A version of the Jensen – Johnsbråten coding at arbitraly level $n \geq 3$

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Abstract

Theorem 1 *Let* $n \geq 2$. *There is a CCC (in* L *) forcing notion* $P = P_n \in \mathrm{L}$ *such that* P*-generic extensions of* L *are of the form* $\mathrm{L}[a]$, *where* $a \subseteq \omega$ *and*

(A) a *is* Δ^1_{n+1} *in* $\mathrm{L}[a]$;

(B) *if* $b \in \mathrm{L}[a]$, $b \subseteq \omega$ *is* Σ^1_n *in* $\mathrm{L}[a]$ *then* $b \in \mathrm{L}$ *and* b *is* Σ^1_n *in* L.

In addition, if a model $\mathcal{M}$ *extends* L *and contains two different* P*-generic sets* $a, a' \subseteq \omega$, *then* $\omega_1^{\mathcal{M}} > \omega_1^{\mathrm{L}}$.

For $n = 2$, this is the result of Jensen and Johnsbråten [4] (in this case, (B) is a corollary of the Shoenfield absoluteness theorem).

In the absense of the additional requirement, the result was proved by Harrington [1] (using a version of the almost disjoint coding of Jensen and Solovay [3]) and, independently, by the author [5, 6] (using a version of the Jensen "minimal Δ^1_3" coding [2]). Our proof is a similar modification of the construction in [4].

Recall that the forcing notion in [4] is the union of a certain increasing ω_1-sequence of its countable initial segments. We choose another such a sequence, which is more complicated (leading to $a \in \Delta^1_{n+1}$ instead of $a \in \Delta^1_3$ in $\mathrm{L}[a]$), but bears an appropriate amount of "symmetry", sufficient for (B).

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1 Preliminaries

By a *normal tree* we shall understand a tree T, which consists of sequences (so that every $t \in T$ is a function with $\mathtt{dom}\, t \in \mathtt{Ord}$ and the order $<_T$ is the extension order $\subset$) and satisfies conditions i) – iv): *normal tree*

i) the empty sequence Λ does *not* belong to T; Λ

ii) if $t \in T$ and $1 \le \alpha < \mathtt{dom}\, t$ then $t \restriction \alpha \in T$.

Let $|t| = \mathtt{dom}\, t$ for any sequence t. It follows from i), ii) that, for any $\alpha \ge 1$, $T(\alpha) = \{t \in T : |t| = \alpha\}$ is just the α-th level of T. (We start counting levels with level 1; the missed, for the sake of convenience, level 0 would consist of Λ.) $|t|$

Let $|T|$ be the least ordinal > 0 and $>$ all $|t|$, $t \in T$ (the *height* of T). $|T|$

For $\alpha < |T|$, let $T \restriction_{<\alpha} = \bigcup_{\gamma<\alpha} T(\gamma)$ (the *restriction*). $T \restriction_{<\alpha}$

iii) each non-maximal $t \in T$ has infinitely many immediate successors;

iv) each level $T(\alpha)$ is at most countable.

Let $1 \le \lambda \le \omega_1$. A *normal λ-tree* is a normal tree T satisfying *normal λ-tree*

v) $|T| = \lambda$, and, if $t \in T$ and $|t| < \alpha < \lambda$ then t has successors in $T(\alpha)$.

(Thus the only normal 1-tree is the empty tree. Normal 0-trees do not exist.)

1.1 Iterated sequence of Souslin trees

We are going to define, in L, a sequence of normal ω_1-trees T_n, and, for all n and $t \in T_n$, a subtree $\mathcal{T}_{(t)} \subseteq T_{n+1}$, satisfying the following requirements (1) through (8), and some extra conditions, to be formulated later. T_n, $\mathcal{T}_{(t)}$

Fix once and for all a recursive partition $\mathbb{Q}^+ = \bigcup_n Q_n$ of the set $\mathbb{Q}^+$ of all positive rationals onto countably many countable topologically dense sets Q_n. Q_n

(1) $T_n(\alpha) \subseteq (\mathbb{Q}^+)^\alpha$ for all n and $1 \le \alpha < \omega_1$.

(2) $T_n(1) = \{\langle q \rangle : q \in Q_n\}$ for all n.

(3) If $t \in T_n(\alpha)$ and $q \in \mathbb{Q}^+$ then $t^\wedge q \in T_n(\alpha+1)$.

($t^\wedge q$ denotes the extension of a sequence t by q as the rightmost term.) Thus any element $t \in T_n(\alpha)$ is a sequence $t = \langle t_\gamma \rangle_{\gamma<\alpha}$ of positive rationals, and the trees T_n do not intersect each other. $t^\wedge q$

(4) If $t \in T_n(\alpha)$ then $\mathcal{T}_{(t)} \subseteq T_{n+1}$ is a normal α-tree, and $T_{n+1} = \bigcup_{t \in T_n} \mathcal{T}_{(t)}$.

(5) If $t,\, t_1 \in T_n$ and $t < t_1$ then $\mathcal{T}_{(t)} = \mathcal{T}_{(t_1)} \restriction_{<|t|}$.

(6) Suppose that $t \in T_n(\alpha)$, $\alpha \geq 1$, and $q_1 \neq q_2 \in \mathbb{Q}^+$. Then $\mathcal{T}_{(t_1)} \cap \mathcal{T}_{(t_2)} = \mathcal{T}_{(t)}$.

We observe that $\mathcal{T}_{(t)} = \emptyset$ whenever $t \in T_n(1)$.

It follows that for any $s \in T_{n+1}(\alpha)$ (here $\alpha \geq 1$) there is unique $s \in T_n(\alpha+1)$ such that $s \in \mathcal{T}_{(t)}$. This t will be denoted by $t = \mathfrak{f}(s)$. We have $\mathfrak{f}$

$\langle * \rangle$ if $t,\, t' \in T_{n+1}$ and $t \subset t'$ then $\mathfrak{f}(t) \subset \mathfrak{f}(t')$.

The next requirement will imply that every T_n is a Souslin tree in L .

(7) Suppose that $\lambda < \omega_1$ is a limit ordinal. Let $\vartheta < \omega_1$ be the least ordinal such that L_ϑ models $\mathbf{ZFC}^-$ (minus the power set axiom), λ is countable in L_ϑ, and both the sequence $\langle T_n \restriction_{<\lambda} \rangle_{n\in\omega}$ and the map which sends every $t \in \bigcup_n T_n \restriction_{<\lambda}$ to $\mathcal{T}_{(t)}$ belong to L_ϑ. *We require that* every $t \in T_n(\lambda)$ satisfies $\exists\, s \in D\, (s \subset t)$ whenever $D \in \mathrm{L}_\vartheta$ is a pre-dense subset of $T_n \restriction_{<\lambda}$.

