STABLE CURVES AND SCREENS ON FATGRAPHS

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Abstract. The mapping class group invariant ideal cell decomposition of the Teichmüller space of a punctured surface times an open simplex has been used in a number of computations. This paper answers a question about the asymptotics of this decomposition, namely, in a given cell of the decomposition, which curves can be short? Screens are a new combinatorial structure which provide an answer to this question. The heart of the calculation here involves Ptolemy transformations and the triangle inequalities on lambda lengths.

1. Introduction

Throughout this paper, $F = F_g^s$ will denote a fixed smooth oriented surface of genus g with $s \geq 1$ punctures, where $2g - 2 + s > 0$, with mapping class group $MC(F)$.

Let $\mathcal{T}(F)$ denote the Teichmüller space of $F = F_g^s$ and $\widetilde{\mathcal{T}}(F)$ denote the trivial $\mathbb{R}^s_{>0}$ bundle over it. Let $\mathcal{M}(F) = \mathcal{T}(F)/(MC(F)$ denote Riemann's moduli space with its Deligne-Mumford compactification $\overline{\mathcal{M}}(F)$. $MC(F)$ also acts on $\widetilde{\mathcal{T}}(F)$ by permuting the numbers assigned to punctures.

There is a $MC(F)$-invariant ideal cell decomposition [7, 10, 15, 22] of $\widetilde{\mathcal{T}}(F)$ which has found wide application in geometry and physics [1, 8, 9, 11, 12, 13, 14, 16]. Cells in this decomposition are in one-to-one correspondence with homotopy classes of "fatgraph spines" of F, that is, a homotopy class of embedded graph in F in the usual sense together with cyclic orderings on the half-edges about each vertex. (See the next section for further precision)

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Thus, to each fatgraph spine G of F, there is a corresponding cell $C(G) \subset \widetilde{\mathcal{T}}(F)$. In the interests of understanding $\overline{\mathcal{M}}(F)$ combinatorially, it is natural ask:

Question 1.1. Given G and given a collection K of non-parallel and non-puncture-parallel disjointly embedded and essential simple closed curves in F, when is there $\tilde{\Gamma} \in C(G)$ so that the curves in K are the only curves of hyperbolic length less than ϵ for the underlying hyperbolic metric, for some small $\epsilon > 0$? That is, which curves can be short in $C(G)$?

We give in this paper a complete answer to this question, as follows, where we shall concentrate in this introduction on the case that G is trivalent for simplicity.

Let E denote the set of edges of G and consider any proper subset $A \subset E$. There is a smallest (not necessarily connected) subgraph G_A of G containing A, and we say that A is "recurrent" if G_A has no univalent vertices. (Again, see the next section for a more detailed discussion of recurrence.) Suppose A is recurrent and G_A is connected, and get rid of all bivalent vertices of G_A in the usual way to produce either a simple cycle in G or another trivalent fatgraph G'. A neighborhood of $G_A \subset G$ in F is a subsurface of F, an annulus in the former case and a punctured surface of negative Euler characteristic in the latter. Define the "relative boundary" of A to be the edge-path in G of the simple cycle itself in the former case and those of the boundary components of this subsurface in the latter case, where you discard any such cycles that are puncture-parallel in F itself.

The new combinatorial structure which provides the answer to Question 1.1 (and was introduced in [19]), a "screen on a fatgraph G" is a subset $\mathcal{A}$ of the power set (i.e., the set of subsets) of the set E of edges of G with the following properties:

i) $E \in \mathcal{A}$;

ii) each $A \in \mathcal{A}$ is recurrent;

iii) if $A, B \in \mathcal{A}$ with $A \cap B \neq \emptyset$, then either $A \subseteq B$ or $B \subseteq A$;

iv) for each $A \in \mathcal{A}$, $\cup\{B \in \mathcal{A} : B \text{ is a proper subset of } A\}$ is a proper subset of A.

Condition i) is simply a convenient convention, conditions iii-iv) are familiar from Fulton-MacPherson [5], and here we impose the further condition ii) of recurrence. Notice that the properness condition iv) and recurrence condition ii) together imply that if G_A is a simple cycle in G, then for any screen $\mathcal{A}$ on G with $A, B \in \mathcal{A}$ and $A \cap B \neq \emptyset$, we must have $A \subseteq B$, i.e., simple cycles are necessarily atomic in any screen.

Each element $A \in \mathcal{A}$ other than $A = E$ has an immediate predecessor $A' \in \mathcal{A}$, and regarding A as a set of edges in $G_{A'}$ in the natural way, has its relative boundary $\partial_{\mathcal{A}} A$ defined before. Finally, the "boundary" of the screen itself is $\partial \mathcal{A} = \bigcup_{A \in \mathcal{A} - \{E\}} \partial_{\mathcal{A}} A$.

Here is the answer to Question 1.1, our main result:

Theorem 1.2. *For any fatgraph G, the cell $C(G)$ admits as short curves a family K of non-parallel and non-puncture parallel disjointly embedded and essential simple closed curves in F if and only if $K = \partial \mathcal{A}$ for some screen $\mathcal{A}$ on G.*

Let us immediately do several examples, where it is typically easiest to study the quotient of $\widetilde{\mathcal{T}}(F)$ by the natural $\mathbb{R}_{>0}$-action, the projectivized space, which we shall denote

$$P\widetilde{\mathcal{T}}(F) = \widetilde{\mathcal{T}}(F)/\mathbb{R}_{>0} \approx \mathcal{T}(F) \times \Delta^{s-1},$$

where Δ^p denotes the open p-dimensional simplex. In particular for a once-puntured surface, we have $P\widetilde{\mathcal{T}}(F) = \mathcal{T}(F)$.

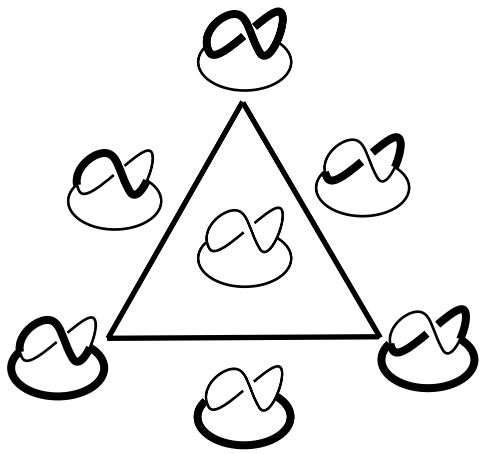

Figure 1 Screens on the once-punctured torus.

Example 1.3. For the once-punctured torus $F = F_1^1$, the ideal cell decomposition of $\mathcal{T}(F)$ is the Farey tesselation of the disk [15]. In Figure 1 we depict a typical top-dimensional 2-cell, which is indexed by a non-planar fatgraph G with two 3-valent vertices as is also illustrated. The codimension-one cells arise by collapsing any one of the three edges shown as darkened in the figure, and the codimension-two cells at infinity are indexed by the three possible recurrent subgraphs of G as likewise illustrated. In this example, non-empty screens are always singletons, and the boundary of a screen always consists of a single curve.

