : V \to \mathbb{P}^1$ on a smooth completion V of X with an SNC boundary divisor D. The pseudominimality means that none of the (-1)-curves in $D - D_\infty$, where D_∞ is the horizontal component of D, can be contracted without loosing the SNC property, see [Za, 3.4]. Since $e(X) = 1$ all fibers of $\pi : X \to \mathbb{A}^1$ are irreducible. We let $\bar{\pi}^{-1}(t_j)$, $j = 1, \ldots, n$ ($n \ge 0$) be the degenerate fibers of $\bar{\pi}$ and E_j be the component of the fiber $\bar{\pi}^{-1}(t_j)$ such that $E_j \cap X = \pi^{-1}(t_j) \simeq \mathbb{A}^1$. By the pseudominimality assumption, E_j is the only (-1)-curve in the fiber $\bar{\pi}^{-1}(t_j)$. Therefore V is obtained from a Hirzebruch surface Σ_m by blowing up process which enjoys the properties of a comb attachment. Performing, if necessary, elementary transformations in the fiber F_∞ we may assume that $\bar{D}_\infty^2 = 0$, where $\bar{D}_\infty$ is the image of D_∞ in Σ_m and so, $\Sigma_m = \Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$. □

Remark 2. Every surface X as considered above can actually be obtained from affine plane $\mathbb{A}^2$ via a suitable affine modification that is [KaZa, §1], by blowing up with center in a zero dimensional subscheme $V(I)$ of $\mathbb{A}^2$ located on a principal divisor D and deleting the proper transform of D. Indeed X contains a cylinder $U \times \mathbb{A}^1$, where $U \subseteq \mathbb{A}^1$ is a Zariski open subset, see [MiSu] or [Mi, Ch. 3, 1.3.2]. The canonical projections of $U \times \mathbb{A}^1$ to the factors regarded as rational functions on

X, say, f, h, can be made regular by multiplying h by an appropriate polynomial $q \in \mathbb{C}[t]$. Then $\varphi = (f, g) : X \to \mathbb{A}^2$, where $g = qh$, yields a birational morphism. Since every birational morphism between affine varieties is an affine modification [KaZa, Prop. 1.1] the claim follows.

For instance the following example from [Be] can be treated in terms of affine modifications.

Example 1. ([Be, Ex. 2.6.1], [KaZa, 7.1]) The *Bertin surfaces* are surfaces in $\mathbb{A}^3$ with equations

$$x^e z = x + y^d .$$

Every such surface X appears as affine modification of the plane $\mathbb{A}^2 = \operatorname{Spec}\mathbb{C}[x, y]$ with center $(I, (x^e))$, where $I = (x^e, x + y^d) \subseteq \mathbb{C}[x, y]$. Actually X is a $\mathbb{Q}$-homology plane with $\operatorname{Pic}(X) \cong H_1(X; \mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$, and $\pi = x|X : X \to \mathbb{A}^1$ gives an $\mathbb{A}^1$-ruling on X with a unique multiple fiber of multiplicity d over $x = 0$. Whereas the $\mathbb{A}^1_*$-fibration $f = x^{e-1}z : X \to \mathbb{A}^1$ appears as the orbit map of a $\mathbb{C}^*$-action on X (cf. Remark 1.1 below).

Proposition 4. *Any two disjoint homology lines Γ_0 and Γ_1 on a $\mathbb{Q}$-homology plane X appear as two different fibers of an $\mathbb{A}^1$-ruling $\pi : X \to \mathbb{A}^1$. In particular, X arises as in the above construction. If, moreover, X is a $\mathbb{Z}$-homology plane then there exists an isomorphism $X \simeq \mathbb{A}^2$ sending Γ_0, Γ_1 to two parallel lines.*

Proof. The second assertion is proven in [Za, §9]. The first one can be deduced by a similar argument. Namely, $\operatorname{Pic}(X)$ being a torsion group, $m\Gamma_0$ is a principal divisor and so $m\Gamma_0 = q^*(0)$ for some $m \in \mathbb{N}$, $q \in \mathcal{O}[X]$. Applying Stein factorisation we may assume that the general fibers of q are irreducible. Since Γ_0 and Γ_1 are disjoint, Γ_1 is a component of a fiber, say, $F_1 = q^{-1}(1)$. Let the degenerate fibers of q be among the fibers $F_0 = \Gamma_0, F_1, \ldots, F_n$, and denote F a general fiber of q. By the Suzuki formula *loc. cit.*

$$\sum_{j=0}^{n} (e(F_j) - e(F)) = 1 - e(F) ,$$

where all summands are non-negative by [Za, 3.2]. Since $e(F_0) = e(\Gamma_0) = 1$ we have $e(F_j) = e(F)$ $\forall j = 1, \ldots, n$. It follows by [Za, 3.2] or [Gu] that either

