On Complete families of curves with a given fundamental group in positive characteristic

Mohamed Saïdi

Abstratct

We prove in this paper, that complete families of smooth, and projective curves, of genus $g \geq 2$, in characteristic $p > 0$, with a constant geometric fundamental group, are isotrivial.

0. Introduction. Let k be an algebraically closed field, and let X be a complete, irreducible, and smooth curve over k, of genus g. The structure of the étale fundamental group $\pi_1(X)$ of X is well understood, if $\text{char}(k) = 0$, thanks to the Riemann existence Theorem. Namely, it is isomorphic to the profinite completion Γ_g, of the topological fundamental group of a compact, orientable, topological surface, of genus g. In particular, the structure of $\pi_1(X)$ depends only on g, in this case. In the case, where $\text{char}(k) = p > 0$, the structure of the full $\pi_1(X)$ is far from being understood. However, we understand, in this case, the structure of some quotients of $\pi_1(X)$. Assume $\text{char}(k) = p > 0$. Let $\pi_1^p(X)$ (resp. $\pi_1(X)^{p'}$) be the maximal pro-p-quotient of $\pi_1(X)$ (resp. its maximal prime-to-p quotient). The following results are well known:

(1) The fundamental group $\pi_1(X)$, in characteristic $p > 0$, is a quotient of the group Γ_g. In particular, $\pi_1(X)$ is topologically finitely generated.

(2) The structure of $\pi_1(X)^{p'}$ is well known, by Grothendieck's specialization theory for fundamental groups (cf. [SGA-1], X). Namely, it is isomorphic to the maximal prime-to-p quotient of Γ_g.

(3) The structure of $\pi_1^p(X)$ is well known, by Shafarevich theorem (cf. [Sh]). Namely, it is a free pro-p-group on $r := r_X$ generators, where r_X is the p-rank of the curve X.

Apart from these results, very little is known about the structure of the (geometric) fundamental group of curves, in positive characteristic.

The anabelian geometry (or philosophy), as initiated by Grothendieck (cf. [Gr]), predicted that the structure of the arithmetic fundamental group, of hyperbolic curves over number fields, should depend on the isomorphy type of the curve in discussion. It came as a surprise when Tamagawa proved, in [Ta-3], such an anabelian statement, for hyperbolic affine curves, defined over a finite field of characteristic $p > 0$. Tamagawa's result suggests, that anabelian phenomena may even hold, for the geometric fundamental group of complete curves, over arbitrary algebraically closed fields of characteristic $p > 0$, which would explain to some extend, the complexity of π_1 in positive characteristic. In this paper we give a new evidence to these expectations.

In order to get an idea about the complexity of π_1 of proper curves, in positive characteristic, we introduce the notion of fundamental group, for points in the moduli space of curves. Let $\mathcal{M}_g \to \mathbb{F}_p$ be the coarse moduli space of proper and smooth curves, of genus $g \geq 2$, in charactersitic $p > 0$. It is well known that $\mathcal{M}_g$ is a quasi-projective, and geometrically irreducible variety. Let L be an algebraically closed field, of characteristic p. Then $\mathcal{M}_g(L)$ is the set of isomorphism classes of irreducible, proper, and smooth curves of genus g, over L. For a point $\overline{x} \in \mathcal{M}_g(L)$, let $C_{\overline{x}} \to \operatorname{Spec} L$ be a curve classified by $\overline{x}$, and let $x \in \mathcal{M}_g$ be a point, such that $\overline{x} : \operatorname{Spec} L \to \mathcal{M}_g$, factors through x. We define the geometric fundamental group $\pi_1(x) := \pi_1(C_{\overline{x}})$, of the point x, as the fundamental group of the curve $C_{\overline{x}}$. We remark that the structure of $\pi_1(x)$, as a profinite group, depends only on x, and not on the concrete geometric point $\overline{x} \in \mathcal{M}_g(L)$ used to define it. Indeed, if $\overline{k(x)}$ is an algebraic closure of the residue field $k(x)$ at x, and C_x is a curve classified by $\operatorname{Spec} \overline{k(x)} \to \mathcal{M}_g$, then $C_{\overline{x}} \simeq C_x \times_{\overline{k(x)}} L$ is the base change of C_x to L. Hence, $\pi_1(C_{\overline{x}}) \simeq \pi_1(C_x)$ by the geometric invariance of the fundamental group for proper varieties.

A key tool in the study of π_1, is Grothendieck's specialization theory for fundamental groups (cf. [SGA], X). Let $y \in \mathcal{M}_g$ be a point, which specializes in $x \in \mathcal{M}_g$. Then, by Grothendieck's specialization theorem, there exists a surjective, continuous, homomorphism $\operatorname{Sp}_{y,x} : \pi_1(y) \to \pi_1(x)$. Concerning this specialization homomorphism, we have the following result:

Theorem (Saïdi, Pop, Raynaud, Tamagawa): *Let $\overline{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$. Let $x \in \mathcal{M}_g \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ be a* **closed** *point, and let $y \in \mathcal{M}_g \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ be a point which specializes in x. Then, the specialization homomorphism $\operatorname{Sp}_{y,x} : \pi_1(y) \to \pi_1(x)$ is not an isomorphism.*

The above theorem was proven by Pop and Saïdi, in [Po-Sa], in the special case where the point x corresponds to a curve, having an absolutely simple jacobian, with p-rank equal to g or $g - 1$, and by Raynaud, in [Ra-1], in the case $g = 2$, and the case of supersingular curves of arbitrary genus $g > 2$ (i.e. curves whose jacobian is isogenous to a product of supersingular elliptic curves), and finally by Tamagawa, in [Ta-1], in the general case.

This result suggests that the structure, of the geometric fundamental group π_1, is far from being constant on the moduli space $\mathcal{M}_g$, in characteristic $p > 0$, much contrary to the characteritic 0 case. Let k be an algebraically closed field, of characteristic $p > 0$. Let $\mathcal{S} \subset \mathcal{M}_g \times_{\mathbb{F}_p} k$ be a k-subvariety. We say that the geometric fundamental group π_1 is *constant* on $\mathcal{S}$, if for any two points x and y of $\mathcal{S}$, such that y specializes in x, the corresponding specialization homomorphism $\mathrm{Sp}_{y,x} : \pi_1(y) \to \pi_1(x)$ is an isomorphism. This, in particular, would imply that all points of $\mathcal{S}$ have isomorphic geometric fundamental groups. We say that π_1 is *not constant* on $\mathcal{S}$, if the contrary holds, namely: there exists two points x and y of $\mathcal{S}$, such that y specializes in x, and such that the corresponding specialization homomorphism $\mathrm{Sp}_{y,x} : \pi_1(y) \to \pi_1(x)$, is not an isomorphism. The above Theorem implies, in particular, that in the case $k = \overline{\mathbb{F}}_p$, the moduli space $\mathcal{M}_g \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ does not contain, positive dimensional $\overline{\mathbb{F}}_p$-subvarieties, on which π_1 is constant. It is thus natural to ask the following question:

Question 6.3. Let k be an algebraically closed field of characteristic $p > 0$. Does $\mathcal{M}_g \times_{\mathbb{F}_p} k$ contain k-subvarieties, of positive dimension > 0, on which π_1 is constant?

