ON A CONJECTURE OF LANGE.

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Introduction

Let C be a projective non-singular curve of genus $g \geq 2$. Let E be a vector bundle of rank r and degree d. Fix a positive integer $r' < r$. Define

$$s_{r'}(E) = r'd - r \max_{E'}\{degE' | rkE' = r', E' \subset E\}$$

Notice that E is stable if and only if $s_{r'}(E) > 0$ for every $r' < r$. On the other hand, for a generic stable E

$$r'(r-r')(g-1) \leq s_{r'}(E) \leq r'(r-r')(g-1) + r$$

(cf [L] Satz 2.2 p.452 and [Hi] Th.4.4). One can then stratify the moduli space $U(r,d)$ of vector bundles of rank r and degree d according to the value of s. Define

$$U_{r',s}(r,d) = \{E \in U(r,d) | s_{r'}(E) = s\}$$

We want to study this stratification. A vector bundle $E \in U_{r',s}(r,d)$ can be written in an exact sequence

$$0 \to E' \to E \to E'' \to 0$$

with E', E'' vector bundles of ranks r', r'' and degrees d', d'' satisfying $r = r' + r'', d = d' + d'', r'd - rd' = r'd'' - r''d' = s$. Note that the condition $s > 0$ is equivalent to the inequality of slopes $\mu(E') < \mu(E'')$. One expects that when this condition is satisfied, a generic such extension will yield a stable E. We call this statement Lange's conjecture (cf. [L]).

The conjecture is now solved and a great deal is known about the geometry of the strata : the rank two case is treated in [L,N], the case $s \leq \min(r', r'')(g-1)$ in [B,B,R]. In [T1], the result is proved for the generic curve and for every curve if E is assumed to be only semistable. This apparently implies the result also for E stable (cf.[B]). In [B,L], a proof is provided for $g \geq (r+1)/2$.

The purpose of this paper is to give a simpler proof of the result valid without further assumptions. The method is somehow the converse of the one used by Brambila-Paz and Lange in [B,L]. They start with the most general E in $U(r,d)$ and then show that a suitable transformation of E gives a new bundle with smaller s. They need to check then that such an E is in fact stable. Here, we start with an E with the smallest possible s and produce an E' with larger s. Stability then comes for free because E' is more general than E. The drawback is that one needs to prove existence of stable vector bundles with small s but this is surprisingly easy.

Our method of proof provides additional information on the geometry of the strata. We can prove that $U_{r',s}(r,d)$ is contained in the closure of $U_{r',s+r}(r,d)$ as well as the unicity of the subbundle (see also [T2])

Our results can be stated in the following

Theorem 0.1. *Assume that* $0 < s \leq r'(r-r')(g-1), s \equiv r'd(r)$. *Write* $d' = \frac{r'd-s}{r}$. *If* $g \geq 2$, *then* $U_{r's}(r,d)$ *is non-empty, irreducible of dimension*

$$dim U_{r's}(r,d) = r^2(g-1)+1+s-r'(r-r')(g-1)$$

Moreover, a generic $E \in U_{r',s}(r,d)$ *can be written in an exact sequence*

$$0 \to E' \to E \to E'' \to 0$$

with both E', E'' *stable and* E' *is the unique subbundle of* E *of rank* r' *and degree* d'.

Theorem 0.2. *If* $s \geq r'(r-r')(g-1)$, *every stable vector bundle has subbundles of rank* r' *and degree* d'. *Denote by*

$$A_{r',d'}(E) = \{E' | rkE' = r', degE' = d', E' \subset E, E' \text{saturated}\}.$$

Then, for generic E, $A_{r',d'}(E)$ *is irreducible of dimension*

$$dim A_{r',d'}(E) = r'(r-r')(g-1) - s.$$

These results and our methods of proof can be used to study twisted Brill-Noether loci. We can show the following

Theorem 0.3. *(twisted Brill-Noether for one section). Let* E *be a generic vector bundle of rank* r_E *and degree* d_E. *Consider the twisted Brill-Noether loci* $W^0_{r_F,d_F}(E)$. *This is defined as the subset of the moduli space* $U(r_F,d_F)$ *consisting of those* F *such that* $h^0(F^* \otimes E) \geq 1$. *Then the dimension of* $W^0_{r_F,d_F}(E)$ *is the expected dimension given by the Brill-Noether number*

$$\rho^0_{r_F,d_F}(E) = r_F(r_F-r_E)(g-1) + r_F d_E - r_E d_F$$

if this number is positive and is empty otherwise. Moreover, when non-empty it is irreducible and its generic elements considered as maps $F \to E$ *have maximal rank.*

We also include a proof of Hischowitz's Theorem that states that the tensor product of two generic vector bundles is non-special.