(i) the fibers F_j are general $\forall j = 1, \ldots, n$, or
(ii) $F \simeq F_j \simeq \mathbb{A}^1_*$ $\forall j = 1, \ldots, n$, or
(iii) $F \simeq F_j \simeq \mathbb{A}^1$ $\forall j = 1, \ldots, n$.

The case (ii) must be excluded since $\Gamma_1 \subseteq F_1$ and $e(\Gamma_1) = 1$. If (i) holds then $F_1 \simeq \pi^*(1)$ is a general fiber of q, hence $F \simeq F_1 \simeq \mathbb{A}^1$. Thus in any case all fibers of π are isomorphic to $\mathbb{A}^1$ and so, Γ_0, Γ_1 are fibers of the $\mathbb{A}^1$-ruling $\pi = q : X \to \mathbb{A}^1$. □

Now one can easily deduce Theorem 1(a).

Proof of Theorem 1(a). $X \setminus \Gamma$ being a smooth affine surface with $\bar{k}(X \setminus \Gamma) = -\infty$, there exists an $\mathbb{A}^1$-ruling on $X \setminus \Gamma$ [Mi, 2.1.1]. The curve $\Gamma_0 = \Gamma$ and a general fiber, say, Γ_1 of this ruling provide two disjoint homology lines on X. By Proposition 4 Γ_0 and Γ_1 are two different fibers of an $\mathbb{A}^1$-ruling $\pi : X \to \mathbb{A}^1$ on X and so, $\Gamma = \Gamma_0$ is an affine line stable under an effective $\mathbb{C}_+$-action on X along this ruling, see e.g. [FlZa$_3$, 1.6]. Now the assertion follows easily. □

3. $\mathbb{Q}$-homology planes with a $\mathbb{C}^*$-action

To deduce Theorem 1(b) we recall first Example 1 in [KiKo], cf. also Examples 3.8, 3.9 in [GMMR].

Construction. The construction begins with a divisor $D_0 = M_a + \bar{M}_a + F_0 + F_1 + F_\infty$ on a Hirzebruch surface Σ_a, where F_0, F_1, F_∞ are 3 distinct fibers of the standard projection $\pi_0 : \Sigma_a \to \mathbb{P}^1$ and M_a, $\bar{M}_a$ are two disjoint sections with $M_a^2 = -a$. We may suppose that $F_j = \pi_0^{-1}(j)$ ($j = 0, 1, \infty \in \mathbb{P}^1$). Besides, the construction involves a sequence of *inner* blowups $\mu : V \to \Sigma_a$ over D_0 i.e., successive blowups with centers at double points on D_0 or on its total transforms. This results in a $\mathbb{Q}$-homology plane $X = V \setminus D$, where $D \subseteq \mu^{-1}(D_0)$ is a suitable SNC tree of rational curves on V (see a description below). The induced $\mathbb{P}^1$-ruling $\bar{\pi} : V \to \mathbb{P}^1$ restricts to an untwisted $\mathbb{A}^1_*$-fibration $\pi : X \to \mathbb{A}^1$.

More precisely μ replaces the fiber F_j ($j = 0, 1$) by a linear chain of smooth rational curves with a unique (-1)-curve E_j, which is a multiple component of the corresponding divisor $\bar{\pi}^*(j)$. The dual graph of the chain $\bar{\pi}^{-1}(0)$ has a sequence of weights $[[-n, -1, \underbrace{-2, \ldots, -2}_{n-1}]]$, where for the strict transform F_0' of F_0 on V one has ${F'}_0^2 = -n \le -2$. The boundary divisor D appears as the total transform of D_0 in V with the components E_0, E_1 and F_0' being deleted. In the affine part $X = V \setminus D$ these deleted components form the only degenerate fibers of π, namely $\pi^*(0) = \Gamma + n(E_0 \cap X)$ and $\pi^*(1) = m(E_1 \cap X)$, where $\Gamma = F_0' \cap X$ and $m, n \ge 2$. Thus the only reducible affine fiber $\pi^{-1}(0) = (F_0' + E_0) \cap X$ is isomorphic to the cross $\mathbb{A}^1 \wedge \mathbb{A}^1 = \{xy = 0\} \subset \mathbb{A}^2$. Furthermore $\pi^{-1}(1) = E_1 \cap X \simeq \mathbb{A}^1_*$ is an irreducible multiple fiber.