Our main result answering the above question is the following:

Theorem (6.4). *Let k be an algebraically closed field, of characteristic p. Let $\mathcal{S} \subset \mathcal{M}_g \times_{\mathbb{F}_p} k$ be a* **complete** *k-subvariety of $\mathcal{M}_g \times_{\mathbb{F}_p} k$. Then, the fundamental group π_1 is not constant on $\mathcal{S}$.*

Note, that it is well known that $\mathcal{M}_g \times_{\mathbb{F}_p} k$ contains complete subvarieties (cf. [Oo-1], and [Fa-Lo], for example). In the case where, the generic points of the subvariety $\mathcal{S}$, are contained in the locus of ordinary curves, i.e. curves having maximal p-rank equal to g, this result is well known, and due to Szpiro, Raynaud, and Moret-Bailly (cf. [Sz], and [Mo]). It can also be deduced from Oort's result, on complete families of abelian varieties of dimension g, with constant p-rank equal to $g-1$ (cf. [Oo], 6.2), if the generic points of the subvariety $\mathcal{S}$, correspond to curves with p-rank equal to $g-1$. One may wonder, whether there exists non-isotrivial complete smooth families of curves, of genus g, with constant p-rank. For otherwise, this would directly imply the above theorem, since the p-rank is encoded in the isomorphy type of the fundamental group. It turns out that such families exist. In [Oo-1], Oort constructed an example of a non-isotrivial complete smooth family of curves, of genus 3, having constant p-rank equal to 0. In section 5, we extend Oort's argument, in order to construct such examples, for any genus $g \geq 2$.

For the proof of Theorem 6.4, it is easy to reduce to the case where $\mathcal{S}$ is a complete curve. In this case we prove, the following more precise result:

Theorem 6.6. *Let k be an algebraically closed field, of characteristic $p > 0$. Let S be a smooth, complete, and irreducible k-curve. Let $f : X \to S$ be a* **non-isotrivial***, proper and smooth family of curves, of genus $g \geq 2$. Then, there exists a finite étale cover $S' \to S$*

of S, an étale cover $Y' \to X' := X \times_S S'$, of degree prime to p, and a closed point $s_0 \in S'$, such that the p-rank of the geometric fibre $Y'_{k(\bar{s}_0)} \to k(\bar{s}_0)$, of Y' above the point s_0, is strictly smaller, that the p-rank of the generic geometric fibre $Y'_{k(\bar{\eta})} \to k(\bar{\eta})$ of Y' above the generic point η of S'.

The main ingredients we use, in order to prove Theorem 6.6, are: first, Raynaud's theory of theta divisors in positive characteristic. Secondly, the Theorem of Szpiro, Raynaud, and Moret-Bailly, on the isotriviality of complete families of ordinary abelian varieties. And finally, a recent result of Tamagawa, on the equi-characteristic deformation of generalized Prym varieties. Finally, the statement of Theorem 6.4 can be easily generalized to the case where we consider the (geometric) tame fundamental group, see Theorem 6.10.

This paper is organized as follows. In sections 1, and 2, we review Raynaud's theory of theta divisors in characteristic $p > 0$, and its application to the study of the p-rank of cyclic, of order prime-to-p, étale covers of curves. In section 3, we explain Tamagawa's result on the equi-characteristic deformation of generalized Prym varieties. In section 4, we recall the results of Szpiro, Raynaud, Moret-Bailly, on complete families of abelian varieties with constant maximal p-rank. In section 5, we extend an argument of Oort, in order to construct a complete family of smooth curves, for every genus $g \geq 2$, which has constant p-rank equal to 0. In section 6, we prove our main result, using the results exposed in sections 1, 2, 3, and 4.

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1. The sheaf of locally exact differentials in characteristic $\mathbf{p > 0}$ and its theta divisor.

In this section we review, mainly following Raynaud, the definition of the sheaf of locally exact differentials associated to a smooth algebraic curve in positive characteristic and its theta divisor (cf. [Ra], 4, and [Ta], 1, for futher generalisations). Let X be a proper smooth and connected algebraic curve of genus $g_X := g \geq 2$, over an algebraically closed field k of characteristic $p > 0$. Consider the following cartesian diagram:

$$\begin{array}{ccc} X^1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{F} & \operatorname{Spec} k \end{array}$$

where F denotes the absolute Frobenius morphism. The projection $X^1 \to X$ is a scheme isomorphism. In particular, X^1 is a smooth and proper curve of genus g. The absolute Frobenius morphism $F : X \to X$ induces in a canonical way a morphism $\pi : X \to X^1$ called the *relative Frobenius*, which is a radicial morphism of degree p. The canonical differential $\pi_* d : \pi_* \mathcal{O}_X \to \pi_* \Omega^1_X$ is a morphism of $\mathcal{O}_{X^1}$-modules, its image $B_X := B := \mathrm{Im}(\pi_* d)$ is the *sheaf of locally exact differentials.* One has the following exact sequence:

$$0 \to \mathcal{O}_{X^1} \to \pi_* \mathcal{O}_X \to B \to 0$$

and B is a vector bundle on X^1 of rank $p-1$.

Consider the *Cartier operation* $c : \pi_*(\Omega^1_X) \to \Omega^1_{X^1}$, which is a morphism of $\mathcal{O}_{X^1}$-modules. The kernel $\ker(c)$ of c is equal to B, and the following sequence of $\mathcal{O}_{X^1}$-modules is exact (cf. [Se], 10):

$$0 \to B \to \pi_*(\Omega^1_X) \to \Omega^1_{X^1} \to 0$$

Let $\mathcal{L}$ be a *universal Poincaré bundle* on $X^1 \times_k J^1$, where $J^1 := \mathrm{Pic}^0(X^1)$ is the Jacobian variety of X^1. The restriction of $\mathcal{L}$ to $X^1 \times \{a\}$, for any $a \in J^1(k)$, is isomorphic to the degree zero line bundle $\mathcal{L}_a$ which is the image of a under the natural isomorphism $J^1(k) \simeq \mathrm{Pic}^0(X^1)$ ($\mathcal{L}$ is normalized in such a way that $\mathcal{L}_{0_{J^1}} \simeq \mathcal{O}_{X^1}$). Let $h : X^1 \times_k J^1 \to X^1$, and $f : X^1 \times_k J^1 \to J^1$, be the canonical projections. As $R^i f_*(h^* B \otimes L) = 0$ for $i \geq 2$, the total direct image $Rf_*(h^* B \otimes L)$, of $(h^* B \otimes L)$ by f, can be realized by a complex $u : \mathcal{M}^0 \to \mathcal{M}^1$ of length 1, where $\mathcal{M}^0$ and $\mathcal{M}^1$ are vector bundles on J^1, $\ker u = R^0 f_*(h^* B \otimes L)$, and $\operatorname{coker} u = R^1 f_*(h^* B \otimes L)$. Moreover, as the Euler-Poincaré characteritic $\chi(h^* B \otimes L) = 0$, the vector bundles $\mathcal{M}^0$ and $\mathcal{M}^1$ have the same rank. In [Ra], Théorème 4.1.1, Raynaud proved the following theorem:

1.1. Theorem (Raynaud): *The determinant* $\det u$ *of* u *is not identically zero on* J^1.

In particular, one can consider the divisor $\theta := \theta_B$ on J^1, which is the positive Cartier divisor locally generated by $\det u$. This is the *theta divisor* associated to the vector bundle B. By definition a point $a \in J^1(k)$ lies on the support of θ if and only if $H^0(X^1, B \otimes \mathcal{L}_a) \neq 0$.

2. p-Rank of cyclic étale covers of degree prime to p.

We use the same notation as in 1. We will only discuss in this section the p-rank and the notion of *new-ordinariness* for cyclic covers of degree l := a prime integer distinct from p. This is the only case we use in this paper. For the general case of any integer prime to p see [Ra-1], 2, and [Ta], 3.

The absolute Frobenius morphism $F : X \to X$ induces a semi-linear map F :

$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$, and we have a canonical decomposition:

$$H^1(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X)^{\mathrm{ss}} \oplus H^1(X, \mathcal{O}_X)^{\mathrm{n}}$$

where $H^1(X, \mathcal{O}_X)^{\mathrm{ss}}$ is the semi-simple part on which F is bijective, and $H^1(X, \mathcal{O}_X)^{\mathrm{n}}$ is the nilpotent part on which F is nilpotent. The *p-rank* $r_X := r$ of X is the dimension of the k-vector space $H^1(X, \mathcal{O}_X)^{\mathrm{ss}}$. By duality, it is also the dimension of the subspace of $H^0(X, \Omega^1_X)$ on which the Cartier operation c is bijective (cf. [Se], 10). The p-rank r_X of X is also the rank of the maximal pro-p-quotient $\pi_1^p(X)$ of the fundamental group $\pi_1(X)$ of X, which is known to be a finitely generated free pro-p-group (cf. [Sh]). If A is an abelian variety of dimension d over k, then the rank of the étale part of the kernel of the morphism $[p] : A \to A$ of multiplication by p is p^h, where $0 \le h \le d$ is the p-rank of A. The abelian variety A is said to be *ordinary* if is has maximal p-rank equal to d, which is also equivalent to the fact that the Frobenius F is bijective on $H^1(A, \mathcal{O}_A)$. With the above notation, if $J = \mathrm{Pic}^0(X)$ is the jacobian variety of X, then it is well known that the p-rank of X equals the p-rank of J.

