Local ADHM construction and holomorphic local vector bundles on the twistor space

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Abstract

Local ADHM theory has been discussed, after making some general remark about Penrose transform and methods of monad we construct holomorphic vector bundles on neighbourhooh of a projective line in the twistor space. By inverse Ward transformation this corresponds to local solution space of self dual Yang-Mills equation. In the final section we discuss some possible applications of this theorem.

1 Introduction

Suppose E is a Hermitian vector bundle over a compact Riemannian four manifold and E has a unitary connection ∇ whose curvature is fundamental form F, a two form with values in the endomorphism bundle of E i.e.

$$F \in \wedge^2(M) \otimes End(E).$$

The Riemannian metric allows us to decompose F into two components F_+ and F_-, due to the Hodge decomposition of $\wedge^2(M)$. The total energy of the field F is given by the Yang - Mills action

$$YM(F) = \int_M ||F||^2 d\mu = -\int_M tr(F \wedge^* F) d\mu.$$

The Euler-Lagrange equation for this action gives us the Yang - Mills equation

$$\nabla \wedge^* F = 0.$$

The conformal invariance of the Hodge star $*$ operator on $\wedge^2$ shows that Yang-Mills equations are conformally invariant in four dimensions. The quantity

$$\int_M tr(F \wedge F) = \int_M (||F_-||^2 - ||F_+||^2) d\mu$$

is a topological invariant of the bundle E, whose value is $8\pi^2 k$, where k is the characteristic number $c_2 - \frac{1}{2}c_1^2$. This action will be a minimum when either

$$F_+ = 0 \text{ i.e. } {}^*F = -F$$

or

$$F_- = 0 \text{ i.e. } {}^*F = F$$

depending on whether $k \geq 0$ or $k \leq 0$, such connections are called anti-instanton or instanton respectively. From the Bianchi identity, $\nabla \wedge F = 0$, one can readily see that instanton satisfy the Yang - Mills equation.

When G is $SU(2)$ and $k = 1$, the spherically symmetric solutions about the origin in $\mathbf{R}^4$ were discovered by Belavin et.al.[BPST]. For $k > 1$, this has been extended by t'Hooft [unpublished] and Jackiw et.al. [JNR]. These solutions can be imagined as superpositions of

k single instantons located at different points of $\mathbf{R}^4$ and the superpositions are achieved through some ansatz. But this ansatz failed to yield solutions for general k instantons. Penrose twistor theory ([At],[WW])provides a complete solution of the instanton problem for all classical groups.

The Penrose fibration (see[PR1],[PR2],[At],[BE],[WW]) $\pi : \mathbf{CP}^3 \longrightarrow \mathbf{S}^4$ tells us that each point of $\mathbf{S}^4$ corresponds to $\mathbf{CP}^1$ in $\mathbf{CP}^3$ and the anti-self dual (or self-dual) solution of the Yang-Mills equation in the conformally compactified Euclidean 4-space in $\mathbf{S}^4$ corresponds to certain global real algebraic bundles on the complex projective space $\mathbf{CP}^3$. The Atiyah-Ward correspondence ([At],[Wa1]) says, giving an $SU(2)$ anti instanton (solutions of anti self dual Yang-Mills equation) bundle on $\mathbf{S}^4$ is equivalent to giving a holomorphic rank 2 vector bundle $\mathcal{E}$ whose restriction to each projective line is trivial and carrying a suitable real structure. In a celebrated paper Atiyah et.al. [AHMD] have shown using Ward correspondence and algebro-geometric techniques 'methods of monads' introduced by Horrocks and Barth [OSS] that all instantons have a unique description in terms of linear algebra for any arbitrary compact classical group.

Soon after the discovery of (global) ADHM construction [AHMD] Hartshorne [Ha1] put forwarded a list of problems about the algebraic vector bundles on projective spaces. In that list he also stated the problem of local ADHM as the problem of understanding vector bundles on a tubular neighbourhood of a projective line in the twistor space $\mathbf{CP}^3$. As a hint he stated that this problem could also be tackled via Penrose transformations. The local problem is different from the global problem in a number of ways: for example one loses the second Chern class and the moduli space becomes infinite dimensional.

Earlier Witten [Wi1] studied this problem in two different approches. The first part of his paper heavily dependent on physics and it is difficult to follow. Of course he used mathematical techniques in the second part but in many ways this part is incomplete and incorrect, for example his twistor argument, the vector bundles of the monad etc. But we must admit the first part of the paper is correct although it is hard to follow. We present a correct mathematical proof in this chapter.