Finally $\bar{\pi}^{-1}(F_\infty) = F'_\infty \subseteq D$. A computation in [KiKo, Example 1] shows that $\bar{k}(X \setminus \Gamma) = 0$ if $m = n = 2$ and $\bar{k}(X \setminus \Gamma) = 1$ otherwise.

Remark 3. One could consult e.g. [FlZa$_1$] for a construction giving all $\mathbb{Q}$-homology planes X with an $\mathbb{A}^1_*$-fibration $\pi : X \to B$. In the terminology of [FlZa$_1$], such a surface with a twisted (untwisted) $\mathbb{A}^1_*$-fibration over $B = \mathbb{A}^1$, $B = \mathbb{P}^1$, respectively, is said to be of type $A1$, $A2$ ($B1$, $B2$), respectively. Thus the surface X as in the Construction above is of type $B1$, with a comb attachment applied at F_0 and with F_1 replaced by a fiber of a *broken chain* type in the terminology of [FlZa$_1$].

In the following lemma we prove the first assertion of Theorem 1(b). We recall that a $\mathbb{C}^*$-action on X is *hyperbolic* if its general orbits are closed, *elliptic* if it possesses an attractive or repelling fixed point in X, and *parabolic* if its fixed point set is one-dimensional.

Lemma 5. *If (X, Γ) satisfies the assumptions of Theorem 1(b) then Γ is an orbit closure of an effective hyperbolic $\mathbb{C}^*$-action on X.*

Proof. According to Theorem 1.1 in [KiKo] (cf. Theorem 3.10 in [GMMR]), under our assumptions (X, Γ) is one of the pairs as in the above Construction. There exists an effective $\mathbb{C}^*$-action on Σ_a along the fibers of π_0 with the fixed point set equal to $M_a \cup \bar{M}_a$. By induction on the number of blowups this $\mathbb{C}^*$-action lifts to V stabilizing the total transform $\mu^{-1}(D_0)$. Indeed the centers of successive inner blowups in μ are fixed under the $\mathbb{C}^*$-action constructed on the previous step, and so Lemma 2.2(b) in [FKZ] applies. It follows that the curve $D \subseteq \mu^{-1}(D_0)$ as in the Construction above is stable under the lifted $\mathbb{C}^*$-action, so this action restricts to a hyperbolic $\mathbb{C}^*$-action on $X = V \setminus D$. In turn, the affine line Γ on X as in the Construction is an orbit closure for this restricted action, as required. □

The resulting surface X with a hyperbolic $\mathbb{C}^*$-action admits the following description in terms of the DPD orbifold presentation[8] as elaborated in [FlZa$_2$].

DPD presentation. Let $C = \operatorname{Spec} A_0$ be a smooth affine curve and (D_+, D_-) be a pair of $\mathbb{Q}$-divisors on C with $D_+ + D_- \leq 0$. Letting $A_{\pm k} = H^0(C, \mathcal{O}(kD_\pm))$, $k \geq 0$ we consider the graded A_0-algebra $A = A_0[D_+, D_-] = \bigoplus_{k\in\mathbb{Z}} A_k$ and the associated normal affine surface $X = \operatorname{Spec} A$. The grading determines, in a usual way, a graded semisimple Euler derivation δ on A, where $\delta(a_k) = ka_k$ $\forall a_k \in A_k$, and,

[8] i.e. the Dolgachev-Pinkham-Demazure presentation.

in turn, an effective hyperbolic $\mathbb{C}^*$-action on X. Vice versa, any effective hyperbolic $\mathbb{C}^*$-action on a normal affine surface with the orbit space C arises in this way [FlZa$_2$, 4.3].