The relative Frobenius morphism $\pi : X \to X_1$ induces (because it is a radicial morphism) a "canonical" isomorphism $\pi_1(X) \to \pi_1(X_1)$ between fundamental groups (cf. [SGA-1], IX, Théorème 4. 10). In particular, for any **prime** integer l, which is distinct from p, one has a one-to-one correspondence between μ_l-torsors of X and those of X^1. More precisely, the canonical homomorphism $H^1_{\mathrm{et}}(X^1, \mu_l) \to H^1_{\mathrm{et}}(X, \mu_l)$, induced by π, is an isomorphism. Consider a μ_l-torsor $f : Y \to X$ with Y connected. By Kummer theory, the torsor f is given by an invertible sheaf $\mathcal{L}$ of order l on X, and $Y := \mathrm{Spec}(\oplus_{i=0}^{l-1} \mathcal{L}^{\otimes i})$. There exists then an invertible sheaf $\mathcal{L}^1$ on X^1, of order l, such that if $f' : Y^1 \to X^1$ is the associated μ_l-torsor, we have a cartesian diagram:

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ Y^1 & \xrightarrow{f'} & X^1 \end{array}$$

Let $J_Y := \mathrm{Pic}^0(Y)$ (resp. $J_X := \mathrm{Pic}^0(X)$) denotes the Jacobian variety of Y (resp. the Jacobian of X). The morphism $f : Y \to X$ induces a natural homomorphism $f^* : J_X \to J_Y$ between Jacobians, which has a finite kernel (f^* is given by the pull-back of degree zero invertible sheaves). Let $J^{\mathrm{new}} := J_{Y/X}$ denotes the quotient of J_Y by the image $f^*(J_X)$ of J_X. The variety J^{new} is an abelian variety of dimension $g_Y - g_X$, and p-rank equal to $r_Y - r_X$, it is called the *new part* of the Jacobian J_Y of Y with respect to the morphism f.

2.1. Definition. The μ_l-torsor $f : Y \to X$ is said to be *new-ordinary*, if the new part

$J^{\rm new}$ of the Jacobian of Y, with respect to the morphism f, is an ordinary abelian variety, i.e. if the equality $g_Y - g_X = r_Y - r_X$ holds.

Raynaud's theory of theta divisors allows another important geometric interpretation of new-ordinariness, which we explain below. This interpretation has allowed significant recent progress in the study of fundamental groups of curves in positive characteristics.

There exists an isomorphism $H^1(J_Y, \mathcal{O}_{J_Y}) \simeq H^1(Y, \mathcal{O}_Y)$ (cf. [Se-1], VII, théorème 9), and $H^1(Y, \mathcal{O}_Y) = H^1(X, f^*\mathcal{O}_Y) = H^1(X, \oplus_{i=0}^{l-1}\mathcal{L}^{\otimes i})$, from which we deduce that $H^1(J^{\rm new}, \mathcal{O}_{J^{\rm new}}) \simeq H^1(X, \oplus_{i=1}^{l-1}(\mathcal{L})^{\otimes i})$. Note that the above identifications are compatible with the action of Frobenius. Hence, the kernel of Frobenius on $H^1(J^{\rm new}, \mathcal{O}_{J^{\rm new}})$ is isomorphic to the kernel of Frobenius acting on $H^1(X, \oplus_{i=1}^{l-1}\mathcal{L}^{\otimes i})$. On the other hand, as f' is étale, we have $(f')^*(B_X) = B_Y$. Thus, also $(f')_*(B_Y) = B_X \otimes (f')_*(\mathcal{O}_{Y^1}) = \oplus_{i=0}^{l-1}(B_X \otimes (\mathcal{L}^1)^{\otimes i})$. Now, by duality, the kernel of the Frobenius acting on $H^1(X^1, \oplus_{i=1}^{l-1}\mathcal{L}^{1^{\otimes i}})$ is isomorphic to the kernel of the Cartier operator acting on $H^0(X^1, \pi_*\Omega^1_X \otimes (\oplus_{i=1}^{l-1}(\mathcal{L}^1)^{\otimes i}))$, which is $\oplus_{i=1}^{l-1}H^0(X^1, B_X \otimes (\mathcal{L}^1)^{\otimes i})$. Thus, we see that the above μ_l-torsor $f : Y \to X$ is new ordinary, if and only if the Frobenius F is injective (hence bijective) on $H^1(J^{\rm new}, \mathcal{O}_{J^{\rm new}})$, i.e. if and only if $H^0(X^1, B \otimes (\mathcal{L}^1)^{\otimes i}) = 0$ for all $i \in \{1, ..., l-1\}$. Finally, this last statement is equivalent, by the very definition of the theta divisor θ_X associated to the vector bundle B_X, to the following:

2.2. Proposition: *The μ_l-torsor $f : Y \to X$ is new-ordinary, if and only if the subgroup $< \mathcal{L}^1 >$, generated by $\mathcal{L}^1$ in J^1, intersects the support of the theta divisor θ_X at most at the zero point 0_{J^1} of J^1.*

Using the above interpretation of new-ordinariness, and with an input form intersection theory, one can prove that for $l >> 0$ "most" μ_l-torsors are new-ordinary. More precisely, one has the following result which is essencially due to Serre and Raynaud (see [Ra], théorème 4.3.1, and [Ta], corollary 3.10, for a proof):

2.3. Theorem: *There exists a constant c, depending only on g and p, such that for each prime integer $l \neq p$, the set of elements of $J[l](k)$ whose corresponding μ_l-torsor is not new-ordinary, has cardinality $\leq c(l-1)l^{2g-2}$ (Here $J[l]$ denotes the kernel of multiplication by l in J). Moreover, one can take $c = (p-1)3^{g-1}g!$.*

In particular, if $l >> 0$ we can find an element of $J[l](k)$ such that the corresponding μ_l-torsor is new-ordinary, since Card $J[l](k) = l^{2g}$.

3. Equi-characteristic deformation of generalized Prym varieties.

In this section, we state the theorem of Tamagawa, on the local infinitesimal Torelli problem for generalized Prym varieties. This Theorem is an essential tool in the proof of the main result of this paper.

Let k be an algebraically closed field, of arbitrary characteristic. Denote by $\mathcal{C}_k$ the category of artinian local rings, with residue field k. For a proper and smooth k-variety X_0, one defines the (equi-characteristic) deformation functor M_{X_0} of X_0, to be the functor:

$$M_{X_0} : \mathcal{C}_k \to (\text{Sets})$$

which to an element R of $\mathcal{C}_k$, associates the set of isomorphism classes of pairs (X, φ), where X is a proper and smooth R-scheme, and φ is an isomorphism $X \times_R k \simeq X_0$. The functor M_{X_0} is well understood in the case where X_0 has dimension 1, and genus $g > 1$, (resp. if X_0 is an abelian variety of dimension d). In this case the functor M_{X_0} is pro-representable by a ring of formal power series of $3g - 3$, (resp. d^2), variables over k. We will be mainly interested in this two cases.

