Stable maps and Quot schemes

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Contents

1. Introduction

Let C be a smooth projective curve of genus g over an algebraically closed field of arbitrary characteristic, and E a vector bundle on C of rank r. The main theme of this paper is the relationship between two different compactifications of the space of vector bundle quotients of E of fixed rank and degree. The first is Grothendieck's Quot scheme $\mathrm{Quot}_{k,d}(E)$ of coherent quotients of rank k and degree d, while the second is a compactification $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ via stable maps into the relative Grassmannian $\mathbb{G}(E,k)$ of k-dimensional quotient spaces of E. Quot schemes have played a central role in the "classical" study of vector bundles on curves, in particular for constructing moduli spaces (see e.g. [23]) and for understanding their geometry (see [11]). In some general sense, any construction or computation in this area depends on understanding geometric or dimensional properties of (subsets of) Quot schemes.

Before the availability of stable maps, Quot schemes have also been used in Gromov-Witten theory as a way to compactify the space of maps into Grassmannians and thereby compute intersection numbers (most notably in

[3], [4], [5] and subsequent extensions [8], [9], [10], [15]). The introduction by Kontsevich of the moduli space of stable maps now gives an alternate setting for these calculations; however in this paper we follow in some respect an orthogonal direction by showing how an understanding of the stable map space leads to good results about Quot schemes. For instance, we establish an essentially optimal upper bound on the dimension of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$, which gives as an immediate consequence an identical upper bound on the dimension of $\mathrm{Quot}_{k,d}(E)$. We prove that for every vector bundle E, $\mathrm{Quot}_{k,d}(E)$ is irreducible and generically smooth for d sufficiently large, and we precisely describe all the vector bundles for which the same thing holds for $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$. We show that, despite their similar behaviour, there are in general no natural morphisms between these two compactifications. Finally, as an application of the dimension bound, we obtain new cases of a conjecture on the effective base point freeness of linear series on moduli spaces of vector bundles on C.

We now turn to a more detailed discussion of the results of the paper.

In recent literature there has been some interest in finding good dimension bounds for families of sheaves or vector bundles on curves, most notably for Quot schemes (see e.g. [7], [14], [21] or [24]). Our first goal in this paper is to give a good answer to this question for an arbitrary vector bundle E. We do this by considering the second compactification of the space of quotient bundles of E of rank k and degree d, namely the Kontsevich moduli space $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ of stable maps of arithmetic genus g to the relative Grassmann bundle $\mathbb{G}(E,k)$ over C. Here $\beta_d\in H_2(\mathbb{G}(E,k),\mathbb{Z})$ denotes the homology class of a section corresponding to such a quotient of E. Set

$$d_k=d_k(E):=\min_{\mathrm{rk}\,F=k}\{\deg F\mid F\text{ a quotient of }E\},$$

i.e. the minimal degree of a quotient bundle of rank k. Our first main result is:

Theorem.

$$\dim\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)\le k(r-k)+(d-d_k)r,\ \textit{for all}\ d\ge d_k.$$

The advantage of using the stable map space comes from the possibility of applying methods characteristic to the study of families of curves. The idea is that once the space of maps is large enough, the Bend and Break Lemma of Mori ensures the existence of degenerate curves related to quotient bundles of smaller degree, and this gives an inductive bound. We expect that this approach might be useful with other fibrations over curves.

Going back to Quot schemes, the theorem above implies almost immediately the analogous statement:

Theorem.

$$\dim \mathrm{Quot}_{k,d}(E) \leq k(r-k) + (d-d_k)r, \ \text{for all } d \geq d_k.$$

The case $d = d_k$ of this result was first proved by Mukai-Sakai [19] and was generalized to the setting of G-bundles by Holla-Narasimhan [14]. The general statement improves a similar result found in [21] by different methods.

We next look at some examples showing certain aspects of the behaviour of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ and $\mathrm{Quot}_{k,d}(E)$. These show that the bounds above are essentially optimal if we are not willing to impose extra conditions on the bundle E (and close to optimal even if we are). An illustration of this is the fact that the difference between our upper bound and the usual deformation theoretic lower bound $rd - ke - k(r-k)(g-1)$ gives:

$$rd_k - ke \leq k(r-k)g,$$

which is precisely a theorem of Lange and Mukai-Sakai generalizing a well known result of Segre and Nagata on minimal sections of ruled surfaces (see §5).

The dimension estimates also allow us to understand the component structure of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ and $\mathrm{Quot}_{k,d}(E)$. For an arbitrary vector bundle, the picture turns out to be as nice as one could possibly hope for.

Theorem. *For all large d there is a unique component of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ whose generic point corresponds to a smooth section (i.e. a vector bundle quotient); this component is of dimension $rd - ke - k(r-k)(g-1)$.*

Our main result in this part of the paper gives an even stronger statement in the case of Quot schemes:

Theorem. *For any vector bundle E on C, there is an integer $d_Q = d_Q(E,k)$ such that for all $d \geq d_Q$, $\mathrm{Quot}_{k,d}(E)$ is irreducible. For any such d, $\mathrm{Quot}_{k,d}(E)$*

is generically smooth, has dimension $rd-ke-k(r-k)(g-1)$, and its generic point corresponds to a vector bundle quotient.

In the case when E is the trivial bundle (and over the complex numbers), this result was proved by Bertram-Daskalopoulos-Wentworth (in [5] Theorem 4.8), and it was used to produce an effective way of computing Gromov-Witten invariants on the space of holomorphic maps into Grassmannians.

The proofs of the two theorems rely on the estimates above and on some vanishing results of independent interest, involving stable quotients of arbitrary vector bundles. These are quite elementary, but seem to have been overlooked in the literature. Note also that an immediate consequence of the theorem is that every vector bundle can be written as an extension of generic stable bundles (see Corollary 6.3).

The next natural question to ask is if it is possible to give a precise description of all the bundles for which the spaces $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ are irreducible for all k and all large d. It turns out that, in contrast with the Quot scheme situation, they are all stable bundles of a very special kind. It is shown in [24] that there exists a nonempty open subset of the moduli space $U_C^s(r,e)$ of stable bundles of rank r and degree e whose elements E satisfy the properties that $d_k(E)$ is the smallest value making the expression $d_k r-ke-k(r-k)(g-1)$ nonnegative, and that dim $\mathrm{Quot}_{k,d}(E)=rd-ke-k(r-k)(g-1)$ (this also applies to $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$). This subset is contained in the better known open subset of Lange generic stable bundles (for precise details see Example 5.4). Our result is:

Theorem. *For any vector bundle E the spaces $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ are connected for all large d. They are irreducible for all k and all large d if and only if E is stable bundle, generic in the above sense.*

For stable bundles the invariants involved in the proof can be controlled, and in fact a d which guarantees irreducibility can be expressed effectively. This is the content of Theorem 6.8. A number of other effective statements are given at the end of §6.

Our methods show that in most situations it is more convenient to work with stable maps with reducible domain rather than quotients with torsion. In the hope for a best possible picture, we can ask whether there is always a morphism $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)\to \mathrm{Quot}_{k,d}(E)$ extending the identification on the locus corresponding to vector bundle quotients (or, equivalently, smooth sections). We prove that in general this question has a negative answer, and so a relationship between the two compactifications, if any, has to be more indirect. More precisely:

Theorem.
(a) *If* $k = (r-1)$ *then there is a surjective morphism from* $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ *to* $\mathrm{Quot}_{k,d}(E)$ *which extends the map on the locus where the domain curve is smooth. Such a morphism also exists for any* k *if* $d = d_k$ *or* $d = d_k + 1$*.*
(b) *If* $k \neq (r-1)$ *then in general there is no morphism from* $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ *to* $\mathrm{Quot}_{k,d}(E)$ *extending the map on smooth curves.*

Our final application of the dimension bound dicussed above is to derive bounds on base point freeness for linear series on the moduli spaces of vector bundles on curves. The fact that uniform (i.e., independent of special conditions on E) estimates of this type produce base point freeness bounds is explained in [21] §4. The bound here allows us to show that the linear series $|\mathcal{L}^p|$ is base point free on $SU_C(r)$ for $p \geq [\frac{r^2}{4}]$, where $SU_C(r)$ is the moduli space of semistable bundles of rank r with trivial determinant and $\mathcal{L}$ is the determinant line bundle. This can then be used to show a few new cases of the Conjecture 5.5 in [21] on optimal effective base point freeness on the moduli space $U_C(r,0)$ of semistable bundles of rank r and degree 0, generalizing the classical case of Jacobians. Namely, if Θ_N is the generalized theta divisor on $U_C(r,0)$ associated to a line bundle $N \in \mathrm{Pic}^{g-1}(X)$, then we have:

Theorem.

$$|k\Theta_N| \textit{ is base point free on } U_C(r,0) \textit{ for } k \geq r+1 \textit{ if } r \leq 5.$$

The paper is organized as follows. In §2 we describe the basic setup for moduli spaces of stable maps in the case of relative Grassmannians. The main dimension estimate for $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ is proved in §3. In §4 we show the analogous bound for Quot schemes. In §5 we give a series of examples leading to a discussion of the optimality of the bound. A study of the irreducibility of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ and $\mathrm{Quot}_{k,d}(E)$ appears in §6, and in §7 we show that there are in general no natural morphisms between the two compactifications, except in the case $k = (r-1)$. We conclude in §8 by using the dimension estimate to prove the result on base point freeness on moduli spaces.

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2. Basic Setup

Let C be a smooth projective curve of genus g . Suppose that E is a vector bundle of rank r and degree e on C. For any k, $1 \leq k < r$, let $\mathbb{G}(E, k)$ be the relative Grassmannian of k-dimensional quotient spaces of E; the fibre of $\mathbb{G}(E, k)$ over a point p of C is the Grassmannian $\mathbb{G}(E_p, k)$ of k-dimensional quotient spaces of E_p. Let π be the projection

$$\pi : \mathbb{G}(E, k) \longrightarrow C.$$

The quotient bundles F of E of rank k are in one to one correspondence with sections σ_F of π. We want to compactify the space of quotient bundles by compactifying the space of sections. One thing we will do to begin with is see how the data of the section σ_F determines the only other piece of numerical data about F, its degree.

Let T_{V} be the vertical tangent bundle relative to the map π. Also, for any quotient bundle F, let S_F be the induced subbundle of E:

$$0 \longrightarrow S_F \longrightarrow E \longrightarrow F \longrightarrow 0.$$

The pullback $\sigma_F^* T_{\mathrm{V}}$ is the bundle $\mathcal{H}om(S_F, F)$ on C. If F has degree d, then $\sigma_F^* T_V \cong S_F^* \otimes F$ has degree $(rd - ke)$. This shows that:

$$d = \deg(F) = \frac{\deg(\sigma_F^* T_{\mathrm{V}}) + ke}{r}.$$

Alternatively, if β_F is the homology class $\sigma_{F*}[C] \in H_2(\mathbb{G}(E, k), \mathbb{Z})$ of the section, then we can also write this as:

$$d = \deg(F) = \frac{c_1(T_{\mathrm{V}}) \cdot \beta_F + ke}{r}.$$

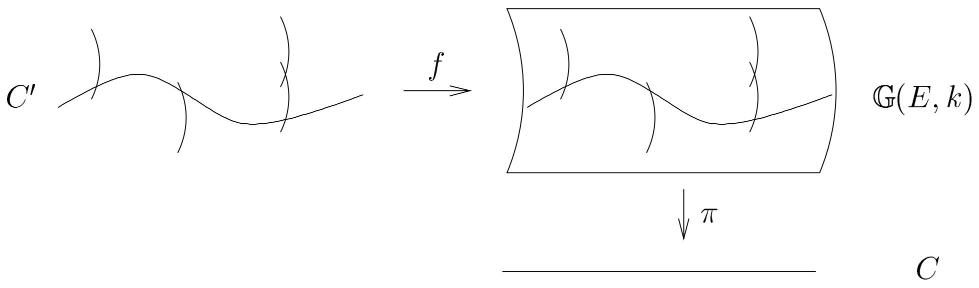
The second cohomology group $H^2(\mathbb{G}(E,k),\mathbb{Q})$ is generated (over $\mathbb{Q}$) by $c_1(T_{\mathrm{V}})$ and the class of a fibre of π. The class of any curve β in $H_2(\mathbb{G}(E,k),\mathbb{Q})$ is therefore determined by the intersection of β with these two classes. Any section of π will have intersection number one with the class of a fibre, and we have seen above that the degree of the quotient bundle F determines the intersection with $c_1(T_{\mathrm{V}})$ for the corresponding section class β_F. The conclusion is that if we fix the degree of the quotient bundle d, then there is a *unique* class β_d such that $\beta_d = \sigma_{F*}[C]$ for all sections σ_F corresponding to quotient bundles F of degree d.

From the point of view of sections of the relative Grassmannian, one natural way to compactify the space of quotient bundles of degree d is to look at the Kontsevich space of stable maps $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$.