1.2 Coding idea of the construction of Jensen and Johnsbråten

Now suppose that, in a generic extension of L, for $n \in \omega$, C_n is a branch in T_n, so that $C_n \in (\mathbb{Q}^+)^{\omega_1}$ and $C_n \restriction \alpha \in T_n$ for all $1 \leq \alpha < \omega_1$. Suppose further that

$\langle \dagger \rangle$ $\mathfrak{f}(C_{n+1} \restriction \alpha) \subset C_n$ for all $n \in \omega$ and $1 \leq \alpha < \omega_1$, or, in other words, if $1 \leq \alpha < \omega_1$ then $C_{n+1} \restriction \alpha \in \mathcal{T}_{(C_n \restriction (\alpha+1))}$.

In this case there is a straightforward procedure of "decoding" the branches C_n from the sequence $\langle q_n \rangle_{n\in\omega} \in (\mathbb{Q}^+)^\omega$, where $q_n = C_n(0) \in \mathbb{Q}^+$:

$\langle \ddagger \rangle$ We begin with the values $C_n \restriction 1 = \langle q_n \rangle$, put $C_n \restriction \alpha + 1 = \mathfrak{f}(C_{n+1} \restriction \alpha)$ (by induction on α simultaneously for all n), and take unions at all limit steps.

Thus $\langle C_n \rangle_{n\in\omega}$ is constructible from $\langle q_n \rangle_{n\in\omega}$!

We are going to define such an extension of the universe, in which there exists only one sequence $\vec{q} = \langle q_n \rangle_{n\in\omega} \in (\mathbb{Q}^+)^\omega$ for which the procedure "converges" in the sense that $C_n \restriction \alpha$ is an extension of $C_n \restriction \beta$ whenever $1 \leq \beta < \alpha < \omega_1$. Note that the meaning the "convergence" is

$\langle \S \rangle$ *First*, every q_n must be the 1st term of the 2-term sequence $\mathfrak{f}(\langle q_{n+1} \rangle)$. *Second*, the unions at limit steps, in the inductive computation of $C_n \restriction \alpha$, must remain in the trees T_n .

1.3 The uniqueness in the construction of Jensen and Johnsbråten

The principal idea of [4] is to arrange things so that in any extension of L, if there are two different sequences of rationals for which the procedure $\langle\ddagger\rangle$ "converges" then $\omega_1^{\mathrm{L}} < \omega_1$. Technically, the collapse will be realised in the form of an increasing ω_1^{L}-sequence of rationals.

Assume that $\{q_\gamma\}_{\gamma<\alpha}$ is a sequence of non-negative rationals. Set $\sum_{\gamma<\alpha} q_\gamma$ to be the supremum of finite partial sums (including the case of $+\infty$).

If $s, t \in T_n(\alpha)$ then define $\sum(s,t) = \sum_{\gamma<\alpha} |s_\gamma - t_\gamma|$. We require the following: $\sum(s,t)$

(8) Suppose that T is T_0 or $\mathcal{T}_{(t)}$ for some $t \in T_n$ ($n \in \omega$), $\alpha < \lambda < |T|$, λ is limit, and $s, t \in T(\lambda)$. Then $\sum(s{\restriction}\alpha, t{\restriction}\alpha) < \sum(s,t) < +\infty$.

(It can be easily shown that there must be $s \in T_0$ – as well as in any T_n – satisfying $\sum s = \infty$, so that some "series" diverge to infinity. However by (8) they diverge in "almost parallel" fascion.)

1.4 The "limit" generic extension

To summarize the consideration, suppose that, in L, we have a system $\mathfrak{T} = \langle\langle T_n\rangle_{n\in\omega}, \mathcal{T}_{(\cdot)}\rangle$ satisfying (1) – (8) in L. $\mathfrak{T}$

Define $\mathfrak{f}$ as above. $\mathfrak{f}$

Define $\mathbb{P}_{\mathfrak{T}} = \lim T_n$ to be the "limit" of the sequence, that is, the set of all "tuples" $\vec{t} = \langle t_0, \dots, t_n\rangle$, where $n \in \omega$, and $t_i \in T_i$ and $t_i = \mathfrak{f}(t_{i+1})$ for all $i \le n$. $\mathbb{P}_{\mathfrak{T}} = \lim T_n$ We order $\mathbb{P}_{\mathfrak{T}}$ as follows: $\vec{t} = \langle t_0, \dots, t_n\rangle \le \vec{s} = \langle s_0, \dots, s_m\rangle$ ($\vec{s}$ is *stronger* than $\vec{t}$) iff $n \le m$ and $t_i \subset s_i$ in T_i for all i.

Theorem 2 [4] 1) $\mathbb{P}_{\mathfrak{T}}$ *is a CCC forcing in* L. *Each* T_n *is a Souslin tree in* L.

2) *In a* $\mathbb{P}_{\mathfrak{T}}$*-generic extension of* L, *there is a sequence* $\langle q_n\rangle_{n\in\omega} \in (\mathbb{Q}^+)^\omega$ *for which the procedure* $\langle\ddagger\rangle$ *"converges" in the sense of* $\langle\S\rangle$.

3) *In any extension of* L, *if there are two different sequences* $\langle q_n\rangle_{n\in\omega}$ *for which the procedure* $\langle\ddagger\rangle$ *"converges" in the sense of* $\langle\S\rangle$, *then* ω_1^{L} *is countable.*

Proof 1) Follow classical patterns, with the help of (7).

2) It is clear that any $\mathbb{P}_{\mathfrak{T}}$-generic extension of L has the form $\mathrm{L}[\langle C_n\rangle_{n\in\omega}]$, where $C_n = \bigcup_{\langle t_0,\dots,t_n\rangle\in\mathbb{P}_{\mathfrak{T}}} t_n$ is an ω_1^{L}-branch in T_n. Moreover, the branches C_n C_n satisfy $\langle\dagger\rangle$, hence each C_{n+1} is a branch in the subtree ${}^*T_{n+1} = \bigcup_{1\le\alpha<\omega_1} \mathcal{T}_{(C_n\restriction\alpha)} \in \mathrm{L}[C_n]$ of T_n. (By the way $C_n \in \mathrm{L}[C_{n+1}]$ because $C_n = \bigcup_{\alpha<\omega_1^{\mathrm{L}}} \mathfrak{f}(C_{n+1}{\restriction}\alpha)$ by $\langle\dagger\rangle$.) ${}^*T_n$

Now the procedure $\langle\ddagger\rangle$ "converges" (just to the chains C_n) for the sequence of the rationals $q_n = C_n(0)$. q_n

3) Suppose that $\langle q_n\rangle_{n\in\omega}$ and $\langle q'_n\rangle_{n\in\omega}$ are two different sequences of positive rationals for which the procedure $\langle\ddagger\rangle$ "converges", to resp. branches C_n and C'_n in T_n ($n\in\omega$). Now either $C_0\neq C'_0$ or there is n such that $C_{n+1}\neq C'_{n+1}$ but $C_k=C'_k$ for all $k\le n$. (Otherwise $q_n=q'_n$ for all n.) q_n, q'_n C_n, C'_n

In the "either" case C_0 and C'_0 are two different branches in T_0, which implies, by (8), that there exists a strictly increasing ω_1^{L}-sequence of rationals, namely the sequence of sums $\sum(C_0\restriction\alpha, C'_0\restriction\alpha)$, $\alpha<\omega_1^{\mathrm{L}}$, hence ω_1^{L} is countable.