Now, assume that X_0 is a proper, connected, and smooth algebraic curve over k, with genus $g \geq 2$. Let l be a prime integer distinct from the characterisic of k, and let $f_0 : Y_0 \to X_0$ be a μ_l-torsor, with Y_0 connected. The torsor f_0 corresponds to an element $\mathcal{L}_0 \in J_0[l](k)$, where $J_0[l](k)$ denotes the k-subgroup, of l-torsion points, in the jacobian J_0 of X_0. Let J_0^{new} be the new part of the jacobian of Y_0, with respect to the morphism f_0. Then, for any element R of $\mathcal{C}_k$, there is a natural map:

$$T_{\mathcal{L}_0}(R) : M_{X_0}(R) \to M_{J_0^{\text{new}}}(R)$$

defined as follows: let (X, φ) be an element of $M_{X_0}(R)$. The μ_l-torsor $f_0 : Y_0 \to X_0$ lifts uniquely, by the theorems of lifting of étale covers (cf. [SGA-1], I, 8), to a μ_l-torsor $f : Y \to X$. Let $J_X := \text{Pic}^0(X)$ (resp. $J_Y := \text{Pic}^0(Y)$) be the relative jacobian of X (resp. the relative jacobian of Y), which is an abelian scheme over R, and let $f^* : J_X \to J_Y$ be the natural homomorphism, which is induced by the pull back of invertible sheaves. Define $J^{\text{new}} := J_Y / f^*(J_X)$, to be the quotient of J_Y by the image $f^*(J_X)$ of J_X. Then J^{new} is an abelian R-scheme, and there exists a natural isomorphism $\psi : J^{\text{new}} \times_R k \simeq J_0^{\text{new}}$, which is induced by φ. Thus, the pair (J^{new}, ψ) is an element of $M_{J_0^{\text{new}}}(R)$, which we define to be the image under $T_{\mathcal{L}_0}(R)$ of (X, φ) in $M_{J_0^{\text{new}}}(R)$. The infinitesimal Torelli problem asks whether or not the above natural map:

$$T_{\mathcal{L}_0} : M_{X_0} \to M_{J_0^{\text{new}}}$$

is an immersion. More precisely, let $k[\epsilon]$ $(\epsilon^2 = 0)$ be the ring of dual numbers on k. Then, the question is whether the natural map: $T_{\mathcal{L}_0}(k[\epsilon]) : M_{X_0}(k[\epsilon]) \to M_{J_0^{\text{new}}}(k[\epsilon])$, between tangent spaces, is injective. This is also equivalent asking, whether if A (resp. B) is the pro-representing object of the functor M_{X_0} (resp. of the functor $M_{J_0^{\text{new}}}$), then the natural homomorphism $B \to A$, induced by $T_{\mathcal{L}_0}$, is surjective. If this is the case, it would in particular imply the following: for every element $R \in \mathcal{C}_k$, and $(X, \varphi) \in M_{X_0}(R)$, if the image (J^{new}, ψ) of (X, φ) in $M_{J_0^{\text{new}}}(R)$, via the map $T_{\mathcal{L}_0}(R)$, is a trivial deformation of J_0^{new}, then X is a trivial deformation of X_0. This is a generalization of the classical infinitesimal Torelli problem, which asks whether the natural map from the space of deformations of the curve X_0, to the space of deformations of its jacobian, is an immersion, in which case one knows that the answer is yes, if the curve X_0 is not hyperelliptic. Concerning the above generalization, Tamagawa proves the following:

3.1. Theorem (Tamagawa): *Let X_0 be a proper, connected, and smooth algebraic curve, over an algebraically closed field k, with genus $g \geq 2$. Let $d_{X_0} := \{\min \deg(f) \ / \ f : X_0 \to \mathbb{P}^1_k$ non constant$\}$, be the gonality of X_0 (cf. [Ta-1], 1). Assume that $d_{X_0} \geq 5$. Then, there exists a constant c_1, which depends only on X_0, and such that for each prime integer $l \neq \operatorname{char}(k)$, the subset of the elements $\mathcal{L}_0$ of $J[l](k)$, such that the corresponding natural map $T_{\mathcal{L}_0} : M_{X_0} \to M_{J_0^{\text{new}}}$ is not an immersion, has cardinality $\leq c_1 l^{2g-2}$.*

In particular, if $l >> 0$, then one can find an element $\mathcal{L}_0 \in J[l](k)$, such that the corresponding map: $T_{\mathcal{L}_0} : M_{X_0} \to M_{J_0^{\text{new}}}$ is an immersion. For a proof of the above result see [Ta-1], corollary 4.16. Concerning the gonality of curves, Tamagawa also proves the following:

3.2. Theorem (Tamagawa): *Let X be a proper, connected, and smooth algebraic curve, over an algebraically closed field k, with genus $g \geq 2$. Then, there exists an étale cover $f : Y \to X$, such that the gonality d_Y of Y satisfies $d_Y \geq 5$. Moreover, the cover f can be chosen to be a composition of two cyclic étale covers, of (suitable) degree prime to the characteristic of k.*

For the proof of Theorem 3.2, combine Theorem 2.7, Proposition 2.14, and Corollary 2.19 from [Ta-1].

Combining both the Theorems 3.2 and 2.3 above, we obtain the following result, which we will use in the proof of our main Theorem in section 6. This result was also used by Tamagawa, in [Ta-1], in order to prove Theorem 6.1 in that paper.

3.3. Theorem: *Let X be a proper, connected, and smooth algebraic curve, of genus $g \geq 2$, over an algebraically closed field k of characteristic $p > 0$. Assume that the gonality d_X of X is ≥ 5. Then, if $l \neq p$ is a prime integer, such that $l > 1 + c_1 + (p-1)3^{g-1}g!$, where c_1 is the constant in theorem 3.2, there exists a non zero element $\mathcal{L} \in J[l](k)$, such that the following two conditions are satisfied:*

(i) The μ_l-torsor $f : Y \to X$ corresponding to $\mathcal{L}$ is new ordinary.

(ii) The natural map $T_{\mathcal{L}} : M_X \to M_{J^{\text{new}}}$ is an immersion.

4. Complete families of ordinary abelian varieties in characteristic $p > 0$.

In this section, we state the theorem of Raynaud, Szpiro, and Moret-Bailly, on the isotriviality of complete families of ordinary abelian varieties, in characteristic $p > 0$. This result will be used in section 6. In all what follows we fix a prime integer $p > 0$.

Let $g \geq 1$, and $d \geq 1$ be integers. Let $\mathcal{A}_{g,d} \to \operatorname{Spec} \mathbb{F}_p$ denotes the coarse moduli scheme of polarized abelian varieties (with a degree d polarization), of dimension g, in characteristic p. The scheme $\mathcal{A}_{g,d}$ is a quasi-projective variety of dimension $g(g+1)/2$, and has the following property: for every scheme S in characteristic p, and $X \to S$ a polarized abelian S-scheme (with a degree d polarization), of relative dimension g, there exists a unique natural map $S \to \mathcal{A}_{g,d}$, defined by the family $X \to S$. This map sends a point $s \in S$, to the moduli point corresponding to the fibre $X_s \to \operatorname{Spec} k(s)$, of X above the point s.

4.1. Definition: Let k be a field of characteristic $p > 0$. Let S be a normal and integral k-variety, with generic point η, and let $X \to S$ be an abelian S-scheme of relative dimension g. Assume that the generic fibre $X_\eta \to \operatorname{Spec} k(\eta)$ of X, has a polarization of degree d. Then the family $X \to S$ is called *isotrivial*, if the natural map $\operatorname{Spec} k(\eta) \to \mathcal{A}_{g,d}$, defined by $X_\eta \to \operatorname{Spec} k(\eta)$, factorizes through $\operatorname{Spec} k(\eta) \to \operatorname{Spec} k' \to \operatorname{Spec} \mathcal{A}_{g,d}$, where k' is a finite extension of k.