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1. Existence and dimensionality

In this section we prove the existence of extensions with central term stable and we compute the dimension of the set of vector bundles that fit in such exact sequences. We need several preliminary results.

Lemma 1.1. *Let E be a stable vector bundle. Assume that we have an exact sequence*

$$0 \to E' \to E \to E'' \to 0.$$

Then $h^0(E''^* \otimes E') = 0$.

Proof. A non-zero map $E'' \to E'$ induces an endomorphism of E that is not an homothethy. This is impossible if E is stable. $\square$

Theorem 1.2 (Hirschowitz)**.** *The tensor product of two generic vector bundles is not special.*

This result was stated and proved in [Hi], 4.6. As this is , unfortunately, still unpublished, we provide an alternative proof below.

Proof. We shall denote by r_G, d_G the rank and degree of a given sheaf say G.

By Serre duality, it is enough to show that if E, F are generic vector bundles, then $h^0(F^* \otimes E) > 0$ implies $\chi(F^* \otimes E) > 0$.

Assume $h^0(F^* \otimes E) \neq 0$. Then, there is a non-zero map $F \to E$. Denote by F' its kernel, I its image, E'' its cokernel. Let T be the torsion subsheaf of E'' and $\bar{E} = E''/T$. We then have the following exact sequences of sheaves

$$0 \to F' \to F \to I \to 0$$

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & T & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & I & \to & E & \to & E'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bar{I} & \to & E & \to & \bar{E} & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & T & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

As T is a torsion sheaf, I is determined by $\bar{I}$, the support of T and for every point in the support a map from the fiber of $\bar{I}$ at the point to the basefield. Hence,

$$dim\{\bar{I}\} \leq dim\{I\} + r_I degT$$

As any vector bundle can be deformed to a stable vector bundle, (cf.[N,R]Prop.2.6), $F', \bar{I}, \bar{E}$ depend at most on $r_{F'}^2(g-1)+1, r_{\bar{E}}^2(g-1)+1, r_{\bar{I}}^2(g-1)+1$ moduli respectively. From 1.1 and the stability of E, F, $h^0(I^* \otimes F) = 0, h^0(\bar{E}^* \otimes \bar{I}) = 0$. Notice that F is determined by F', I and an extension class in $H^1(I^* \otimes F')$ up to homotethy. Similarly, E is determined by $\bar{I}, \bar{E}$ and an extension class in $H^1(\bar{E}^* \otimes \bar{I})$ up to homotethy. From the genericity of the pair E, F, we find

$$\begin{gathered}
r_F^2(g-1)+1+r_E^2(g-1)+1 = dimU(r_F, d_F) + dimU(r_E, d_E) \leq \\
\leq dimU(r_{F'}, d_{F'}) + dimU(r_I, d_I) + dimU(r_{\bar{E}}, d_{\bar{E}}) + r_I degT + \\
+h^1(I^* \otimes F') - 1 + h^1(\bar{E}^* \otimes \bar{I}) - 1
\end{gathered}$$

$$\leq (r_{F'}^2+r_I^2+r_{\bar{E}}^2+r_{F'}r_I+r_I r_{\bar{E}})(g-1)+1+r_{F'}d_I-r_I d_{F'}+r_I d_{\bar{E}}-r_{\bar{E}}d_I+r_I degT$$

This condition can be written as

$$(*) r_{F'}d_I - r_I d_{F'} + r_I d_{\bar{E}} - r_{\bar{E}} d_I + r_I degT - r_I(r_I + r_{\bar{E}} + r_{F'})(g-1) - 1 \geq 0$$

From the genericity of E, F and [L] Satz 2.2, we obtain

$$\mu(I) - \mu(F') \geq g-1, \mu(\bar{E}) - \mu(\bar{I}) \geq g-1$$

Hence

$$\mu(\bar{E}) - \mu(F') \geq 2(g-1) + degT/r_I$$

Equivalently

$$r_{F'}d_{\bar{E}} - r_{\bar{E}}d_{F'} \geq 2r_{F'}r_{\bar{E}}(g-1) + \frac{r_{F'}r_{\bar{E}}}{r_I}degT$$