A point of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ corresponds to the data of a stable map (C',f), where C' is a complete connected curve of arithmetic genus g, with at most nodes as singularities, and $f : C' \to \mathbb{G}(E,k)$ is a map with $f_*[C'] = \beta_d$. The map is *stable* in the following sense (see e.g. [12] 1.1):
– if a rational component of C' is contracted by f, then it must contain at least three points of intersection with other components
– if a component of arithmetic genus 1 is contracted by f, then it must contain at least one point of intersection with other components (this will not occur in our situation).

For more details about spaces of stable maps, see [12] or [13] 2.E, and for a construction in characteristic p see [1].

The fact that β_d is the class of a section, and that C' is of the same arithmetic genus as C puts strong conditions on C' and on the map f. First of all there must be one component, C'_0, of C' mapping isomorphically to C under the composition $\pi \circ f$. The condition on the genus now means that the other components of C' can only have genus 0, forming trees of rational curves hanging off of this distinguished component. The map f must send C'_0 to a section over C, and map the rational tails into the Grassmannians $\mathbb{G}(E_{p_i},k)$ over various points p_i of C. Also, *it cannot completely collapse any tree of rational curves* (by the stability condition) but it can map a rational component multiply onto its image. A basic picture of one of these maps is shown below.



The previous description is slightly more confusing than it needs to be. Nodes connecting two rational curves can always be smoothed, owing to the fact that the tangent bundle to the Grassmannian is generated by global sections. This means that the general point of any component of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ consists of a section C_0' mapping isomorphically to C, along with a certain number of rational tails, each one hanging off of C_0'. None of these tails are collapsed under the map to the relative Grassmannian, and if the fibres of the relative Grassmannian have dimension greater than two the map will be an immersion at the general point of the component.

On each $\mathbb{G}(E_{p_i},k)$ above a point p_i of C, the vertical tangent bundle T_{V} restricts to the tangent bundle of this Grassmannian. The determinant line bundle of this tangent bundle on the fibre is *ample* (more precisely, it is r times the line bundle corresponding to the Plücker embedding). If C_j' is any rational component of C', this shows that $f_*[C_j']\cdot c_1(T_{\mathrm{V}})$ is a *positive* multiple of r. Since $f_*[C']\cdot c_1(T_{\mathrm{V}}) = (rd-ke)$, this means that

$$f_*[C_0']\cdot c_1(T_{\mathrm{V}}) \leq (r(d-1)-ke),$$

and so the quotient bundle F_0 represented by the section C_0' is of degree *less* than d.

As a conclusion, from the discussion above we get the following useful

Observation: The boundary points of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ are constructed from sections σ_{F_0} corresponding to quotient bundles of degree *strictly smaller* than d, along with attached rational tails which function to increase the total degree of intersection with $c_1(T_{\mathrm{V}})$.

Remark. The terminology that a stable map with reducible domain is a *boundary point* is standard but potentially slightly misleading. It is possible (and common) that there are components of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ such that every point of the component is a boundary point.

For later reference, we end this section by recalling the existence of the more general moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{G}(E,k),\beta_d)$ of stable maps with n marked points (see [12] §1). This comes with a natural forgetful morphism to $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$.

3. Dimension Estimates for Spaces of Stable Maps

For each fixed rank k, there is a lower bound on the degrees d of possible quotient bundles F of that rank. For instance, it is a simple consequence of Riemann-Roch that $d \geq (e+(r-k)(1-g)-h^0(C,E))$; if F were a quotient bundle of smaller degree, then the kernel S_F of the map to F would have at least a

$$h^0(C,S_F) \geq (e-d)+(r-k)(g-1) > h^0(C,E)$$

dimensional space of sections, which is a contradiction.

For each rank k, let

$$d_k = d_k(E) := \min_{\mathrm{rk}F=k} \{\deg F \mid F \text{ a quotient of } E\}.$$

This number will depend on the vector bundle E, and is lower semi-continuous in families of vector bundles.

The purpose of this section is to prove the dimension estimate:

Theorem 3.1

$$\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d) \leq k(r-k)+(d-d_k)r, \textit{ for all } d \geq d_k.$$

This estimate includes those components of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ whose general points do not correspond to quotient bundles. For these the inequality is actually strict, as the proof will show.

The method of obtaining the bound is very simple. We will show how the *bend and break* lemma of Mori guarantees that once any component of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ has dimension strictly greater than $k(r-k)$, there is a divisor in that component corresponding to reducible maps. From the previous analysis, these correspond to quotients of lower degree, and this will inductively give us the dimension bound. This application of bend and break is the content of the following lemma.

Lemma 3.1 *If X is a component of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ whose generic point corresponds to a map with irreducible domain, and* $\dim X > k(r-k)$*, then there is a nonempty divisor $Y \subset X$ consisting of reducible maps.*

Before proving the lemma, let us use it to prove the dimension bound.

Proof of Theorem 3.1 (using lemma): First of all, suppose that $d = d_k$. Every point of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d_k})$ corresponds to an irreducible map – since a reducible map would give a quotient bundle of strictly smaller degree. The lemma then shows that no component of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d_k})$ can have dimension greater than $k(r-k)$ (the key is that Y is nonempty). This gives the base case for the induction.

For $d > d_k$, suppose that X is a component of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ satisfying the conditions of the lemma. The divisor Y produced by the lemma consists of reducible maps. Let Y' be a component of Y of the same dimension. By the discussion in the previous section, we can assume that the generic point of Y' consists of a reducible map maps where the component C_0' corresponds to a quotient bundle of degree $(d-\delta)$, along with a rational component which makes up the difference when intersected with $c_1(T_V)$; i.e., if C_1' is the rational tail, then $f_*[C_1']\cdot c_1(T_V) = r\delta$.

This set of reducible maps can be built up in the usual way by using smaller spaces of stable maps, and this will give us the inductive dimension bound.

Let α_δ be the homology class in $H_2(\mathbb{G}(E,k),\mathbb{Z})$ corresponding to curves of the same type as these rational tails – these are curves contained in a fibre of the map $\pi : \mathbb{G}(E,k) \longrightarrow C$, and which have intersection $r\delta$ with the vertical tangent bundle. The space of maps $\overline{\mathcal{M}}_{0,1}(\mathbb{G}(E,k),\alpha_\delta)$ of rational tails with a marked point is of dimension $r\delta + k(r-k) - 1$.

The component Y' of Y is in the image of

$$\overline{\mathcal{M}}_{g,1}(\mathbb{G}(E,k),\beta_{d-\delta}) \times_{\mathbb{G}(E,k)} \overline{\mathcal{M}}_{0,1}(\mathbb{G}(E,k),\alpha_\delta) \longrightarrow \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$$

where we glue together the maps whose marked points are sent to the same point of $\mathbb{G}(E,k)$. This fibre product has dimension

$$\begin{aligned}
&\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d-\delta}) + 1 + \dim \overline{\mathcal{M}}_{0,1}(\mathbb{G}(E,k),\alpha_\delta) - \dim \mathbb{G}(E,k) \\
&= \dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d-\delta}) + 1 + r\delta + k(r-k) - 1 - k(r-k) - 1 \\
&= \dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d-\delta}) + r\delta - 1.
\end{aligned}$$

Since Y' is in the image of this map, the dimension of Y' is less than or equal to the dimension above. Hence:

$$\dim X = \dim Y' + 1 \le \dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d-\delta}) + r\delta.$$

Using the inductive dimension bound for the dimension of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d-\delta})$ this gives us

$$\dim X \le k(r-k) + (d-\delta-d_k)r + r\delta = k(r-k) + (d-d_k)r,$$

which is exactly what we wanted.

This shows why the lemma gives us the dimension bound, at least as long as we consider components X of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ whose generic map is irreducible. In the case that X is a component whose generic point *does* correspond to a reducible map, we can express X (as above) as being in the image of a product of spaces parametrizing rational tails, and a space $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d'})$ where d' is of smaller degree. If the generic point of X consists of a map with l rational tails, then the same kind of argument as above shows that

$$\dim X \le k(r-k) + (d-d_k)r - l.$$

This finishes the proof of the theorem using Lemma 3.1 ■.

The only step remaining is to prove the lemma.

Proof of Lemma: Let X be a component satisfying the conditions of the lemma. It is easier to pass to the space $\overline{\mathcal{M}}_{g,1}(\mathbb{G}(E,k),\beta_d)$ with a single marked point; let $\widetilde{X}$ be the inverse image of X under the forgetful map $\overline{\mathcal{M}}_{g,1}(\mathbb{G}(E,k),\beta_d) \longrightarrow \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$. It is sufficient to prove that there is a nonempty divisor $\widetilde{Y}$ in $\widetilde{X}$ consisting of reducible maps. (In fact it would be sufficient to prove that this component has at least one point corresponding to a reducible map, since the locus of nodal curves in a family is always of codimension at least one, provided it is nonempty.)

The advantage of passing to the space with a marked point is that we have an evaluation map $ev : \widetilde{X} \longrightarrow \mathbb{G}(E,k)$. By hypothesis, $\widetilde{X}$ has dimension strictly greater than $k(r-k)+1$. Since $\mathbb{G}(E,k)$ is of dimension $k(r-k)+1$, we see that all the fibres of the evaluation map are at least one dimensional. To show the existence (and dimension) of $\widetilde{Y}$, it is sufficient to show that each fibre has a nonempty set of reducible maps, of codimension zero or one.

Pick a point p of C, and let F_p be a k dimensional quotient of the vector space E_p. Let $Z_{F_p} = ev^{-1}([F_p])$ be the fibre of the evaluation map over this point in $\mathbb{G}(E,k)$. If Z_{F_p} is empty there is nothing to prove. If Z_{F_p} consists completely of reducible maps, then there is still nothing to prove. The remaining possibility, which has to happen generically over the image, is that Z_{F_p} has some points which correspond to maps with an irreducible domain curve. If the locus of reducible maps is *not* a nonempty divisor, then we can find an irreducible complete curve B'' in Z_{F_p} such that every point of B'' corresponds to a map with irreducible domain curve. (Also note, by the discussion in §1, that these curves are all isomorphic to C). It is now easy to show that the existence of B'' leads to a contradiction.

Let B' be the normalization of B'', so that we can work with a smooth base curve. We have a map $B' \longrightarrow B'' \longrightarrow \overline{\mathcal{M}}_{g,1}(\mathbb{G}(E,k),\beta_d)$. Since the moduli space of maps is only a coarse moduli space, this might not correspond to an actual family. However, there is a finite cover $B \longrightarrow B'$ so that the induced map $B \longrightarrow \overline{\mathcal{M}}_{g,1}(\mathbb{G}(E,k),\beta_d)$ *does* come from a family $\mathcal{C}$ over B. This family is isotrivial, since all fibres are isomorphic to C.

At this stage we have the family $\mathcal{C}$ over B, along with a section $s : B \longrightarrow \mathcal{C}$ corresponding to the marked point, and a map $f : \mathcal{C} \longrightarrow \mathbb{G}(E,k)$ inducing the stable maps on the fibres. The combination of the map f followed by the projection π from $\mathbb{G}(E,k)$ to C gives an isomorphism of each fibre of the family with C, and so we see that $\mathcal{C}$ is the trivial family $\mathcal{C} = C \times B$ over B.

The section s is sent to the point F_p under f, since B'' is contained in the fibre Z_{F_p} of the evaluation map. This shows that the images of the maps $C \to \mathbb{G}(E,k)$ parametrized by B go through the fixed point F_p. Now this family of stable maps over B induced by f is not the constant family, since B'' was a nontrivial curve in the moduli space. This contradicts the bend and break lemma of Kollár and Mori (see [16] 1.7), which asserts that in the given situation at least one of the image curves should "break" and have a rational component passing through F_p. We conclude that the curve B''

could not exist, and therefore that the locus of reducible curves in Z_{F_p} is a nonempty divisor $\blacksquare$.

Remark. Often in arguments using degeneration the difficult part is proving that a one parameter family actually has to degenerate. In our situation, the degeneration argument is very easy, the advantage coming from the fact that the curves in our families do not vary in moduli.

Some variations on this bound. Let m_k be the dimension of the space of stable maps in the case of rank k quotients of minimal degree. i.e.:

$$m_k := \dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_{d_k}).$$

In order to use induction in the proof of Theorem 3.1, we need that at each stage the dimension of the component we are considering be greater than $k(r-k)$. Therefore, whenever $m_k \geq k(r-k)-r$, the same argument also gives the slightly better result

$$\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d) \leq m_k + (d-d_k)r. \tag{1}$$

There is also a lower bound on the dimension of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ involving the same number m_k (and which does not require that $m_k \geq k(r-k)$). The bound is obtained by taking sections C_0' corresponding to minimal quotients and gluing on a single rational curve which maps to one of the fibres, with the degree of the rational curve chosen to make up the difference between d and d_k. There are m_k parameters for the choice of C_0', 1 parameter for the choice of where to glue the rational curve onto C_0', and $(d-d_k)r-2$ parameters for the choice of the map of the rational curve into the Grassmanian fibre. This gives:

Proposition 3.2 $\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d) \geq m_k + (d-d_k)r - 1.$

The curves constructed are all reducible curves, and it is typically the case that they represent an entire component (or components) of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$.