Let $\pi : X \to C$ be the orbit map. Given a point $p \in C$ we let $m_\pm(p)$ denote the minimal positive integer such that $m_\pm(p)D_\pm(p) \in \mathbb{Z}$. In case where $(D_+ + D_-)(p) = 0$ we set $m(p) = m_\pm(p)$. If $(D_+ + D_-)(p) < 0$ then the fiber $\pi^{-1}(p)$ is reducible, isomorphic to the cross $\mathbb{A}^1 \wedge \mathbb{A}^1$ in $\mathbb{A}^2$ and consists of two orbit closures $\bar{O}_p^\pm$. Its unique double point p' is a fixed point; p' is smooth on X if and only if $(D_+ + D_-)(p) = -1/m_+m_-$ [FlZa$_2$, 4.15]. Actually $m_\pm(p)$ are the multiplicities of the curves $\bar{O}_p^\pm$, respectively, in the divisor $\pi^*(p)$.

In case where $(D_+ + D_-)(p) = 0$ the fiber $O_p = \pi^{-1}(p) \simeq \mathbb{A}^1_*$ is irreducible of multiplicity $m(p)$ in $\pi^*(p)$.

The inversion $\lambda \longmapsto \lambda^{-1}$ in $\mathbb{C}^*$ results in interchanging D_+ and D_-, respectively, $\bar{O}_p^+$ and $\bar{O}_p^-$. Passing from the pair (D_+, D_-) to another one $(D'_+, D'_-) = (D_+ + D_0, D_- - D_0)$ with a principal divisor D_0 on C results in passing from A to an isomorphic graded A_0-algebra A', so the corresponding $\mathbb{C}^*$-surfaces are equivariantly isomorphic over C.

Lemma 6. *Given a normal affine surface $X = \operatorname{Spec} A$ with a hyperbolic $\mathbb{C}^*$-action determined by a pair (D_+, D_-) of $\mathbb{Q}$-divisors on the affine curve $C = \operatorname{Spec} A_0$ with $D_+ + D_- \le 0$, we denote by $p_1, \ldots, p_l$, $q_1, \ldots, q_k$ the points of C with $(D_+ + D_-)(p_j) < 0$, $(D_+ + D_-)(q_i) = 0$ and $m(q_j) \ge 2$, respectively. Letting $\pi : X \to C$ be the orbit map we assume that $C \simeq \mathbb{A}^1$ and that X is smooth that is, $(D_+ + D_-)(p_j) = -1/m_+(p_j)m_-(p_j)$ $\forall j = 1, \ldots, l$. Then the following hold.*

(a) $e(X) = l$.
(b) $\operatorname{Pic}(X) \otimes \mathbb{Q} = 0$ *if and only if* $l \le 1$ *that is,* π *has at most one reducible fiber.*
(c) Moreover X is $\mathbb{Q}$-acyclic if and only if $l = 1$. In the latter case $\pi : X \to \mathbb{A}^1$ is an untwisted $\mathbb{A}^1_*$-fibration[9].

Proof. (a) holds by the additivity of the Euler characteristic, and (b) follows from the description of the Picard group $\operatorname{Pic}(X)$ in [FlZa$_2$, 4.24]. For a smooth rational affine surface X we have $b_3 = b_4 = 0$ and $b_1 = \rho(X)$, where $\rho(X)$ is the Picard number of X [Mi, Ch. 3, 4.2.1]. Thus X is $\mathbb{Q}$-acyclic if and only if $e(X) = 1$ and $\operatorname{Pic}(X) \otimes \mathbb{Q} = 0$, whence (c) follows. □

[9]It is of type B1 in the classification of [FlZa$_1$].

Lemma 7. *Every $\mathbb{Q}$-homology plane X as in the above Construction is isomorphic to a $\mathbb{C}^*$-surface* $\operatorname{Spec} A_0[D_+, D_-]$, *where $A_0 = \mathbb{C}[t]$ and*

$$D_+ = \frac{e}{m}[1], \quad D_- = -\frac{1}{n}[0] - \frac{e}{m}[1] \ \text{ with } 0 < e < m,\ \gcd(e,m) = 1,\ m, n \geq 2\,.$$

Conversely, every $\mathbb{C}^*$-surface with such a DPD-presentation appears via the above Construction.