Consider the "or" case. Define the subtrees ${}^*T_{k+1}=\bigcup_{\alpha<\omega_1}\mathcal{T}_{(C_k\restriction\alpha)}$ and ${}^*T'_{k+1}=\bigcup_{\alpha<\omega_1}\mathcal{T}_{(C'_k\restriction\alpha)}$ of T_{k+1} for all k. Then ${}^*T'_{k+1}={}^*T_{k+1}$ for all $k\le n$, in particular, ${}^*T'_{n+1}={}^*T_{n+1}$. Thus C_{n+1} and C'_{n+1} are two different ω_1^{L}-branches in ${}^*T_{n+1}$, which, as above, implies that ω_1^{L} is countable. $\square$ ${}^*T_n$, ${}^*T'_n$

2 Construction of the trees

Let us now describe how a collection of trees and a map $t\longmapsto\mathcal{T}_{(t)}$ satisfying (1) through (8) can be constructed in L.

The following requirement will facilitate the construction.

(9) Suppose that $T=T_0$ or $T=\mathcal{T}_{(t)}$ for some $t\in T_n$ and $n\in\omega$. Let further $r\in\mathbb{Q}^+$, $t,t'\in T(\beta)$, and $s\in T(\alpha)$, $\beta<\alpha<|T|$, $t\subset s$. Then there exists $s'\in T(\alpha)$ such that $t'\subset s'$ and $\sum(s,s')-\sum(t,t')<r$.

This looks weaker than (9) in [4], but implies the latter by the triangle inequality.

Definition 3 An *embrion* of height λ ($\lambda\le\omega_1$) is a system $\mathfrak{T}=\langle\langle T_n\rangle_{n\in\omega},\mathcal{T}_{(\cdot)}\rangle$ of normal λ-trees T_n and a map $t\longmapsto\mathcal{T}_{(t)}$ which satisfy (1) through (8) of Section 1 plus (9) below λ. *embrion*

An embrion $\mathfrak{T}'$ *extends* $\mathfrak{T}$, symbolically $\mathfrak{T}\preceq\mathfrak{T}'$, when $\lambda\le\lambda'$, $T_n=T'_n\restriction_{<\lambda}$, and $\mathcal{T}_{(\cdot)}$ is the restriction of $\mathcal{T}'_{(\cdot)}$ on $\bigcup_n T_n$. *extends* $\preceq$

$|\mathfrak{T}|$ denotes the height of the embrion $\mathfrak{T}$. `Emb` is the set of all embrions of <u>countable</u> height. $|\mathfrak{T}|$ `Emb`

If $\mathfrak{T}=\langle\langle T_n\rangle_{n\in\omega},\mathcal{T}_{(\cdot)}\rangle$ is an embrion and $\lambda<|\mathfrak{T}|$ then we define the restriction $\mathfrak{T}\restriction\lambda=\langle\langle T_n\restriction_{<\lambda}\rangle_{n\in\omega},\mathcal{T}_{(\cdot)}\restriction\lambda\rangle$, where $\mathcal{T}_{(\cdot)}\restriction\lambda$ is the restriction of $\mathcal{T}_{(\cdot)}$ on the domain $\bigcup_n T_n\restriction_{<\lambda}$. Obviously $\mathfrak{T}\restriction\lambda$ is an embrion of height λ. $\square$ $\mathfrak{T}\restriction\lambda$

Lemma 4 *Let $\mathfrak{T}=\langle\langle T_n\rangle_{n\in\omega},\mathcal{T}_{(\cdot)}\rangle$ be an embrion of a countable height λ. There is an embrion $\mathfrak{T}'=\langle\langle T'_n\rangle_{n\in\omega},\mathcal{T}'_{(\cdot)}\rangle$ of height $\lambda+1$ extending $\mathfrak{T}$.*

Proof [1] We have to define the levels $T_n(\lambda)$, $m\in\omega$, and extend the map $t\longmapsto\mathcal{T}_{(t)}$ on $\bigcup_n T_n(\lambda)$. This depends on the form of the ordinal λ.

[1] A brief form of the proof in [4].

Case 1: λ is a limit ordinal. Note that possible elements of $T_n(\lambda)$ are sequences $s \in (\mathbb{Q}^+)^\lambda$ such that $s \restriction \alpha \in T_n(\alpha)$ for all $\alpha < \lambda$. So the problem is to choose countably many of them for any n.

Let ϑ be defined as in (7), and $\mathcal{M} = \mathrm{L}_\vartheta$. $\vartheta, \mathcal{M}$

Let us start with $T_0(\lambda)$. Thus we have to define a countable set $T_0(\lambda) = S \subseteq (\mathbb{Q}^+)^\lambda$ satisfying

(i) if $s \in S$ and $\gamma < \lambda$ then $s \restriction \gamma \in T_0(\gamma)$;

(ii) if $s \in S$ and $D \in \mathcal{M}$ is a dense subset of T_0 then $\exists\, \gamma < \lambda\, (s \restriction \gamma \in D)$;

(iii) if $s, s' \in S$ then $\sum(s \restriction \alpha, s' \restriction \alpha) < \sum(s, s') < +\infty$ for all $\alpha < \lambda$;

(iv) if $r \in \mathbb{Q}^+$, $\alpha < \lambda$, $t, t' \in T_0(\alpha)$, $s \in S$, and $t \subset s$, then there is $s' \in S$ such that $t' \subset s'$ and $\sum_\alpha(s, s') < r$.

The construction can be carried out by a rather cumbersome forcing over $\mathcal{M}$, described in [4]. To present the idea but avoid most of technicalities, let us conduct a simpler construction. Namely, suppose that $r \in \mathbb{Q}^+$, $\alpha < \lambda$, t and t' belong to $T_0(\alpha)$, and a sequence $s \in (\mathbb{Q}^+)^\lambda$ satisfies $t \subset s$, (i), and (ii). Find a sequence $s' \in (\mathbb{Q}^+)^\lambda$ satisfying $t' \subset s'$, (i), (ii), and $\sum_\alpha(s, s') < r$.

Since λ and $\mathcal{M}$ are countable, the following is sufficient:

Claim *Assume that* $\varepsilon \in \mathbb{Q}^+$, $\gamma < \lambda$, $t \in T_0(\gamma)$, *and* $D \in \mathcal{M}$ *is dense in* T_0. *There exist an ordinal* γ', $\gamma < \gamma' < \lambda$, *and* $t' \in T_0(\gamma') \cap D$, *such that* $t \subset t'$ *and* $\sum_\gamma(\sigma, t') < \varepsilon$, *where* $\sigma = s \restriction \gamma'$.

Proof The set $D' = \{\sigma \in T_0 : \exists\, t' \in T_0 \cap D\, [\, t \subseteq t' \,\&\, |t'| = |\sigma| \,\&\, \sum_\gamma(\sigma, t') < \varepsilon\,]\}$ belongs to $\mathcal{M}$ and is dense in T_0 by (9), therefore there is an ordinal γ', $\gamma < \gamma' < \lambda$, such that $\sigma = s \restriction \gamma' \in D'$. $\dashv$

Now suppose that $S_n = T_n(\lambda)$ has been defined, and define $S_{n+1} = T_{n+1}(\lambda)$. S_n, S_{n+1}
We assume that S_n satisfies (i) and (ii) (for T_n rather than T_0, of course).