Our main result is:

Theorem .1 *Let E be a vector bundle defined locally on the neighbourhood of a projective line L in $\mathbf{CP}^3$ such that the bundle E is trivial when it is restricted to the line. Then bundle E is realized from the cohomology of following monad*

$$V(-1) \xrightarrow{a} W \xrightarrow{b} U(1)$$

where

$$V = H^1(\Omega^2(1)/H^1(E(-2))$$
$$W = H^1(E \otimes \Omega^1)$$
$$U = H^1(E(-1))$$

i.e $E = Ker\ b/\ Im\ a$

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2 Preliminaries

In this section we discuss some basic features of twistor geometry ([PR1],[PR2],[WW]) connected to our problem and some definitions regarding monads. So for convenience we split this section into two parts, first part is related to twistor theory and the second one deals with methods of monads.

2.1 Some features of twistor theory

The idea of twistor theory is quite old and goes back to the famous Plücker-Klein relationship [WW] where it describes the straight lines in $\mathbf{CP}^3$ by the points of a quadric hypersurface Q in $\mathbf{CP}^5$ In the Penrose twistor programme one uses the holomorphic geometry of the twistor space to produce solutions to differential equations. Recall the

Penrose fibration defined by $\pi : \mathbf{CP}^3 \longrightarrow \mathbf{S}^4$ with fibre $\pi^{-1}(x)$ at each point $x \in \mathbf{S}^4$ is $\mathbf{CP}^1$ which precisely gives the compatible complex structure in $T\mathbf{S}^4$. We can pull back an $SU(2)$ bundle $\tilde{E}$ on $\mathbf{S}^4$ by π to obtain an associated rank 2 bundle E on $\mathbf{CP}^3$. The connection ∇ on $\tilde{E}$ is anti self dual if and only if the pulled back connection determines a holomorphic structure on $E = \pi^*(\tilde{E})$ This is the basis of the Ward transformation [Wa]. A connection with anti self dual curvature on the original $SU(2)$ bundle gives an almost complex structure on E and the anti self duality condition provides the integrability condition needed for E to be a holomorphic rank 2 vector bundle on $\mathbf{CP}^3$ and since the bundle comes from the bundle over $\mathbf{S}^4$ it carries a real structure. The relevant anti-holomorphic involution is given by $k : \mathbf{CP}^3 \longrightarrow \mathbf{CP}^3$,

$$k(z_1, z_2, z_3, z_4) = (-\bar{z}_2, +\bar{z}_1, -\bar{z}_4, +\bar{z}_3)$$

where z_is are homogeneous coordinates on $\mathbf{CP}^3$. This map is conjugate linear in the sense that $k(\lambda \mathbf{z}) = \bar{\lambda} k(\mathbf{z})$ for any $\lambda \in \mathbf{C}$ and $z \in \mathbf{C}^4$. Each fibre $\pi^{-1}(x)$ is a k- invariant projective line and the restriction of the pullback bundle E on each real line $\pi^{-1}(x)$ is trivial. Also one can easily see that the induced automorphism of the space of lines can be realized as the complex conjugate of Plücker co-ordinates of the quadric, thus real points Q_R of the quadric Q correspond to real lines in $\mathbf{CP}^3$ (cf. [WW]).

So the holomorphic vector bundles E coming from instantons over $\mathbf{CP}^3$ have zero first Chern class (which is clear since the structure group is $SU(2)$ so $\det E$ is trivial) and the instanton number k is the second Chern class $c_2(E)$. We know from GAGA [Se] that all holomorphic bundles on the projective spaces have unique algebraic structures. Fixing $c_1 = 0$ and $c_2 = k$ (say) we can define the moduli space M_k of stable algebraic rank 2 vector bundles on $\mathbf{CP}^3$.The bundles coming from instantons have some characteristic features which we will discuss in the next part.

2.2 Methods of monads

There are two main ways of studying vector bundles over complex projective spaces. One is via curves and jumping line, the other is by monads [OSS]. The idea is twist the bundle E by $\mathcal{O}(n)$ such that the new bundle $E(n)$ has plenty of global sections. If s is a generic section then the set of points in $\mathbf{CP}^3$ where s becomes zero will be a curve, C, in $\mathbf{CP}^3$. With the given curve and some algebraic data and machinery one can recover E. The second method is the most successful and widely used technique.