Proof. By our Construction, the degenerate fibers of the induced $\mathbb{A}^1_*$-family $\pi : X \to \mathbb{A}^1$ are $\pi^*(0) = n(E_0 \cap X) + \Gamma$ and $\pi^*(1) = m(E_1 \cap X)$, where $\Gamma = F_0' \cap X$, $E_0 \cdot \Gamma = 1$ and E_j (j=0,1) is the unique (-1)-curve in the fiber $\bar\pi^{-1}(j)$ of the induced $\mathbb{P}^1$-ruling $\bar\pi : V \to \mathbb{P}^1$. Clearly, all these curves are orbit closures for the $\mathbb{C}^*$-action on X as in Lemma 5. We may suppose that, in the notation as above, $\Gamma = O_0^+$, $E_0 \cap X = O_0^-$ and $E_1 \cap X = O_1$ so that

$$k = l = 1,\ p_1 = 0,\ q_1 = 1,\ m_+(0) = 1,\ m_-(0) = n \text{ and } m(1) = m\,.$$

Since every integral divisor on $C = \mathbb{A}^1$ is principal, passing to an equivalent pair of $\mathbb{Q}$-divisors we may achieve that (D_+, D_-) is a pair as in the lemma. This proves the first assertion. The converse easily follows by virtue of Lemma 6. ☐

Remarks 1. 1. According to [FlZa$_3$, 5.5], for $e = 1$ and $m \mid n$ the above surfaces actually coincide with the Bertin surfaces from Example 1.

2. The formula for the canonical divisor in [FlZa$_2$, 4.25] gives in our case

$$K_X = -(e(n-1)+1)[O_1], \qquad \text{where} \quad m[O_1] = 0\,.$$

Therefore $K_X = 0$ if and only if $e(n-1) \equiv -1 \bmod m$. The question arises whether, among the $\mathbb{C}^*$-surfaces from Lemma 7 satisfying the latter condition, the Bertin surfaces are the only hypersurfaces.

3. For $n > 1$ the fractional part $\{D_-\}$ in Lemma 7 is supported on 2 points, hence by [FlZa$_3$, 4.5] the surface X as in Lemma 7 admits a unique $\mathbb{A}^1$-ruling $X \to \mathbb{A}^1$ (i.e., X is of class ML_1 in the terminology of [GMMR]).

In contrast, for $n = 1$ there is a second $\mathbb{A}^1$-ruling $X \to \mathbb{A}^1$, so X has trivial Makar-Limanov invariant. In particular for $e/m = 1/2$ and $n = 1$ by virtue of [FlZa$_3$, 5.1], $X \simeq \mathbb{P}^2 \setminus \Delta$, where Δ is a smooth conic in $\mathbb{P}^2$.

4. Following [FlZa$_2$, 4.8] it is possible to define, by explicit equations, a family of surfaces in $\mathbb{A}^4$, not necessarily complete intersections, whose normalizations are the $\mathbb{Q}$-homology planes in the above Construction.

Now we are ready to complete the proof of Theorem 1(b).

Proof of Theorem 1(b). Since the fractional part $\{D_+\}$ of the divisor D_+ as in Lemma 7 is supported on one point, there exists a graded locally nilpotent derivation on A of positive degree, see [FlZa$_3$, 2.2, 3.23]. This derivation generates an effective $\mathbb{C}_+$-action on X, and also an action of a semidirect product $G = \mathbb{C}^* \ltimes \mathbb{C}_+$ with an open orbit $U \simeq \mathbb{A}^1 \times \mathbb{A}^1_*$. Moreover by [FlZa$_3$, 3.25], the orbit map $X \to \mathbb{A}^1$ of the associate $\mathbb{C}_+$-action has a unique irreducible multiple fiber $\Gamma' = \bar{O}_0^-$ $(= E_0 \cap X)$ of multiplicity $m \geq 2$. General orbits of this $\mathbb{C}_+$-action on X being transversal to Γ, the action moves Γ, as stated. □

4. Isotrivial families of curves and $\mathbb{C}^*$-actions

To indicate a proof of Theorem 1(c) let us recall first a necessary result from [LiZa, Za]. For the sake of completeness we sketch the proof.

Lemma 8. ([LiZa, Lemma 5]) *Let X^* be a smooth affine surface and $\pi : X^* \to \mathbb{A}^1_*$ be a family of curves without degenerate fibers which is not a twisted $\mathbb{A}^1_*$-family. Then π is equivariant with respect to a suitable effective $\mathbb{C}^*$-action on X^* and a nontrivial $\mathbb{C}^*$-action on $\mathbb{A}^1_*$.*

Proof. Let F denote a general fiber of π. In the case where $F \simeq \mathbb{A}^1$ the surface X^* admits a completion which is a Hirzebruch surface Σ_a with the boundary divisor $D = \Sigma_a \setminus X^*$ consisting of a section and two fibers. It follows that π is a trivial family, which implies the assertion. The same argument applies if $F \simeq \mathbb{A}^1_*$ since in this case by our assumption π is untwisted.