Define $\mathcal{T}_{(s)} = \bigcup_{\gamma<\lambda} \mathcal{T}_{(s \restriction \gamma)}$ for all $s \in S_n$. Thus $\mathcal{T}_{(s)}$ is a subtree of T_n.

Claim *If* $D \in \mathcal{M}$ *is a dense subset of* T_{n+1} *then, for any* $s \in T_n(\lambda)$, *the intersection* $D \cap \mathcal{T}_{(s)}$ *is dense in* $\mathcal{T}_{(s)}$.

Proof Let us fix $t_0 \in \mathcal{T}_{(s)}$. Then $|t| = \gamma < \lambda$ and $\sigma_0 = \mathfrak{f}(t_0) \in T_n(\gamma + 1)$. Now

$$D' = \{\sigma \in T_n : |\sigma| > \gamma \;\&\; [\, \sigma_0 \subseteq \sigma \Longrightarrow \exists\, t \in \mathcal{T}_{(\sigma)}\, (t \in D \;\&\; t_0 \subseteq t)\,]\}$$

belongs to $\mathcal{M}$ and is dense in T_n, so $\sigma = s \restriction \gamma' \in D'$ for some $\gamma' < \lambda$. $\dashv$

This allows to define S_{n+1} as the union of separate parts, each part being defined within $\mathcal{T}_{(s)}$ for some $s \in S_n$ in the same way as $S = T'_0(\lambda)$ above.

This completes the definition of $T'_n(\lambda)$ for all n and $\mathcal{T}_{(s)}$ for all $s \in \bigcup_n T'_n(\lambda)$.

Case 2: $\mathfrak{T}$ is an embrion of height $\lambda + 1$, $\lambda < \omega_1$ still being a limit ordinal. Thus $T_n(\lambda)$ is defined (and is the maximal level in each T_n). Then $T_n(\lambda + 1) = \{s^\wedge q : s \in T_n(\lambda) \ \&\ q \in \mathbb{Q}^+\}$ by (3), so the task is to define $\mathcal{T}_{(\sigma)}$ for $\sigma \in T_n(\lambda+1)$.

Let $S^t_{n+1} = \{s \in T_n(\lambda) : \forall \gamma < \lambda \ (s \restriction \gamma \in \mathcal{T}_{(t)})\}$. By the construction the tree $\mathcal{T}_{(t)} \cup S^t_{n+1}$ is a normal $(\lambda + 1)$-tree satisfying (9). Moreover we can divide S^t_{n+1} onto countably many infinite pairwise disjoint parts, $S^t_{n+1} = \bigcup_k S^t_{n+1,k}$, so that still each $\mathcal{T}_{(t)} \cup S^t_{n+1,k}$ is a normal $(\lambda + 1)$-tree satisfying (9). Now fix a recursive enumeration $\mathbb{Q}^+ = \{q_k : k \in \omega\}$ and set $\mathcal{T}_{(t^\wedge q_k)} = \mathcal{T}_{(t)} \cup S^t_{n+1,k}$ for all k. *enumeration q_k*

Case 3: $\mathfrak{T}$ is an embrion of height $\lambda + j + 1$, $\lambda < \omega_1$ being limit or 0, and $j \geq 1$. Put $T_n(\lambda+j+1) = \{s^\wedge q : s \in T_n(\lambda+j) \ \&\ q \in \mathbb{Q}^+\}$, as above. Now define $\mathcal{T}_{(\sigma)}$ for $\sigma = s^\wedge q \in T_n(\lambda + j + 1)$. The tree $\mathcal{T}_{(s)}$ is a normal $(\lambda + j)$-tree, hence it has the maximal level $T = \mathcal{T}_{(s)}(\lambda + j - 1)$. Now set $\mathcal{T}_{(s^\wedge q_k)} = \mathcal{T}_{(s)} \cup \{t^\wedge q : t \in T \ \&\ q \in Q_k\}$ for all k, where Q_k are the sets introduced in Subsection 1.1. $\square$

Note that there exists an embrion of height 2 : put $T_n(1) = \{\langle q \rangle : q \in Q_n\}$ for all n, according to (2). (Obviously this is the only embrion of height 2.)

Corollary 5 [4] (assuming $\mathrm{V} = \mathrm{L}$) *There exists an increasing Δ^{HC}_1 sequence $\langle \mathfrak{T}_\alpha \rangle_{2 \leq \alpha < \omega_1}$ such that each $\mathfrak{T}_\alpha$ is an embrion of height α and $\mathfrak{T}_\beta$ extends $\mathfrak{T}_\alpha$ whenever $\alpha < \beta < \omega_1$.* $\square$

Set $T_n = \bigcup_{2 \leq \alpha < \omega_1} T_n(\alpha)$ for all n and define the map $t \longmapsto \mathcal{T}_{(t)}$ accordingly. Then both the map $\mathcal{T}_{(\cdot)}$ and the trees T_n uniformly on n belong to Δ^{HC}_1. Put $\mathfrak{T} = \langle \langle T_n \rangle_{n \in \omega}, \mathcal{T}_{(\cdot)} \rangle$ and define the forcing $\mathbb{P}_{\mathfrak{T}}$ as in Subsection 1.4.

Theorem 6 [4] *Any $\mathbb{P}_{\mathfrak{T}}$-generic extension of* L *has the form* $\mathrm{L}[a]$, *where a is a Δ^1_3 real in* $\mathrm{L}[a]$.

Proof The extension has the form $\mathrm{L}[\langle C_n \rangle_{n \in \omega}$, where each C_n is an ω^{L}_1-branch in T_n. Let $q_n = C_n(0)$. Then, the sequence of positive rationals $\langle q_n \rangle_{n \in \omega}$ is, in the extension, the only sequence in $(\mathbb{Q}^+)^\omega$ such that the procedure $\langle \ddagger \rangle$ "converges" in the sense of $\langle \S \rangle$, by Theorem 2. It remains to demonstrate that the condition $\langle \S \rangle$ can be expressed by a Π_1 formula in HC. But this is rather clear: the formula says that *any sequence of $\alpha < \omega^{\mathrm{L}}_1$ steps in the "procedure" $\langle \ddagger \rangle$ starting from $\langle q_n \rangle_n$ and satisfying $\langle \S \rangle$ can be extended by one more step so that $\langle \S \rangle$ is not violated.* $\square$

3 Proof of the theorem: part 1

Theorem 6 is equal to the main theorem (Theorem 1) for $n = 2$. The proof of the general case, presented in this section, follows the scheme of Jensen and Johnsbråten, but contains one more idea: the final ω_1-trees and the map $\mathcal{T}_{(\cdot)}$ (or, what is equivalent, the increasing ω_1-sequence of embrions which generates the former) must be "generic" in a sense relevant to the level Δ^1_{n+1} : roughly, it will intersect all dense subsets in the collection $\mathtt{Emb}$ of all (countable) embrions.