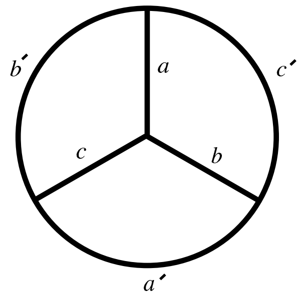

Figure 2 A fatgraph for the four-punctured sphere.

Example 1.4. For the four-times punctured sphere $F = F_0^4$, consider the Mercedes sign fatgraph G depicted in Figure 2. Both screens $\mathcal{A}_1 = \{\{a, b, a', b'\}\}$ and $\mathcal{A}_2 = \{\{a, b, c, a', b'\}, \{a, c, b'\}\}$ correspond to pinching to zero the closed edge-path $a - b - a' - b'$, and both screens have this same edge-path as boundary.

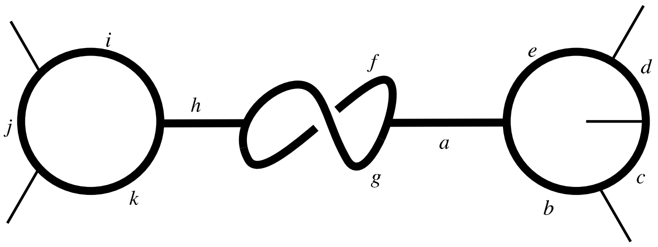

Figure 3 Typical example.

Example 1.5. Consider the sub-fatgraph of a fatgraph G with edges E depicted in Figure 3 and the screen

$$\mathcal{A} = \big\{E, \{a - k\}, \{a - g\}, \{f - g\}, \{b - e\}, \{i - k\}\big\}$$

on G. The boundary of $\mathcal{A}$ is comprised of the four edge-paths $f-g$, $b-c-d-e$, $h-i-j-k-h-f-g$, and $a-b-c-d-e-a-f-g$.

We shall rely on "lambda length" coordinates from [15] (recalled in §3) on the *decorated Teichmüller space* $\widetilde{\mathcal{T}}(F)$, where the fiber over a point is taken to be the set of all s-tuples of horocycles, one horocycle about each puncture; one may take the hyperbolic lengths of the distinguished horocycles as a convenient coordinate on the fiber.

In effect, we shall record the rates of divergence of lambda lengths regarded as projective coordinates on $P\widetilde{\mathcal{T}}(F)$, and the crucial point is that in the cell $C(G)$, the lambda length coordinates on G must satisfy all three strict triangle inequalities at each vertex of G (cf. Lemma 3.6). This is what forces the recurrence condition.

The proof of Theorem 1.2 involves the explicit calculation of holonomies using "path-ordered products" of matrices (due to Bill Thurston and Volodya Fock [3] independently and recalled in §3). The proof further requires estimates on the absolute traces of the representing matrices. To this end, we find a condition weaker than the triangle inequalities which satisfies two properties: 1) the condition is invariant under certain "Whitehead moves" (see the next section for a definition) sufficient to simplify the path-ordered product; and 2) the condition guarantees the required estimates on the absolute traces. This is the heart of the paper (in §5), and the techniques involve only "Ptolemy transformations" (cf. Lemma 3.1a), path-ordered products, and the triangle inequality.

Because the argument at heart only depends upon these formulae, we are optimistic that the current paper may have ramifications more generally for cluster algebras [6] and cluster ensembles [4]. Since Wolpert has recently announced [21] that lambda lengths are strictly convex along Weil-Peterssen geodesics, we are likewise optimistic about applications to the asymptotic WP geometry.

There is furthermore a program to extend the cell decomposition of moduli space to the Deligne-Mumford compactification using screens, which is already well underway (as discussed in the closing remarks §7).

2. Fatgraphs and recurrence

A *graph* is a finite one-dimensional CW complex with no isolated vertices whose 1-cells are *edges* and whose 0-cells are *vertices*. The set of edges of G will be denoted $E(G)$. A *half-edge* of an edge $e \in E(G)$

is either one of the two components of the interior of e with an interior point removed, and the *valence* of a vertex is the number of half-edges containing the vertex in their closures, said to be *incident* on the vertex. A *fatgraph* is a graph together with a cyclic ordering on the half-edges incident on each vertex. In particular, a finite CW decomposition of a circle is an example of a fatgraph, as is a planar tree where the cyclic ordering is induced by the counter-clockwise orientation on the plane.

A fatgraph G determines a punctured surface $F'(G)$ gotten by assigning to each k-valent vertex an oriented ideal k-gon, whose sides correspond to the incident half-edges, and finally identifying in the natural way pairs of sides of these polygons associated to pairs of half-edges contained in a common edge of G.

The vertices of the ideal polygons are identified to the punctures of $F'(G)$. Each edge $e \in E(G)$ gives rise to its dual ideal arc $\alpha_{(G,e)}$ connecting punctures in $F'(G)$.

An *ideal triangulation* of F_g^s is the homotopy class of a set of arcs connecting punctures in F_g^s, called *ideal arcs*, which decompose the surface into a collection of triangles with vertices at the punctures. More generally, an *ideal cell decomposition* is the homotopy class of a subset of an ideal triangulation which decomposes the surface into polygons.

Provided each vertex of G has valence at least three, $\{\alpha_{(G,e)} : e \in E(G)\}$ is an ideal cell decomposition of $F'(G)$ said to be *dual* to G. Conversely, the Poincaré dual of an ideal cell decomposition of F_g^s is a fatgraph embedded in F_g^s each of whose vertices has valence at least three, where the cyclic ordering in the fatgraph structure is induced by the clockwise order in the oriented surface F_g^s.

A fatgraph G also determines a corresponding oriented surface $F(G)$ with boundary constructed by assigning to each k-valent vertex an oriented $(2k)$-gon, whose alternating sides correspond to the incident half-edges, and as before, identifying pairs of sides of these polygons corresponding to pairs of half-edges contained in a common edge of G. The alternating unpaired edges of these polygons comprise the boundary of $F(G)$. We may regard $F(G) \subseteq F'(G)$ as a strong deformation retraction in the natural way.

In particular, G is a strong deformation retraction or *spine* of $F(G)$ or $F'(G)$. It follows that any free homotopy class of essential curve in $F(G)$ or $F'(G)$ gives rise to a closed edge-path in G, which is uniquely determined up to its starting point provided we demand that the edge-path is *efficient* in the sense that it never consecutively traverses the same edge of G with opposite orientations.

A closed edge-path in G corresponding to a boundary component of $F(G)$ will be called a *boundary component* of G itself. An efficient boundary component of G must have edges of G incident on only one side. Put another way for a trivalent fatgraph, an efficient edge-path is a boundary component if and only if it consists entirely of left turns or consists entirely of right turns.