Adding (*) and this last inequality, we find

$$\chi(F^* \otimes E) \geq 1 + r_{F'}r_{\bar{E}}(g-1) + \frac{r_{F'}r_{\bar{E}}}{r_I}degT + r_{F'}degT \geq 1$$

□

Lemma 1.3. *Denote by $V_{r',s}(r,d)$ the set of stable E that can be written in an exact sequence of vector bundles*

$$0 \to E' \to E \to E'' \to 0$$

with $rkE' = r', degE' = d'$. Assume that $V_{r's}(r,d)$ is non-empty. If $s \le r'(r-r')(g-1)$, then the generic such E has only a finite number of subbundles of rank r' and degree d' such that $r'd - rd' = s$. If $s \ge r'(r-r')(g-1)$, then the dimension of the space of subbundles of rank r' and degree d' of the generic E is at most $s - r'(r-r')(g-1)$

Proof. Several proofs of this fact appear in the literature . We sketch a proof here for the convenience of the reader.

The set of subbundles of rank r' and degree d' of E is parametrised by the quotient scheme of E of the corresponding rank and degree. The tangent space to this quotient scheme at the point corresponding to a bundle E with subbundle E' and quotient E'' is $H^0(E'^* \otimes E'')$. As E is generic, we can assume E', E'' generic. Then, from 1.2, $E'^* \otimes E''$ is non-special. Hence, if $s \le r'(r-r')(g-1), h^0(E'^* \otimes E'') = 0$ while if $s \ge r'(r-r')(g-1)$, then $h^1(E'^* \otimes E'') = 0$ and so $h^0(E'^* \otimes E'') = s - r'(r-r')(g-1)$. □

Proposition 1.4. *With the notations of 1.3, if $V_{r',s}(r,d)$ is non-empty, then, it is irreducible and the generic $E \in V_{r',s}(r,d)$ can be written in an exact sequence as above with E', E'' stable. Moreover, $dimV_{r',s}(r,d) = \min[r^2(g-1)+1, r^2(g-1)+1+s-r'(r-r')(g-1)]$*

Proof. This proof appears in [T]. We give a sketch here for the convenience of the reader .

Consider an extension

$$0 \to E_0' \to E_0 \to E_0'' \to 0$$

with E_0 stable. From [N,R] Prop.2.6, there are irreducible families of vector bundles $\mathcal{M}', \mathcal{M}''$ containing E_0', E_0'' respectively and whose generic member is stable. Consider the universal family of extensions $\mathbf{P}$ of an $E'' \in \mathcal{M}''$ by an $E' \in \mathcal{M}'$. Consider the open subset $U \subset \mathcal{M}' \times \mathcal{M}''$ consisting of those pairs (E', E'') such that $h^0(E''^* \otimes E') = 0$. As $\mu(E') < \mu(E'')$, U contains all pairs in which both E', E'' are stable. From 1.1, $(E_0', E_0'') \in U$. As $h^1(E''^* \otimes E')$ is constant on U, the inverse image $\mathbf{P}(U)$ of U in $\mathbf{P}$ is irreducible. This proves that the given extension can be deformed to an extension with both E', E'' stable. Consider the canonical map $\pi : \mathbf{P}(U) \to U(r,d)$. By definition $V_{r',d'}(r,d)$ is the image of this map. Hence, it is irreducible.The dimension of $\mathbf{P}$ can be computed as

$$dim\mathbf{P} = dim\mathcal{M} + dim\mathcal{M}' + h^1(E^{''*} \otimes E') - 1 = (r^2 - r'r'')(g-1) + 1 + s$$

From 1.3, the fibers of π have dimension $\max[0, s - r'(r-r')(g-1)]$. Hence, the result follows. □

Lemma 1.5. *Let $\pi : X \to Y$ be a map of reduced quasiprojective irreducible schemes defined over an algebraically closed field. If the generic fiber of π has positive dimension, then it is irreducible.*