3.1 Formulas

We argue in L .

Let $\mathfrak{T} = \langle \langle T_n \rangle_{n\in\omega}, \mathcal{T}_{(\cdot)} \rangle$ be an embrion of height $\lambda < \omega_1$. Define and order $\mathbb{P}_{\mathfrak{T}} = \lim T_n$ as in Subsection 1.4. $\mathbb{P}_{\mathfrak{T}}$

Let $\mathcal{M}(\mathfrak{T}) = \mathrm{L}_\vartheta$, where $\vartheta < \omega_1$, as in (7), is the least ordinal such that L_ϑ models $\mathbf{ZFC}^-$, λ is countable in L_ϑ, and $\mathfrak{T} \in \mathrm{L}_\vartheta$. We observe that $T \in \mathcal{M}(\mathfrak{T})$. $\mathcal{M}(\mathfrak{T})$

Define $\mathtt{Trm}(\mathfrak{T})$ to be the set of all $\mathfrak{T}$-*terms* for subsets of ω, that is, all countable sets $\tau \subseteq \mathbb{P}_{\mathfrak{T}} \times \omega$. Put $\mathtt{Trm}^*(\mathfrak{T}) = \mathtt{Trm}(\mathfrak{T}) \cap \mathcal{M}(\mathfrak{T})$. $\mathtt{Trm}(\mathfrak{T})$ $\mathtt{Trm}^*(\mathfrak{T})$

We use a special language to facilitate the study of analytic phenomena in T-generic extensions. Let $\mathcal{L}$ be the language containing variables l, m, i, j of type 0 (for natural numbers) and x, y, z of type 1 (for subsets of ω), arithmetical predicates for type 0 and the membership $i \in x$. $\mathcal{L}$

Let a $\mathfrak{T}$-*formula* be a formula of $\mathcal{L}$ some (or all) free variables of which, of types 0 and 1, are substituted by resp. natural numbers and elements of $\mathtt{Trm}^*(\mathfrak{T})$. $\mathfrak{T}$-*formula*

If φ is a $\mathfrak{T}$-formula and $G \subseteq \mathbb{P}_{\mathfrak{T}}$ then $\varphi[G]$ will denote the formula obtained by substitution, in φ, of each term $\tau \in \mathtt{Trm}^*(\mathfrak{T})$ by the set $\tau[G] = \{l \in \omega : \exists \vec{t} \in G\, (\langle \vec{t}, l\rangle \in \tau)\}$. Thus $\varphi[G]$ is a formula of $\mathcal{L}$ containing subsets of ω as parameters. $\varphi[G]$

Let $\mathfrak{T}\Sigma^0_\infty$-*formula* be any $\mathfrak{T}$-formula which does not contain quantifiers over variables of type 1. Formulas of the form $\mathfrak{T}\Sigma^0_\infty$-*formula*

$$\exists\, x_1\, \forall\, x_2\, \exists\, x_3\, \dots\, \forall\,(\exists)\, x_k\, \psi\,, \quad \forall\, x_1\, \exists\, x_2\, \forall\, x_3\, \dots\, \exists\,(\forall)\, x_k\, \psi\,, \quad \text{where} \quad \psi \in \mathfrak{T}\Sigma^0_\infty\,,$$

will be called resp. $\mathfrak{T}\Sigma^1_k$-*formulas* and $\mathfrak{T}\Pi^1_k$-*formulas*. $\mathfrak{T}\Sigma^1_k$-*formula* $\mathfrak{T}\Pi^1_k$-*formula*

3.2 "Approximations" of the forcing

We introduce the relation $\vec{t}\ \mathtt{forc}_{\mathfrak{T}}\ \varphi$. Here it is assumed that $\mathfrak{T} \in \mathtt{Emb}$, $\vec{t} \in \mathbb{P}_{\mathfrak{T}}$, and φ is a closed $\mathfrak{T}$-formula of one of the classes $\mathfrak{T}\Sigma^1_k$, $\mathfrak{T}\Pi^1_k$. The definition goes on by induction on k.

(F1) If $\varphi \in \mathfrak{T}\Sigma^0_\infty \cup \mathfrak{T}\Sigma^1_1 \cup \mathfrak{T}\Pi^1_1$ then $\vec{t}\ \mathtt{forc}_{\mathfrak{T}}\ \varphi$ iff ($\mathfrak{T}, \vec{t}, \varphi$ are as above and) $\vec{t} \Vdash_{\mathfrak{T}} \varphi$, where $\Vdash_{\mathfrak{T}}$ is the ordinary forcing in the sense of $\mathcal{M}(\mathfrak{T})$ as the initial model and $\mathbb{P}_{\mathfrak{T}}$ as the notion of forcing.

(F2) Let $k \geq 1$, $\varphi(x) \in \mathfrak{T}\Pi^1_k$. Define $\vec{t}\,\mathtt{forc}_{\mathfrak{T}}\; \exists x\; \varphi(x)$, iff there is a term $\tau \in \mathtt{Trm}^*(\mathfrak{T})$ such that $\vec{t}\,\mathtt{forc}_{\mathfrak{T}}\; \varphi(\tau)$.

(F3) Let $k \geq 2$, φ is a closed $\mathfrak{T}\Pi^1_k$ formula. Put $\vec{t}\,\mathtt{forc}_{\mathfrak{T}}\; \varphi$ if $\neg\, \vec{s}\,\mathtt{forc}_{\mathfrak{S}}\; \varphi^-$ for any countable embrion $\mathfrak{S} \in \mathtt{Emb}$ which extends $\mathfrak{T}$ and any $\vec{s} \in \mathbb{P}_{\mathfrak{S}}$, $\vec{s} \geq \vec{t}$, where φ^- is the result of the transformation of $\neg\,\varphi$ to the $\mathfrak{T}\Sigma^1_k$ form.

The following statements are true for the usual forcing, hence true for the relation $\mathtt{forc}$ restricted on formulas φ in $\mathfrak{T}\Sigma^0_\infty \cup \mathfrak{T}\Sigma^1_1 \cup \mathfrak{T}\Pi^1_1$, while the extension on more complicated formulas is easily carried out by induction.

(a) *If* $\vec{t}\,\mathtt{forc}_{\mathfrak{T}}\; \varphi$ *and* $\mathfrak{S} \in \mathtt{Emb}$ *extends* $\mathfrak{T}$, $\vec{s} \in \mathbb{P}_{\mathfrak{S}}$, $\vec{s} \geq \vec{t}$, *then* $\vec{s}\,\mathtt{forc}_{\mathfrak{S}}\; \varphi$.

(b) $\vec{t}\,\mathtt{forc}_{\mathfrak{T}}\; \varphi$ *and* $\vec{t}\,\mathtt{forc}_{\mathfrak{T}}\; \varphi^-$ *are incompatible.*

Now consider the *complexity* of the relation $\mathtt{forc}$.