Suppose that G is a fatgraph with set $E = E(G)$ of edges and corresponding surface $F = F(G)$. Any subset $A \subseteq E$ determines a subgraph by including all vertices of G on which edges in A are incident. Furthermore by restriction, the fattening on G induces a fattening on this subgraph, which thus determines a well-defined sub-fatgraph G_A. We may regard $F(G_A)$ as a subsurface embedded in the interior of F in the natural way. Define the *boundary* of A to be the collection ∂A of (unoriented) efficient closed edge-paths in G corresponding to the relative boundary of $F(G_A)$ in $F = F(G)$, that is, the collection of closed edge-paths corresponding to the components of the boundary $\partial F(G_A)$ which are not homotopic to boundary components of F itself. In particular, if G_A is a circle, then ∂A is the closed edge path of G_A if this circle is not boundary parallel in F, and ∂A is empty if this circle is boundary parallel in F.

We say that $A \subseteq E$ is *recurrent* if for every edge $a \in A$, there is an efficient closed edge-path γ_a in G so that γ_a traverses a and traverses only edges in A. Any subset $A \subseteq E$ has a (possibly empty, e.g., in the case of a planar tree) *maximal recurrent subset* $R(A)$, namely, the set of edges of A traversed by an efficient closed edge-path in G_A.

Lemma 2.1. *Suppose that G is a fatgraph and $A \subseteq E = E(G)$. Then the following are equivalent:*

i) *A is recurrent;*

ii) *there is a function $\mu : E \to \mathbb{Z}_{\geq 0}$ whose support is A so that for each vertex of G with incident half-edges $e_1, \ldots, e_k$ and extending the function μ to be defined on half-edges in the natural way, we have that $\sum_{i=1}^{k} \mu(e_i)$ is even, and the generalized weak triangle inequalities hold, i.e., for each $j = 1, \ldots, k$,*

$$\mu(e_j) \leq \sum_{i \neq j} \mu(e_i);$$

iii) *every vertex of G_A has valence at least two.*

Proof. First suppose that A is recurrent, and let $\mu_a(e)$ be the number of times that a chosen γ_a traverses e for each $a \in A$ and $e \in E$. Each $\mu_a : E \to \mathbb{Z}_{\geq 0}$ satisfies the restrictions of condition (ii), hence so too does their sum $\mu = \sum_{a \in A} \mu_a$, which has full support on A. Thus, (i) implies (ii).

Conversely, suppose that μ is a function supported on A satisfying the properties of condition (ii). For each k-valent vertex of G, there is a dual ideal k-gon in the corresponding punctured surface $F'(G)$, and we shall construct a family of arcs properly embedded in this k-gon realizing the values of μ on the dual edges of G as the geometric intersection numbers. These arc families in the k-gons then combine uniquely to produce disjointly embedded curves in the natural way, whose component simple closed curves in F have corresponding edge-paths which satisfy the required properties.

The construction in each k-gon proceeds by induction on $k \geq 2$ with notation for incident edges as in condition (ii). In case $k = 2$, simply take a collection of $\mu(e_1) = \mu(e_2)$ arcs crossing the bigon. For the case $k = 3$, take

$$\frac{1}{2}[\mu(e_{i_1}) + \mu(e_{i_2}) - \mu(e_{i_3})] = \frac{1}{2}[\mu(e_{i_1}) + \mu(e_{i_2}) + \mu(e_{i_3}) - 2\mu(e_{i_3})] \in \mathbb{Z}_{\geq 0}$$

parallel copies of the arc joining edges e_{i_1} to e_{i_2}, where $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. For the induction step, take a consecutive pair of edges e_i, e_{i+1} so that $\mu(e_i) + \mu(e_{i+1})$ is least among all consecutive pairs of edges, here taking the indices modulo n so that $e_{n+1} = e_1$. Cutting along the diagonal separating e_i and e_{i+1} from the rest decomposes the k-gon into a $(k-1)$-gon and a triangle. Extend μ to a function defined on the edges of these regions by taking value $\mu(e_i)+\mu(e_{i+1})$ on the diagonal, so the generalized triangle inequalities hold on each region by our choice of consecutive edges, and the parity condition holds by construction. By the inductive hypothesis, appropriate arc families exist in each region, and they combine in the natural way to give the required arc family in the k-gon itself. It follows that (i) is equivalent to (ii).

If G_A has a univalent vertex, say with incident edge $a \in A$, then there can be no efficient edge-path in G_A traversing a, so (i) implies (iii). To see that (iii) implies (ii), define μ to take value 2 on the edges in A and vanish otherwise, and note that μ satisfies condition (ii) provided G_A has no univalent vertices. □

Suppose that e is an edge of a fatgraph G with distinct endpoints. We may *collapse* e to a vertex to produce a new fatgraph G', where the cyclic ordering at the resulting vertex arises by combining the cyclic orderings on the half-edges incident on e in the natural way. Dually, one removes the ideal arc $\alpha_{(G,e)}$ from the dual ideal cell decomposition.

If G is a trivalent fatgraph and e is an edge of G with distinct endpoints, then a *Whitehead move* on e is the fatgraph that results by collapsing e and then un-collapsing the resulting four-valent vertex in the unique distinct manner. A Whitehead move along an edge e is depicted in Figure 4, which furthermore indicates the notation near an edge e which we shall adopt in many of the calculations of this paper.

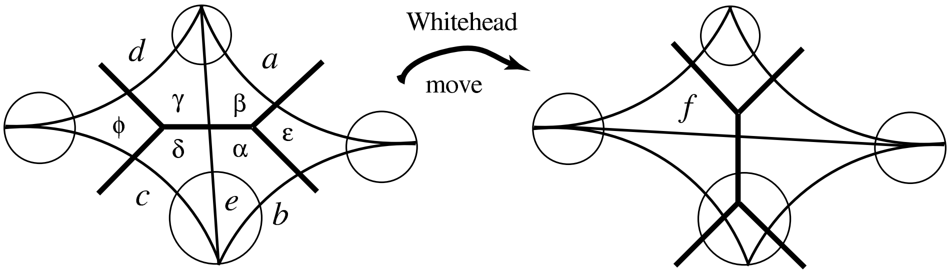

Figure 4 Standard notation for Whitehead moves.

Using the characterization Lemma 2.1iii), it follows directly that recurrence is invariant under Whitehead moves on trivalent fatgraphs and is furthermore in any case invariant under collapse of edges e with distinct endpoints neither of which is univalent.

3. Coordinates

The reader is referred to [15] or the more recent treatment [20] for proofs and further details on the material which is recalled in this section. We begin with several formulae on horocycles in the hyperbolic plane.

If h, h' are horocycles in the hyperbolic plane with distinct centers in the circle at infinity, then consider the unique geodesic $\gamma(h, h')$ connecting their centers. The horocycles h, h' truncate $\gamma(h, h')$ to a geodesic segment of some finite signed length taken to be positive if and only if h and h' are disjoint. Define the *lambda length* of h, h' to be $\lambda(h, h') = \sqrt{\exp\, \delta}$. (This is a different normalization for lambda lengths than in [15], for instance, where the lambda length is taken as $\sqrt{2\, \exp \delta}$, cf. [20].)