Proof. The intersection of an irreducible X of dimension at least two with a generic divisor in an ample linear system is irreducible (cf. [J] Th.6.5 4)). A point in Y can be obtained as intersection of a finite number of divisors in an ample linear system. The pull-back to X of the ample linear system is again ample. Then the result follows by induction on the number of hyperplanes in the intersection. □

Proposition 1.6. *Assume $s > r'(r-r')(g-1)$ and E is a generic stable vector bundle. If $A_{r'd'}(E)$ is non-empty, then it is irreducible of dimension $s - r'(r-r')(g-1)$.*

Proof. Let $E' \in A_{r'd'}(E), E'' = E/E'$. We use the notations in the proof of 1.4. From 1.1, $(E', E'') \in U$. From the proof of 1.4, $\mathbf{P}(U)$ is irreducible. Then the result follows from 1.5. □

Definition 1.7. *Let E be a vector bundle. A vector bundle $\tilde{E}$ is called an elementary transformation of E if there is an exact sequence*

$$0 \to \tilde{E} \to E \to \mathbf{C}_P \to 0$$

Here $\mathbf{C}_P$ denotes the skyscraper sheaf isomorphic to the base field with support on the point P.

A vector bundle $\bar{E}$ is called a dual elementary transformation of E if E is an elementary transformation of $\bar{E}$. Equivalently, the dual of $\bar{E}$ is an elementary transformation of the dual of E or equivalently there is an exact sequence

$$0 \to \bar{E} \to E(Q) \to \mathbf{C}_Q^{r-1} \to 0.$$

Lemma 1.8. *Let*

$$0 \to E' \to E \to E'' \to 0$$

be an exact sequence of vector bundles. Then, for a generic elementary transformation $\tilde{E}$ of E, we have an exact sequence of vector bundles

$$0 \to \tilde{E}' \to \tilde{E} \to E'' \to 0$$

where $\tilde{E}'$ is a generic elementary transformation of E'.

Proof. There is an injective map $0 \to E'_P \to E_P$. The elementary transformation depends on the choice of a map $E_P \to \mathbf{C}_P \to 0$. If this map is generic, it induces a non-zero map $E'_P \to \mathbf{C}_P$.

Hence, we have a diagram

$$\begin{array}{ccccccccc}
0 & \to & \tilde{E}' & \to & \tilde{E} & \to & \tilde{E}'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E' & \to & E & \to & E'' & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbf{C}_P & \to & \mathbf{C}_P & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

This proves the statement.

□

Lemma 1.9. *Let*

$$0 \to E' \to E \to E'' \to 0$$

be an exact sequence of vector bundles. Then, for a generic dual elementary transformation $\bar{E}$ of E, we have an exact sequence of vector bundles

$$0 \to E' \to \bar{E} \to \bar{E}'' \to 0$$

where $\bar{E}''$ is a generic dual elementary transformation of E''.

Proof. : Dualise the proof above □

Proposition 1.10. *Let*

$$0 \to E' \to E \to E'' \to 0$$

be an exact sequence of vector bundles. Assume that E is stable. Then, there exists an exact sequence of vector bundles

$$0 \to \hat{E}' \to \hat{E} \to \hat{E}'' \to 0$$

satisfying

i): $deg\hat{E}' = degE' + 1, deg\hat{E} = degE$

ii): *$\hat{E}$ is stable.*

Proof. Take first an elementary transformation of the exact sequence based at a point P. Take next a dual elementary transformation based at a point Q. From the two Lemmas above, $deg\hat{E}' = degE + 1$. We now construct a family of these transformations which contains E as one of its members: let the point Q vary until it coincides with P. Then, with a suitable choice of the dual transformation, one can go back to E. The existence of this family of vector bundles together with the stability of E, implies the stability of the generic $\hat{E}$. □

Corollary 1.11. *If $U_{r's}(r,d)$ is non empty, then it is contained in the closure of $U_{r',s+r}(r,d)$.*

Proof. Take $E \in U_{r',s}(r,d)$ and consider an exact sequence

$$0 \to E' \to E \to E'' \to 0$$

with E' of rank r' and maximal degree d'. In the proof above, we construct a family with special member E and generic member $\tilde{E}$ that fits in an exact sequence

$$0 \to \tilde{E}' \to \tilde{E} \to \tilde{E}'' \to 0$$

with $deg(\tilde{E}') = d' + 1$. Hence, this $\tilde{E} \in V_{r',s+r}(r,d)$. From 1.4, $V_{r',s+r}(r,d)$ is irreducible and from the dimensionality statement in 1.4, $V_{r',s+r}(r,d) \not\subseteq V_{r',s-kr}(r,d), k \geq 0$. Hence the generic element in $V_{r',s+r}(r,d)$ is in $U_{r',s+r}(r,s)$. □