Suppose that $\varphi(x_1, \dots, x_m, l_1, \dots, l_\mu)$ is a parameter–free formula of $\mathcal{L}$. Put

$$\begin{aligned}\mathtt{Forc}(\varphi) = \;& \{\langle \mathfrak{T}, \vec{t}, \tau_1, \dots, \tau_m, l_1, \dots, l_\mu\rangle : \mathfrak{T} \in \mathtt{Emb}\ \&\ \tau_1, \dots, \tau_m \in \mathtt{Trm}^*(\mathfrak{T})\ \&\\ & \vec{t} \in \mathbb{P}_{\mathfrak{T}}\ \&\ l_1, \dots, l_\mu \in \omega\ \&\ \vec{t}\,\mathtt{forc}_{\mathfrak{T}}\; \varphi(\tau_1, \dots, \tau_m, l_1, \dots, l_\mu)\}\,.\end{aligned}$$

Theorem 7 *If* φ *is a formula of one of the classes* Σ^0_∞, Σ^1_1, Π^1_1, *then* $\mathtt{Forc}(\varphi) \in \Delta^{\mathrm{HC}}_1$. *If* $k \geq 2$ *and* $\varphi \in \Pi^1_k$, *then* $\mathtt{Forc}(\varphi) \in \Sigma^{\mathrm{HC}}_{k-1}$. *Esli* $k \geq 2$ *i* $\varphi \in \Sigma^1_k$, *to* $\mathtt{Forc}(\varphi) \in \Pi^{\mathrm{HC}}_{k-1}$.

Proof The result for $\varphi \in \Sigma^0_\infty \cup \Sigma^1_1 \cup \Pi^1_1$ follows from the definability of the usual forcing $\Vdash_{\mathfrak{T}}$ in the model $\mathcal{M}(\mathfrak{T})$, which is uniformly $\Delta^{\mathrm{HC}}_1(\mathfrak{T})$. The induction step is clear. $\square$

We now formulate a theorem which states that the relation $\vec{t}\,\mathtt{forc}_{\mathfrak{T}}\; \varphi$ actually does not depend on the choice of $\vec{t}$ and $\mathfrak{T}$, provided φ is a parameter–free formula.

Theorem 8 *Let* φ *be a closed* $\mathfrak{T}\Sigma^1_k$*-formula, which does not contain parameters of type* 1, $\mathfrak{T}, \mathfrak{T}' \in \mathtt{Emb}$, *and* $\vec{t} \in \mathbb{P}_{\mathfrak{T}}$, $\vec{t}' \in \mathbb{P}_{\mathfrak{T}'}$. *Then* $\vec{t}\,\mathtt{forc}_{\mathfrak{T}}\; \varphi$ *is inconsistent with* $\vec{t}'\,\mathtt{forc}_{\mathfrak{T}'}\; \varphi^-$.

The proof (see below) is based on a system of automorphisms of $\mathtt{Emb}$.

3.3 Isomorphisms between the embrions

Suppose that $\mathfrak{T} = \langle\langle T_n\rangle_{n\in\omega}, \mathcal{T}_{(\cdot)}\rangle$ and $\mathfrak{T}' = \langle\langle T'_n\rangle_{n\in\omega}, \mathcal{T}'_{(\cdot)}\rangle$ are two embrions, of equal heigth $\lambda < \omega_1$. An *isomorphism of* $\mathfrak{T}$ *onto* $\mathfrak{T}'$ is a collection $h = \langle h_n\rangle_{n\in\omega}$ of order isomorphisms $h_n : T_n$ onto T'_n such that, for all n and $t \in T_n$, the map h_{n+1} restricted on $\mathcal{T}_{(t)}$ is an order isomorphism of $\mathcal{T}_{(t)}$ onto $\mathcal{T}'_{(h(t))}$. [2] *isomorphism of embrions*

In this case, if $\vec{t} = \langle t_0, \dots, t_n\rangle \in \mathbb{P}_{\mathfrak{T}}$ (so that $t_i \in T_i$ and $t_i = \mathfrak{f}(t_{i+1})$ for all i), then we put $h\vec{t} = \langle h_0(t_0), \dots, h_n(t_m)\rangle$; then $h\vec{t} \in \mathbb{P}_{\mathfrak{T}'}$. $h\vec{t}$

Let $\mathtt{Isom}(\mathfrak{T}, \mathfrak{T}')$ denote the set of all isomorphisms of $\mathfrak{T}$ onto $\mathfrak{T}'$. $\mathtt{Isom}(\mathfrak{T}, \mathfrak{T}')$

Theorem 9 *Suppose that* $\lambda < \omega_1$ *is a limit ordinal and* $\mathfrak{T}$, $\mathfrak{T}'$ *are embrions of height* $\lambda + 1$, $\vec{t} = \langle t_0, \dots, t_n\rangle \in \mathbb{P}_{\mathfrak{T}}$, $\vec{t}' = \langle t_0, \dots, t_n\rangle \in \mathbb{P}'_{\mathfrak{T}}$, *and* $|t_0| = |t'_0|$. (Then $|t_k| = |t'_k| = |t_0| - k$ for all k.) *Then there is an isomorphism* $h \in \mathtt{Isom}(\mathfrak{T}, \mathfrak{T}')$ *such that* $h\vec{t} = \vec{t}'$.

Proof Let us define an order isomorphism $h_0 : T_0$ onto T'_0 such that $h_0(t_0) = t'_0$.

Fix $\sigma_0 \in T_0(\lambda)$ and $\sigma'_0 \in T'_0(\lambda)$ such that $t_0 \subset \sigma_0$ and $t'_0 \subset \sigma'_0$.

A function H will be called *a correct map* if $\mathtt{dom}\, H$ and $\mathtt{ran}\, H$ are subsets of $(\mathbb{Q}^+)^\lambda$, H is $1-1$, and, for all $s \neq t \in \mathtt{dom}\, H$, if $s' = g(s)$ and $t' = g(t)$, then the maximal $\alpha < \lambda$ such that $s \restriction \alpha = t \restriction \alpha$ is equal to the maximal $\alpha < \lambda$ such that $s' \restriction \alpha = t' \restriction \alpha$. *correct map*

Since all elements of T and T', except for those of the maximal levels $T(\lambda)$ and $T'(\lambda)$, have infinitely (countably) many successors, there is a correct map $g : T(\lambda)$ onto $T'(\lambda)$ such that $H(\sigma_0) = \sigma'_0$. Set $h_0(t) = g(t)$ for $t \in T_0(\lambda)$. If $s \in T_0 \restriction_{<\lambda}$ then pick any $t \in T_0(\lambda)$ satisfying $s \subset t$, and put $h_0(s) = g(t) \restriction |s|$.

Let us demonstrate how to get h_1.

We observe that, for any $t \in T_0(\lambda)$, $S_t = \mathcal{T}_{(t)}$ is a normal λ-tree, a subtree of $T_1 \restriction_{<\lambda}$. Let $L_t = \{t_1 \in T_1(\lambda) : \forall\, \alpha < \lambda\, (t_1 \restriction \alpha \in S_t)\}$. Then $\overline{S}_t = S_t \cup L_t$ is normal $(\lambda+1)$-tree, a subtree of T_1, having L_t as its λ-th (and the upper) level, and $T_1(\lambda)$ is a pairwise disjoint union of L_t.