Remark. Going back to the case $m_k \geq k(r-k)-r$, the upper bound there is one more then this lower bound, and this leaves only two possibilities for the dimension of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$. Which possibility occurs depends on whether or not it is possible to smooth the nodes in the families above.

The same trick of gluing on a rational curve to make up the difference in degree gives an inequality between the dimensions of any two of the stable map spaces, namely

$$\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d_2}) \geq \dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d_1}) + (d_2-d_1)r - 1, \text{ for } d_2 \geq d_1.$$

Combining this information about the growth of the dimensions with the same inductive argument of Theorem 3.1 gives the following other variation: If d' is the first degree so that $\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d'}) > k(r-k) - r$, then

$$\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d) \leq k(r-k) + (d-d')r, \text{ for all } d \geq d_k. \tag{2}$$

Our basic picture in these upper (and lower) bounds is that the dimension grows by r when the degree increases by 1, at least once the dimension of the space is large enough. It is tempting to wonder if this is always the case (even when the dimensions are small), and if, for example, inequality (1) might be true in all cases. This is in fact not true, as Example 5.3 will show.

4. Dimension Estimates for Quot Schemes

We now switch to looking at the other, more familiar, compactification of the space of vector bundle quotients of a fixed vector bundle E, namely Grothendieck's Quot scheme. Let $\mathrm{Quot}_{k,d}(E)$ be the the scheme parametrizing coherent quotients of E of generic rank k and degree d. A general introduction to the theory of Quot schemes is given e.g. in [18] §4 and §8. We would like to prove the identical dimension estimate:

Theorem 4.1

$$\dim \mathrm{Quot}_{k,d}(E) \leq k(r-k) + (d-d_k)r, \textit{ for all } d \geq d_k.$$

Proof. If Q is a component of $\mathrm{Quot}_{k,d}(E)$ whose generic point parametrizes vector bundle quotients, then there is nothing to prove, since it is birational to a component of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ and Theorem 3.1 applies. As in the case of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$, however, there are usually components whose general point does not correspond to a locally free quotient, and, as in the case of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$, we will show that for these the inequality is always strict.

Let Q be a component of this type. Every non locally free quotient F determines a diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_F \arrow[r] \arrow[d, "\cong"'] & S_{F'} \arrow[r] \arrow[d] & \tau \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_F \arrow[r] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & F' \arrow[r, "\cong"] \arrow[d] & F' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where $S_{F'}$ is the saturation of the kernel S_F, F' is a quotient vector bundle of degree $(d-a)$ and τ, the torsion subsheaf of F, is given by a nontrivial zero-dimensional subscheme, say of length $a > 0$. We can stratify the set of all such F's in Q according to the value of the parameter a, which runs over a finite set. If we denote by $\{F\}_a$ the subset of Q corresponding to a fixed a, this gives:

$$\dim\ \{F\}_a \leq \dim\ \mathrm{Quot}_{k,d-a}(E) + (r-k)a.$$

Assuming the dimension bound true for smaller d, we get the inequality:

$$\dim\{F\}_a \leq k(r-k) + (d-a-d_k)r + (r-k)a =$$

$$= k(r-k) + (d-d_k)r - ka.$$

Letting a run over all possible values (or the minimum value) which occur for the component Q finishes the proof of the bound $\blacksquare$.

Remark. With only slight rewording the proof also gives the stronger

$$\dim \mathrm{Quot}_{k,d}(E) \leq \dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d) \quad \text{for all } d \geq d_k.$$

As in the case of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ (see 3.2), we can easily produce a lower bound for the dimension of $\mathrm{Quot}_{k,d}(E)$, this time by starting with vector bundle quotients of lower degree and going up via elementary transformations. Again, it may well be the case that the components achieving such a bound consist entirely of non-locally free quotients.

Proposition 4.2 *Let $d \geq d' \geq d_k$ and $\delta = (d - d')$. Then we have*

$$\dim \operatorname{Quot}_{k,d}(E) \geq \dim \operatorname{Quot}_{k,d'}(E) + \delta(r-k).$$

In particular $\dim \operatorname{Quot}_{k,d}(E) \geq m_k + (d-d_k)(r-k)$*, for all* $d \geq d_k$*.*

Proof. We only need to consider the diagram in the proof of Theorem 4.1 when F' runs over the family of quotients of degree d'. This gives at most $\dim \operatorname{Quot}_{k,d'}(E)$ parameters, and the other $\delta(r-k)$ parameters come from the fact that S_F is an (inverse) elementary transformation of $S_{F'}$ of length δ. The last statement follows by making $d' = d_k$.

Remark. Propositions 3.2 and 4.2 show that in general there is more flexibility in dealing with stable maps with reducible domain rather than non-locally free quotients. In section §7 we will show that there is always a surjective morphism $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d) \longrightarrow \operatorname{Quot}_{k,d}(E)$ extending the map on smooth curves if $k = (r-1)$, and no such map if $k \neq (r-1)$. In general the relation between the two compactifications seems a bit mysterious.

5. Examples and Optimality of the Bound

Example 5.1: Let E be the trivial bundle of rank r on C, and let k be any number from 1 to $(r-1)$. A "section of the relative Grassmann bundle" in this case is just the graph of a map from C to the usual Grassmannian $\mathbb{G}(r,k)$. The degree of the corresponding quotient bundle is the degree of the linear series associated to the map from C to $\mathbb{P}^N$ $\left(\text{with } N = \binom{r}{k} - 1\right)$ induced by the Plücker embedding.

The minimal degree of a quotient bundle is zero (i.e. $d_k = 0$ for all k) and in this case we have $\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_0) = k(r-k)$. For $d = 1$, there will be no vector bundle quotients unless the genus of the curve is zero, since for $g \geq 1$ there are no maps from C to $\mathbb{P}^N$ of degree 1. If $g \geq 1$, all the points of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_1)$ correspond to maps mapping C to a point, and mapping an attached rational curve in as a line. In the Quot scheme case, all the degree 1 quotients are torsion.

For degree $d = 2$, there will again be no vector bundle quotients, unless the curve is hyperelliptic, but there will always be reducible maps or torsion quotients as above. For other low d, the geometry of both the stable map and Quot scheme depend on the particular special linear series that live on the curve.

For large d, the behaviour of linear series on C is regular, and there will be components of both spaces corresponding to vector bundle quotients, of dimension $dr-k(r-k)(g-1)$. The behaviour of the largest components, however, are different in the two cases. Following the constructions of Proposition 3.2, $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ has a locus of dimension $rd+k(r-k)-1$ constructed by mapping C to a point, and mapping an attached rational curve as a curve of degree d. If $g \neq 0$, this is larger than the dimension of points parameterizing vector bundle quotients, and so constitutes a separate component. On the other hand, if we apply Proposition 4.2 to the Quot scheme, we get a torsion locus of dimension $d(r-k)+k(r-k)$. For large d this is a sublocus of the component of $\mathrm{Quot}_{k,d}(E)$ whose general point parametrizes vector bundle quotients (see Theorem 6.4) and not a new component at all.

This example shows that there are vector bundles E where the bound is exact for $d = d_k$, and gives an example of a case where the families constructed in Proposition 3.2 form components of the moduli spaces, i.e. where it is impossible to smooth the nodes of the family from Proposition 3.2.

Example 5.2: On a curve C of genus $g \geq 2$, let L be a line bundle of degree 1 with no global section, and let G_1 be a generic extension of the type

$$0 \to \overset{r-1}{\oplus}\,\mathcal{O} \to G_1 \to L \to 0.$$

If $g \geq (r-1)$ then the usual dimension counting shows that the generic extension of this type is a stable bundle (see also [6] §5 for more general counts of this kind).

The minimum degree of any quotient is $d_k = 1$, and for any k from 1 to $(r-1)$ there is at least a $(k-1)(r-k)$ dimensional family of degree $d = 1$ quotients obtained by dividing out by a rank $(r-k)$ subbundle of the direct sum (if $g \geq r$ then these are the only quotients of degree 1).

In most uses of a bound for the dimension of the quotients of a fixed bundle E, what is really needed is a bound for the *moduli* of the bundles F, i.e. the dimension of the isomorphism classes of F, throwing away the data of the particular morphisms $E \to F$ which express F as a quotient.

In this example, ($d = 1$, $g \geq (r-1)$, and the extension generic) it is not hard to check that all the quotients are nonisomorphic, and this provides an example of a family of quotients with maximal variation in moduli (so that the dimension of the space of quotients is the same as the dimension of the moduli of F's) as well as an example of a stable bundle where the dimension in the case $d = d_k$ is close to the upper bound.

A similar example is to take a generic extension

$$0 \to L^* \to G_0 \to G_1 \to 0$$

with L^* the dual of the line bundle L above. For $g \geq r$ this extension is also stable, and provides an example of a stable bundle with trivial determinant with a large family of quotients of degree 1 (which also vary maximally in moduli, since they are the same quotients as above).

Example 5.3: For any $r \geq 11$, pick k with $r/2 < k < r$ such that $(r-k)(2k-r-1) > r$ (this works out to be any k between $(r+5)/2$ and $(r-2)$). Let G_1 be a bundle of rank k of the type from Example 5.2, and let H be a stable bundle of rank $(r-k)$ and degree 1. Set

$$E = H \oplus G_1,$$

which is a bundle of rank r and degree 2.

By the stability of G_1 and H, E can have no quotients of degree 0. If F is a quotient bundle of rank k and degree 1, then the kernel bundle S_F fits in the following diagram (where the vertical arrows are injections):

$$\begin{tikzcd}
0 \arrow[r] & S'_F \arrow[r] \arrow[d] & S_F \arrow[r] \arrow[d] & S''_F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] & E \arrow[r] & G_1 \arrow[r] & 0
\end{tikzcd}$$

This shows that the slopes of S'_F and S''_F are less than $1/(r-k)$ and $1/k$ respectively. Since $1/k < 1/(r-k)$ and the slope of S_F *is* $1/(r-k)$, the only possibility is $S'_F = H$ and $S''_F = 0$, so that G_1 is the only quotient bundle of rank k and degree 1. Or, in the notation of §3, we have $d_k(E) = 1$, $m_k(E) = 0$.

At the end of §3 we wondered if the bound (1)

$$\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d) \leq m_k + (d-d_k)r$$

might be true in all cases. In this example, for $d = 2$, this would give the bound $\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_2) \leq r$.

One way to get quotient bundles of degree 2, rank k is by taking quotients of G_1 which have degree 1 and rank $(2k-r)$. We know the dimension of this family by the calculation in Example 5.2, and this shows that $\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_2) \geq (2k-r-1)(r-k)$, which we chose at the beginning to be strictly larger than r.

Remark. The idea of the proof of the bound is to use degeneration to relate bundles of higher degree to bundles of lower degree, which allows us to see that the dimension goes up by r as the degree increases. The only way that (1) might not be true in general is if there were families of quotients which have no relation to each other at all, and that is what this example provides.

Example 5.4: Let E be a generic stable bundle of degree e and rank r on a curve C of genus $g \geq 2$. The behaviour of the dimensions of the Quot schemes (which also applies to $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$) in this situation has been worked out in [24], and a summary is the following:

- The invariant d_k is the smallest value which makes the expression $d_k r - ke - k(r-k)(g-1)$ nonnegative.
- The number m_k is the value of $d_k r - ke - k(r-k)(g-1)$ (i.e. the residue class of $k^2(g-1) - ke$ mod r).
- $\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d) = \dim \mathrm{Quot}_{k,d}(E) = dr - ke - k(r-k)(g-1)$ $= m_k + (d-d_k)r$ for all $d \geq d_k$.

The bundles satisfying the first condition are sometimes called *Lange generic* and were first studied in [17]. If a bundle satisfies all three conditions above, we will say that it is *generic in the sense of example 5.4*, or sometimes just *5.4 stable.*

The expression $dr - ke - k(r-k)(g-1)$ appearing repeatedly above is the cohomological lower bound for a quotient bundle F of rank k and degree d:

$$\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d) \geq \chi(S_F^* \otimes F) = dr - ke - k(r-k)(g-1).$$

This example shows both that there are situations where this lower bound is achieved, and where formula (1) in §3 may be valid without the condition $m_k \geq k(r-k)-r$. Most importantly, in §6 we will see that these are precisely the bundles for which $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ is irreducible for large d.

For special choice of the parameters, the dimension of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ in this example exactly matches our upper bound. If $k=1$ and r divides $(g-e)$, or $k=(r-1)$ and r divides $(g+e)$, then $\dim \overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d) = (r-1)+(d-d_k)r$ for all $d \geq d_k$, which is the upper bound (similarly for $\mathrm{Quot}_{k,d}(E)$). For all other choices of parameters the upper bound will be strictly larger.