Proposition 1.12. *Let C be a projective non-singular curve of genus $g \geq 2$. Consider an exact sequence of vector bundles on C*

$$0 \to E' \to E \to E'' \to 0.$$

Denote by r', r'', r, d', d'', d the ranks and degrees of E', E'', E. Assume that E', E'' are generic stable vector bundles of their ranks and degrees. If $0 < r'd - rd' \leq r$, then, the generic such E is stable.

Proof. Assume that E is not stable. Let F be a subbundle of E such that $\mu(F) \geq \mu(E)$. Up to replacing F by a subbundle of smaller rank or by its saturation, we can assume F stable and E/F without torsion. As E' is stable and $\mu(E') < \mu(E)$, F gives rise to a non-zero map $\phi : F \to E''$. Denote by F' its kernel, F'' its image. Denote by $r_{F'}, r_F, r_{F''}, d_{F'}, d_F, d_{F''}$ the ranks and degrees of the bundles F', F, F''.

Claim 1. $r_{F''} = r''$.

Proof of Claim 1: Assume $r_{F''} < r''$. By the genericity of E'' this implies $r_{F''}d'' - r''d_{F''} \geq r_{F''}(r'' - r_{F''})(g-1)$ (cf.[L] Satz 2.2). Equivalently

$$\mu(F'') \leq \mu(E'') - (1 - (r_{F''}/r''))(g-1)$$

By the initial assumption $r'd - rd' \leq r$,

$$(*)\mu(E'') \leq \mu(E) + 1/r''.$$

As F is a destabilizing subbundle,

$$\mu(E) \leq \mu(F)$$

and from the stability of F

$$\mu(F) \leq \mu(F'')$$

with equality if and only if $F = F''$. Notice that

$$1/r'' - (1 - (r_{F''}/r''))(g-1) \leq 1/r'' - (1 - (r_{F''}/r'')) \leq 0$$

With equalities if and only if $g = 2, r_{F''} = r'' - 1$. Puting together all of the above inequalities, we find that they are all equalities. This proves Claim 1 except in the case when all of the following properties are satisfied:

i): $g = 2, r_{F''} = r'' - 1$
ii): $\mu(E) = \mu(F)$, $F' = 0$ and $F'' = F$ is a subsheaf of E''
iii): $(r'' - 1)d'' - r''d_F = r'' - 1$

We shall see at the end of the proof that this situation does not correspond to a generic E. This will finish the proof of Claim 1.

Claim 2. $F' = 0$

Proof of Claim 2. Note that E^* satisfies the hypothesis in 1.12 . If F is a maximal destabilising subbundle of E and we write $G = E/F$, then G^* is a maximal destabilising subbundle of E^*. Then, Claim 2 follows from Claim 1.

From now on, we assume that F is a subbundle of E'' and $r_F = r''$

As F is a destabilising subbundle, $\mu(F) \geq \mu(E)$. As F is a subbundle of E'', $\mu(F) \leq \mu(E'')$. Using (*) and $r_F = r''$, we obtain either $d_F = d''$ or $d_F = d'' - 1$ and $\mu(F) = \mu(E) = \mu(E'') - 1/r''$. If $F = E''$, the sequence splits and the extension is not generic. If $d_F = d'' - 1$, we have an exact sequence

$$0 \to F \to E'' \to T \to 0$$

where T is a torsion sheaf of degree one supported at one point, say P. Then, F is determined by the choice of P and a map from E_P to the base field defined up to homothety. Therefore the number of moduli for such F is at most r''.