Using the same construction, we define $S'_{t'}$, $L'_{t'}$, $\overline{S'}_{t'}$ for all $t' \in T'_0(\lambda)$. Then, similarly to the case of T_0 and T'_0 above, we cad define a correct map $g : T_1(\lambda)$ onto $T'_1(\lambda)$ which maps each $\overline{S}_t$ onto $\overline{S'}_{h_0(t)}$. This leads to an order isomorphism $h_1 : T_1$ onto T'_1, as above. A separate point is to guarantee that $h_1(t_1) = t'_1$. Pick $\sigma_1 \in L_{\sigma_0}$ and $\sigma'_1 \in L'_{\sigma'_0}$ so that $t_1 \subset \sigma_1$ and $t'_1 \subset \sigma'_1$. Now it suffices to arrange the action of g on L_t so that $g(\sigma_1) = \sigma'_1$.

The same argument allows to obtain, by induction, all other order isomorphisms $h_n : T_n$ onto T'_n satisfying $h_n(t_n) = t'_n$. $\square$

[2] The definition contains nothing to match the "metric" properties (8) and (9).

3.4 Extensions of isomorphisms on higher embrions

Let $\mathfrak{T} = \langle\langle T_n\rangle_{n\in\omega}, \mathcal{T}_{(\cdot)}\rangle$ and $\mathfrak{T}' = \langle\langle T'_n\rangle_{n\in\omega}, \mathcal{T}'_{(\cdot)}\rangle$ be embrions of one and the same height $\eta+1$, $\eta<\omega_1$, and $h = \langle h_n\rangle_{n\in\omega} \in \mathtt{Isom}(\mathfrak{T},\mathfrak{T}')$. η

In this case, the action of h can be correctly defined for any embrion $\mathfrak{S} = \langle\langle S_n\rangle_{n\in\omega}, \mathcal{S}_{(\cdot)}\rangle \in \mathtt{Emb}$ which extends $\mathfrak{T}$. Indeed suppose that $n\in\omega$ and $s\in S_n(\gamma)$. If $\gamma\le\eta$ then $s\in T_n(\gamma)$, and we put $h^+_n(s) = h_n(s)$. If $\eta<\gamma<|\mathfrak{S}|$ then $s\restriction\eta \in T_n(\eta)$ (the maximal level of T_n), so that $h_n(s\restriction\eta)\in T'_n(\eta)$ is defined. Define $s' = h^+_n(s)\in(\mathbb{Q}^+)^\gamma$ so that $s'\restriction\eta = h_n(s\restriction\eta)$ while $s'(\alpha)=s(\alpha)$ for all $\eta\le\alpha<|\mathfrak{S}|$. Thus $h^+_n(s)$ is defined for all n and $s\in S_n$. h^+

We let $S'_n = \{h^+(s) : s\in S_n\}$ for each n. Now define the associated map $\mathcal{S}'_{(\cdot)}$. Suppose that $s' = h^+(s)\in S'_n$, so that $s\in S_n$. Put $\mathcal{S}'_{(s')} = \{h^+_{n+1}(t) : t\in\mathcal{S}_{(s)}\}$.

This ends the definition of $\mathfrak{S}' = \langle\langle S'_n\rangle_{n\in\omega}, \mathcal{S}_{(\cdot)}\rangle$. We shall write $\mathfrak{S}' = h\mathfrak{S}$. $h\mathfrak{S}$

Lemma 10 *In this case, if* $\mathcal{M}(\mathfrak{S}')\subseteq\mathcal{M}(\mathfrak{S})$ *and* $h\in\mathcal{M}(\mathfrak{S})$ *then* $\mathfrak{S}'$ *is an embrion extending* $\mathfrak{T}'$ *and* $h^+\in\mathtt{Isom}(\mathfrak{S},\mathfrak{S}')$.

Proof It suffices to check only (7), (8), and (9) for $\mathfrak{S}'$ below $\eta+1$; the rest of requirements is quite obvious.

Consider (7). Let $\lambda<|\mathfrak{S}|=|\mathfrak{S}'|$ be a limit ordinal, $\mathcal{M}'=\mathcal{M}(\mathfrak{S}'\restriction\lambda)$, and $D'\in\mathcal{M}'$ be a pre-dense subset of $S'_n\restriction\lambda$. Prove that any $s'=h^+(s)\in S'_n(\lambda)$ satisfies $\exists\,\alpha<\lambda\,(s'\restriction\alpha\in D)$. The case $\lambda\le\eta$ is clear: apply (7) for $\mathfrak{T}'$. Thus we assume that $\eta<\lambda<|\mathfrak{S}'|$. Then $\mathcal{M}(\mathfrak{S}'\restriction\lambda)\subseteq\mathcal{M}(\mathfrak{S}\restriction\lambda)$ because $h\in\mathcal{M}(\mathfrak{T})$ and $\mathcal{M}(\mathfrak{T}')\subseteq\mathcal{M}(\mathfrak{T})$. It follows that $D=\{t\in S_n\restriction_{<\lambda} : h^+(t)\in D'\}$ belongs to $\mathcal{M}(\mathfrak{S}\restriction\lambda)$. Moreover, D is a pre-dense subset of $S_n\restriction_{<\lambda}$. Therefore $s\restriction\alpha\in D$ for some $\alpha<\lambda$. Then $s'\restriction\alpha\in D'$, as required.

Consider (8). Suppose that S' is S'_0 or $\mathcal{S}'_{(s')}$ for some $s'=h^+(s)\in S'_n$, $\alpha<\lambda<|\mathfrak{S}'|$, λ is limit, and $s'_1=h(s_1)$, $s'_2=h(s_2)$ belong to $S'(\lambda)$. (Then s_1 and s_2 belong to $S=\mathcal{S}_{(s)}$.) Prove that $\sum(s'_1\restriction\alpha, s'_2\restriction\alpha)<\sum(s'_1,s'_2)<+\infty$. Assume $\eta<\lambda$ (the nontrivial case). To prove the right inequality note that

$$\textstyle\sum(s'_1,s'_2)=\sum(s'_1\restriction\lambda, s'_2\restriction\lambda)+\sum_{\lambda\le\gamma<|\mathfrak{S}|}|s_1(\gamma)-s_2(\gamma)|\,,$$

by definition, so the result follows from the fact that $\mathfrak{S}$ and $\mathfrak{T}'$ are embrions. The left inequality is demonstrated similarly.

Finally prove (9). Suppose that S' is S'_0 or $\mathcal{S}'_{(s')}$ for some $s'=h^+(s)\in S'_n$, $r\in\mathbb{Q}^+$, $\beta<\eta<\alpha<|S'|$ (the nontrivial case), and $t'_1=h(t_1)$, $t'_2=h(t_2)$ belong to $S'(\beta)$, and $s'_1=h(s_1)\in S'(\alpha)$, $t'_1\subset s'_1$. We have to find $s'_2\in S'(\alpha)$ such that $t'_2\subset s'_2$ and $\sum(s'_1,s'_2)-\sum(t'_1,t'_2)<r$.