A monad is a pair of maps of holomorphic vector bundle over a complex manifold $\mathcal{M}$

$$L(-1) \xrightarrow{a} M \xrightarrow{b} N(1)$$

such that a is injective and b is surjective and the composite map $ba = 0$ everywhere. The bundle $E = \mathrm{Ker}\ b/\mathrm{Im}\ a$ is the 'cohomology' of the monad. The word monad was used by Horrocks. The idea of this method is to construct complicated bundles from three simpler bundles L, M and N over the $\mathcal{M}$. The process of taking cohomology of a complex is in general functorial, so that two monads which are isomorphic (in the categorical sense) define isomorphic vector bundles.

In order to see how the connection arises from the monad we shall follow Donaldson [Do]. Let X be the two dimensional vector space underlying $\mathbf{P}(X) = \mathbf{P}^1$. Here "$a$" is an element of $X^* \otimes Hom(L, M)$ and "b" is an element of $X^* \otimes Hom(M, N)$. So the composite map will be $X^* \otimes X^* Hom(L, N)$. Since $ba = 0$ satisfies everywhere and this requirement allows us to say that this is a skew symmetric on X^*. When we impose the condition that the bundle E be holomorphically trivial on the projective line we obtain an isomorphism

$$\wedge^2 X^* \otimes Hom(L, N) \cong Hom(L, N).$$

Following Donaldson, this triviality condition can be re-expressed by choosing two distinct points m,n in the projective line. Thus we obtain four linear subspaces of the vector space M, given by

Ima_m	$kerb_m$
Ima_n	$kerb_n$

Elementary linear algebra shows that the restriction of E to is naturally isomorphic to the subspace

$$kerb_m \cap kerb_n \subset M$$

and also to the quotient,

$$M/Ima_m + Ima_n.$$

Above two description mean that the fibre of the associated bundle E' comes as a projective subspace of the fixed vector space M equipped with maps

$$E' \overset{i}{\rightleftharpoons} M$$

Now using these projective maps we know how to define define connection of a subbundle of a fixed vector space. Suppose M has the flat connection ∇ and we have a smooth bundle projection $\pi : M \longrightarrow E'$, which is a left inverse to the inclusion map i. Then we get an induced connection A on E' with covariant derivative:

$$\pi \circ \nabla \circ i(s).$$

Thus we get a connection on any bundle E associated to monad on the twistor space.

In principle the use of monads reduces the study of vector bundles to linear algebra. Once we obtain a vector bundle from the monad then the inverse Ward correspondence [Wa] gives the general ADHM description of all self dual gauge fields over $\mathbf{S}^4$.

When the bundles have some additional structure then this additional structure goes into the monad automatically [Ha2]. Let $\mathcal{E}$ be the sheaf of holomorphic (or algebraic) sections of $\pi^*(\tilde{E})$ over $\mathbf{CP}^3$. Suppose the coherent sheaves have following vanishing cohomologies

$$H^0(\mathcal{E}(m)) = 0 \text{ for } m \leq -1$$

$$H^1(\mathcal{E}(m)) = 0 \text{ for } m \leq -2$$

$$H^2(\mathcal{E}(m)) = 0 \text{ for } m \geq -2$$

$$H^3(\mathcal{E}(m)) = 0 \text{ for } m \geq -3$$

The coherent sheaves on $\mathbf{CP}^3$ with these property are called admissible sheaves(see [MD],[Ha1]) and the corresponding monad will be a special monad. There is a functorial equivalence between the category of special monads and the category of admissible sheaves. In order to prove the vanishing of this cohomologies it is suffices to show the vanishing of the first two cohomologies. The other two follow from the Serre duality,

$$H^i(\mathcal{E}(m))^* \cong H^{3-i}(\mathcal{E}^*(-4 - m))$$

The fibres of $\pi : \mathbf{CP}^3 \longrightarrow \mathbf{S}^4$ are the projective lines in $\mathbf{CP}^3$ and the restriction of $\mathcal{E}$ to them is holomorphically trivial and for that reason $H^0(CP^1, \mathcal{O}(m)) = 0$ when $m < 0$, hence $H^0(CP^1, \mathcal{E}(m)) = 0$.

Since in the entire calculation we have used the local version of Beilinson's spectral sequence ([Be],[OSS]). To keep the paper self concise we give the statement of the theorem without proof (for proof see [OSS]).

Theorem .2 (Beilinson) *(see 3.1.4 [OSS]) Let E be a m dimensional holomorphic bundle over the Zariski open subset U of $\mathbf{CP}^n$ then there exist a spectral sequence E_m^{pq} with E_1 - term*

$$E_1^{pq} = H^q(U, E \otimes \Omega^{-p}(-p)) \otimes \mathcal{O}_{P_n}(q)$$

which converges to

$$E^j = E \; for \; j = 0$$

and otherwise 0. This means that

$$E_\infty^{pq} = 0 \; for \; p + q \neq 0$$

and

$$\bigoplus_{p=0}^{n} E_\infty^{-p,p}$$

is the associated graded sheaf of a filtration of E.