How good is this bound? Viewed as functions of d, both our upper bound (3.1 or 4.1) and the cohomological lower bound mentioned above are linear functions with the same slope r, and therefore this part of the bound is certainly sharp. There are also examples (5.1 and 5.4) where the values of the "constant term" are achieved and so without extra conditions on the bundle this cannot be improved either. Further evidence that the constant term is good is that the difference between upper and lower bounds gives the inequality:

$$rd_k - ke \leq k(r-k)g,$$

which is a theorem of Lange [17] and Mukai-Sakai [19] generalizing a rank 2 result of Segre and Nagata [20] on minimal sections of ruled surfaces.

One appealing aspect of the upper bound is that it depends on very little data. The rank and degree (k and d) of the quotients and the rank r of the bundle are the basic numerical invariants of the problem, and the only thing which really ties the formula to the particular vector bundle E is the (admittedly sometimes hard to understand) invariant $d_k(E)$. In particular, the formula is independent of the genus of the curve, the characteristic of the field, or any other conditions on the bundle E (including its degree).

In conclusion, it seems that the bound is the best possible, especially given the limited amount of information being put into the formula. Even with extra conditions on the bundle E, the only possible improvement is better (and somewhat limited) control over the behaviour of the constant term. On the other hand, it is exactly the lack of extra conditions on E that is one of the strongest features of this bound. This lack of restrictions makes it very useful in computations; Lemma 6.3 in the next section is an illustration of this.

6. Components of Quot and $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ for large d

Let E be a bundle of rank r and degree e on a curve of genus g. We start by proving a slightly weaker version of the theorem we are aiming for.

Proposition 6.1 *For all large d there is a unique component of* $\mathrm{Quot}_{k,d}(E)$ *and $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ whose generic point corresponds to a stable bundle quotient; this component is of dimension $rd - ke - k(r-k)(g-1)$.*

We will need two vanishing lemmas.

Lemma 6.1 *There is a number $d_1 = d_1(E)$ such that for all stable bundles F of rank k, degree $d \geq d_1$, $\mathcal{H}om(E,F)$ is generated by its global sections and $H^1(E^* \otimes F) = 0$.*

Proof. Let L be a line bundle of degree 1. For any bundle F', there is a number n so that $E^* \otimes F' \otimes L^{\otimes m}$ is globally generated, and so that $H^1(E^* \otimes F' \otimes L^{\otimes m})$ vanishes for all $m \geq n$.

The condition we want is open, and so given any family X of vector bundles F', the data of the minimum possible n_x for each member F'_x of the family stratifies X into locally closed subsets. In particular, this function has a maximum value on X, and so taking n equal to this shows we can choose an n which works for all members of the family.

Apply this to the moduli spaces $U^s_C(k,j)$ of rank k stable bundles for degrees $j = 0, \ldots, (k-1)$, and choose an n which works for all of these families. Set $d_1 = nk$.

Any stable bundle F of rank k and degree $d \geq d_1$ can be written as $F = F' \otimes L^{\otimes m}$ for $m \geq n$ (the n above) with F' a stable bundle of rank k and degree between 0 and $(k-1)$, which proves the lemma ■.

Remark. Considering filtrations shows that it is possible to choose a d_1 so that Lemma 6.1 holds for all *semi*-stable bundles F of rank k, degree $d \geq d_1$.

Lemma 6.2 *There is a number $d_2 = d_2(E)$ so that for any quotient bundle F of rank k, degree $d \geq d_2$ with either F or S_F (the kernel) semistable, $h^1(S_F^* \otimes F) = 0$ (and so $h^0(S_F^* \otimes F) = rd - ke - k(r-k)(g-1)$).*

Proof. Since the proofs in the two cases are very similar, we only look at the case when F (as opposed to S_F) is semistable. Having $h^1(S_F^* \otimes F) \neq 0$ is equivalent by Serre duality to having a non-zero map $F \to S_F \otimes \omega_C$. Let F_1 be the image of this map. Since F is semistable, we have:

$$\mu(F) \leq \mu(F_1).$$

On the other hand we have the inclusions

$$F_1 \hookrightarrow S_F \otimes \omega_C \hookrightarrow E \otimes \omega_C,$$

which imposes an upper bound on the slope of F_1. Therefore, once the slope of F is large enough, no such map is possible ∎.

Proof of Proposition 6.1. Let F be a stable bundle of rank k and degree $d \geq \max(d_1(E), d_2(E))$. Lemma 6.1 guarantees that $\mathcal{H}om(E, F)$ is globally generated, and since $k < r$ it is not hard to see that the generic homomorphism is a surjection, which means that both $\mathrm{Quot}_{k,d}(E)$ and $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ contain a point corresponding to the quotient

$$0 \longrightarrow S_F \longrightarrow E \longrightarrow F \longrightarrow 0.$$

Lemma 6.2 now says that this point is a smooth point of the component it lies on, and that the dimension of this component is $rd - ke - k(r-k)(g-1)$. Since a small deformation of a stable bundle is again a stable bundle, the general point of this component also corresponds to a stable quotient, and this proves all the statements of the proposition except for the uniqueness.

By Lemma 6.1 $h^1(E^* \otimes F) = 0$ for all F in the moduli space $U_C^s(k,d)$ of of stable rank k degree $d \geq d_1$ bundles, and therefore the vector spaces $\mathrm{Hom}(E, F)$ form a bundle over $U_C^s(k,d)$. Since $U_C^s(k,d)$ is irreducible, the total space of the bundle is as well. The points corresponding to surjections from E onto a stable bundle F of rank k, degree d form an open subset of this total space. The points of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ and $\mathrm{Quot}_{k,d}(E)$ corresponding to stable quotients are in one to one correspondence with the points of this irreducible open set, and this proves uniqueness ∎.

Remark. The proof shows in particular that, as long as d is large enough, *every* semistable bundle of degree d is a quotient of E and moreover, all such quotients correspond to points in the smooth locus.

Remark. The proposition is not as useful in the case $g = 0$ because there are not many semistable bundles (if d is not a multiple of r then the bundle cannot be semistable). In the genus zero case it is useful to have a "good" class of bundle which exists for all possible ranks and degrees. This will let us (for example) prove Theorem 6.2 in the $g = 0$ case too. The bundles in this case are the following. For each d and k, let $F_{k,d}$ be the unique (up to isomorphism) bundle of rank k and degree d on $C \cong \mathbb{P}^1$ such that $F_{k,d} = \oplus \mathcal{O}(a_i)$ with $|a_i - a_j| \leq 1$ for all $1 \leq i, j \leq k$. This class of bundles has the two key properties that we use in our arguments: First, if L is a line bundle of degree 1 then $F_{k,d} \otimes L \cong F_{k,d+k}$, so that this set of bundles is "periodic" with respect to twisting – this lets us prove the vanishing in Lemma 6.1. Second, any bundle of rank k and degree d on $\mathbb{P}^1$ can be generalized to $F_{k,d}$ – this property will be used to compare different components parametrizing vector bundle quotients and show that they are the same. In short, Proposition 6.1 and Theorems 6.2 and 6.4 are still true in the genus zero case, the only changes being the words "stable bundle" used in the statements and arguments should be changed to "$F_{k,d}$ bundle".

The uniqueness result is true even if we ask just that the general point of the component parametrize vector bundle quotients (without necessarily being stable). To prove this we will use Proposition 6.1 and the following lemma on avoiding quotients.

Lemma 6.3 *Given any d_0 and $k_0 < k$ there is a number $d_3 = d_3(E, d_0, k_0)$ such that if F is a general quotient of E of rank k and degree $d \geq d_3$, then F has no quotient of degree d_0 and rank k_0.*

Remark. The expression "general quotient" means that F is a general point of any component of either $\overline{\mathcal{M}}_g(\mathbb{G}(E, k), \beta_d)$ or $\mathrm{Quot}_{k,d}(E)$ (and such that the component actually has points corresponding to vector bundle quotients).

Proof. If we count dimensions for the space of the possible F's which *do* have a quotient $F \to F_0$ to a bundle F_0 of degree d_0 and rank k_0, then (viewed as a function of the degree d) we have:

- The dimension of the choice of bundle F_0 is constant as a function of d. Any quotient of such an F is also a quotient of E, so the choice of F_0 is at most the dimension of $\overline{\mathcal{M}}_g(\mathbb{G}(E, k_0), \beta_{d_0})$.

- Fixing an F_0 and the kernel S_{F_0} of the surjection $E \to F_0$, we can construct an F by taking a subbundle of S_{F_0} of rank $(r-k)$ and degree $(e-d)$, and then dividing E by this subbundle. Applying theorem 3.1 to the bundle $S^*_{F_0}$ we see that the dimension of the choice of subbundles is bounded above by a linear function of d with slope $(r-k_0)$.

In summary, the dimension of possible F's with the kind of quotient we are trying to avoid is bounded above by a linear function of d with slope $(r-k_0)$. On the other hand, the dimension of the components parametrizing vector bundle quotients is bounded below by a linear function of d with slope r (the cohomological lower bound). Therefore, for large enough d, no general point of any of these components can have a quotient of degree d_0 and rank k_0 ■.

Remark. This proof shows the utility of Theorems 3.1 and 4.1. It is important that the theorems put no special condition on the bundle, since no matter what conditions we put on E we have little control over S_{F_0}, and it is important that the "slope" of the upper bound be the best possible, since it is exactly a small difference in slopes which leads to the proof of the lemma.

Theorem 6.2 *For all large d there is a unique component of* $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ *and* $\mathrm{Quot}_{k,d}(E)$ *whose generic point corresponds to a vector bundle quotient; this component is of dimension* $rd - ke - k(r-k)(g-1)$.

Proof. The key observation is that there are only a finite number of possible pairs (d_0, k_0) such that E has a quotient bundle F_0 of degree d_0 and rank k_0 which also injects into $E \otimes \omega_C$. If we fix k_0, then the degrees of possible quotients of E of rank k_0 are bounded from below, and similarly the degrees of possible subbundles of $E \otimes \omega_C$ are bounded above. This shows that the number of possible d_0's for a fixed k_0 is finite, and therefore there are only a finite number of possible pairs, since k_0 can only run between 1 and $(r-1)$.

The reason that this is useful is that if F is any quotient bundle of E, then any map from F into $E \otimes \omega_C$ must factor through one of the bundles F_0 of the type above. If F has no such F_0 quotient, then the only map from F to $E \otimes \omega_C$ is the zero map, and this amounts to a vanishing theorem as in Lemma 6.2.

Apply Lemma 6.3 to all the possible pairs (d_0, k_0) above, and pick a d_3 which works for all of them. For any $d \geq d_3$, pick any component of $\mathrm{Quot}_{k,d}(E)$

or $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ whose general point corresponds to a vector bundle quotient. By Lemma 6.3, the general point F of this component has no map to $E\otimes\omega_C$, and so (by Serre duality) both $h^1(E^*\otimes F)$ and $h^1(S_F^*\otimes F)$ are zero.

Pick a deformation (as an abstract bundle) of F over an irreducible base B so that the generic member of the deformation is a stable bundle. Throw away the closed set of points of B where either $h^1(E^*\otimes F_b)$ or $h^1(S_{F_b}^*\otimes F_b)$ jumps. We now have $h^1(E^*\otimes F_b)=0$ for all b in B, so that the vector spaces $\mathrm{Hom}(E,F_b)$ form a bundle over B. The condition of being a surjection is open, so that we can also deform the surjection $E\to F$ to a surjection $E\to F_b$ for any general point b of B. The deformation was constructed so that for general b, the bundle F_b is stable.

Since $h^1(S_{F_b}\otimes F_b)=0$ for all b in B, this deformation always stays in the smooth locus of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ or $\mathrm{Quot}_{k,d}(E)$, therefore in deforming this surjection we remain on the same component that we started on.

This shows that as long as $d\geq d_3$, the condition that the general point of a component correspond to a vector bundle quotient is enough to imply that the general quotient on that component is actually stable. From Proposition 6.1 we know that once d is large enough, there is a unique such component, of dimension $rd-ke-k(r-k)(g-1)$, and this proves the theorem $\blacksquare$.

Remark. The condition $d\geq d_3$ is already enough to guarantee the uniqueness of the component parametrizing vector bundle quotients, since (as the proof shows) every such component has quotients which form an open set of the set of rank k, degree d, stable bundles. Since this set is irreducible, and these quotients lie in the smooth loci of the components, the uniqueness statement follows.

The theorem immediately implies the following useful Corollary:

Corollary 6.3 *Every vector bundle E on C can be written as an extension*

$$0\longrightarrow E_1\longrightarrow E\longrightarrow E_2\longrightarrow 0,$$

with E_1 and E_2 generic stable bundles.

Note that we are free to choose what we mean by generic, that is, we may specify ahead of time any open condition on the space of stable bundles (for example 5.4 stable) and then express E as an extension where both E_1 and E_2 satisfy that open condition.