4.2. Theorem (Raynaud, Szpiro, Moret-Bailly): *Let k be a field of characteristic $p > 0$. Let S be a normal and integral k-variety, which is* **projective**. *Let $X \to S$ be an abelian S-scheme, of relative dimension g. Assume that all the geometric fibres $X_{\bar{s}} \to \operatorname{Spec} k(\bar{s})$, of X over S, are ordinary abelian varieties. Then the family $X \to S$ is isotrivial.*

Next, we would like to explain the stratification, in characteristic p, of the moduli space of polarized abelian varieties, by the p-rank. Let k be an algebraically closed field of characteristic p. For each fixed integer $0 \leq f \leq g$, let $\mathcal{V}_f \subset \mathcal{A}_g \times_{\mathbb{F}_p} k$ be the subset of points corresponding to abelian varieties having a p-rank $\leq f$. Concerning the subsets $\mathcal{V}_f$, we have the following:

4.3. Theorem: *For each fixed integer $0 \leq f \leq g$, and any algebraically closed field k of characteristic p, let $\mathcal{V}_f \subset \mathcal{A}_g \times_{\mathbb{F}_p} k$ be the subset of points corresponding to abelian varieties having a p-rank $\leq f$. Then, the subset $\mathcal{V}_f$ is a closed subscheme of $\mathcal{A}_g \times_{\mathbb{F}_p} k$, and every irreducible component of $\mathcal{V}_f$ has dimension equal to $g(g+1)/2 - g + f$. Moreover, the closed subscheme $\mathcal{V}_0$ is a complete subvariety of $\mathcal{A}_g \times_{\mathbb{F}_p} k$, of dimension $g(g-1)/2$.*

For the proof of the fact that $\mathcal{V}_f$ is closed see [Oo-1], Corollary 1.5. For the statement concerning the dimension of the above strata see [No-Oo], Theorem 4.1. Finally, for the fact that $\mathcal{V}_0$ is complete, see [Oo-1], proof of Theorem 1.1 a).

5. Complete families of curves with constant p-rank in characteristic $p > 0$.

In this section, we want to extend the argument of Oort, in [Oo-1], in order to show the existence, for every integer $g > 2$, of a non-isotrivial complete family of smooth and proper curves of genus g, with constant p-rank equal to 0. Before stating the main result, we will explain, and adopt some notation. In all what follows we fix a prime integer $p > 0$.

Let $g \geq 2$ be an integer, and let $\mathcal{M}_g \to \operatorname{Spec} \mathbb{F}_p$ be the coarse moduli scheme, of projective, smooth, and irreducible curves of genus g, in characteristic p. The scheme $\mathcal{M}_g$ is a quasi-projective irreducible variety, of dimension $3g-3$. Let $\overline{\mathcal{M}}_g \to \operatorname{Spec} \mathbb{F}_p$ be the Deligne-Mumford compactification of $\mathcal{M}_g$, which is a coarse moduli scheme, of projective, and stable curves of genus g, in characteristic p. The scheme $\overline{\mathcal{M}}_g$ is an irreducible projective variety, which contains $\mathcal{M}_g$ as an open subscheme (cf. [De-Mu] for more details. We will also consider $\mathcal{M}'_g \to \operatorname{Spec} \mathbb{F}_p$, which is the coarse moduli scheme of projective, and stable curves of genus g, in characteristic p, whose jacobian is an abelian variety (these are exactly the projective, and stable curves of genus g, whose configuration is a tree like cf. [Bo-Lu-Ra], 9, corollary 12).

Let S be a scheme of characteristic p. By a smooth relative (or a family of) curve(s) $f : X \to S$ over S, of genus g, we mean that f is a proper and smooth equidimensional morphism, with relative dimension 1, whose fibres are curves of genus g. We say that the relative curve $f : X \to S$ is *complete*, if S is a complete, and irreducible variety, in characteristic p. The moduli scheme $\mathcal{M}_g$ has the following property: if $f : X \to S$ is a smooth relative curve over S, of genus g, then there is a natural map $S \to \mathcal{M}_g$, which is uniquely determined by f. This map sends a point $s \in S$, to the moduli point corresponding to the fibre $X_s \to \operatorname{Spec} k(s)$, of X above the point s. We also have the following:

5.1. Proposition: *Let k be a field of characteristic p, and let $\mathcal{S}$ be a subvariety of $\mathcal{M}_g \times_{\mathbb{F}_p} k$. Then, there exists a finite étale cover $h : \mathcal{S}' \to \mathcal{S}$, and a smooth relative $\mathcal{S}'$-curve $f' : X' \to \mathcal{S}'$ of genus g, such that the natural morphism $\mathcal{S}' \to \mathcal{M}_g$, induced by f', factorizes $\mathcal{S}' \to S \to \mathcal{M}_g$ through h.*

Proof: Standard, by passing to the fine moduli scheme of smooth, projective, and irreducible curves of genus g, with a symplectic level n-structure.

5.2. Definition: Let S be a scheme of characteristic p, and let $f : X \to S$ be a smooth relative S-curve, of genus g. The curve $f : X \to S$ is said to be *isotrivial*, if the corresponding map $S \to \mathcal{M}_g$, has an image which consists of a point.

5.3. Definition: Let S be a scheme of characteristic p, and let $f : X \to S$ be a smooth relative S-curve of genus g. Let $0 \leq r \leq g$ be an integer. We say that the family $f : X \to S$ has constant p-rank, equal to r, if each geometric fibre $X_{\bar{s}} \to \operatorname{Spec} k(\bar{s})$ of f has a p-rank equal to r.

In [Oo-1], Oort showed in the proof of theorem 1.1. b), the existence over any algebraically closed field k of characteristic p, of a complete curve contained in $\mathcal{M}_3 \times_{\mathbb{F}_p} k$, which is contained in the locus of curves having p-rank equal to 0. This indeed corresponds by Proposition 5.1, to a non-isotrivial complete family of smooth curves, of genus 3, with constant p-rank equal to 0. Next, we want to extend Oort's argument, in order to prove the following:

5.4. Theorem: *Let $g \geq 3$ be an integer. Let k be an algebraically closed field, of characteristic p. Then $\mathcal{M}_g \times_{\mathbb{F}_p} k$ contains a complete irreducible curve, which is contained in the locus of curves having p-rank equal to* 0.

Proof: The proof consists in considering the Torelli map, together with a dimension argument. More precisely, let:

$$t : \mathcal{M}'_g \times_{\mathbb{F}_p} k \to \mathcal{A}_{g,1} \times_{\mathbb{F}_p} k$$

be the Torelli morphism, which sends the class of a stable curve, whose jacobian is an abelian variety, to the class of its jacobian, endowed with its canonical principal polarization coming from the theta divisor. Torelli's theorm (cf. [Mi], 12) says that the map t is injective on geometric points. In particular, the image $\mathcal{J}'_g := t(\mathcal{M}'_g \times_{\mathbb{F}_p} k)$ (resp. $\mathcal{J}_g := t(\mathcal{M}_g \times_{\mathbb{F}_p} k)$) of $\mathcal{M}'_g \times_{\mathbb{F}_p} k$ (resp, of $\mathcal{M}_g \times_{\mathbb{F}_p} k$), which is called the *jacobian locus* (resp. *open jacobian locus*), is a subvariety of $\mathcal{A}_{g,1}$ of dimension $3g-3$. Moreover, $\mathcal{J}'_g$ is closed in $\mathcal{A}_{g,1}$ (cf. [Mu], lecture IV, p. 74). For each fixed integer $0 \leq f \leq g$, let $\mathcal{V}_f$ be the closed subscheme of $\mathcal{A}_{g,1} \times_{\mathbb{F}_p} k$ as defined in 4.3. Then, every irreducible component of $\mathcal{J}'_{f,g} := \mathcal{J}'_g \cap \mathcal{V}_f$, has dimension at least $3g - 3 - g + f = 2g - 3 + f$ (cf. [Oo-1], lemma 1.6). Moreover, it is well known, that every irreducible component of $\mathcal{J}'_{0,g} := \mathcal{J}'_g \cap \mathcal{V}_0$ has dimension $2g - 3$ (cf. [Fa-Lo], 11, p. 36). We claim, that $\{\mathcal{J}'_g - \mathcal{J}_g\} \cap \mathcal{V}_0$, has codimension at least two, in $\mathcal{J}'_{0,g} := \mathcal{J}'_g \cap \mathcal{V}_0$. Indeed, $\{\mathcal{J}'_g - \mathcal{J}_g\} \cap \mathcal{V}_0$ is contained in the images of $\mathcal{J}_{0,g_1} \times ... \times \mathcal{J}_{0,g_t}$, for all possible family of integers $\{g_1, ..., g_t\}$, such that $g_1 + ... + g_t = g$, with $t \geq 2$, via the natural morphism $\mathcal{J}'_{0,g_1} \times ... \times \mathcal{J}'_{0,g_t} \to \mathcal{V}_{0,g}$. Now, counting the dimension of $\mathcal{J}'_{0,g_1} \times ... \times \mathcal{J}'_{0,g_t}$, which is $2g_1 - 3 + ... + 2g_t - 3 < 2g - 5$, we conclude that $\{\mathcal{J}'_g - \mathcal{J}_g\} \cap \mathcal{V}_0$, has codimension at least two in $\mathcal{J}'_{0,g} := \mathcal{J}'_g \cap \mathcal{V}_0$.