Suppose further that $e(F) < 0$ i.e. that F is a hyperbolic curve. By Bers' Theorem the Teichmuller space corresponding to F, with its natural complex structure, is biholomorphic to a bounded domain in $\mathbb{C}^M$ for some $M > 0$, hence is as well hyperbolic. Therefore the family π over a non-hyperbolic base $\mathbb{A}^1_*$ is isotrivial i.e., its fibers are all pairwise isomorphic. Since $\mathrm{Aut}(F)$ is a finite group the monodromy $\mu \in \mathrm{Aut}(F)$ of the family π has finite order, say, N. After a cyclic étale base change $z \longmapsto z^N$ we obtain a trivial family $F \times \mathbb{A}^1_* \to \mathbb{A}^1_*$, which is a cyclic étale covering of the given family π. The standard $\mathbb{C}^*$-action on its base lifts to a free $\mathbb{C}^*$-action on $F \times \mathbb{A}^1_*$ commuting with the monodromy $\mathbb{Z}/N\mathbb{Z}$-action. Therefore the lifted $\mathbb{C}^*$-action descends to X^* so that π becomes equivariant with respect to the $\mathbb{C}^*$-action $\lambda.z = \lambda^N z$ on $\mathbb{A}^1_*$, as needed. □

Remark 4. The $\mathbb{A}^1_*$-family of orbits of a hyperbolic $\mathbb{C}^*$-action on an affine surface is always untwisted [FKZ]. Hence the conclusion of Lemma 8 does not hold for twisted $\mathbb{A}^1_*$-families.

Proof of Theorem 1(c). Let Γ be a non-smooth homology line on a $\mathbb{Q}$-homology plane X, and let $m\Gamma = f^*(0)$ for a suitable $m \in \mathbb{N}$ and a primitive regular function $f \in \mathcal{O}(X)$ with irreducible general fiber F (cf. the proof of Proposition 4). Let $p' \in \Gamma$ be a singular point of Γ with Milnor number $\mu > 0$. In a suitable small spherical neighborhood B of p', the function $f^{1/m}$ is holomorphic and its general fiber say R (which is the Milnor fiber of (Γ, p')) is a Riemann surface with boundary of positive genus $g = \mu/2$ [Mil, 10.2]. For a fixed general fiber F of f sufficiently close to Γ, the intersection $F \cap B$ is a disjoint union of m copies of the Milnor fiber R, hence F as well is of positive genus.

Therefore $e(F) < 0$. By Lemma 3.2 in [Za], since $e(X \setminus \Gamma) = 0$ the family $\pi = f|(X \setminus \Gamma) : X \setminus \Gamma \to \mathbb{A}^1_*$ has no degenerate fiber, and so Lemma 8 applies.

As a matter of fact, the $\mathbb{C}^*$-action on $X \setminus \Gamma$ as in Lemma 8 extends to an elliptic $\mathbb{C}^*$-action on X making f equivariant and p' an attractive or repelling fixed point. For a $\mathbb{Z}$-homology plane X, the existence of such an extension was shown in [LiZa] and in [Za] in two different ways. The both proofs work *mutatis mutandis* in our more general setting. We choose below to follow the lines of the proof of Lemma 6 in [LiZa].

Let $\bar{F}$ be a smooth projective model of F. The cyclic étale covering $\rho : F \times \mathbb{A}^1_* \to X \setminus \Gamma$ as in the proof of Lemma 8 extends to an equivariant rational map which fits into the commutative diagram

$$\begin{tikzcd} \bar{F} \times \mathbb{P}^1 \arrow[r, dashed, "\rho"] \arrow[d, "\mathrm{pr}_2"] & V \arrow[d, "\bar{\pi}"] \\ \mathbb{P}^1 \arrow[r, "z \longmapsto z^N"] & \mathbb{P}^1 \end{tikzcd} \tag{1}$$