Lemma 3.1. *Suppose h_1, h_2, h_3, h_4 are horocycles with distinct centers occurring in this clockwise order in the circle at infinity, and let $\lambda_{ij} = \lambda(h_i, h_j)$ for distinct $i, j \in \{1, 2, 3, 4\}$. Then:*

a) [Ptolemy's equation] $\lambda_{13}\lambda_{24} = \lambda_{12}\lambda_{34} + \lambda_{14}\lambda_{23}$;

b) [Cross Ratios] *the Möbius transformation that takes the centers of h_3, h_2, h_1 respectively to $0, 1, \infty$ also maps the center of h_4 to $-\frac{\lambda_{23}\lambda_{14}}{\lambda_{12}\lambda_{34}}$;*

c) [h-lengths] *the hyperbolic length of the horocyclic segment in h_i with endpoints $h_i \cap \gamma(h_i, h_j)$ and $h_i \cap \gamma(h_i, h_k)$ is given by $\frac{\lambda_{jk}}{\lambda_{ij}\lambda_{ik}}$, where $\{i, j, k\} = \{1, 2, 3\}$;*

d) [Affine duality] *taking the upper sheet $\mathbb{H}$ of the hyperboloid in Minkowski 3-space as the model for the hyperbolic plane, there is a unique isotropic vector u_i with positive z-coordinate so that $h_i = \{w \in \mathbb{H} : w \cdot u_i = -2^{-\frac{1}{2}}\}$, for $i = 1, 2, 3, 4$ where $\cdot$ denotes the pairing with quadratic form $x^2 + y^2 - z^2$, and $\lambda_{ij} = \sqrt{-u_i \cdot u_j}$ for distinct $i, j \in \{1, 2, 3, 4\}$;*

e) [Simplicial coordinates] *in the notation of part d), the signed volume of the Euclidean tetrahedron in Minkowski 3-space spanned by u_1, u_2, u_3, u_4 is given by $2\sqrt{2}\ \lambda_{12}\lambda_{23}\lambda_{34}\lambda_{14}$ times*

$$\frac{\lambda_{12}^2 + \lambda_{23}^2 - \lambda_{13}^2}{\lambda_{12}\lambda_{23}\lambda_{13}} + \frac{\lambda_{14}^2 + \lambda_{34}^2 - \lambda_{13}^2}{\lambda_{14}\lambda_{34}\lambda_{13}},$$

where the sign is positive if and only if the edge of the tetrahedron connecting u_1, u_3 lies below the edge connecting u_2, u_4.

f) [Ellipticity] *in the notation of part d), the affine plane containing u_1, u_2, u_3 determines an elliptic conic section if and only if $\lambda_{12}, \lambda_{13}, \lambda_{23}$ satisfy the three strict triangle inequalities.*

Given a point $\tilde{\Gamma} \in \widetilde{\mathcal{T}}(F)$ and given the homotopy class of an ideal arc α in F, we may straighten α to the geodesic for the underlying hyperbolic structure and truncate this geodesic by cutting it at the horocycles centered at its endpoints coming from the decoration. This geodesic segment has a signed hyperbolic length δ taken with a positive sign if and only if the horocycles are disjoint. The basic coordinate of an ideal arc in a decorated hyperbolic surface is the *lambda length* (also sometimes called the "Penner coordinate") defined by $\lambda(\alpha; \tilde{\Gamma}) = \sqrt{\exp\ \delta}$.

Theorem 3.2. *Fix any trivalent fatgraph G. Then the assignment of lambda lengths*

$$\widetilde{\mathcal{T}}(F'(G)) \to \mathbb{R}_{>0}^{E(G)}$$
$$\tilde{\Gamma} \mapsto (e \mapsto \lambda(\alpha_{(G,e)}; \tilde{\Gamma}))$$

is a real-analytic homeomorphism onto.

For convenience when the fatgraph G is fixed or understood, we shall refer to the lambda length of an edge e of G rather than that of its dual arc $\alpha_{(G,e)}$. We shall also often identify an arc with its lambda length for convenience.

Suppose that G is a trivalent fatgraph. Consider an edge e of G and adopt the notation of Figure 4, where e has distinct endpoints with incident half-edges a, b and c, d occurring in the alphabetic counter-clockwise order about e. (If e does not have distinct endpoints or if a, b, c, d are not distinct, then adopt the corresponding notation for nearby edges in the universal cover.) Dual to each vertex of e is an ideal triangle, and each such triangle has three vertices, denoted by Greek letters in Figure 4. To each such triangle/vertex pair is naturally associated a *sector* of G, that is, a pair of consecutive half-edges of G incident on a common vertex, namely, the pair of half-edges adjacent to the given vertex in the given triangle.

In fact, one can conveniently calculate the holonomies of based closed curves in $F'(G)$ as follows. Define the matrices

$$R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \; L = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in PSL_2(\mathbb{R}).$$

According to Lemma 3.1b), the cross ratio of the ideal quadrilateral with edges $\alpha_{(G,a)}$, $\alpha_{(G,b)}$, $\alpha_{(G,c)}$, $\alpha_{(G,d)}$ is given by $-bd/ac$, where we have identified an edge of G with its lambda length for convenience, and we further define the matrix

$$X_e = \begin{pmatrix} 0 & \sqrt{ac/bd} \\ -\sqrt{bd/ac} & 0 \end{pmatrix}.$$

Choosing a vertex of G as basepoint, consider a closed edge-path γ in G representing an essential based closed curve in F. We may as well assume that γ is efficient (though this is not necessary since $RL = R^3 = L^3 = X_e^2 = 1 \in PSL_2(\mathbb{R})$), so that it alternately traverses edges and sectors of G and makes turns, right or left, at each sector. Suppose that γ serially makes turns t_i at the sectors, then traverses edges e_i, for $i = 1, \ldots, n$, and associate the *path-ordered product*

$$M = T_1 X_{e_1} T_2 \cdots X_{e_n}$$

of matrices, where $T_i = R$ or L if t_i is a right or left turn respectively. The matrix $M \in PSL_2(\mathbb{R})$ gives the holonomy of the based curve γ. Of course by conjugacy invariance of trace, the absolute value of the trace of M is independent of the basepoint. We shall use these path-ordered products to detect the short curves that occur on a path in $C(G) \subseteq \widetilde{\mathcal{T}}(F'(G))$.

The quadrilateral in Figure 4 is realized as a geodesic ideal quadrilateral with horocycles centered at each vertex. We define the *h-length* of a sector of G to be the hyperbolic length of the corresponding horocyclic segment. According to Lemma 3.1c), the h-length of a sector is the opposite lambda length divided by the product of adjacent lambda lengths.