Finally, since $\mathcal{V}_0$ is projective, we can find a closed immersion $\mathcal{V}_0 \to \mathbb{P}^N_k$ into a projective k-space, of suitable dimension. Further, we can find a general linear subspace L

of $\mathbb{P}^N_k$, of suitable dimension, such that $L \cap \mathcal{J}'_{0,g}$ is a (necessarily complete) curve S', and such that $L \cap \{\mathcal{J}'_{0,g} - \{\mathcal{J}_g \cap \mathcal{V}_0\}\}$ is empty (cf. [Da-Sh], II, Chapter 3, 1.2). Now the inverse image $S := t^{-1}(S')$, of S' via the Torelli map t, is a complete curve, contained in $\mathcal{M}_g \times_{\mathbb{F}_p} k$, and which by construction, is contained in the locus of curves having p-rank equal to 0.

5.5. Corollary: *Let $g \geq 3$ be an integer. Let k be an algebraically closed field, of characteristic p. Then there exists a complete, and smooth algebraic k-curve S, and a non iso-trivial smooth S-curve $f : X \to S$, of genus g, with constant p-rank equal to* 0.

5.6. Remark: It is tempting to try to construct an S-curve X, as in 5.5, for special values of g, by considering a Galois cover $f : X \to Y$, with group $\mathbb{Z}/p\mathbb{Z}$, where Y is a ruled surface over S, and such that f is étale outside an S-section of Y. The Deuring-Shafarevich formula, comparing the p-rank in Galois p-covers, would then imply that all fibres of X over S have p-rank equal to 0. However, by a result of Pries, all such covers $f : X \to Y$ are necessarily isotrivial (cf. [Pr], Theorem 3.3.4).

5.7. Question: Let $g \geq 3$ be an integer. Let k be an algebraically closed field, of characteristic p. Let $\tilde{r}$ be the maximum of the integers r, such that $\mathcal{M}_g \times_{\mathbb{F}_p} k$ contains a complete curve S, which is contained in $\mathcal{V}_r - \mathcal{V}_{r-1}$. We have $0 \leq \tilde{r} < g$. What is the value of $\tilde{r}$? Does the value of $\tilde{r}$ depends only on g? Or does it depend on p as well?

6. Complete families of curves with a given fundamental group in characteristic $p > 0$.

This is our main section, in which we prove the main result of this paper, which asserts that complete families of curves with a constant geometric fundamental group are isotrivial. In all what follows we fix a prime intger $p > 0$.

First, we will explain how to define the fundamental group of points, in the moduli space of curves. Let $\mathcal{M}_g \to \operatorname{Spec} \mathbb{F}_p$ be the coarse moduli scheme of smooth, and projective curves of genus g, in characteristic $p > 0$. Let k be an algebraically closed field of characteristic $p > 0$. For a geometric point $\overline{x} \in \mathcal{M}_g(k)$, let $C_{\overline{x}} \to \operatorname{Spec} k$ be a smooth curve, of genus g, which is classified by $\overline{x}$, and let $x \in \mathcal{M}_g$ be the point such that $\overline{x} : \operatorname{Spec} k \to \mathcal{M}_g$ factors through x. We define the *geometric fundamental group* $\pi_1(x) := \pi_1(C_{\overline{x}})$, of the point x, as the étale fundamental group of the curve $C_{\overline{x}}$ (we assume of course the choice of a base point). We remark that the structure of $\pi_1(x)$, as a profinite group, depends only on the point x, and not on the concrete geometric point $\overline{x} \in \mathcal{M}_g(k)$ used to define it. Indeed, first, if $\overline{k(x)}$ is an algebraic closure of the residue field $k(x)$ at x, and C_x is the curve classified by $\operatorname{Spec} \overline{k(x)} \to \mathcal{M}_g$, then $C_{\overline{x}} \simeq C_x \times_{\overline{k(x)}} k$, is the base change of C_x to k. Hence, $\pi_1(C_{\overline{x}}) \simeq \pi_1(C_x)$, by the geometric invariance of the fundamental group for proper varieties (cf. [SGA-1], X, Corollaire 1.8). Second, the isomorphy type of C_x as an $\mathbb{F}_p$-scheme does not depend on the choice of $\overline{k(x)}$: a geometric point of M_g dominating x. We thus have a map:

$$\pi_{1,\mathrm{geom}} : \mathcal{M}_g \to \{\mathrm{ProfGrp}\}$$
$$x \to \pi_1(x)$$

Next, we recall the specialization theory of Grothendieck for fundamental groups. Let $y \in \mathcal{M}_g$ be a point which specializes into the point $x \in \mathcal{M}_g$. Then Grothendieck's specialization theorem shows the existence, of a surjective continuous homomorphism $\mathrm{Sp} : \pi_1(y) \to \pi_1(x)$ (cf. [SGA-1], X). In particular, if η is the generic point of $\mathcal{M}_g$, then C_η is the generic curve of genus g, and every point x of $\mathcal{M}_g$ is a specialization of η. Hence, for every $x \in \mathcal{M}_g$, there exists a surjective homomorphism $\mathrm{Sp}_x : \pi_1(\eta) \to \pi_1(x)$. For every such an x, we fix such a map once for all. In particular, if y specializes to x, we also fix a surjective homomorphism $\mathrm{Sp}_{y,x} : \pi_1(y) \to \pi_1(x)$, such that $\mathrm{Sp}_{y,x} \circ \mathrm{Sp}_y = \mathrm{Sp}_x$.

6.1. Definition. Let $\mathcal{S} \subset \mathcal{M}_g$ be a subscheme of $\mathcal{M}_g$. We say that the (geometric) fundamental group π_1 is *constant* on $\mathcal{S}$, if for any two points x and y of $\mathcal{S}$, such that y specializes in x, the corresponding specialization homomorphism $\mathrm{Sp}_{y,x} : \pi_1(y) \to \pi_1(x)$ is an isomorphism. We say that π_1 is *not constant* on $\mathcal{S}$, if the contrary holds, namely: there exists two points x and y of $\mathcal{S}$, such that y specializes in x, and such that the corresponding specialization homomorphism $\mathrm{Sp}_{y,x} : \pi_1(y) \to \pi_1(x)$ is not an isomorphism.

For every field k of characteristic p, we define in a similar way, the geometric fundamental group of points in $\mathcal{M}_g \times_{\mathbb{F}_p} k$, as well as the notion of a subvariety $\mathcal{S} \subset \mathcal{M}_g \times_{\mathbb{F}_p} k$, on which the geometric fundamental group π_1 is constant.