We now study when the spaces $\mathrm{Quot}_{k,d}(E)$ or $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ can be irreducible for large d. In both cases our arguments are based on the following:

Observation. The families of nodal curves constructed in Proposition 3.2 vary maximally in moduli, given that the curves involved must have a node. Similarly, the families of torsion quotients constructed in Proposition 4.2 vary maximally in moduli, given that the quotients have torsion of length δ. Therefore, if either family sits inside a larger family, the general member of the larger family is simpler, being either a map from a smooth curve, or being a quotient with smaller torsion.

Our main result says that in the case of the Quot scheme the best possible result is true:

Theorem 6.4 *For any vector bundle E on C, there is an integer $d_Q = d_Q(E,k)$ such that for all $d \geq d_Q$, $\mathrm{Quot}_{k,d}(E)$ is irreducible.*

Remark. Of course, this unique component is generically smooth, of dimension $rd - ke - k(r-k)(g-1)$, and the general point corresponds to a vector bundle quotient, all of which follows immediately from Theorem 6.2.

Proof of Theorem 6.4. For simplicity, let us use $\mathrm{Quot}^0_{k,d}(E)$ for the open subset of $\mathrm{Quot}_{k,d}(E)$ parametrizing vector bundle quotients (note that for some d this might miss entire components of $\mathrm{Quot}_{k,d}(E)$).

Also set:

$$w_d = \dim \mathrm{Quot}^0_{k,d}(E) - (dr - ke - k(r-k)(g-1)),$$

i.e. the difference between the dimension of this subset and the expected dimension (the lower bound). We have $w_d \geq 0$ for all $d \geq d_k$, and the w_d are eventually zero (by Theorem 6.2) for large d.

Let W be the set of $d \geq d_k$ so that $w_d \neq 0$, and define M by:

$$M := \max_{d \in W} \left\{ d + \frac{w_d}{k} \right\}.$$

Finally, set

$$d_Q = \max(\lceil M \rceil, d_3(E,k)),$$

where $\lceil M \rceil$ is the smallest integer greater than M and $d_3(E, k)$ is the integer d_3 appearing in the proof of Theorem 6.2 (i.e., such that there is a unique component corresponding to vector bundle quotients for $d \geq d_3$).

The claim is that for all $d > M$, the general point of every component of $\mathrm{Quot}_{k,d}(E)$ corresponds to a vector bundle quotient. Since, by Theorem 6.2 there is only one of these as soon as $d \geq d_3$, this proves the irreduciblity of the Quot scheme for all $d \geq d_Q$.

Suppose we pick a degree d' smaller than d and construct, as in Proposition 4.2, torsion quotients of degree d by introducing torsion of order $\delta = (d - d')$ (all torsion loci are produced by this kind of construction). The torsion quotients we get this way have dimension

$$\dim \mathrm{Quot}^0_{k,d'}(E) + \delta(r - k)$$

or, another way to write this (using the w's) is

$$w_{d'} + d'r - ke - k(r - k)(g - 1) + \delta(r - k).$$

The difference between this dimension and the lower bound $rd - ke - k(r - k)(g - 1)$ is:

$$-(d - d')r + w'_d + \delta(r - k) = w'_d - \delta k.$$

But, by assumption, $d > M$, which means that $d > d' + \frac{w'_d}{k}$, or equivalently $w'_d - \delta k < 0$ (or, if $d' \notin W$, then $w'_d = 0$ and what we want is automatic). In other words, the dimension of the space of quotients with torsion is strictly lower than the lower bound, so the general point must be an actual vector bundle quotient, which finishes the proof $\blacksquare$.

Remark. We will use Theorem 6.4 at the end of the next section to show that $\overline{\mathcal{M}}_g(\mathbb{G}(E, k), \beta_d)$ is connected for large d (see Proposition 7.5).

We now study the irreducibility of the spaces $\overline{\mathcal{M}}_g(\mathbb{G}(E, k), \beta_d)$. Unlike the case of $\mathrm{Quot}_{k,d}(E)$, it is typical (even for large d) that there are components of larger dimension consisting entirely of reducible maps. If we want to ensure that $\overline{\mathcal{M}}_g(\mathbb{G}(E, k), \beta_d)$ has no other components, then we must put strong conditions on the vector bundle E. (We assume in what follows that $g \geq 2$, see the remark after the proof for a discussion of $g = 0, 1$.)

Proposition 6.5 *If $\overline{\mathcal{M}}_g(\mathbb{G}(E, k), \beta_d)$ is irreducible for all k and all large d, then E is a stable bundle, generic in the sense of Example 5.4.*

Proof of Proposition 6.5. Let E be a vector bundle of rank r and degree e satisfying the conditions of the proposition. Consider k fixed; we will show that (at least as far as rank k quotients are concerned) E is a generic stable bundle in the sense of Example 5.4. Since it holds for all k this will prove the proposition.

For d large we know that there is a component of dimension $rd - ke - k(r - k)(g-1)$, and since we are assuming $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ irreducible, this is the only component. Let m_k be (as in §3) the dimension of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d_k})$.

In Proposition 3.2 we constructed a family of reducible curves of dimension $m_k + (d-d_k)r - 1$. This family has to lie in the unique component, and since the curves constructed have generically only one node, it is a divisor in this component, which gives us

$$m_k + (d - d_k)r = rd - ke - k(r-k)(g-1), \text{ or}$$

$$m_k = d_k r - ke - k(r-k)(g-1).$$

Since m_k is nonnegative, the right hand side of the equation is as well.

For the fixed numerical data of r, e, g, and k, let D_k be the smallest integer so that $D_k r - ke - k(r-k)(g-1)$ is nonnegative. By the results in [24] this is the smallest degree in which a generic stable bundle of rank r, degree e has a rank k quotient.

The fact that m_k is nonnegative gives us $d_k \geq D_k$, but since the number $d_k(E)$ is *lower* semicontinuous in families of vector bundles, and since any bundle can be deformed to a generic stable bundle, we always have $d_k(E) \leq D_k$, and this gives $d_k = D_k$.

This also shows that E has no rank k destabilizing quotient bundles, either from the fact that $(d_k r - ke)$ is positive, or from the fact that the degrees of its rank k quotients are the same as that of a generic stable bundle of the same rank and degree.

To see that E is 5.4 stable (that is, has the same properties listed in example 5.4) we need to show the following additional two things:

1. $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ has a component of dimension $dr - ke - k(r-k)(g-1)$ for all $d \geq d_k$.

2. There are no components of larger dimension.

Remark. We really only need to show this for "small d", since for large d this is a consequence of the assumptions and of Proposition 6.1.

The families of nodal curves constructed in Proposition 3.2 are of dimension $m_k + (d - d_k)r - 1$ which is $dr - ke - k(r-k)(g-1) - 1$ in our case.

The deformation theoretic lower bound for this family of curves is $dr - ke - k(r-k)(g-1)$, which is one larger. This shows that the family sits inside a larger family, and so by the observation there is a component of dimension $dr - ke - k(r-k)(g-1)$ for all $d \geq d_k$. We will be finished once we establish condition 2.

Suppose for some d_1 that there is a component of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_{d_1})$ of dimension strictly greater than $d_1 r - ke - k(r-k)(g-1)$. Again by attaching a single rational curve mapping in with large degree we can produce nodal families of dimension at least $dr - ke - k(r-k)(g-1)$ for all $d \geq d_1$.

For large d, this is greater than or equal to the dimension of the unique component. The family must be properly contained in this component, since the general member of the component corresponds to a smooth map, and not a nodal curve. This contradiction gives condition 2 ∎.

The fact that the components are always of the same dimension as the deformation theoretic lower bound has an important consequence. Combined with Theorem 6.2 this consequence proves the converse of Proposition 6.5.

Proposition 6.6 *For a stable vector bundle E, generic in the sense of example 5.4, the generic point of every component of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ corresponds to a map from a smooth curve (i.e., vector bundle quotient).*

Proof. Suppose that we have a component consisting entirely of reducible maps. We consider this to be built up (as usual) from a smaller component by attaching rational curves. We know the dimension of the smaller component, and if we compute the dimension of the family we obtain by gluing on the curves, we find that it is of *smaller* dimension than it is supposed to be ∎.

Another way to see how this property (having all components exactly the dimension of the lower bound) contributes to having vector bundle quotients is the following: If we start with a component parametrizing maps of degree d_1 and glue on rational curves to form a larger family, of maps of larger degree (say d_2), we find that the dimension of this family is smaller than the deformation theoretic lower bound for maps of degree d_2. This means that this nodal family must sit in a larger family. By the same kind of consideration as in the observation following Proposition 6.5, the general member of this family corresponds to a smooth curve.

Putting together Propositions 6.5 and 6.6, and Theorem 6.2, we arrive at

Theorem 6.7 *The spaces $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ are irreducible for all k and all large d if and only if E is stable bundle, generic in the sense of Example 5.4.*

Remark. In the proof of Proposition 6.5 we showed that if for a fixed k, $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ is irreducible for all large d, then E is 5.4 stable "in rank k" (i.e. at least as far as rank k quotients are concerned). On the other hand, if we assume that E is 5.4 stable in rank k, then the result of Proposition 6.6 is still true for rank k quotients, and so applying Theorem 6.2 we get a slight generalization of Theorem 6.7: For fixed k, $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ is irreducible for all large d if and only if E is 5.4 stable in rank k.

Remark. In the case that $g=0$ or $g=1$, the proof breaks down at the following point: when we wanted to show that $D_k \geq d_k(E)$ we used the fact that d_k is lower semicontinuous in families, and that we could deform E to a more general bundle E' with $d_k(E') = D_k$. In the case that $g<2$, the gap is that we don't know that such a bundle E' exists. The rest of the argument still goes through, and shows that $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ is irreducible for large d if and only if all components of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ have exactly the dimension of the lower bound, $dr - ke - k(r-k)(g-1)$. The only difference in the result is that we don't know when E starts to have quotients of rank k, i.e., we don't know the value of $d_k(E)$ is – it could be strictly larger than D_k. As an illustration, in Example 5.1 in the case that $g=0$ the value of D_k is negative, but $d_k(E)=0$ for all k. However, the dimension of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ is the predicted value for all $d \geq 0$, and these spaces are easily seen to be irreducible.

Since in the case of 5.4 stable bundles one has good numerical control over the invariants involved, all the steps in the proof can be made effective, and we have:

Theorem 6.8 *Let E be a 5.4 stable bundle. Then, for a fixed k, $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ and $\mathrm{Quot}_{k,d}(E)$ are irreducible for*

$$d \geq \frac{ke + k(r-k)(g-1)}{r} + 2(k-1)(g-1) + (r-k)(k-1).$$

In particular

$$d \geq (r-1)\mu(E) + \left(\frac{9}{4}r - 4\right)(g-1) + \frac{r^2}{4} - r + 1$$

works for all k.

Proof. As noted after the proof of Theorem 6.2, all we need to do is to find an effective expression for the number d_3 appearing in Lemma 6.3 which works for all the possible pairs (k_0, d_0) which are the rank and degree of quotients of E that are also subsheaves of $E \otimes \omega_C$ and such that $k_0 < k$.

Fixing such a pair (k_0, d_0), we first bound the dimension of the space of quotients F of E which in turn have a quotient F_0 of rank k_0 and degree d_0. These are themselves quotients of E, and since E is 5.4 stable, we have

$$\dim\{F_0\} \leq \dim \mathrm{Quot}_{k_0,d_0}(E) = rd_0 - k_0 e - k_0(r-k_0)(g-1).$$

Now each such F_0 determines an exact sequence

$$0 \longrightarrow S_{F_0} \longrightarrow E \longrightarrow F_0 \longrightarrow 0,$$

and so the space of F's which have F_0 as a quotient is determined by the space of subbundles of S_{F_0} of rank $r-k$ and degree $e-d$. Its dimension is bounded above by the formula given in Theorem 4.1, and after a small computation we get that the dimension of the space of all F's we are looking at is bounded by:

$$\dim\{F\} \leq rd_0 - k_0 e - k_0(r-k_0)(g-1) + (r-k)(k-k_0) + (d - d_k(E))(r-k_0).$$

On the other hand (since E is 5.4 stable) the dimension of degree d rank k quotients is exactly $rd - ke - k(r-k)(g-1)$, and so we want d large enough so that

$$\begin{aligned} k_0 d \quad > \quad & ke + k(r-k)(g-1) + rd_0 - k_0 e \\ & - k_0(r-k_0)(g-1) - d_k(E)(r-k_0) + (r-k)(k-k_0). \end{aligned}$$

Again using the fact that E is 5.4 stable, we have

$$-d_k(E) \leq \frac{-ke - k(r-k)(g-1)}{r}.$$

Plugging this into the previous inequality and computing further, we obtain

$$d > \frac{ke + k(r-k)(g-1)}{r} + r\mu(F_0) - e - (r-k_0)(g-1) + (r-k)\left(\frac{k}{k_0} - 1\right).$$

In the proof of Theorem 6.2 we are assuming that F_0 is a subsheaf of $E \otimes \omega_C$ (which is also 5.4 stable), giving the additional condition:

$$\mu(F_0) \leq \mu(E) + \frac{r+k_0}{r}(g-1).$$

And this gives us:

$$d > \frac{ke + k(r-k)(g-1)}{r} + 2k_0(g-1) + (r-k)\left(\frac{k}{k_0} - 1\right).$$

Using the inequalities $1 \leq k_0 \leq k-1$, we see that this is satisfied whenever:

$$d \geq \frac{ke + k(r-k)(g-1)}{r} + 2(k-1)(g-1) + (r-k)(k-1),$$

which is the desired bound on d depending only on k, r and e.