Furthermore in the notation of Figure 4, we define the *simplicial coordinate* of the edge e to be the quantity

$$\frac{a^2+b^2-e^2}{abe} + \frac{c^2+d^2-e^2}{cde} = \frac{a}{be} + \frac{b}{ae} - \frac{e}{ab} + \frac{c}{de} + \frac{d}{ce} - \frac{e}{cd}.$$

According to Lemma 3.1e), the simplicial coordinate is a multiple of the signed volume of the corresponding tetrahedron, and by inspection, it is a linear combination of the nearby h-lengths. From the definition, the simplicial coordinate is the sum of two terms each of which is associated to a vertex of the corresponding edge.

Consider a trivalent fatgraph G with set E of edges for the surface F_g^s together with an assignment of lambda lengths $\lambda : E \to \mathbb{R}_{>0}$. We say that λ satisfies the *no vanishing cycle condition* provided that all the corresponding simplicial coordinates are non-negative and there is no cycle in G all of whose simplicial coordinates vanish.

Theorem 3.3. *For any surface $F = F_g^s$ with $s \geq 1$, there is a $MC(F)$-invariant ideal cell decomposition of $\widetilde{\mathcal{T}}(F)$, where the cells in this decomposition are in one-to-one correspondence with homotopy classes of embeddings of fatgraph spines of F each of whose vertices has valence at least three. The face relation in this cell decomposition is generated by Whitehead collapse.*

In particular, if G is a trivalent fatgraph spine of F, then the closed cell $C(G) \subseteq \widetilde{\mathcal{T}}(F)$ corresponding to it is described in lambda length coordinates with respect to G by the no vanishing cycle condition.

Furthermore, suppose G' arises from G by collapsing to a point each component of a forest in G. Then the corresponding closed cell $C(G') \subseteq C(G) \subseteq \widetilde{\mathcal{T}}(F)$ is described by taking all simplicial coordinates on edges in the forest to vanish. Finally, any assignment of non-negative real *numbers to the edges of G' with no vanishing cycles is realized as the simplicial coordinates of a uniquely determined collection of positive lambda lengths on G.*

Corollary 3.4. *There is a $MC(F)$-invariant ideal cell decomposition of $P\widetilde{\mathcal{T}}(F)$ for any surface $F = F_g^s$ with $s \geq 1$, where the cells in this decomposition are in one-to-one correspondence with homotopy classes of embeddings of fatgraph spines of F whose vertices have valence at least three.*

Proof. This follows immediately from the previous theorem and homogeneity of the formula for simplicial coordinates. □

Lemma 3.5. *Suppose that γ is an efficient edge-path in G serially traversing edges e_i alternating with sectors t_i, for $i = 1, \dots, n$. Let E_i denote the simplicial coordinate of e_i and α_i the h-length of the sector t_i. Then $\sum_{i=1}^n E_i = 2\sum_{i=1}^n \alpha_i$.*

Proof. The proof follows from the definition of simplicial coordinates in terms of h-lengths. □

Lemma 3.6. [15] *The no vanishing cycle condition implies that the lambda lengths at any vertex of G satisfy the three strict triangle inequalities.*

Proof. Adopt the notation of Figure 4 for the half-edges near an edge e (again, in the universal cover if the edges a, b, c, d are not distinct or if e does not have distinct endpoints). If $c + d \leq e$, then $c^2 + d^2 - e^2 \leq -2cd$, so the non-negativity of the simplicial coordinate E of e gives $0 \leq cd[(a-b)^2 - e^2]$, and we find a second vertex so that the triangle inequality fails. This is a basic algebraic fact about simplicial coordinates. It follows that if there is any such vertex so that the triangle inequalities do fail for the lambda lengths of incident half-edges, then there must be an efficient closed edge-path γ passing through such triangles. Letting e_i denote the consecutive edges of G serially traversed by γ and b_i denote the half-edge of G incident on the common endpoint of e_i and e_{i+1}, we find $e_{j+1} \geq b_j + e_j$, for $j = 1, \dots n$.

Upon summing and canceling like terms, we find $0 \geq \sum_{j=1}^{n} b_j$, which is absurd since lambda lengths are positive. $\square$

4. Screens

Suppose that G is a trivalent fatgraph with set E of edges and corresponding surface F, and suppose that $\lambda_t : E \to \mathbb{R}_{>0}$, i.e., $\lambda_t \in \mathbb{R}_{>0}^E$, is a continuous one-parameter family of lambda lengths for $t \geq 0$. We shall typically apply Theorem 3.2 to regard such a one-parameter family as a path in $\widetilde{\mathcal{T}}(F'(G))$ itself. There is an induced $\bar{\lambda}_t \in P(\mathbb{R}_{>0}^E)$, where P denote projectivization, and by compactness, there is an accumulation point of $\lim_{t\to\infty} \bar{\lambda}_t$ in $P(\mathbb{R}_{\geq 0}^E)$.

Say that λ_t *stable* if there is a unique such limit point denoted $\bar{\lambda}_\infty \in P(\mathbb{R}_{\geq 0}^E)$. If λ_t is any path, then any accumulation point of $\bar{\lambda}_t$ is also the limit of some stable path since decorated Teichmüller space is path connected.

Suppose that $\lambda_t \in P(\mathbb{R}_{>0}^E)$ is stable with limit $\bar{\lambda}_\infty \in P(\mathbb{R}_{\geq 0}^E)$. Set $E^0 = E$ and make the following recursive definition for $k \geq 1$. If $E^{k-1} \neq \emptyset$, then there is $e \in E^{k-1}$ so that for all $f \in E^{k-1}$, we have $\bar{\lambda}_\infty(e)/\bar{\lambda}_\infty(f) < \infty$, and we set

$$E^k = \{f \in E^{k-1} : \bar{\lambda}_\infty(e)/\bar{\lambda}_\infty(f) = 0\},$$

and set $E_k = \emptyset$ if $E_k = E_{k-1}$. Notice that if two edges $e_1, e_2 \in E^{k-1}$ satisfy the condition that for all $f \in E^{k-1}$, we have $\bar{\lambda}_\infty(e_i)/\bar{\lambda}_\infty(f) < \infty$, for $i = 1, 2$, then we have $0 < \bar{\lambda}_\infty(e_1)/\bar{\lambda}_\infty(e_2) < \infty$, so e_1 and e_2 determine the same subset E^k. Thus, $E = E^0 \supsetneq E^1 \supsetneq \cdots \supsetneq E^N$ is a well-defined nested sequence of finite length N of proper subsets, and we set $E^{N+1} = \emptyset$ for convenience.

Now, suppose that λ_t stays for all finite $t \geq 0$ in the closed cell $C(G)$ corresponding to G, i.e., the lambda lengths satisfy the no vanishing cycle condition by Theorem 3.3. Define

$$\mathcal{A}(\lambda_t) = \{A \subseteq E : A \text{ is the set of edges of a component of some } E^k\},$$

a subset of the power set of E.