Beilinson's work has enabled us to construct an inverse functor, i.e it helps us to construct monads from the admissible sheaves. Consider for example the global ADHM case [AHMD], the monads coming from instantons always have a special structure

$$A(-1) \xrightarrow{\alpha} B \xrightarrow{\beta} C(1)$$

where $A(-1) = A \otimes \mathcal{O}(-1)$ etc and A,B and C are three complex vector spaces. Barth observed that corresponding bundles $\mathcal{E}$ on $\mathbf{CP}^3$ with $c_1 = 0$ and $c_2 = k$ are stable and satisfies $H^1(\mathcal{E}(-2)) = 0$, using Penrose transform we can deduce that it is equivalent to the condition that there are no non-zero solutions of the equation.

$$(\Delta + \frac{1}{6}R)s = 0$$

has no global solutions. Here Δ denotes Laplace - Beltrami operator coupled to the connection, $R > 0$ is the positive scalar curvature of $\mathbf{S}^4$ and s is a section of $\tilde{E}$.

But in the local case this vanishing argument does not apply so the cohomology group $H^1(\mathcal{E}(-2)) \neq 0$ in the local case moreover this will appear in the vector spaces of the monad and since the bundle is supported on a non compact space we can't use Serre duality either. Instead of that we will use some techniques of several complex variable to deduce the vanishing of the cohomologies in the spectral sequence.

3 Construction of monads for local bundles

In this section we construct the monads of holomorphic bundles on a tubular neighbourhood of a projective line in $\mathbf{CP}^3$. It has been known that Penrose transformation deals with double fibration of a generalized flag variety [BE]. This transformation has been used in the local ADHM problem by localization at a point in $\mathbf{S}^4$ which corresponds to localization near a line in $\mathbf{CP}^3$. Let us recall the basic double fibration

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{f} & Q \\ \downarrow{\scriptstyle g} & & \\ \mathbf{CP}^3 & & \end{array}$$

where $\mathbf{F}$ is the flag variety and Q is the complexification of $\mathbf{S}^4$. As we choose the image variety of the transform a *Stein* subset [GR] of the complexification of $\mathbf{S}^4$.

Definition .3 *A closed subset in V of a complex space X is called Stein set (in X) if Cartan's "theorem B " holds good. This says for every coherent analytic sheaf Ξ*

$$H^q(V, \Xi) = 0 \textit{ for all } q \geq 1$$

is valid on V. A complex space which is itself a Stein set is called a Stein space.

Given a open subset of the complexification of $\mathbf{S}^4$ and with the help of the map f we can pull back this Stein set to flag variety. Let

S^a be the stein subset of Q. Suppose $\mathbf{F}^a = f^{-1}\mathbf{S}^a$ is the open subset of flag variety then by pushing down this open subset we obtain the corresponding open subset of the twistor space $\mathbf{P}^a$. So the basic double fibration induces a double fibration among the open subset

$$\begin{array}{ccc} \mathbf{F}^a & \xrightarrow{f} & \mathbf{S}^a \\ \downarrow g & & \\ \mathbf{P}^a & & \end{array}$$

Here $\mathbf{P}^a$ is the open subset of the twistor space and this can be covered by two Stein subsets cutting out the centre and two subsets thickenings at the lower and upper stratum of $\mathbf{P}^a$. In the case of Stein subset which is cut out from the centre, the first cohomology does not vanish.

Let Y_1 and Y_2 be the two open centrally cut out Stein subsets of $\mathbf{P}^a$ and let $\mathbf{P}^a = Y_1 \bigcup Y_2$ such that $H^1(Y_i, \Xi) \neq 0$.