Optimizing the terms involving k individually, we arrive at the final answer which works for all k: all spaces $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ and $\mathrm{Quot}_{k,d}(E)$ are irreducible for:

$$d \geq d_3 = (r-1)\mu(E) + \left(\frac{9}{4}r - 4\right)(g-1) + \frac{r^2}{4} - r + 1 \;\blacksquare.$$

Remark. In the final two steps of the computation above we optimized each term containing k_0 and then k individually, rather than optimizing the whole expression as a function of k. Although the bound obtained is weaker, we chose to do this in order to avoid some extra, not very illuminating, calculations. The reader can certainly find better approximations for any particular example, if needed.

In the case when E is only assumed to be stable, the number d_3 can still be approximated (thus making Theorem 6.2 effective), by simply changing the 5.4 stability condition into usual stability. We skip the computation, which is identical to the one above, but list the result:

$$d \geq (r-1)\mu(E) + \left(\frac{r^2+8r}{4}\right)(g-1) + \frac{r^2}{4} + r.$$

It is worth noting that, at least in characteristic 0, we only need E to be stable in order to perform some other effective calculations. We conclude with a sample such result, which essentially makes Proposition 6.1 effective:

Proposition 6.9 *Assume that C is defined over a field of characteristic 0, and let E be a stable bundle of rank r and degree e on C. Then, for a fixed k, all semistable bundles of rank k and degree $d \geq k(\mu(E)+2g)$ are quotients of E, and in fact smooth points of a unique component of $\mathrm{Quot}_{k,d}(E)$. In particular*

$$d \geq (r-1)(\mu(E)+2g)$$

works for all k.

Proof. Following the proof of Proposition 6.1 and the subsequent Remark, we see that we only need to bound the numbers d_1 and d_2 appearing in Lemmas 6.1 and 6.2 respectively. This involves only the well known facts that the tensor product of two semistable bundles is semistable (in characteristic 0) and that for a semistable bundle V, $\mu(V) \geq 2g-1$ implies $h^1(V)=0$ and $\mu(V) \geq 2g$ implies global generation.

In Lemma 6.1 we had $\mu(E^* \otimes F) = \frac{d}{k} - \mu(E)$, and since E and F are now both stable, $E^* \otimes F$ will be globally generated and have vanishing H^1 as long as $d \geq k(\mu(E)+2g)$. Even simpler, the definition of stability for E implies that the sequence of inclusions in Lemma 6.2 cannot hold if $d \geq k(\mu(E)+2g-2)$. This gives the desired bound ∎.

7. Maps from $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ to $\mathrm{Quot}_{k,d}(E)$

We would like to try and extend the isomorphism between the open sets of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ and $\mathrm{Quot}_{k,d}(E)$ parametrizing vector bundle quotients to a morphism between the two spaces. Since the boundary loci of the stable map space are always of larger dimension than the boundary loci of the Quot scheme, the only possible direction for such a morphism is from $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ to $\mathrm{Quot}_{k,d}(E)$.

The moduli functor for $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ classifies flat families $\mathcal{C} \to B$ of nodal curves over a base B along with a map $f : \mathcal{C} \longrightarrow \mathbb{G}(E,k) \times B$ such that over each point b of B, the fibre $\mathcal{C}_b$ and morphism f_b is a stable map into $\mathbb{G}(E,k)$, with $f_{b*}[\mathcal{C}_b] = \beta_d$.

The corresponding moduli functor for $\mathrm{Quot}_{k,d}(E)$ classifies flat families $\mathcal{Q}$ of quotients of E on $C \times B$, such that over each point b the restriction $\mathcal{Q}_b$ of the family to the fibre C over b is a coherent sheaf of degree d and generic rank k. (We will also use the symbol E for the pullback p_1^*E on $C \times B$).

To give a morphism from $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ to $\mathrm{Quot}_{k,d}(E)$ is to give a functorial way of producing a flat family of quotients $\mathcal{Q}$ on $C \times B$ from a family of stable maps over B.

The Grassmann bundle $\mathbb{G}(E,k)$ comes with the projection $\pi : \mathbb{G}(E,k) \longrightarrow C$, and so given any family of stable maps over B, we can compose the morphism $f : \mathcal{C} \longrightarrow \mathbb{G}(E,k) \times B$ with π to get a map

$$g = (\pi \circ f) : \mathcal{C} \longrightarrow C \times B.$$

One way to produce sheaves on $C \times B$ is to push forward bundles from the family $\mathcal{C}$, and we will use the following lemma (with $g = (\pi \circ f)$ as above).

Lemma 7.1 *If $\mathcal{V}$ is a vector bundle on a family of stable maps over B such that $R^1 g_* \mathcal{V} = 0$, then $g_* \mathcal{V}$ is a flat family of coherent sheaves on $C \times B$, and formation of $g_* \mathcal{V}$ commutes with arbitrary base change $B' \to B$.*

Proof. This is a relative version of the standard theorem on base change and cohomology, and the proof is almost the same. The bundle $\mathcal{V}$ is flat over B since $\mathcal{C}$ is a flat family and $\mathcal{V}$ locally free. We can compute the derived sheaves on $C \times B$ by means of the relative Čech complex, which is a two term complex $\mathcal{V}^0 \to \mathcal{V}^1$ of quasi-coherent flat B modules. The map of complexes is surjective since by hypothesis $R^1 g_* \mathcal{V} = 0$. This gives us the exact sequence

$$0 \longrightarrow g_* \mathcal{V} \longrightarrow \mathcal{V}^0 \longrightarrow \mathcal{V}^1 \longrightarrow 0$$

on $C \times B$. Since both $\mathcal{V}^0$ and $\mathcal{V}^1$ are flat over B, $g_* \mathcal{V}$ is as well. Computation of the kernel commutes with arbitrary base change induced from B, again since $\mathcal{V}^1$ is flat, and finally, $g_* \mathcal{V}$ is coherent over $C \times B$ since g is proper ∎.

The map g from $\mathcal{C}$ to $C \times B$ is the contraction of all the "rational tails" on the curves, and the derived sheaf $R^1 g_* \mathcal{V}$ vanishes whenever the bundle $\mathcal{V}$ has no H^1 when restricted to these fibres. If it happens that $\mathcal{V}$ is trivial when restricted to the fibres, then the pushforward $g_* \mathcal{V}$ is a vector bundle on $C \times B$, a fact which we will use later on.

Given any family of stable maps, the exact sequence of tautological bundles

$$0 \longrightarrow \mathcal{S} \longrightarrow \pi^* E \longrightarrow \mathcal{F} \longrightarrow 0$$

on $\mathbb{G}(E, k)$ pulls back via f to give a similar exact sequence on the family $\mathcal{C}$. For any rational curve D mapping to a fibre of the relative Grassmannian, the pullback $f^*\pi^*E$ is trivial; this means that the tautological subbundle $f^*\mathcal{S}$ restricted to D decomposes as $\oplus_i \mathcal{O}(a_i)$ with all $a_i \leq 0$, and similarly that the pullback of the tautological quotient bundle $f^*\mathcal{F}$ restricted to D decomposes as a sum $\oplus_j \mathcal{O}(b_j)$ with all $b_j \geq 0$.

The decomposition of $f^*\mathcal{F}$ on the rational curves shows that this bundle has no H^1 on any of the fibres of the map $g : \mathcal{C} \to C \times B$, and therefore Lemma 7.1 says that the sheaf $g_*(f^*\mathcal{F})$ forms a flat family (over B) of coherent sheaves on $C \times B$. On each fibre C_b, the restriction is of degree d and rank k.

This family of coherent sheaves unfortunately does *not* in general give a map from $\overline{\mathcal{M}}_g(\mathbb{G}(E, k), \beta_d)$ to $\mathrm{Quot}_{k,d}(E)$. There are two ways to see this; the first is that when the subbundle $f^*\mathcal{S}$ has higher cohomology on the rational fibres, the induced map from $g_*(f^*\pi^*E) = E$ to $g_*(f^*\mathcal{F})$ on $C \times B$ is not surjective. This does not completely rule out this family, since it is conceivable that it might be a quotient of E in some other way. What really shows that this is the wrong construction is the following. If C_b is a fibre where the family $g_*(f^*\mathcal{F})$ has torsion, and x a local coordinate at a torsion point, then multiplication by x kills all torsion at that point. In general for quotients of E with torsion, we expect situations where multiplication by x^2 (or higher powers of x) is necessary to annihilate the torsion submodule.

In situations where the subbundle does *not* have higher cohomology along the rational fibres (for instance, if $d = d_k + 1$, where the a_i's are all zero except for one -1), this does give the right morphism to $\mathrm{Quot}_{k,d}(E)$.

The correct approach is to try and alter the subbundle $f^*\mathcal{S}$ along the contracted rational fibres of g so that it is trivial along these fibres. The push-forward will then be a vector bundle on $C \times B$ which includes into E, and gives a flat family of quotients.

This is always possible in the case that $k = (r - 1)$, when the subbundle is a line bundle.

Theorem 7.1 *If $k = (r-1)$ then there is a surjective morphism from $\overline{\mathcal{M}}_g(\mathbb{G}(E, k), \beta_d)$ to $\mathrm{Quot}_{k,d}(E)$ which extends the map on the locus where the domain curve is smooth.*

Proof. In order to describe a construction to be applied naturally (i.e., functorally) to all families of stable maps, we should describe the construction on $\overline{\mathcal{M}}_{g,1}(\mathbb{G}(E,k),\beta_d)$, the universal curve over $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$. The idea is that the loci of rational tails form divisors in $\overline{\mathcal{M}}_{g,1}(\mathbb{G}(E,k),\beta_d)$ and twisting by these divisors will allow us to make the line bundle $f^*\mathcal{S}$ trivial along all the tails. There is some awkwardness in the description because there can be components of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ where the general map is reducible, and here the loci of rational tails we must twist by in $\overline{\mathcal{M}}_{g,1}(\mathbb{G}(E,k),\beta_d)$ occur as both divisors and components.

Let D be either a divisor or a component of $\overline{\mathcal{M}}_{g,1}(\mathbb{G}(E,k),\beta_d)$ such that the general fibre of the map to $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ is an irreducible rational curve, with the additional condition that this general fibre touches only one other curve in the domain of the map it is associated to. (This extra condition only comes up because of the components where the general map is reducible).

For each such locus D there is a well defined line bundle $\mathcal{O}_{\overline{\mathcal{M}}_{g,1}}(D)$ on $\overline{\mathcal{M}}_{g,1}(\mathbb{G}(E,k),\beta_d)$; this line bundle has degree -1 when restricted to the general fibre of the map from D to $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$. We attach the weight δ to such a D if the degree of $f^*\mathcal{S}$ restricted to the general fibre in D is $-\delta$.

Set
$$\mathcal{S}' = f^*\mathcal{S} \otimes_\delta \mathcal{O}_{\overline{\mathcal{M}}_{g,1}}(-\delta \cdot D_\delta),$$
where the tensor product runs over all possible δ and all possible D_δ with weight δ satisfying the conditions above.

For any family $\mathcal{C}$ of stable maps over a base B, this construction gives us a line bundle $\mathcal{S}'$ on $\mathcal{C}$ of degree 0 on all components of all rational tails. In addition, this bundle has a nonzero map $\mathcal{S}' \to f^*\mathcal{S}$ obtained by multiplying by a section of $\otimes_\delta \mathcal{O}(\delta \cdot D_\delta)$.

By Lemma 7.1 the pushforward $g_*\mathcal{S}'$ is a flat family of line bundles on $C \times B$, and comes with a natural inclusion $g_*\mathcal{S}' \hookrightarrow E$ induced from the map $\mathcal{S}' \to f^*\mathcal{S}$ on the family $\mathcal{C}$. Let $\mathcal{Q}$ be the quotient.

By definition, $\mathcal{Q}$ is a family of quotients of E. If b is a point of the base B where the domain of the map $\mathcal{C}_b$ is an irreducible curve, then the construction with $\mathcal{S}'$ changes nothing near $\mathcal{C}_b$ and the quotient on the fibre C_b is precisely the quotient we expect. The only step remaining is to see that $\mathcal{Q}$ is a flat family over B.