where V is a smooth equivariant SNC completion of $X \setminus \Gamma$. If the $\mathbb{C}^*$-action on $X \setminus \Gamma$ possesses an orbit O which is not closed in X then so are all orbits and the action extends to X. Indeed the closure $\bar{O}$ meets Γ in one point say q, and $n\bar{O} = h^*(0)$ for some regular function h on X and for some $n \in \mathbb{N}$. Let P be the connected component of the polyhedron $|f| \le \varepsilon$, $|h| \le \varepsilon$ which contains q. Since $\lambda.h = \lambda^k \cdot h$ for some $k \in \mathbb{Z}$, for every $\lambda \in \mathbb{C}^*$ with $|\lambda| = 1$ the complement $P \setminus \Gamma$ is stable under the action of λ. The polyhedron P being compact for a sufficiently small $\varepsilon > 0$, such an action on $P \setminus \Gamma$ extends across Γ. Hence it also extends through Γ to an action on the whole X for all $\lambda \in \mathbb{C}^*$ with $|\lambda| = 1$, and then also for all $\lambda \in \mathbb{C}^*$.

The indeterminacy set of ρ being at most 0-dimensional, ρ restricts to the fiber over $0 \in \mathbb{P}^1$ yielding a morphism $\rho : \bar{F} \to \bar{\pi}^{-1}(0)$. The following alternative holds: either $\rho(\bar F) = p \in \Gamma$, or $\rho(\bar F) = \bar\Gamma$, or finally $\rho(\bar F) \cap \Gamma = \emptyset$. Let us show that the last two possibilities cannot occur.

Indeed supposing that $\rho(\bar F) = p \in \Gamma$ ($\rho(\bar F) = \bar\Gamma$, respectively) the general orbits of the $\mathbb{C}^*$-action on $X \setminus \Gamma$ are not closed in X, and the action extends to an elliptic (parabolic, respectively) $\mathbb{C}^*$-action on X. Since the fixed point set of a parabolic $\mathbb{C}^*$-action on a normal affine surface is smooth (see [FlZa$_2$, §3]), the latter case must be excluded.

To exclude the last possibility, suppose on the contrary that $\rho(\bar F) \cap \Gamma = \emptyset$. Letting $p_1, \ldots, p_k \in \bar F \times \{0\}$ be the indeterminacy points of ρ on the central fiber, we observe that under our assumption, all $\mathbb{C}^*$-orbits $\rho(\{p\} \times \mathbb{A}^1_*)$ are closed in X, because general orbits are. The orbits $\rho(\{p_i\} \times \mathbb{A}^1_*)$, $i = 1, \ldots, k$, meet any fiber $F_\xi = f^{-1}(\xi)$, $\xi \in \mathbb{A}^1$, in a finite set, say, T.

Fixing further a general fiber $F = F_\xi$ sufficiently close to Γ and a sufficiently small neighborhood ω of the finite set $T \cup (\bar F \setminus F)$ in $\bar F$, we let $K = \bar F \setminus \omega \subseteq F$. Under our assumptions K is a compact Riemann surface of positive genus with boundary, and $B \cap \lambda.K = \emptyset$ for all sufficiently small $\lambda \in \mathbb{C}^*$. Hence $F \cap \lambda^{-1}.B \subseteq \omega$ is a disjoint union of Riemann surfaces of genus 0. On the other hand

$$F \cap \lambda^{-1}.B \cong B \cap \lambda.F = B \cap F_{\lambda^N \xi}$$

is a disjoint union of m copies of the Milnor fiber R of the analytic plane curve singularity (Γ, p'). This is a contradiction because R is of positive genus.

Thus Γ is stable under the extended elliptic $\mathbb{C}^*$-action on X. So Γ is an orbit closure of this action and the singular point $p' \in \Gamma$ is a fixed point of the action. Consider an equivariant embedding $X \hookrightarrow \mathbb{A}^N$ which sends p' to the origin, where $\mathbb{A}^N$ is equipped with a linear $\mathbb{C}^*$-action, and fix an equivariant linear projection $\mathbb{A}^N \to T$, where $T \simeq \mathbb{A}^2$ is the tangent plane of X at $p' = \bar 0$. This projection restricted to X gives an equivariant isomorphism $X \simeq \mathbb{A}^2$, where the $\mathbb{C}^*$-action on $\mathbb{A}^2$ is linear (indeed, both actions have the origin as an attractive fixed point). In appropriate linear coordinates the latter linear action is diagonal: $\lambda.(x, y) \longmapsto (\lambda^l x, \lambda^k y)$ with $\gcd(k, l) = 1$. So either the image of Γ is one of the axes, which contradicts the assumption that Γ is singular, or it is a curve $\alpha x^k - \beta y^l = 0$ for some $\alpha, \beta \in \mathbb{C}^*$. This proves (c) of Theorem 1. Now the proof of Theorem 1 is completed. $\square$

References

[Be] J. Bertin, *Pinceaux de droites et automorphismes des surfaces affines*, J. Reine Angew. Math. **341** (1983), 32–53.