Lemma .4 *Let X be a complex space and V_1 and V_2 be two Stein subspaces of X. Then $V_1 \cap V_2$ is Stein too.*

For proof one must consult Grauert and Remmert [GR]. □

Proposition .5 *Let $\mathbf{P}^a$ be the open subset of twistor space constructed above. Every coherent analytic sheaf $\mathcal{F}$ on $\mathbf{P}^a$ satisfies*

$$H^q(\mathbf{P}^a, \mathcal{F}) = 0$$

for all $q \geq 2$

Proof :: Since $\mathbf{P}^a = Y_1 \bigcup Y_2$ using Mayer-Vietoris sequence [BT] we obtain

$$\longrightarrow H^{q-1}(Y_1 \bigcap Y_2, \Xi) \longrightarrow H^q(\mathbf{P}^a, \Xi) \longrightarrow H^q(Y_1, \Xi) \oplus H^q(Y_2, \Xi) \longrightarrow \ldots$$

By the theorem B of H.Cartan we already know

$$H^q(Y_i, \Xi) = 0 \text{ for } q \geq 1$$

and

$$H^q(Y_1 \cap Y_2, \Xi) = 0$$

by lemma 4. So $H^q(\mathbf{P}^n, \Xi) = 0$ for any coherent sheaf Ξ when $q > 1$. This completes the proof. □

The result of this proposition will be used to establish the vanishing of relevant cohomology groups in the spectral sequence.

Now we follow Drinfeld and Manin's procedure in [MD] where they have given a nice procedure of constructing vector bundle using monad. Let Ω^1 denotes the cotangent bundle of $\mathbf{CP}^3$ and Ω^n denote the corresponding n-th exterior product of the cotangent bundle. We obtain following sheaf exact sequence

$$[T_{CP^3}(-1)]^\vee \longrightarrow \mathcal{O}_{CP^3} \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

Resolving into locally free modules we obtain the following Koszul complex in our case

$$\Omega^3(3) \longrightarrow \Omega^2(2) \longrightarrow \Omega^1(1) \longrightarrow \mathcal{O}_{CP^3} \longrightarrow \mathcal{O}_P \longrightarrow 0$$

Following Drinfeld-Manin [MD] we tensor the above sequence with an arbitrary vector bundle $E(-1)$ so we obtain following exact sequence from the Koszul complex

$$\Omega^3(2)\otimes E \longrightarrow \Omega^2(1)\otimes E \longrightarrow \Omega^1\otimes E \longrightarrow E(-1)|_{CP^3} \longrightarrow E(-1)|_P \longrightarrow 0$$

In order to extract the information of the bundle E we have to go the for spectral sequence developed by Beilinson [Be]. In our case the spectral sequence associated with the double complex would be the following one

$H^3(\Omega^3(2)\otimes E)$	$H^3(\Omega^2(1)\otimes E)$	$H^3(\Omega^1\otimes E)$	$H^3(E(-1))$
$H^2(\Omega^3(2)\otimes E)$	$H^2(\Omega^2(1)\otimes E)$	$H^2(\Omega^1\otimes E)$	$H^2(E(-1))$
$H^1(\Omega^3(2)\otimes E)$	$H^1(\Omega^2(1)\otimes E)$	$H^1(\Omega^1\otimes E)$	$H^1(E(-1))$
$H^0(\Omega^3(2)\otimes E)$	$H^0(\Omega^2(1)\otimes E)$	$H^0(\Omega^1\otimes E)$	$H^0(E(-1))$

The two cohomologies are related by the operators satisfying

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

such that $d_r^2 = 0$. When $r = 1$ we have the cohomology of the rows above.

Our strategy is to find the monad corresponding to this bundle E. This is possible provided sufficient number of cohomology groups are zero in the spectral sequence. The vanishing of higher cohomologies follow from our earlier result (proposition 5)

$$H^3(\Omega^3(2) \otimes E) = H^3(\Omega^2(1) \otimes E) = H^3(\Omega^1 \otimes E) = H^3(E(-1)) = 0$$

and

$$H^2(\Omega^3(2) \otimes E) = H^2(\Omega^2(1) \otimes E) = H^2(\Omega^1 \otimes E) = H^2(E(-1)) = 0$$

Therefore we conclude that the first two rows of the spectral sequence vanish identically. Our next task is to show the bottom most row also vanishes.

Lemma .6 *Let E be the bundle defined on the tubular neighbourhood of projective line. If it is trivial on the line then it satisfies $H^0(E(-k)) = 0$ for all $k > 0$.*

Proof :: This is a trivial case of Kodaira vanishing theorem [GH], hence we obtain

$$H^0(\mathcal{O}(-k)) = 0$$

for all $k > 0$. So the result follows immediately. □

Claim .7

$$H^0(\Omega^3(2) \otimes E) = H^0(\Omega^3(1) \otimes E) = H^0(\Omega^1 \otimes E) = H^0(E(-1)) = 0$$

Proof :: Restricting to a line the tangent bundle of the $\mathbf{CP}^3$ fits into the exact sequence

$$\mathcal{O}(2) \longrightarrow T_{CP^3}|_L \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1)$$