Proposition 4.1. *For any connected trivalent fatgraph G with set E of edges and any continuous stable one-parameter family $\lambda_t \in \mathbb{R}_{>0}^E$, for $t \geq 0$, which stays for all finite t in the cell $C(G) \subseteq \widetilde{\mathcal{T}}(F(G))$ corresponding to G, the collection $\mathcal{A} = \mathcal{A}(\lambda_t)$ satisfies the following properties:*

i) *$E \in \mathcal{A}$;*

ii) *each* $A \in \mathcal{A}$ *is recurrent;*

iii) *if* $A, B \in \mathcal{A}$ *with* $A \cap B \neq \emptyset$*, then either* $A \subseteq B$ *or* $B \subseteq A$*;*

iv) *for each* $A \in \mathcal{A}$*,* $\cup\{B \in \mathcal{A} : B \text{ is a proper subset of } A\}$ *is a proper subset of* A*.*

A subset of the power set of E satisfying properties i-iv) is called a *screen* on G for any (not necessarily trivalent) recurrent fatgraph G with set E of edges.

Proof. The first condition holds since G is connected and $E = E^0$. Recursively applying Lemmas 2.1 and 3.6, we conclude that E^k is a proper recurrent set in the possibly disconnected fatgraph $G_{E^{k-1}}$, for $k = 1, \ldots, N$, so the second condition holds as well. The third condition holds since two components of a topological space either coincide or are disjoint, and the fourth follows since each inclusion $E^k \subsetneq E^{k-1}$ is proper. □

If $\mathcal{A}$ is a screen, then each $A \in \mathcal{A} - \{E\}$ has an immediate predecessor A', i.e., $A \subsetneq A'$ and if $B \in \mathcal{E}$ and $A \subseteq B \subseteq A'$, then$B = A$ or $B = A'$. The maximum length of a chain $A \subsetneq A' \subsetneq \cdots \subsetneq E$ of immediate predecessors in $\mathcal{A}$ is called the *depth* of A in $\mathcal{A}$, and the depth of $e \in E$ in $\mathcal{A}$ is the maximum depth of $A \in \mathcal{A}$ with $e \in A$.

Lemma 4.2. *Every screen* $\mathcal{A}$ *on every trivalent fatgraph* G *arises as* $\mathcal{A} = \mathcal{A}(\bar{\lambda}_t)$ *for some stable* $\lambda_t \in \mathbb{R}^E_{>0}$ *lying in* $C(G)$*.*

Proof. For any screen $\mathcal{A}$ on any trivalent fatgraph G, define a one-parameter family of lambda lengths by taking $\lambda_t(e) = t^{d_e}$, where d_e is the depth of e in $\mathcal{A}$. For any vertex v of G, the maximum degree of the incident (half-)edges is achieved either twice or thrice by recurrence of elements of $\mathcal{A}$. Thus, the contribution from v to each of the three possible simplicial coordinates of edges incident on v is positive, and so the simplicial coordinate of each edge of G for λ_t is also positive; λ_t thus lies in $C(G)$ by Theorem 3.3, and $\mathcal{A}(\bar{\lambda}_t) = \mathcal{A}$ by construction. □

Let $\partial_{\mathcal{A}} A$ denote the relative boundary of $F(G_A)$ in $F(G_{A'})$, where A' is the immediate predecessor of A in $\mathcal{A}$, and define the *boundary* of

$\mathcal{A}$ itself to be

$$\partial\mathcal{A} = \bigcup_{A\in\mathcal{A}-\{E\}} \partial_{\mathcal{A}} A.$$

Lemma 4.3. *For any trivalent fatgraph G with set E of edges and stable $\lambda_t \in \mathbb{R}^E_{>0}$ lying in $C(G)$, each edge-path in $\partial\mathcal{A}(\bar{\lambda}_t)$ is homotopic to a curve in $F'(G)$ whose hyperbolic length tends to zero as t tends to infinity. Furthermore, these are only such asymptotically short curves for λ_t.*

Proof. Let K be a component of $\partial\mathcal{A}$, so $K = \partial_{\mathcal{A}} A$ for some $A \in \mathcal{A}-\{E\}$ with immediate predecessor A'. Orient K with the subsurface $F(G_A)$ on its left. Consider the universal cover $\tilde{F}$ of $F = F(G)$, let $\tilde{G}$, $\tilde{G}_A$, and $\tilde{G}_{A'}$ respectively denote the full pre-images of G, G_A, and $G_{A'}$ in $\tilde{F}$, and choose a lift $\tilde{K}$ of K to $\tilde{F}$. We shall refer to lambda lengths of edges of $\tilde{G}$, by which we mean the value of λ_t on the projection of the edge to F, and we will as usual identify the lambda length of an edge of $\tilde{G}$ with the edge itself for convenience.

On the right of $\tilde{K}$ since K is homotopic to a boundary component of $F(G_A)$, there are no edges of $\tilde{G}_A$, and since K is not homotopic to a boundary component of $F(G_{A'})$, there is at least one edge of $\tilde{G}_{A'}$ not in $\tilde{G}_A$ on the right. Furthermore on the left of $\tilde{K}$, there is at least one edge of $\tilde{G}_{A'}$ again since K is not homotopic to a boundary component of $F(G_{A'})$.

Since λ_t corresponds to points in $C(G)$, it follows that the triangle inequalities hold on lambda lengths at each vertex of $\tilde{G}$ by Lemma 3.6. We claim that the following further properties of lambda lengths follow from these facts, where all limits are taken as $t \to \infty$:

1) if x is an edge on the right of $\tilde{K}$ and y is an edge of $\tilde{K}$, then we have $\frac{x}{y} \to 0$;

2) if x is an edge on the right of $\tilde{K}$, y_0 is an edge on the left of $\tilde{K}$, and y_1, y_2 are edges of $\tilde{K}$ so that y_0, y_1, y_2 are all incident at a common vertex in $\tilde{K}$, then $\frac{xy_0}{y_1y_2} \to 0$;

3) if y_0, y_1 are consecutive edges of $\tilde{K}$ and x is an edge on the right of $\tilde{K}$ incident on their common endpoint, then $\frac{y_0}{y_1} \to 1$.

The first property follows from the definition of K as a relative boundary component of $F(G_A)$ in $F(G_{A'})$ and the definition of the screen $\mathcal{A}(\lambda_t)$. For property 2, y_0, y_1, y_2 satisfy the triangle inequality $y_0 < y_1 + y_2$, so dividing by $y_1 y_2$ and multiplying by x, we find $\frac{xy_0}{y_1 y_2} < \frac{x}{y_1} + \frac{x}{y_2}$; the right hand side tends to zero by property 1. Finally for property 3 again by the triangle inequalities, we have $y_0 < y_1 + x$ and $y_1 < y_0 + x$. Upon dividing the first by y_1 and the second by y_0 and applying property 1, we conclude $1 \leq \lim \frac{y_0}{y_1} \leq 1$, as required.