[Fi] K.-H. Fieseler, *On complex affine surfaces with $\mathbb{C}^+$-action*, Comment. Math. Helv. 69 (1994), 5–27.

[FlZa$_1$] H. Flenner, M. Zaidenberg, *Q-acyclic surfaces and their deformations*, Proc. Conf. "Classification of Algebraic Varieties", Mai 22–30, 1992, Univ. of l'Aquila, L'Aquila, Italy /Livorni ed. Contempor. Mathem. Vol. 162, Providence, RI, 1994, 143–208.

[FlZa$_2$] H. Flenner, M. Zaidenberg, *Normal affine surfaces with $\mathbb{C}^*$-actions*, Osaka J. Math. 40 (2003), 981–1009.

[FlZa$_3$] H. Flenner, M. Zaidenberg, *Locally nilpotent derivations on affine surfaces with a $\mathbb{C}^*$-action*, Osaka J. Math. 42 (2005), 44p.

[FKZ] H. Flenner, S. Kaliman, M. Zaidenberg, *Completions of $\mathbb{C}^*$-surfaces*, math.AG/0511282, 37p.; Preprint, Institut Fourier de Mathématiques, 693, Grenoble 2005, 36p. Affine algebraic geometry. In honor of Prof. M. Miyanishi, World Sci. 2007, 149-200 (to appear).

[Fu] T. Fujita, *On the topology of noncomplete algebraic surfaces.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 29 (1982), 503–566.

[Gu] R. V. Gurjar, *A new proof of Suzuki's formula*, Proc. Indian Acad. Sci. Math. Sci. 107 (1997), 237–242.

[GMMR] R. V. Gurjar, K. Masuda, M. Miyanishi and P. Russell, *Affine lines on affine surfaces and the Makar-Limanov invariant*, preprint, 2005, 42p.

[GuPa] R. V. Gurjar, A. J. Parameswaran, *Affine lines on $\mathbb{Q}$-homology planes*, J. Math. Kyoto Univ. 35 (1995), 63–77.

[KaZa] S. Kaliman, M. Zaidenberg, *Affine modifications and affine varieties with a very transitive automorphism group*, Transform. Groups 4 (1999), 53-95.

[KiKo] T. Kishimoto and H. Kojima, *Affine lines on $\mathbb{Q}$-homology planes with logarithmic Kodaira dimension $-\infty$* Transform. Groups 11 (2006), no. 4 (to appear).

[LiZa] V. Ya. Lin, M. G. Zaidenberg, *An irreducible simply connected algebraic curve in $\mathbb{C}^2$ is equivalent to a quasihomogeneous curve*, Soviet Math. Dokl. 28 (1983), 200–204.

[MaMi$_1$] K. Masuda, M. Miyanishi, *The additive group actions on $\mathbb{Q}$-homology planes*, Ann. Inst. Fourier (Grenoble) 53 (2003), 429–464.

[MaMi$_2$] K. Masuda, M. Miyanishi, *Affine pseudo-planes and cancellation problem*, Trans. Amer. Math. Soc. 357 (2005), 4867–4883.

[Mil] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. 61 Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1968.

[Mi] M. Miyanishi, *Open algebraic surfaces.* CRM Monograph Series, 12. Amer. Math. Soc., Providence, RI, 2001.

[MiSu] M. Miyanishi, T. Sugie, *Affine surfaces containing cylinderlike open sets*, J. Math. Kyoto Univ. 20 (1980), 11–42.

[Suz] M. Suzuki, *Sur les opérations holomorphes de $\mathbb{C}$ et de $\mathbb{C}^*$ sur un espace de Stein.* Fonctions de plusieurs variables complexes, III (Sém. Norguet, 1975–1977), 394, Lecture Notes in Math., 670, Springer, Berlin, 1978, 80–88.

[Za] M. G. Zaidenberg, *Isotrivial families of curves on affine surfaces and characterization of the affine plane*, Math. USSR Izvestiya 30 (1988), 503–532. Additions and corrections, ibid., 38 (1992), 435–437.

E-mail address: `zaidenbe@ujf-grenoble.fr`

Université Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, BP 74, 38402 St. Martin d'Hères cédex, France