Hence we obtain the following splitting of the tangent bundle

$$T_{CP^3}|_L = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1).$$

Then the dual of this splitting will be the splitting of the cotangent bundle.

$$[T_{CP^3}]^\vee = \Omega^1 = \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$$
$$E \otimes \Omega^1|_L = E(-2) \oplus E(-1) \oplus E(-1)$$

So we conclude from the previous lemma

$$H^0(\Omega^1 \otimes E) = H^0(E(-1)) = 0$$

Since $\Omega^3 = \mathcal{O}(-4)$ i.e the canonical bundle of $\mathbf{CP^3}$ then the first cohomology group reduced to

$$H^0(\Omega^3(2) \otimes E) \cong H^0(E(-2)) = 0$$

Hence we obtain $H^0(\Omega^2(1) \otimes E) = 0$ from the spectral sequence. Then in the spectral sequence the whole 0-th row vanishes Thus we prove the lemma. □

Thus we have left with second row only which is expressed as follows

$$H^1(\Omega^3(2) \otimes E) \longrightarrow H^1(\Omega^2(1) \otimes E) \longrightarrow H^1(\Omega^1 \otimes E) \longrightarrow H^1(E(-1))$$

Observe that the first element in the sequence is $H^1(\Omega^3(2) \otimes E) = H^1(E(-2))$, since Ω^3 stands for canonical line bundle of $\mathbf{CP}^3$ and hence $\Omega^3 = \mathcal{O}(-4)$. This element vanishes identically in the case of bundle over $\mathbf{S}^4$, but in the local case it contributes to the monad. This sequence of four vector spaces can be easily transformed in to standard monad, i.e a pair of morphism and three vector spaces. The monad of the local bundle E is

$$[H^1(\Omega^2(1) \otimes E)/H^1(E(-2))] \otimes \mathcal{O}(-1) \xrightarrow{a} H^1(\Omega^1 \otimes E) \xrightarrow{b} [H^1(E(-1))] \otimes \mathcal{O}(1),$$

where a and b are two morphisms and the bundle is recovered from the cohomology of the monad.

Remark .8 *If we compare our chapter with the earlier paper of Witten [Wi1](he attempted this calculation for Minkowski space time) we find the following replacements* (1) *the first vector space is the quotient space* $H^1(\Omega^2(1)\otimes E)/H^1(E(-2))$ *not the space* $H^1(\Omega^2(1)\otimes E)$ *which was found by Witten. (2) Witten used long exact sequence which is wrong instead of Beilinson's spectral sequence. Moreover he didn't show explicitly why the cohomologies were vanished. (3) Moreover, we want to point out that unlike in the global case, the vector space* A *is not dual to* C *in the local case.*

Putting all the result concerning monad and local vector bundle together we obtain our main theorem.

4 Identification of the cohomologies

In this section we will identify the vector spaces appearing in the monads. In the local case all the vector spaces forming the monads are infinite dimensional vector spaces. They are the solutions of the three auxiliary equations as Witten showed. He showed in the first half of his paper that two of the vector spaces are the solutions of Dirac equations and other one is the solution of some scalar equation.

In order to see this in detail we must apply the Penrose transform. In this section we will demonstrate how to obtain the information about $H^1(\Omega^1)$.

This approach is based on local twistor theory as shown by Lionel Mason. Let us consider the Euler sequence on the twistor space:

$$0 \longrightarrow \overset{\times Z^\alpha}{\longrightarrow} \mathcal{O}^\alpha(1) \longrightarrow T \longrightarrow 0$$

One can regard Z^α is the tautological section. Dualizing the above sequence, we obtain

$$0 \longrightarrow \Omega^1 \longrightarrow \mathcal{O}_\alpha(-1) \overset{\times Z^\alpha}{\longrightarrow} \mathcal{O} \longrightarrow 0.$$

So from the long exact sequence we obtain

$$0 \longrightarrow H^0(\mathbf{P}^a,\mathcal{O}) \longrightarrow H^1(\mathbf{P}^a,\Omega^1) \longrightarrow H^1(\mathbf{P}^a,\mathcal{O}_\alpha(-1)) \longrightarrow H^1(\mathbf{P}^a,\mathcal{O})$$

Penrose transformation of $H^1(\mathbf{P}^a,\mathcal{O}_\alpha(-1))$ satisfies

$$\nabla^{B}_{B'}\phi_{B\alpha} = 0$$

where ∇ is the spin connection and α is the twistor index.