Let $\mathcal{I}$ be an ideal sheaf in $\mathcal{O}_B$. Tensoring the defining sequence for $\mathcal{Q}$ with $(\mathcal{O}_B/\mathcal{I})$ gives the sequence

$$0 \to \mathcal{T}or_1(\mathcal{O}_B/\mathcal{I}, \mathcal{Q}) \longrightarrow g_*\mathcal{S}'\otimes(\mathcal{O}_B/\mathcal{I}) \longrightarrow E\otimes(\mathcal{O}_B/\mathcal{I}) \longrightarrow \mathcal{Q}\otimes(\mathcal{O}_B/\mathcal{I}) \to 0.$$

However, both $g_*\mathcal{S}'$ and E are vector bundles, and at a general point of every fibre over B the map $g_*\mathcal{S}' \to E$ is an injection, therefore the map from $g_*\mathcal{S}' \otimes (\mathcal{O}_B/\mathcal{I})$ to $E \otimes (\mathcal{O}_B/\mathcal{I})$ is an injection and $\mathcal{T}or_1(\mathcal{O}_B/\mathcal{I}, \mathcal{Q}) = 0$ ∎.

It is easy to see what this construction is doing. Suppose that we start with C_0, a section of the relative Grassmannian $\mathbb{G}(E, r-1)$, and let S_0 be the universal sub-line bundle restricted to C_0. Attach a rational curve to C_0 at a point p, and extend the inclusion $C_0 \hookrightarrow \mathbb{G}(E, r-1)$ to this reducible curve by mapping the rational component in so that it accounts for δ in degree. In this situation the construction produces the subsheaf S' of S_0 (on C_0) which vanishes to order δ at p. This bundle includes into E, and gives a quotient with torsion of order δ at that point.

This suggests how the "right" torsion quotient to associate to any stable map into the relative Grassmannian might look.

Possible description of Quotient: Suppose that we have a reducible map consisting of a section C_0 and a single rational curve attached to C_0 at p mapping into $\mathbb{G}(E, k)$. Let S_0 be the restriction of the universal subbundle to the section C_0 and let S_1 be the restriction to the rational curve. The bundle S_1 on the rational curve breaks up into a sum $\oplus_{i=1}^{r-k}\mathcal{O}(a_i)$ with $a_i \leq 0$.

We can identify the fibre of S_0 at p with the fibre of S_1 at the corresponding point of attachment. The picture in the case $k = (r-1)$ suggests that we should be able to choose a local basis $\{s_1, \ldots, s_{(r-k)}\}$ for S_0 at p, compatible with the splitting on S_1, so that if x is a local coordinate at p on C_0, the subsheaf S' of S_0 generated locally by $\{x^{-a_1}s_1, \ldots, x^{-a_{(r-k)}}s_{(r-k)}\}$ gives the correct quotient.

This unfortunately is also not true in most situations (see Example 7.1 below). It is however true in two special cases, and these special cases will be sufficient to show that there is in general *no* map from $\overline{\mathcal{M}}_g(\mathbb{G}(E, k), \beta_d)$ to $\mathrm{Quot}_{k,d}(E)$ (if $k \neq (r-1)$) which extends the map on smooth curves.

Proposition 7.2 *Suppose that $\mathcal{C} \longrightarrow B$ is a one parameter family of stable maps with both $\mathcal{C}$ and B smooth, and that at a point b_0 of B the fibre $\mathcal{C}_{b_0}$*

consists of a section C_0 and a single rational curve joined at a point p. If the pullback $f^*\mathcal{S}$ of the universal subbundle decomposes on the rational curve as $\oplus\mathcal{O}(a_i)$ such that $|a_i - a_j| \le 1$ for all i, j, then the limiting quotient $\mathcal{Q}_{b_0}$ in $\mathrm{Quot}_{k,d}(E)$ is given by the rule above.

Remark. The condition that both $\mathcal{C}$ and B are smooth means that the general fibre is as well, and this gives a generically defined map from B to $\mathrm{Quot}_{k,d}(E)$; it is this map that we use to compute the one parameter limit.

Remark. In this case (that $|a_i - a_j| \le 1$) *any* choice of local generators of the sheaf S_0 at p (compatible with the splitting along the rational curve) gives the same subbundle S'. (See Proposition 7.3 for an example where this choice does matter).

Proof. Let D be the rational curve over b_0 on the smooth surface $\mathcal{C}$. Suppose that the subbundle $f^*\mathcal{S}$ restricted to D decomposes as

$$f^*\mathcal{S}|_D = \overset{m}{\oplus}\ \mathcal{O}_D(a) \overset{n}{\oplus}\ \mathcal{O}_D(a-1)$$

with $a \le 0$ and $m + n = (r - k)$.

The curve D is a (-1) curve on the surface; and so setting $\mathcal{S}' = f^*\mathcal{S}\otimes\mathcal{O}_{\mathcal{C}}(aD)$ we have

$$\mathcal{S}'|_D = \overset{m}{\oplus}\ \mathcal{O}_D \overset{n}{\oplus}\ \mathcal{O}_D(-1).$$

If S_0 is the bundle $f^*\mathcal{S}$ restricted to C_0, then $\mathcal{S}'$ restricted to C_0 is generated by the sections of S_0 vanishing to order $(-a)$ at p.

Shrink B so that b_0 is the only singular fibre. The bundle $\mathcal{S}'$ then satisfies the conditions of Lemma 7.1 and therefore $g_*\mathcal{S}'$ gives a family of vector bundles over $C \times B$. This bundle also has an inclusion into E induced by the map $\mathcal{S}' = f^*\mathcal{S} \otimes \mathcal{O}(aD) \to f^*\mathcal{S}$.

The quotient of E by $g_*\mathcal{S}'$ gives (using the same argument as in Proposition 7.1) a flat family of quotients over B. Since the Hilbert scheme is proper, the quotient at b_0 must be the correct one parameter limit, and all that remains is to compute what this limit is.

To compare the restriction of $g_*\mathcal{S}'$ to C_{b_0} (on $C \times B$) and the restriction of $\mathcal{S}'$ to C_0 (on the family $\mathcal{C}$) we use the fact that formation of $g_*\mathcal{S}'$ commutes with base change from B, and so we only have to compute this for the contraction map

$$g_{b_0} : \mathcal{C}_{b_0} = (C_0 \sqcup_p D) \longrightarrow C_0 = C_{b_0}.$$

Choose a local basis $s'_1, \ldots, s'_{(r-k)}$ at p of $\mathcal{S}'$ restricted to $(C_0 \sqcup_p D)$, compatible with the splitting of $\mathcal{S}'$ along D. Since the bundle $\mathcal{O}_D(-1)$ has no global sections, we see that $g_*\mathcal{S}'$ is generated by $\{s'_1, \ldots, s'_m, x \cdot s'_{m+1}, \ldots, x \cdot s'_{(r-k)}\}$.

The image of $\mathcal{S}'|_{C_{b_0}}$ in S_0 is (as mentioned above) all the sections of S_0 vanishing to order $(-a)$ at p. The image of $g_*\mathcal{S}'|_{C_0}$ is therefore m sections vanishing to order $(-a)$ and $n = (r-k-m)$ sections vanishing to order $(-a+1)$, as claimed in the proposition ■.

Proposition 7.3 *Suppose that we have a family $\mathcal{C} \longrightarrow B$ satisfying the same conditions as in Proposition 7.2 but that this time the restriction of $f^*\mathcal{S}$ to the rational curve splits as $(r-k-1)$ copies of $\mathcal{O}$ and one copy of $\mathcal{O}(-2)$, then the limiting quotient $\mathcal{Q}_{b_0}$ is also given by the rule above.*

This means that it is possible to choose a basis $\{s_1, \ldots, s_{(r-k)}\}$ for $f^*\mathcal{S}$ restricted to C_0 (compatible with the splitting on the rational curve) so that the limit is the quotient by $S' = \{s_1, \ldots, s_{r-k-1}, x^2 s_{(r-k)}\}$.

Remark. In this case the subbundle S' *does* depend on the basis chosen near p. For example, if $(r-k) = 2$ the bases $\{s_1, s_2\}$ and $\{s_1 + xs_2, s_2\}$ can both be compatible with the splitting (at p) along the rational curve, but the subsheaves generated by $\{s_1, x^2 s_2\}$ and $\{s_1 + xs_2, x^2 s_2\}$ are not the same.

Proof. Let D be the rational curve and C_0 the section as before. The restriction of $f^*\mathcal{S}$ to D is

$$f^*\mathcal{S}|_D = \overset{r-k-1}{\oplus} \mathcal{O}_D \oplus \mathcal{O}_D(-2).$$

Let $\mathcal{S}'$ be the kernel (on $\mathcal{C}$) of the projection onto the $\mathcal{O}_D(-2)$ factor, so that we have

$$0 \longrightarrow \mathcal{S}' \longrightarrow f^*\mathcal{S} \longrightarrow \mathcal{O}_D(-2) \longrightarrow 0.$$

We restrict this defining sequence to D to get

$$0 \longrightarrow \mathcal{T}or_1(\mathcal{O}_D(-2), \mathcal{O}_D) \longrightarrow \mathcal{S}'|_D \longrightarrow f^*\mathcal{S}|_D \longrightarrow \mathcal{O}_D(-2) \longrightarrow 0.$$

The map $f^*\mathcal{S}|_D \longrightarrow \mathcal{O}_D(-2)$ is just the projection we started with, and since $\mathcal{T}or_1(\mathcal{O}_D(-2), \mathcal{O}_D) = \mathcal{O}_D(-1)$, we have

$$0 \longrightarrow \mathcal{O}_D(-1) \longrightarrow \mathcal{S}'|_D \longrightarrow \overset{r-k-1}{\oplus} \mathcal{O}_D \longrightarrow 0.$$

This sequence splits, and the splitting is canonical, since the trivial subbundle of $\mathcal{S}'|_D$ is exactly the subbundle generated by global sections.

Choose a basis $\{s'_1, \ldots, s'_{(r-k)}\}$ near p for $\mathcal{S}'$ restricted to $\mathcal{C}_{b_0}$ compatible with the splitting on D (the basis element $s'_{(r-k)}$ should be associated to the $\mathcal{O}(-1)$ subbundle).

The image of $\mathcal{S}'$ in $f^*\mathcal{S}$ restricted to D is $\overset{(r-k-1)}{\oplus} \mathcal{O}_D$. This shows that we can pick a basis $\{s_1, \ldots, s_{(r-k)}\}$ for $f^*\mathcal{S}$ restricted to C_0 so that the induced map $\mathcal{S}'|_{C_0} \to S_0 = f^*\mathcal{S}|_{C_0}$ is given by $s'_i \mapsto s_i$ for $i = 1, \ldots, (r-k-1)$ and $s'_{(r-k)} \mapsto x \cdot s_{(r-k)}$ for the remaining basis element.

We again pushforward the bundle $\mathcal{S}'$ to get a bundle on $C \times B$ which we use to construct a flat family of quotients over B. Following the same type of argument as in Proposition 7.2, we see that $g_*\mathcal{S}'|C_{b_0}$ is the subbundle of $\mathcal{S}'|_{C_0}$ generated by $\{s'_1, \ldots, s'_{(r-k-1)}, x \cdot s'_{(r-k)}\}$.

Using our previous description for the image of $\mathcal{S}'|_{C_0}$ in $S_0 = f^*\mathcal{S}|_{C_0}$ we see that the image S' of $g_*\mathcal{S}'$ in S_0 is $\{s_1, \ldots, s_{(r-k-1)}, x^2 s_{(r-k)}\}$ ■.

Remark. A combination of the arguments in Propositions 7.2 and 7.3 shows that the "rule" for computing the limit also works if the subbundle splits as m_0 copies of $\mathcal{O}_D(a)$, m_1 copies of $\mathcal{O}_D(a-1)$, and m_2 copies of $\mathcal{O}_D(a-2)$.

The actual subbundle S' of S_0 produced depends on the bundle $\mathcal{S}'$ which is in turn dependent on the first order information of the curve $\mathcal{C}_{b_0}$ in the family. Different families *can* produce different limits, and this is the reason that we cannot in general define a morphism from $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ to $\mathrm{Quot}_{k,d}(E)$.

The easiest way to see that the limit varies according to the family is to take a second limit consisting entirely of singular curves. If we stratify the singular curves by the way the restriction of the universal subbundle decomposes on the rational tails, then the reason why we cannot extend the map is very clear: the limit of the rules for computing the quotient on one strata is not the rule for computing the quotient on a limiting strata.

The simplest case of this is to take a degeneration of the decomposition $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ to the decomposition $\mathcal{O} \oplus \mathcal{O}(-2)$.

Theorem 7.4 *If $k \neq (r-1)$ then in general there is no map from the space $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ to $\mathrm{Quot}_{k,d}(E)$ extending the map on smooth curves.*

Proof. We show the case $(r-k)=2$; the idea works for all $k \neq (r-1)$.

Pick d large so that by Theorem 6.2 there is a component of dimension $dr - ke - k(r-k)(g-1)$ whose generic point is a smooth point of the component, and corresponds to a vector bundle quotient.