The first key point about properties 1-3) is that they are invariant under certain Whitehead moves. In each case, we shall perform a Whitehead move along an edge $e \in K$, where one vertex of e has incident half-edges a, b and the other vertex has incident half-edges c, d, and where the edges a, b, c, d occur in this counter-clockwise order about e. We shall refer to properties 1-3) for the fatgraph before the Whitehead move and the corresponding properties $1'$-$3'$) for the resulting fatgraph, and we shall let $f = \frac{ac+bd}{e}$ denote the edge and lambda length of the edge resulting from e under the Whitehead move.

The first case of utility is when b, c, e lie in $\tilde{K}$ and a, d lie on the right of $\tilde{K}$. The properties for this fatgraph imply that: 1) $\frac{x}{y} \to 0$ for $x \in \{a, d\}$ and $y \in \tilde{K}$; 2) does not involve the vertices of e; and 3) $\frac{b}{e} \to 1$ and $\frac{e}{c} \to 1$. Property $3'$) requires $\frac{b}{c} \to 1$, which follows from property 3). Furthermore by the Ptolemy equation,

$$\frac{f}{a} = \frac{ac+bd}{ae} = \frac{c}{e} + \frac{b}{e}\frac{d}{a} \to 1 + \frac{d}{a},$$
$$\frac{f}{d} = \frac{ac+bd}{de} = \frac{b}{e} + \frac{c}{e}\frac{a}{d} \to 1 + \frac{a}{d},$$

since $\frac{c}{e} \to 1$ and $\frac{b}{e} \to 1$. Thus, at least one of $\frac{f}{a}, \frac{f}{d}$ has a finite limit, hence $\frac{f}{y} = \frac{f}{a}\frac{a}{y} = \frac{f}{d}\frac{d}{y} \to 0$ for any $y \in \tilde{K}$ by property 1) proving property $1'$) and likewise for property $2'$).

The second case of utility is when b, d, e lie in $\tilde{K}$ with a on the right and c on the left of $\tilde{K}$. The properties for this fatgraph imply that: 1) $\frac{a}{y} \to 0$ for any y in $\tilde{K}$; 2) $\frac{xc}{de} \to 0$ for any x on the right; and 3) $\frac{b}{e} \to 1$. Property $3'$), namely, $\frac{d}{f} \to 1$, follows from

$$\frac{f}{d} = \frac{ac+bd}{de} = \frac{ac}{de} + \frac{b}{e} \to 1$$

using properties 2-3). Property $1'$) follows from this and property 1). Finally, since

$$\frac{bf}{xc} = \frac{(ac+bd)b}{xce} = \frac{b}{e}\frac{a}{x} + \frac{b^2}{e^2}\frac{de}{xc} \to \infty$$

using the Ptolemy equation and properties 2-3), property 2′) holds as well.

Applying these two types of Whitehead moves along edges in K, we may alter G' to arrange that the edge-path for K in the resulting graph makes exactly one left turn. Furthermore as we have just proved, properties 1-3) continue to hold for the resulting graph.

We shall complete the proof by calculating that the absolute value of the trace of the holonomy of the edge-path K is asymptotic to 2, and the second key point about properties 1-3) is that they are sufficient to guarantee this. To this end, let us adopt the notation that K traverses the consecutive edges $y_1, \ldots, y_{n+1}$, the unique half-edge on the right is x_0, which is incident on the common endpoint of y_{n+1}, y_1, and the consecutive half-edges on the left are $x_1, \ldots, x_n$, where x_k has common endpoint in K with y_k, y_{k+1}, for $k = 1, \ldots, n$. As usual identifying an edge or a half-edge with its lambda length, which depends upon the parameter t, let us define

$$\zeta_1^2 = \frac{y_2 y_{n+1}}{x_1 x_0}, \zeta_{n+1}^2 = \frac{x_0 x_n}{y_1 y_n}, \text{ and } \zeta_k^2 = \frac{y_{k+1} x_{k-1}}{x_k y_{k-1}}, \text{ for } k = 2, \ldots, n,$$

so the cross ratio of edge y_k is ζ_k^{-2}, for $k = 1, \ldots, n+1$. The path-ordered product of matrices to compute the holonomy of K beginning from the unique left turn is given by

$$L \begin{pmatrix} 0 & \zeta_1 \\ -\zeta_1^{-1} & 0 \end{pmatrix} R \begin{pmatrix} 0 & \zeta_2 \\ -\zeta_2^{-1} & 0 \end{pmatrix} \cdots R \begin{pmatrix} 0 & \zeta_{n+1} \\ -\zeta_{n+1}^{-1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \zeta_1^{-1} & 0 \\ -\zeta_1^{-1} & \zeta_1 \end{pmatrix} \begin{pmatrix} \zeta_2^{-1} & -\zeta_2 \\ 0 & \zeta_2 \end{pmatrix} \cdots \begin{pmatrix} \zeta_{n+1}^{-1} & -\zeta_{n+1} \\ 0 & \zeta_{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} \zeta_1^{-1} & 0 \\ -\zeta_1^{-1} & \zeta_1 \end{pmatrix} \begin{pmatrix} (\zeta_2 \cdots \zeta_{n+1})^{-1} & \zeta_2 \cdots \zeta_{n+1} \sum_{k=1}^n \prod_{j=2}^k \zeta_j^{-2} \\ 0 & \zeta_2 \cdots \zeta_{n+1} \end{pmatrix},$$

so the trace is found to be

$$(\zeta_1 \zeta_2 \cdots \zeta_{n+1}) + (\zeta_1 \zeta_2 \cdots \zeta_{n+1})^{-1} - \zeta_1^{-2} (\zeta_1 \zeta_2 \cdots \zeta_{n+1}) \sum_{k=1}^{n} \prod_{j=2}^{k} \zeta_j^{-2}.$$

Finally, direct calculation shows that the product telescopes, and

$$(\zeta_1 \zeta_2 \cdots \zeta_{n+1}) = \frac{y_{n+1}}{y_1} \to 1$$

since $\frac{y_1}{y_{n+1}} \to 1$ by property 3). Furthermore,

$$\zeta_1^{-2} = \frac{x_0 x_1}{y_2 y_{n+1}} \sim \frac{x_0 x_1}{y_1 y_2} \to 0$$

by properties 2-3), and indeed, the general term in the sum also telescopes

$$\zeta_1^{-2}\zeta_2^{-2}\cdots\zeta_k^{-2} \sim \frac{x_0 x_k}{y_k y_{k+1}} \to 0, \text{ for } k = 2, \ldots, n,$$

again by properties 2-3). The absolute value of the trace is thus indeed asymptotic to 2. Since the absolute value of the trace is twice the hyperbolic cosine of half the hyperbolic length, the curve K is asymptotically short as $t \to \infty$.

For the final assertion, we must show that the edge-path of an essential short curve K for λ_t lies in $\partial\mathcal{A}$. We shall use the Collar Lemma [2] that an essential simple closed curve of hyperbolic length ℓ has an embedded collar of width at least the logarithm of the hyperbolic cotangent of $\frac{\ell}{2}$. Since the dual of G is an ideal triangulation and K is essential, it thus follows that if K traverses an edge e of G, then the lambda length of (the ideal arc dual to) e has divergent lambda length. It follows that K shares an edge with E^k, for some $k \geq 1$. Since two essential short curves cannot intersect, again by the Collar Lemma, we conclude that K cannot meet $\partial\mathcal{A}(\lambda_t)$ for large t, so in fact the edge-path for K is contained in $E^k - E^{k+1}$.