6.2. Definition. Let S be a scheme of characteristic p, and let $f : X \to S$ be a relative smooth S-curve of genus g. We say that the (geometric) fundamental group π_1, is *constant* on the family f, if for any two points η and s of S, such that η specializes in s, the corresponding specialization homomorphism $\mathrm{Sp}_{\eta,s} : \pi_1(X_{\bar\eta}) \to \pi_1(X_{\bar s})$ is an isomorphism, where $X_{\bar\eta} := X \times_S k(\bar\eta)$ (resp. $X_{\bar s} := X \times_S k(\bar s)$) is the geometric fibre of X over the point η, (resp. the geometric fibre of X over the point s). If the above condition doesn't hold, we say that the fundamental group π_1 is not constant on the family f.

Let S be a scheme of characteristic p, and let $f : X \to S$ be a relative smooth S-curve of genus g. It is clear, that if the fundamental group is constant on the family f, and if $h : S \to \mathcal{M}_g$ is the map induced by the family f, then the fundamental group is also constant, on the image $h(S)$ of S in $\mathcal{M}_g$. It is quite natural to ask the following question:

6.3. Question. Let k be an algebraically closed field of characteristic p. Does $\mathcal{M}_g \times_{\mathbb{F}_p} k$ contain, k-subvarieties of positive dimension > 0, on which π_1 is constant?

Our main result is the following:

6.4. Theorem (Main Result). *Let k be an algebraically closed field, of characteristic p. Let $\mathcal{S} \subset \mathcal{M}_g \times_{\mathbb{F}_p} k$ be a* **complete** *k-subvariety of $\mathcal{M}_g \times_{\mathbb{F}_p} k$. Then, the fundamental group π_1 is not constant on $\mathcal{S}$.*

For the proof of Theorem 6.4, it is clear, that one can reduce to the case where S is a complete, and irreducible curve. The proof of Theorem 6.4, then follows easily, by using 5.1, from the following Theorem 6.5:

6.5. Theorem. *Let k be an algebraically closed field, of characteristic p. Let S be a smooth, complete, and irreducible k-curve. Let $f : X \to S$ be a* **non-isotrivial** *smooth family of curves of genus $g \geq 2$. Then, the fundamental group π_1 is not constant on the family f. In particular, if $h : S \to \mathcal{M}_g \times_{\mathbb{F}_p} k$ is the map defined by f, then the fundamental group π_1 is not constant on the image $h(S)$ of S in $\mathcal{M}_g \times_{\mathbb{F}_p} k$.*

In the process of proving Theorem 6.5 we prove, in fact, the following more precise result:

6.6. Theorem. *Let k be an algebraically closed field, of characteristic p. Let S be a smooth, complete, and irreducible k-curve. Let $f : X \to S$ be a* **non-isotrivial** *smooth family of curves of genus $g \geq 2$. Then, there exist a finite étale cover $S' \to S$, an étale cover $Y' \to X' := X \times_S S'$ of degree prime to p, and a closed point $s_0 \in S'$ such that the p-rank of the geometric fibre $Y'_{k(\bar{s}_0)} \to k(\bar{s}_0)$, of Y' above the point s_0, is strictly smaller, that the p-rank of the generic geometric fibre $Y'_{k(\bar{\eta})} \to k(\bar{\eta})$, of Y' above the generic point η of S'.*

First, we start with the following lemmas:

6.7. Lemma/Definition. *Let k be an algebraically closed field, of characteristic p. Let S be a smooth, complete, and irreducible k-curve. Let $f : X \to S$ be a smooth family of curves, of genus $g \geq 2$. Let s be a closed point of S, and let $f_s : Y_s \to X_s := X \times_S k(s)$ be a μ_n-torsor, over the fibre of X above the point s, where n is coprime to p. Then, there exists a positive integer d, such that if n is coprime to d, then there exists a finite étale cover $h : S' \to S$, a μ_n-torsor $f' : Y' \to X' := X \times_S S'$, and a closed point $s' \in S'$, with $h(s') = s$, such that the fibre $f'_{s'} : Y'_{s'} := Y' \times_{S'} k(s') \to X'_{s'} := X' \times_{S'} k(s')$, of the torsor f', above the point $s' \in S'$, coincides with the given cover $f_s : Y_s \to X_s$. We call such a pair (f', h) a* **good lifting** *of the cover $f_s : Y_s \to X_s$.*

Proof. In the case where the morphism f has a section, the above lemma follows easily from the homotopy exact sequence of fundamental groups in [SGA-1], XIII, Proposition 4.3 (see Lemma 4.3.1 in Loc.cit). In the general case, let x be a closed point of the genric fibre X_η of X over S, and let Z be the schematic closure of x in X. Denote by Y be the normalization of Z. The canonical morphism $Y \to S$ is finite of degree d, it is a "multisection" of f of degree d. Assume, further, that the integer n is coprime to d. The sheaf $R^1 f_*(\mathbb{Z}/n\mathbb{Z})$ is locally constant on S_{et}. In particular, there exists a finite étale cover $h : S' \to S$, such that $R^1 f_*(\mathbb{Z}/n\mathbb{Z})/S'$ is constant. We denote by $Y' \to X' := X \times_S S' \to S'$, a multisection of $f' : X' \to S'$ above Y. The Leray spectral sequence in étale cohomology, with respect to the morphism $f' : X' := X \times_S S' \to S'$, and the constant sheaf $\mathbb{Z}/n\mathbb{Z}$, gives rise to an exact sequence of terms of low degree: $0 \to H^1(S', f'_*(\mathbb{Z}/n\mathbb{Z})) \to H^1(X', \mathbb{Z}/n\mathbb{Z}) \to H^0(S', R^1 f'_*(\mathbb{Z}/n\mathbb{Z})) \to H^2(S', f'_*(\mathbb{Z}/n\mathbb{Z})) \to H^2(X', \mathbb{Z}/n\mathbb{Z})$. Let s' be a closed point of S', such that $h(s') = s$. The fibre of the sheaf $R^1 f_*(\mathbb{Z}/n\mathbb{Z})$ at s' is isomorphic to $H^1(X_s, \mathbb{Z}/n\mathbb{Z})$. Let $c_s \in H^1(X_s, \mathbb{Z}/n\mathbb{Z})$ be the class corresponding to the torsor $f_s : Y_s \to X_s := X \times_S k(s)$. Then c_s can be lifted to a global section $c \in H^0(S', R^1 f'_*(\mathbb{Z}/n\mathbb{Z}))$, since $R^1 f_*(\mathbb{Z}/n\mathbb{Z})/S'$ is constant. The element c is the image of a class $\tilde{c} \in H^1(X', \mathbb{Z}/n\mathbb{Z})$, via the above sequence, if and only if its image c' in $H^2(S', F'_*(\mathbb{Z}/n\mathbb{Z}))$ vanishes. The element c' is thus the obstruction to lift the μ_n-torsor $f_s : Y_s \to X_s := X \times_S k(s)$ to a μ_n-torosr $f' : Y' \to X'$. We will show that $c' = 0$. The image of c' in $H^2(X', \mathbb{Z}/n\mathbb{Z})$ vanishes, thus it also vanishes in $H^2(Y', \mathbb{Z}/n\mathbb{Z})$ via the canonical map $H^2(X', \mathbb{Z}/n\mathbb{Z}) \to H^2(Y', \mathbb{Z}/n\mathbb{Z})$. We have a canonical map $H^2(S', \mathbb{Z}/n\mathbb{Z}) \to H^2(Y', \mathbb{Z}/n\mathbb{Z})$. We also have a norm map $H^2(Y', \mathbb{Z}/n\mathbb{Z}) \to H^2(S', \mathbb{Z}/n\mathbb{Z})$, and the composite map $H^2(S', \mathbb{Z}/n\mathbb{Z}) \to H^2(Y', \mathbb{Z}/n\mathbb{Z}) \to H^2(S', \mathbb{Z}/n\mathbb{Z})$ is multiplication by d, Hence, we deduce that the class c' is annihilated by d. Since it is also annihilated by n, the group $H^2(S', \mathbb{Z}/n\mathbb{Z})$ being n-torsion, we deduce that $c' = 0$.