The definition of the local twistor and their construction then give us that $\phi_{B\alpha}$ is equivalent to a pair of fields $\xi_{BA'}$, η_{BA} and these are the sections of $\mathcal{O}_{BA'}$ and $\mathcal{O}_{BA}[-1]$ respectively. These satisfy

$$\nabla^{B}_{B'}\xi_{BA'} - i\epsilon_{B'A'}\eta^{B}_{B} = 0$$

$$\nabla^{B}_{B'}\eta_{BA} = 0$$

This tells us $\xi_{BA'}$ is the potential for the left handed Maxwell field.

$$0 \longrightarrow H^0(\mathbf{P}^a, \mathcal{O}) \longrightarrow H^1(\mathbf{P}^a, \Omega^1) \longrightarrow H^1(\mathbf{P}^a, \mathcal{O}_\alpha(-1)) \longrightarrow H^1(\mathbf{P}^a, \mathcal{O}).$$

Then $H^1(\mathbf{P}^a, \mathcal{O})$ is isomorphic to potentials modulo gauge for such fields.

We are interested

$$0 \longrightarrow H^0(\mathbf{P}^a, \mathcal{O}) \longrightarrow H^1(\mathbf{P}^a, \Omega^1) \longrightarrow H^1(\mathbf{P}^a, \mathcal{O}_\alpha(-1)) \longrightarrow H^1(\mathbf{P}^a, \mathcal{O})$$

$$0 \longrightarrow H^1(\mathbf{P}^a, \Omega^1)/H^0(\mathbf{P}^a, \mathcal{O}) \longrightarrow H^1(\mathbf{P}^a, \mathcal{O}_\alpha(-1)) \longrightarrow H^1(\mathbf{P}^a, \mathcal{O}).$$

We want to seek the kernel of the map $H^1(\mathbf{P}^a, \mathcal{O}_\alpha(-1)) \longrightarrow H^1(\mathbf{P}^a, \mathcal{O})$.

$$\xi_{BA'} = \nabla_{BA'} f$$

for some function f. If we write

$$\eta_{BA} = \rho_{BA} + \tau\epsilon_{BA}$$

So we obtain

$$\nabla^{B}_{B'}\rho_{BA} + \nabla_{B'A}\tau = 0$$

$$\Delta f + 4i\tau = 0$$

Then applying once again Δ we obtain

$$\Delta^2 f = 0.$$

Similarly the cohomology groups have been identified by Lionel Mason and Mike Singer [MS] which are nothing but the solutions of some differential equation as predicted by Witten [Wi1].

Hence we can identify the monad with the Witten complex [Wi1].

$$\bar{\Gamma}(E \otimes S^+)/\bar{\Gamma}(E) \longrightarrow \tilde{\Gamma}(E) \longrightarrow \bar{\Gamma}(E \otimes S^-)$$

Where $\bar{\Gamma}(E \otimes S^+)/\bar{\Gamma}(E)$: space of harmonic section to the section Dirac equation.
$\tilde{\Gamma}(E)$: space of solutions of some scalar equation.
$\bar{\Gamma}(E \otimes S^-)$: space of solutions of Dirac equation (opposite parity)

Remark .9 *The first part of the Witten's paper completely agrees with our result.*

Now we are in the position to lay out explicitly the local ADHM theorem. First we must define the data of local ADHM construction which have already gathered from the last two sections.

Data (Local) :: (1) Three infinite dimensional vector spaces A, B and C where A and C are the solution spaces of Dirac equations and B is the solution spaces of scalar equations.
(2)D be another vector space formed by the solution of the Laplace equation on $\mathbf{S}^a$.
(3) The quotient space A/D, solutions of Dirac equation modulo the harmonic solution.
(4) Two linear maps a and b , where $a : A/D \longrightarrow B$ is an injective map and $b : B \longrightarrow C$ is the surjective map and these give us a structure monad. These maps are linear over the complex projective space.
(5) The cohomology of the monad or the quotient space Im a / Ker b gives the bundle from the local monad.

Please note that the equivalence classes of monad means equivalence classes of ADHM (local) data and this give rise to equivalent classes of local vector bundles on the neighbourhood of a line.

Theorem .10 *There exist a one to one correspondence between (a) equivalence classes of local ADHM data or the equivalent classes of local vector bundles on the formal completion of the projective line on* $\mathbf{CP}^3$ *(b) gauge equivalence classes of local solutions of self-dual Yang-Mills equation.*

5 Applications , discussions and open problem

In this section we have attempted to show some applications of local ADHM, particularly from the point of view of reduction of self dual Yang-Mills equation. At the end of this section we have focused on some of the interesting problem concerning local vector bundles.