Consider the pull-back diagram

$$\begin{array}{ccccccccc}
0 & \to & E' & \to & E & \to & E'' & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & E' & \to & E \times_{E''} F & \to & F & \to & 0
\end{array}$$

As F is a subsheaf of E, the bottom row splits. Hence the top row corresponds to an element in the kernel of the map

$$H^1(E''^* \otimes E') \to H^1(F^* \otimes E') \to 0.$$

This kernel has dimension at most $h^0(T \otimes E') = deg(T) \times rk(E') = r'$. Therefore the dimension of the subspace of $H^1(E''^* \otimes E')$ that may correspond to unstable extensions is at most

$$dim\{F\} + r' \leq r'' + r' = r.$$

On the other hand,

$$h^1(E^{''*} \otimes E') = r'd - rd' + r'(r-r')(g-1) = r'(r-r')(g-1) + r$$

where the last equality comes from the condition $\mu(E) = \mu(E'') - 1/r''$. It is then clear that the generic extension is stable.

We now prove that the special situation at the end of Claim 1 does not occur: Notice that from condition iii) and the stability of E'', E'' does not have a subbundle of rank $r''-1$ and degree higher than d_F. Hence, we have a pull-back diagram as above but $E/F = L$ is a line bundle. From condition iii) and 1.3, there is only a finite number of possible F for a given E''. From the genericity of E'', both F and L are generic (and depend only on E'' and not on E'). Hence, it is enough to show that the canonical map

$$H^1(E^{''*} \otimes E') \to H^1(F^* \otimes E')$$

is non-zero. As this map is surjective, this is equivalent to $H^1(F^* \otimes E') \neq 0$. From Riemann-Roch $h^1(F^* \otimes E') \geq r'(r''-1) + r''(r''-1)(\mu(F) - \mu(E')) > 0$ where the last inequality comes from ii) $\mu(E') < \mu(E)$ and $r_F = r'' - 1 > 0$. □

Proof. . We now prove 0.1 except for the unicity that we postpone to next section. From 1.3 , 1.4 and 1.11 , it is enough to prove the non-emptiness of $U_{r's}(r,d)$. Take now any positive s. Write $s = ar + \bar{s}$, $0 < \bar{s} \leq r$. Then, $U_{r',\bar{s}}(r,d)$ is non empty by 1.12. Applying 1.10 a-times, we obtain the non-emptiness of $V_{r',s}(r,d)$. From the definitions of $V_{r',s}(r,d), U_{r',s}(r,d)$, a generic element of $V_{r',s}(r,d)$ belongs to an $U_{r',\tilde{s}}(r,d)$ for some $\tilde{s} \leq s$. In order to prove the non-emptiness of $U_{r',s}(r,d)$, it is enough to see that $V_{r',s}(r,d) \not\subset V_{r',\tilde{s}}(r,d)$, $\tilde{s} < s$. From the dimensionality statement in Lemma 1.4, this is true. □

Proof. The existence part in 0.2 is proved using 1.12, 1.10 and the dimensionality statement in 1.4. To obtain irreducibility, use 1.6. □

2. Brill-Noether for twisted bundles and unicity of subbundles

In this section we prove the result that we stated in the introduction about twisted Brill-Noether Theory. The corresponding result for the untwisted case (i.e. $E = \mathcal{O}$) is well-known (cf[S] Th IV 2.1). We then use this result to show the unicity of the Lange subbundle.

Proof. (of 0.3) Assume that $W^0_{r_F,d_F}(E)$ is non-empty. Consider an element F in this set. This gives rise to a non-zero map $F \to E$. Denote by F' its kernel, F'' its image. Then F'' is a subbundle of E. Moreover, we have an exact sequence

$$0 \to F' \to F \to F'' \to 0.$$

Assume first $r_{F''} < r_E$. From 0.1, the set of saturated subbundles of E of rank $r_{F''}$ and degree $d_{F''}$ is irreducible of dimension

$$dim\{F''\} = r_{F''}(r_{F''} - r_E)(g-1) + r_{F''}d_E - r_E d_{F''}$$

if this number is positive and is empty otherwise. The set of non-saturated subbundles has dimension smaller than this number.

Consider then the case in which $r_{F''} = r_E$. Then, the quotient E/F'' is torsion. The choice of F'' depends on the choice of the support of this quotient and for each point P on the support the choice of a map (up to homothety) to the base field $E_P \to \mathbf{C}$. Hence, $dim\{F''\} \leq r_E(d_E - d_{F''})$. This coincides with the bound above for the case $r_{F''} = r_E$.