If K is not homotopic to a boundary component of G_{E^k}, then its edge-path must make both right and left turns in G_{E^k}. Without loss, we may assume that there is a left turn followed by a right turn and adopt the following notation. Suppose that the edge-path for K traverses edges $y_0, \ldots, y_{n+1}$ in K with half-edge x_j incident on the common endpoint of y_j, y_{j+1} for $j = 1, \ldots, n$, where $x_0 \in E^k$ lies on the right and x_n lies on the left of K. In the extreme case that $n = 1$, the dual arcs to $x_0, y_1, x_1, y_0 \in E^k$ are the consecutive edges of an ideal quadrilateral whose cross ratio is bounded near one by Lemma 3.1 since the lambda lengths x_0, x_1, y_0, y_1 are comparable, i.e., the limit of the ratio of any pair is finite and non-zero. The arcs dual to y_0 and y_1 are therefore a bounded distance apart, contradicting that K is short. This extreme case gives a lower bound to the distance between the arcs dual to y_0 and y_n, so in any case, K cannot be short. This contradiction establishes the final assertion and completes the proof. $\square$

5. Proof of main result

Theorem 5.1. *The cell $C(G)$ in decorated Teichmüller space corresponding to the fatgraph G is asymptotic to a stable curve with pinch curves K if and only if K is homotopic to the collection of edge-paths $\partial\mathcal{A}$ for some screen $\mathcal{A}$ on G.*

Proof. Suppose that $\lambda_t \in \mathbb{R}^0_{>0}$ is a path of lambda lengths in $C(G)$ whose projectivization $\bar{\lambda}_t$ accumulates at some point of $P(\mathbb{R}^0_{\geq 0})$. Since $C(G)$ is path connected, there is a stable path, still denoted $\bar{\lambda}_t$, whose limit point is this accumulation point. By Lemma 4.3, the short curves for this limit point are the components represented by edge-paths in $\partial\mathcal{A}(\lambda_t)$.

Conversely for any trivalent fatgraph G and any screen $\mathcal{A}$ on G, Lemma 4.2 shows that $\partial\mathcal{A}$ is realized as the set of short curves for a stable path in $C(G)$.

For a general not necessarily trivalent fatgraph, we require a further ingredient for the converse, namely:

Theorem 5.2. [15] *For any cyclically ordered tuple $x_1, \ldots, x_n$ of positive real numbers satisfying the generalized strict triangle inequalities $x_j < \sum_{i \neq j} x_i$, for $j = 1, \ldots, n$, there is a cyclic Euclidean planar polygon (i.e., the polygon inscribes in a circle) unique up to orientation-preserving isometry of the plane which realizes these numbers as its consecutive edge lengths.*

To apply this result, let L^+ denote the collection of isotropic vectors in Minkowski space with positive z-coordinate. Given a collection of coplanar points in L^+ lying in an affine plane determining an elliptic conic section, we may apply a Minkowski isometry to arrange that the plane containing these points is horizontal. The restriction of the Minkowski pairing to this horizontal plane is a multiple of the Euclidean metric induced on the plane, so the projectivized lambda lengths of pairs of these points agree with the projectivized Euclidean lengths in the horizontal plane. Furthermore, the intersection of the horizontal plane with L^+ is a round circle in this Euclidean structure.

Now, given any fatgraph G' with vertices at least trivalent and any screen $\mathcal{A}'$ on G', again define lambda lengths on the edges of G' by $\lambda'_t(e) = t^{d_e}$, where d_e is the depth of e in $\mathcal{A}'$. According to Theorem 5.2, the previous paragraph, and Theorem 3.2, this does indeed determine a path in $C(G')$.

Choose any trivalent fatgraph G which collapses to G'. The lambda lengths on the edges of $G - G'$ are thus given by the Euclidean lengths of the corresponding cyclic polygon again according to Theorem 5.2, the lambda lengths on the edges of G' have already been specified, so λ_t is now determined on G.

It follows that there is a unique screen $\mathcal{A}$ on G which restricts to $\mathcal{A}'$ in the natural sense with corresponding lambda lengths λ_t on G, and satisfying $\partial\mathcal{A}' = \partial\mathcal{A}$ as homotopy classes. The proof of Lemma 4.3 applies to λ_t to conclude that the components of $\partial\mathcal{A}$ are precisely the short curves for λ_t. □

6. Closing remarks

In sketch, we have seen that simplicial coordinates are given explicitly in terms of lambda lengths, and the no vanishing cycle condition is necessary and sufficient to guarantee that these formulae are uniquely invertible. Writing the inverse explicitly is the basic "arithmetic problem" in decorated Teichmüller theory [17].

One ingredient, which is related to the asymptotics of this arithmetic problem, towards describing the Deligne-Mumford compactification is:

Theorem 6.1. [18] *Suppose that λ_t is a stable one-parameter family of lambda lengths on the fatgraph G with no vanishing cycles of corresponding simplicial coordinates X_t. Define $I = \{e \in E : \lambda_t(e) \to \infty\}$ and $J = \{e \in E : X_t(e) \to 0\}$. Then $I \subseteq J$ and $R(G_J) = G_I$, where $R(X)$ denotes the maximal recurrent subset of X.*

This result in tandem with Theorem 1.2 has interesting consequences: Take a straight-line path in the natural affine structure of simplicial coordinates on $C(G)$ for some fatgraph G which limits to a point that fails to satisfy the no vanishing cycle condition. Let $E_1 \subseteq E(G)$ denote the subset of edges of G whose simplicial coordinates vanish, and let $R_1 \subseteq E_1$ denote its maximal recurrent subset. Depending upon the affine path, certain lambda lengths of edges in R_1 diverge at various rates, i.e., a screen magically pops out as determined by the arithmetic problem.

The set of all screens on all fatgraphs of a fixed topological type g, s forms a partially ordered set in the natural way, where the face relation is generated as usual by collapsing edges and now also by inclusion of screens. The mapping class group $MC(F_g^s)$ acts on the geometric realization of this screen poset in the natural way. As will be described in a forthcoming paper by the first-named author, a quotient of this screen poset is homotopy equivalent to the Deligne-Mumford compactification $\bar{\mathcal{M}}(F_g^s)$, where the equivalence relation is finer than simply taking $MC(F_g^s)$-orbits; more specifically, the co-operad structure of $\bar{\mathcal{M}}(F_g^s)$ can be described explicitly using "partially decorated surfaces" (cf. [15, 20]) leading to the usual graphical diagrams of irreducible components for stable curves, and the finer equivalence relation is generated also by combinatorial isomorphisms of these diagrams.

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