6.8. Lemma. *Let k be an algebraically closed field, of characteristic p. Let S be a smooth, complete, and irreducible k-curve. Let $f : X \to S$ be a smooth family of curves, of genus $g \geq 2$. Assume that the fundamental group π_1 is constant on the family f. Then for every finite cover $S' \to S$, and every finite étale cover $Y' \to X' := X \times_S S'$, the fundamental group π_1, is also constant on the family $Y' \to S'$.*

Proof. Standard, using the functorial properties of fundamental groups.

6.9. Lemma. *Let k be an algebraically closed field, of characteristic p. Let S be a smooth, complete, and irreducible k-curve. Let $f : X \to S$ be a smooth family of curves, of genus $g \geq 2$. Let $S' \to S$ be a finite cover, and let $Y' \to X' := X \times_S S'$ be an étale cover. Assume that the smooth relative S'-curve $Y' \to S'$ is isotrivial. Then, the smooth relative S-curve $X \to S$ is also isotrivial.*

Proof. Use Lemma 1.32, in [Ta-2].

Next, we will prove the main Theorem 6.5.

Proof of Theorems 6.5 and 6.6. Fix a closed point s of S, and let $X_s := X \times_S k(s)$ be the fibre of X above the point $s \in S$. By Tamagawa's result, Theorem 3.2, we can find an étale cover $Y_s \to X_s$, such that the gonality d_{Y_s} of Y_s is ≥ 5. Moreover, the cover $Y_s \to X_s$ can be chosen to be a composition of two cyclic, of order prime to p, covers. In particular, we can find, by Lemma 6.7, a finite étale cover $h : S_1 \to S$, and an étale cover $f_1 : Y \to X_1 := X \times_S S_1$, such that the pair (f_1, h) is a good lifting of the cover $Y_s \to X_s$. Now, consider the smooth family of curves $Y \to S_1$. Let s_1 be a closed point of S_1, such that $h(s_1) = s$. Then, by construction, the gonality of the fibre $Y_{s_1} := Y \times_S k(s_1)$ of Y above s_1 is ≥ 5. In particular, by using Theorem 3.3, we can find a prime integer $l >> 0$, distinct from p, and a non trivial μ_l torsor $Z_{s_1} \to Y_{s_1}$, such that the following two conditions hold:

i) The μ_l-torsor $Z_{s_1} \to Y_{s_1}$ is new ordinary.

ii) The natural map, $T : M_{Y_{s_1}} \to M_{J^{\text{new}}_{s_1}}$, induced by the μ_l-torsor $Z_{s_1} \to Y_{s_1}$, where $J^{\text{new}}_{s_1}$ is the new part of the jacobian of Z_{s_1}, with respect to the above torsor, is an immersion (cf. 3).

Also, by applying Lemme 6.7, we can find a finite étale cover $h' : S_2 \to S_1$, and a μ_l-torsor $f_2 : Z' \to Y' := Y \times_{S_1} S_2$, such that the pair (f_2, h'), is a good lifting of the μ_l-torsor $Z_{s_1} \to Y_{s_1}$. Let $J_{Y'}$ (resp. $J_{Z'}$) be the relative jacobian of Y' over S_2 (resp. the relative jacobian of Z' over S_2) which is an S_2-abelian scheme. Let $f_2^* : J_{Y'} \to J_{Z'}$ be the canonical homomorphism, which is induced by the pull back of invertible sheaves. Let $J^{\text{new}} := J_{Z'}/f_2^*(J_{Y'})$, be the new part of the jacobian $J_{Z'}$, with respect to the μ_l-torsor $Z' \to Y'$. Let η' be the generic point of S_2, and let s_2 be a point of S_2 such that $h'(s_2) = s_1$. The fibre $J^{\text{new}}_{s_2} := J^{\text{new}} \times_{S_2} k(s_2)$ of J^{new}, above the point s_2, is by construction an ordinary abelian variety. This implies, a fortiori, that the generic fibre $J^{\text{new}}_{\eta'} := J^{\text{new}} \times_{S_2} k(\eta')$ of J^{new}, where η' is the generic point of S_2, is also ordinary, since $J^{\text{new}}_{\eta'}$ specializes to $J^{\text{new}}_{s_2}$. The following two cases can occur:

Case 1: The abelian scheme $J^{\text{new}} \to S_2$ has constant p-rank, i.e. all fibres of J^{new} over S_2 are ordinary abelian varieties. Then, since S_2 is complete, we deduce from Theorem 4.2, that the abelian scheme $J^{\text{new}} \to S_2$ is isotrivial. Note, that the deformation J^{new} of $J^{\text{new}}_{s_2}$, induces a first order infinitesimal deformation of $J^{new}_{s_2}$, which is a trivial deformation since J^{new} is isotrivial, and which by construction is the image of the first order infinitesimal deformation of Y_{s_1} induced by Y', via the above natural map $T : M_{Y_{s_1}} \to M_{J^{\text{new}}_{s_1}}$. Since, by construction, the map T is an immersion, we conclude that the deformation $Y' \to S_2$ is isotrivial, as well. A fortiori, the family $X \to S$ is also isotrivial, by Lemma 6.9. But this contradicts our hypothesis that the family $X \to S$ is not isotrivial. So, case 1 can not occur.

Case 2: The abelian scheme $J^{\text{new}} \to S_2$ does not have constant p-rank, i.e. there exists a closed point $\tilde{s} \in S_2$, such that the p-rank of the fibre $J^{\text{new}}_{\tilde{s}} := J^{\text{new}} \times_{S_2} \tilde{s}$ of J^{new} over the point $\tilde{s}$, is strictly smaller than the p-rank of the generic fibre $J^{\text{new}}_{\eta'}$ of J^{new}. This, in particular, implies that the p-rank of the fibre $Z'_{\tilde{s}} := Z' \times_{S_2} \tilde{s}$, of Z' above the point $\tilde{s}$, is strictly smaller than the p-rank of the generic fibre $Z'_{\eta'} := Z' \times_{S_2} \eta'$ of Z'. This already proves Theorem 6.6. Now, this implies in particular that the geometric fundamental group π_1 is not constant on the family $Z' \to S_2$. Thus, by Lemma 6.8, we deduce that the geometric fundamental group π_1 is not constant on the family $X \to S$. This finishes the proof of Theorem 6.5.

Theorem 6.5, can be generalized, to the situation where we consider the full tame fundamental group. Note, first, that given a family of curves $f : X \to S$ of genus $g \geq 0$, and n sections $\{s_1, s_2, ..., s_n\}$, of the morphism f, we can extend the notion of isotriviality, as in Definition 5.2, to the pair $(f, \{s_1, s_2, ..., s_n\})$, by considering the moduli scheme $\mathcal{M}_{g,n}$, of n-pointed smooth and projective curves of genus g. Now, it is easy to deduce from 6.5 the following theorem:

6.10. Theorem. *Let k be an algebraically closed field, of characteristic p. Let S be a smooth, complete, and irreducible k-curve, with generic point η. Let $f : X \to S$ be a smooth family of curves, of genus $g \geq 0$. Let $\{s_1, s_2, ..., s_n\}$ be n sections of f, with disjoint support. Assume that the pair $(f, \{s_1, s_2, ..., s_n\})$ is not isotrivial. Then, the tame fundamental group is not constant on the pair $(f, \{s_1, s_2, ..., s_n\})$, i.e. there exists a closed point $s \in S$, such that the specialization homomorphism* $\mathrm{Sp} : \pi_1^t(X_\eta - \{s_1(\eta), s_2(\eta), ..., s_n(\eta)\}) \to \pi_1^t(X_s - \{s_1(s), s_2(s), ..., s_n(s)\})$ *between tame fundamental groups, where $X_\eta := X \times_S \eta$ (resp. $X_s := X \times_S s$), is not an isomorphism.*

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Mohamed Saïdi

Departement of Mathematics
University of Durham
Science Laboratories
South Road
Durham DH1 3LE United Kingdom
Mohamed.Saidi@durham.ac.uk