Let $\sigma'_1=s'_1\restriction\eta$, so that $\sigma'_1=h(\sigma_1)\in S'(\eta)$, where $\sigma_1\in S(\eta)$ while either $S=S_0$ or $S=\mathcal{S}_{(s)}$. Since $\mathfrak{T}'$ is an embrion, there exists $\sigma'_2=h(\sigma_2)\in S'(\eta)$

(where $\sigma_2 \in S(\eta)$) such that $t_2' \subset \sigma_2'$ and $\sum(\sigma_1', \sigma_2') - \sum(t_1', t_2') < r/2$. Since $\mathfrak{S}$ is an embrion, there is $s_2 \in S(\alpha)$ such that $\sigma_2 \subset s_2$ and $\sum(s_1, s_2) - \sum(\sigma_1, \sigma_2) < r/2$. Now $s_2' = h(s_2)$ is as required because by definition

$$\textstyle\sum(s_1', s_2') - \sum(\sigma_1', \sigma_2') = \sum(s_1, s_2) - \sum(\sigma_1, \sigma_2)\,. \qquad \square$$

3.5 Extensions of isomorphisms on terms and formulas

Suppose that $h \in \mathtt{Isom}(\mathfrak{T}, \mathfrak{T}')$. Then h induces an order isomorphism $\vec{t} \longmapsto h\vec{t}$ from $\mathbb{P} = \mathbb{P}_{\mathfrak{T}}$ onto $\mathbb{P}' = \mathbb{P}_{\mathfrak{T}'}$, hence if $\tau \in \mathtt{Trm}(\mathfrak{T})$ then $h\tau$

$$h\tau = \{\langle h\vec{t}, l\rangle : \langle \vec{t}, l\rangle \in \tau\} \in \mathtt{Trm}(\mathfrak{T}')\,,$$

We shall assume that

$\langle **\rangle$ $\mathfrak{M}(\mathfrak{T}) = \mathfrak{M}(\mathfrak{T}')$ and $h \in \mathfrak{M}(\mathfrak{T})$.

Then $h\tau \in \mathtt{Trm}^*(\mathfrak{T}')$ whenever $\tau \in \mathtt{Trm}^*(\mathfrak{T})$. Furthermore, if, assuming $\langle **\rangle$, Φ is a $\mathfrak{T}$-formula then the formula $h\Phi$, obtained by changing of every term $\tau \in \mathtt{Trm}^*(\mathfrak{T})$ in Φ by $h\tau$, is a $\mathfrak{T}'$-formula.

Note finally that $h^{-1} \in \mathtt{Isom}(\mathfrak{T}', \mathfrak{T})$, and the consecutive action of h and h^{-1} on conditions, terms, and formulas, is idempotent.

Lemma 11 *Let $\mathfrak{T}$ and $\mathfrak{T}'$ be embrions of equal height $\lambda < \omega_1$. Suppose that $h \in \mathtt{Isom}(\mathfrak{T}, \mathfrak{T}')$ and $\langle **\rangle$ holds. Assume finally that $\vec{t} \in \mathbb{P}_{\mathfrak{T}}$ and Φ is a $\mathfrak{T}$-formula. Then $\vec{t}\,\mathtt{forc}_{\mathfrak{T}}\,\Phi$ iff $h\vec{t}\,\mathtt{forc}_{\mathfrak{T}'}\,h\Phi$.*

Proof is carried out by induction on the complexity of Φ .

Let Φ be a formula in $\mathfrak{T}\Sigma^0_\infty \cup \mathfrak{T}\Sigma^1_1 \cup \mathfrak{T}\Pi^1_1$ (case (F1) in the definition). Then h defines, in $\mathcal{M}(\mathfrak{T}) = \mathcal{M}(\mathfrak{T}')$, an order isomorphism $\mathbb{P}_{\mathfrak{T}}$ onto $\mathbb{P}_{\mathfrak{T}'}$, such that $\varphi[G]$ is equal to $(h\varphi)[h\text{"}G]$ for any set $G \subseteq \mathbb{P}_{\mathfrak{T}}$ and any $\mathfrak{T}$-formula φ. This implies the result by the ordinary forcing theorems. ($p\text{"}G$ is the p-image of G .)

The induction steps (F2) and (F3) do not cause any problem. (However Lemma 10 participates in the induction step (F3).) $\square$

3.6 Proof of Theorem 8

Suppose, towards the contrary, that $\vec{t}\,\mathtt{forc}_{\mathfrak{T}}\,\varphi$ but $\vec{t'}\,\mathtt{forc}_{\mathfrak{T}'}\,\varphi^-$. We may assume that $\mathfrak{T}$ and $\mathfrak{T}'$ are embrions of one and the same length $\eta + 1$, $\eta < \omega_1$ being a limit ordinal. Moreover we can suppose that $\vec{t} = \langle t_0, \dots, t_n\rangle$ and $\vec{t'} = \langle t_0', \dots, t_n'\rangle$ for one and the same n, and $|t_0| = |t_0'|$ (then $|t_k| = |t_k'|$ for all k).

We observe that by definition $\mathcal{M}(\mathfrak{T}) = \mathrm{L}_\vartheta$ and $\mathcal{M}(\mathfrak{T}') = \mathrm{L}_{\vartheta'}$, where ϑ and ϑ' are countable (limit) ordinals. Let, for instance, $\vartheta' \le \vartheta$. Then, in $\mathcal{M} = \mathcal{M}(\mathfrak{T})$,

$\mathfrak{T}$ and $\mathfrak{T}'$ remain embrions of length $\eta+1$, countable in $\mathcal{M}$. Moreover, applying Theorem 9 in $\mathcal{M}$, we get an isomorphism $h \in \mathtt{Isom}(\mathfrak{T}, \mathfrak{T}') \cap \mathcal{M}$, satisfying $h\vec{t} = \vec{t'}$.

Since we do not assume $\mathcal{M}(\mathfrak{T}) = \mathcal{M}(\mathfrak{T}')$, Lemma 11 cannot be applied directly. However take any embrion $\mathfrak{S}$ of length ϑ, extending $\mathfrak{T}$. Then, by Theorem 10, $\mathfrak{S}' = h\mathfrak{S}$ is an embrion of the same length ϑ and $\mathcal{M}(\mathfrak{S}) = \mathcal{M}(\mathfrak{S}')$. Furthermore there is an extension $h^+ \in \mathcal{M}(\mathfrak{S}) \cap \mathtt{Isom}(\mathfrak{S}, \mathfrak{S}')$ of h.

To complete the proof note that $\vec{t}\ \mathtt{forc}_{\mathfrak{S}}\ \varphi$ and $\vec{t'}\ \mathtt{forc}_{\mathfrak{S}'}\ \varphi^-$ by (a). Applying Lemma 11 to the first statement, we obtain $\vec{t'}\ \mathtt{forc}_{\mathfrak{S}'}\ h\varphi$. However $h\varphi$ coincides with φ, because φ does not contain terms. Thus $\vec{t'}\ \mathtt{forc}_{\mathfrak{T}'}\ \varphi$, which is a contradiction with the assumption $\vec{t'}\ \mathtt{forc}_{\mathfrak{T}'}\ \varphi^-$ by (b).

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