Let C_0 be a section corresponding to one of these points, and S_0 the restriction of the universal subbundle to C_0. Pick a point p of C_0 and glue on a rational curve D. Extend the inclusion $C_0 \hookrightarrow \mathbb{G}(E,k)$ by mapping the curve in so that the universal subbundle restricts to D as $\mathcal{O} \oplus \mathcal{O}(-2)$.

Because the point on $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ corresponding to C_0 was a smooth point of the component, a tangent space calculation shows that this new stable map is a smooth point of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d+2})$ and lies on a component of dimension $(d+2)r - ke - k(r-k)(g-1)$. The dimension of the locus of nearby nodal curves is one less than this, so the general point of this component corresponds to a smooth map.

Take a smooth one parameter family B_1 containing our stable map as a fibre b_0. Proposition 7.3 now shows that there is a basis $\{s_1, s_2\}$ of S_0 near p so that the limiting quotient is given by dividing out by the subsheaf $S_1' = \{s_1, x^2 s_2\}$.

Starting with the same stable map, we now construct another family B_2, by fixing the section C_0, the point of attachment p and varying only the map of the rational curve D into the Grassmannian fibre. For a general variation, the decomposition of the universal subbundle will be $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Let b_0 again stand for the fibre with the original stable map.

Since all of the curves in our family can also be deformed to smooth curves, Proposition 7.2 shows that for points where the decomposition is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ the corresponding point in $\mathrm{Quot}_{k,d}(E)$ must be the quotient by $S_2' = \{xs_1, xs_2\}$. As the point b varies in B_2 the subbundle S_2' does not change, and the inclusion $S_2' \hookrightarrow E$ is clearly the same away from p, and therefore the same at p as well. Therefore this is a *constant* family of quotients, and the limit at b_0 in this family is *also* the quotient by S_2'.

The two quotients are different, and this shows that (unless we blow up) we cannot extend the map on smooth curves to all of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_{d+2})$ ■.

Remark. It is not clear if the fact that the splitting can change is the only obstruction to extending the map. In the two cases where the decomposition *cannot* change (when $k=(r-1)$ or when $d=d_k$ or $d=d_k+1$) the map can always be defined.

Remark. For a general rational curve in a Grassmannian, the splitting of the restriction of the universal subbundle is as "democratic" as possible – no a_i will differ by more than one from any other. Proposition 7.2 is therefore a reflection of the fact that rational maps extend into codimension one subsets.

Example 7.1: Let L be any line bundle, and p any point on a curve C. Set

$$S = L \oplus L(p), \quad \text{and} \quad E = L \oplus L(p) \oplus L(3p).$$

Let b be a coordinate on the affine line, and define a family of inclusions $S \xrightarrow{i_b} E$ by

$$\begin{array}{ccccccc} L & \xrightarrow{i_b} & (& b^2 \cdot \mathrm{Id}_L, & \cdot\sigma_p, & 0 &) \\ L(p) & \xrightarrow{i_b} & (& 0, & b \cdot \mathrm{Id}_{L(p)}, & \cdot\sigma_p^2 &) \end{array}$$

where σ_p is the section of $\mathcal{O}_C(p)$ and "$\cdot$" indicates multiplication by the section. If x is a coordinate at p, then locally the inclusion is given by the matrix

$$\begin{pmatrix} b^2 & x & 0 \\ 0 & b & x^2 \end{pmatrix}$$

If $b \neq 0$ then the inclusion is rank 2 everywhere, and gives a vector bundle quotient. When $b = 0$ the quotient has torsion, and looks like E divided by a bundle of the type $\{xs_1, x^2 s_2\}$.

For $b \neq 0$ we can consider this as a family of stable maps, and take the one parameter limit in $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$. To do this, blow up the family $C \times \mathbb{A}^1_b$ at $(p, 0)$ and pull back both bundles to this surface. The inclusion of S into E now drops rank along the exceptional divisor D, and if we take the saturation of S in E on the blow up, this gives the subbundle of the limiting stable map. Computation in coordinates on the blowup shows that this subbundle restricted to D decomposes as $\mathcal{O}_D \oplus \mathcal{O}_D(-3)$.

Since we know that the corresponding torsion quotient is *not* of the type where we divide out by something of the form $\{s_1, x^3 s_2\}$, this shows that the decomposition of the subbundle on the rational tails has no relation in general to the corresponding limiting quotient in one parameter families.

Remark. In the proof of Proposition 7.3 there was one step where an exact sequence of bundles on D split. It is the failure of this kind of sequence to split in general which prevents the rule from being true. The example above is the simplest case where this splitting (and therefore the rule) fails.

To finish, here is a result that properly belongs in the previous section, but appears here because we need to use our picture of the relationship between $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ and $\mathrm{Quot}_{k,d}(E)$, especially pushing forward vector bundles, and computing one parameter limits.

Proposition 7.5 *There is a $d_{\mathcal{M}} = d_{\mathcal{M}}(E,k)$ so that $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ is connected for all $d \geq d_{\mathcal{M}}$.*

Proof. Choose $d_{\mathcal{M}}$ so that it is larger than the d_Q of Theorem 6.4 and also large enough so that every component of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ has dimension at least $k(r-k)+1$.

For any $d \geq d_{\mathcal{M}}$, let X be a component of $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$. By Lemma 3.1 X has a nonempty divisor Y of reducible maps. Suppose that a point of a component of this divisor is given by a section C_0 corresponding to a quotient of degree $(d-\delta)$ and an attached rational curve making up the difference in degree. By deforming the rational curve alone in the Grassmannian fibre, we can break it into a tree of δ rational curves, each accounting for one in degree (this is a degeneration, and the resulting stable map is still a point of X).

Now move the δ rational curves "off" the tree so that they are connected to C_0 at δ different points (it is at this stage that we may be moving from one component to another). The result is a section C_0 with δ rational curves attached at $\{p_1,\ldots,p_\delta\}$ on C_0, and so that each curve accounts for one in degree. Let (C',f) be the resulting stable map into the relative Grassmannian. The claim is that this point is in the same connected component as the locus parametrizing vector bundle quotients.

To see this, we use the fact that the pushforward $g_*(f^*\mathcal{F})$ of the pullback of the tautological quotient bundle on $\mathbb{G}(E,k)$ gives a quotient of E. This happens since (in this situation) the pullback of the canonical subbundle $f^*\mathcal{S}$ decomposes as the sum of $\mathcal{O}$'s and one $\mathcal{O}(-1)$ on each of the δ rational tails (see the discussion after Lemma 7.1). The resulting pushforward F' is a quotient with torsion. The torsion free part is the vector bundle quotient corresponding to the section C_0, and the torsion part is supported at the points $p_1,\ldots,p_\delta$, and is of length one at each of those points.

Since $d_{\mathcal{M}}$ is larger than d_Q, $\mathrm{Quot}_{k,d}(E)$ is irreducible, and so we can find a smooth one parameter family B of quotients so that the general point is a vector bundle quotient and the special point $q \in B$ is the torsion quotient F'. Since the general point of B is a vector bundle quotient, we can also consider this a curve in $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$, and we take the one parameter limit approaching $q \in B$. To compute the limit we just need to blow up the family $C \times B$ at the points $\{(p_1,q),\ldots,(p_\delta,q)\}$ since this is where the quotient map drops rank, and take the saturation of the universal subbundle pulled back to the blown up surface. The limit is not necessarily the stable map (C',f) but it *is* a stable map with section C_0 (the *same* C_0) and δ rational curves attached at $p_1,\ldots,p_\delta$ each one accounting for one in degree.

This curve can be deformed (by moving the rational curves in the Grasmannian fibres) to (C',f), and so we see that the component X is in the same connected component as the locus parametrizing vector bundle quotients. Since X was arbitrary, this means that $\overline{\mathcal{M}}_g(\mathbb{G}(E,k),\beta_d)$ is connected $\blacksquare$.

8. Application to Base Point Freeness on Moduli Spaces

In this section we assume that C is a smooth projective complex curve of genus $g \geq 2$. Let $SU_C(r)$ denote the moduli space of semistable rank r vector bundles on C of trivial determinant. The Picard group of $SU_C(r)$ is isomorphic to $\mathbb{Z}$ (see [11] Theorem B), with ample generator $\mathcal{L}$ (called the *determinant* bundle).

For any bundle E in $SU_C(r)$, if there is a bundle E' of rank p and slope $(g-1)$ such that $h^0(E \otimes E') = 0$, then E is *not* a basepoint for $|\mathcal{L}^p|$. (It is not known if the converse is true – this is the content of the geometric formulation of the Strange Duality conjecture, see e.g. [2] §8).

If we treat an element of $h^0(E \otimes E')$ as a homomorphism from E^* to E', then we can factor the homomorphism as $E^* \longrightarrow F \hookrightarrow E'$, where the map to F is a surjection.

Suppose that we fix a quotient bundle F of rank k and degree d, then we can ask for the dimension of stable bundles E' of rank p and slope $(g-1)$ which fit in the diagram

$$0 \longrightarrow F \longrightarrow E' \longrightarrow Q \longrightarrow 0.$$

Counting dimensions of the quotient bundle Q and the extension space $\mathrm{Ext}^1(Q,F)$ shows that these bundles are of codimension at least $dp+1$ in the moduli space of bundles of rank p, slope $(g-1)$. (For details and further references related to all these facts, please see [21], especially §4).

This gives the following lemma on basepoint freeness:

Lemma 8.1 *For a bundle E in $SU_C(r)$, if p is a number so that*

$$\dim \left\{ \begin{array}{l} \textit{Moduli of rank } k, \textit{ degree } d, \\ \textit{quotient bundles } F \textit{ of } E \end{array} \right\} \leq dp$$

for all $k = 1, \dots, (r-1)$, and all d, then E is not a basepoint of $|\mathcal{L}^p|$.

The dimension above is the dimension of the moduli of the quotient bundles F, throwing away the data of the surjections $E \to F$. The space of quotients (i.e., this time including the data of the surjections) provides an upper bound for the dimension of this moduli. We have $d_k(E) \geq 1$ for all k since E is stable of degree 0, and using our upper bound 4.1 gives

Theorem 8.1 *The series $|\mathcal{L}^p|$ on $SU_C(r)$ is base point free for*

$$p \geq \max\{[\frac{r^2}{4}], r\}.$$

Remark. This is a small improvement on a similar bound in [21]. In the same paper, the following question appeared:

Question. Is the series $|\mathcal{L}^p|$ base point free on $SU_C(r)$ for $p \geq (r-1)$?

Even though we have examples of bundles with large families of quotients, we know of no potential counterexample (i.e. modulo Strange Duality) to this conjecture. There are two difficulties in constructing good examples in this situation. The first is that we would like the family of quotients to have maximal (or at least large) variation in moduli. The second (and more difficult) is that the dimension count in Lemma 8.1 does not take into account the fact that different extensions involving different Q's may produce the same stable bundle E'. This contribution is difficult to estimate.

For instance, the bundles G_0 constructed in Example 5.2 are stable bundles of rank r and trivial determinant with at least a $(k-1)(r-k-1)$ dimensional family of rank k, degree 1 quotient bundles. As stated in the example, these families of quotients vary maximally in moduli, and so seem as if they should provide a potential counterexample to the conjecture. However, all the quotients F are extensions of the form

$$0 \to \overset{k-1}{\oplus} \mathcal{O} \to F \to L \to 0,$$

where L is a fixed line bundle of degree 1. The general stable bundle E' of slope $(g-1)$ has no global sections, and also will not accept a nonzero map from L, and therefore any morphism of the type $F \to E'$ is the zero morphism.

In any construction we can make, either the dimension of the quotients is too small, or the quotient bundles produced are too special. The vague principle seems to be that quotients of a fixed bundle cannot be too general, and the Question above is a quantitative expression of this.

The improved upper bound also gives a few new cases of a conjecture made in [21] 5.5 about linear series on the moduli space $U_C(r,0)$ of semistable vector bundles of rank r and degree 0 on C.

Once a line bundle $N \in \mathrm{Pic}^{g-1}(C)$ is fixed, we can define (see [11] 7.4.2) a *generalized theta divisor* Θ_N on $U_C(r,0)$, given set-theoretically by:

$$\Theta_N = \{E \mid h^0(E \otimes N) \neq 0\}.$$

The conjecture mentioned above asserts that the linear series $|k\Theta_N|$ should be base point free for $k \geq r+1$. In the case $r=1$, this is the classical statement that $|2\Theta_N|$ is base point free on the Jacobian of C, and in [21] 5.4 a proof is also given for $r=2$ and 3. By arguments in [22] 5.3, $|k\Theta_N|$ is base point free as long as $k \geq r+1$ and $|\mathcal{L}^k|$ is base point free on $SU_C(r)$. Concluding, Theorem 8.1 provides a solution to the conjecture up to the case of rank 5 vector bundles:

Corollary 8.2 *$|k\Theta_N|$ is base point free on $U_C(r,0)$ for $k \geq r+1$ if $r \leq 5$.*

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