During the last few years Ward, Mason (see for example [Wa2],[MS]) and others have shown that many integrable systems particularly in 1+1 dimensions are symmetry reductions of self-dual Yang-Mills equation. The motivation of these 'phenomenological' works show that it could be possible to view the self-dual Yang-Mills equation is the universal integrable system. But it is to early to say since so far mathematical physicists have failed to show famous equations like KP or Davey-Stewartson are the reductions of the self-dual Yang-Mills equation. But it would be rather interesting to know how the geometry of self dual Yang - Mills equation is related to the geometry of the reduction equations. Here we picked up KdV as an example to show how its geometry fits with local ADHM construction. We choose to work on $\mathbf{R}^4$ with coordinates (x, y, z, t) and the metric

$$ds^2 = dx^2 - dy^2 - 4dzdt$$

The Yang-Mills connection $D := \partial - A$ where A takes values in the Lie algebra of $SL(2, C)$ and these are defined upto gauge tranformation

$$A \longrightarrow hAh^{-1} - (\partial h)h^{-1}$$

Following Belavin and Zakharov[BZ], the self-duality conditions becomes

$$[D_x - D_y, D_t] = 0$$

$$[D_x + D_y, D_t] = 0$$

$$[D_x - D_y, D_x + D_y] + [D_z, D_t] = 0$$

Then performing two dimensional reduction, one null and the other timelike and by imposing gauge fixing condition, Mason-Sparling [MS] and Bakas-Depireux [BD]showed that SDYM equation reduces to KdV equation.

We have already encountered one dimensional reduction (here only one non null translation symmetry along ∂_y is imposed) in the case of Bogomolny equation in $\mathbf{R}^3$ where Hitchin[Hi1] and Nahm [Na] have shown this is equivalent in $\mathbf{R}^4$, which is in addition invariant under the action of the additive group $\mathbf{R}$ of translation in the z direction. By means of twistor correspondances Hitchin showed that the $SU(2)$ Bogomolny equation on $\mathbf{R}^3$ corresponds to a holomorphic rank 2 vector bundle E on $T\mathbf{P}_1$ which is quaternionic and trivial on every real section of $\pi : T\mathbf{P}_1 \longrightarrow \mathbf{P}_1$.

In the KdV case we have gone one step further, KdV in $\mathbf{R}^2$ is equivalent to a solution of the self duality equation in $\mathbf{R}^4$ which is in addition invariant under the action of the additive group $\mathbf{R}+\mathbf{R}$ of translation which is a pair of orthogonal space time translation one timelike and one null direction. On top of that, it satisfies some gauge fixing conditions which we have listed below.

Proposition .11 *If we reduce the self-dual Yang-Mills equation by the pair of two orthogonal Killing vectors (one is space like and other is timelike) ∂_y and ∂_z and fixing the gauge $A_z = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ and $A_x + A_y = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}$ and $A_x - A_y = \begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix}$ we obtain the KdV equation as the reduction of self-dual Yang-Mills equation.*

Let us call this data as a reduction data. Recall the monad of the local vector bundle

$$H^1(\Omega^2(1)\otimes E)/H^1(E(-2)) \xrightarrow{a} H^1(\Omega^1\otimes E) \xrightarrow{b} H^1(E(-1))$$

and the morphisms a and b are linear over the projective space. Now in the reduced case these morphisms must be two translation invariant and the corresponding vector bundle is also two translational invariant. As Mason and Sparling [MS] showed, a solution of $SU(2)$ KdV equation on $\mathbf{R}^2$ corresponds to a holomorphic rank 2 holomorphic vector bundles over $T\mathbf{P}^1$ on which we have the action of an additional symmetry, corresponds to extra symmetry.

In the reduction case, one important point should be noted which tells us not every two transltion invariant solution of self dual Yang-Mills equation are the solutions of KdV, since we have imposed a null translation along ∂_z and the gauge fixing in the same direction. This is finiteness condition which is similar to what Hitchin [Hi3] showed in the harmonic case.

There are some open problems in the case of local vector bundle. As Hartshorne [Ha1] pointed out, a global bundle on $\mathbf{CP}^3$ is determined by its restriction to the formal neighbourhood of a projective line so the local problem gives us a new perspective on the global problem.

There is another celebrated problem in the gauge theory that is also a local problem. This is the construction of vector bundles from the full fledge Yang-Mills sheaves (see [HM],[IYG],[Wi2]) in the neighbourhood of some $\mathbf{CP}^1 \times \mathbf{CP}^1$ inside the hypersurface lying inside $\mathbf{CP}^3 \times \mathbf{CP}^3$.

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