Any family of vector bundles can be embedded in a family with generic member stable(cf [NR] Prop.2.6). Hence, F' varies in a parameter space of dimension at most

$$dim\{F'\} = r_{F'}^2(g-1) + 1.$$

From 1.1, $h^0(F*''*\otimes F') = 0$. Using Riemann-Roch,

$$h^1(F''^* \otimes F') = r_{F'}r_{F''}(g-1) + r_{F'}d_{F''} - r_{F''}d_{F'}.$$

The choice of F depends on the choice of the pair F', F'' and the class of the extension up to scalar. Therefore, the dimension of all possible F is bounded by

$$dim\{F\} \leq [r_{F'}^2 + r_{F''}^2 - r_{F''}r_E + r_{F'}r_{F''}](g-1) + r_{F'}d_{F''} - r_{F''}d_{F'} + r_{F''}d_E +$$
$$-r_E d_{F''} = \rho^0_{r_F,d_F}(E) - [r_{F'}d_{E''} - d_{F'}r_{E''} - r_{F'}r_{E''}(g-1)]$$

where E'' denotes the quotient of E by F'' Let us check that

$$(*)[r_{F'}d_{E''} - d_{F'}r_{E''} - r_{F'}r_{E''}(g-1)] \geq 0.$$

By the genericity of E, if F'' exists, then $\mu(E'') - \mu(F'') \geq g-1$ (cf Prop. 2.4) By the stability of F, $\mu(F') < \mu(F'')$. Hence, $\mu(E'') - \mu(F') \geq (g-1)$. This is equivalent to the inequality (*) and proves the upper bound for the dimension. Notice also that (*) vanishes if and only if either $F' = 0$ or $F'' = E$. In both these cases, the map has maximal rank. From the theory of determinantal varieties, any component of the Brill-Noether locus has dimension at least equal to the expected dimension. As the set of saturated subbundles of given rank and degree is irreducible (from 0.2), this is enough to show irreducibility.

It only remains to prove existence in case the Brill-Noether number is positive. If $r_F \neq r_E$, this is equivalent to the existence of a stable subbundle or quotient of E and is contained in 0.2. If $r_F = r_E$, one needs to check that the generic elementary transformation of a stable vector bundle is again stable. This is well known. □

Proposition 2.1. *Let E be a stable vector bundle obtained as a generic extension*

$$0 \to E' \to E \to E'' \to 0.$$

Assume that $0 < s = r'd - rd' \leq r'(r - r')(g - 1)$. Then the only subbundle of E of rank r' and degree d' is E'.

Proof. Assume that there were another subbundle F' of rank r' and degree d'. Denote by F'' the quotient sheaf E/F'.

Claim: If E is general, both F', F'' are generic vector bundles of the given ranks and degrees(i.e. as E varies, F', F'' vary in an open dense subset of the corresponding moduli spaces).

Proof of the claim: If F'' had torsion, then E would have a subbundle of higher degree and from 0.1 it could not be general. If F' or F'' were not general or were not stable, then they would move in varieties of dimension strictly smaller than those parametrising E', E''. From 1.1, $h^1(E''^* \otimes E') = h^1(F''^* \otimes F')$. From 1.3, every E appears in at most a finite number of extensions of an E'' by an E'. Hence, E could not be general. This proves the claim.

We obtain non-zero maps $F' \to E''$ and $E' \to F''$. From the genericity of E', E'' and 0.3 , the dimension of the sets of these F', F'' is at most

$$dim\{F'\} = r'(r' - r'')(g - 1) + r'd'' - r''d' = r'(r' - r'')(g - 1) + s$$

$$dim\{F''\} = r''(r'' - r')(g - 1) + r'd'' - r''d' = r''(r'' - r')(g - 1) + s$$

and both these numbers are positive. From 1.1 and the stability of E, $h^0(F''^* \otimes F') = 0$. Hence, from Riemmann-Roch

$$h^1(F''^* \otimes F') = r'r''(g - 1) - [r''d' - r'd''] = r'r''(g - 1) + s$$

We obtain then a bound for the dimension of the set of E for which there is an exact sequence

$$0 \to F' \to E \to F'' \to 0$$

given by

$$dim\{E\} \leq (r'^2 + r''^2 - r'r'')(g - 1) + 3s - 1$$

On the other hand, from 1.4,

$$dim\{E\} = (r'^2 + r''^2 + r'r''(g - 1) + s + 1$$

It follows then that $2r'r''(g - 1) \leq 2s - 2$. This contradicts our assumption on s. □

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