NON-LINEARIZABLE CR-AUTOMORPHISMS, TORSION-FREE ELLIPTIC CR-MANIFOLDS AND SECOND ORDER ODE

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ABSTRACT. The existence of non-linearizable isotropic CR-automorphisms of Levi non-degenerate hypersurfaces in complex space is characteristic for quadrics. In this paper we discover elliptic CR manifolds of CR-dimension and codimension two that are not related to the quadric and that have non-linearizable isotropic automorphisms. This is a new and unexpected phenomenon. A complete description of such manifolds is given.

0. INTRODUCTION

The interest to invariants and automomorphisms of real submanifolds in complex space goes back to 1907, when Poincaré [32] realized that there are "more" hypersurfaces in $\mathbb{C}^2$ than biholomorphic mappings. He asked for a classification of hypersurfaces with respect to holomorphic coordinate changes. This problem was solved by E. Cartan [10] (in 1932 for $\mathbb{C}^2$) and by N. Tanaka [39] (in 1967 for $\mathbb{C}^N$) and independently by S. Chern and J. Moser [11] (1974), who obtained a complete description of the invariants of real hypersurfaces. The celebrated papers by Cartan and by Chern and Moser initiated fruitful developments in different directions including theory of normal forms of CR-manifolds [43, 42, 30, 18, 19] and CR-geometry [4, 5, 3, 2, 29, 31, 20]. The theory of invariants of real hypersurfaces is a significant part of the parabolic theory of invariants, which was introduced in 1979 by C. Fefferman [22, 1, 24, 8, 7, 34].
A natural global CR-invariant with a clear geometric meaning is the *automorphism group* of the submanifold. Its punctual version is the *isotropy group* at a given point.
Let M be a real submanifold in $\mathbb{C}^N$ that contains the origin. The isotropy group $\mathrm{Aut}_0 M$ is defined as the group of germs of biholomorphic maps ϕ with $\phi(0) = 0$ and $\phi(M) \subset M$. The following theorem provides an upper bound for the dimension of this group

Theorem 0. *Let M be a real-analytic, Levi non-degenerate, generic submanifold in $\mathbb{C}^N$ passing through the origin. Then any isotropic automorphism $\phi \in \mathrm{Aut}_0 M$ is uniquely determined by its first and second order derivatives at* 0.

This theorem was proved by Chern and Moser for hypersurfaces [11] and by Beloshapka [3] for submanifolds of higher codimension.
It turns out that the manifold with maximal isotropy group in the class of manifolds with given Levi-form is the quadric

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$$Q:\ \operatorname{Im} w = \langle z, z\rangle \tag{1}$$

where (z, w) are suitable coordinates in the ambient space $\mathbb{C}^{n+k}$ and $\langle z, z\rangle$ is the non-degenerate, vector-valued Levi form. For non-quadratic submanifolds (i.e., submanifolds that are not locally equivalent to the quadric) the dimension may significantly drop.
The isotropy group of a hyperquadric has dimension $(n+1)^2+1$. For non-quadratic hypersurfaces this dimension does not exceed n^2 - the dimension of the group of pseudo-unitary $n \times n$ matrices. In this case any isotropic automorphism is uniquely determined by the restriction of its Jacobian on the maximal complex subspace $T_0^{CR}M \subset T_0M$. If M is strictly pseudoconvex then, moreover, there are coordinates such that $\operatorname{Aut}_0 M$ is a subgroup of the linear unitary group. This is a particular case of a general phenomenon that is called *linearizability* of the isotropy group. Though, for non-quadratic hypersurfaces with indefinite Levi form the isotropy group is not always linearizable, every isotropic automorphism can be individually linearized in suitable coordinates. The isotropy group of the quadric contains non-identical automorphisms with identical linear part and also automorphisms whose linear part implies the presence of higher order terms that cannot be eliminated by changing coordinates.
In the study of the automorphisms of higher codimensional submanifolds new phenomena occur. If the codimension k is between 2 and $n^2 - 1$ (except for the case $n = k = 2$) then even for almost all quadrics the isotropy group is linearizable [18]. In the exceptional case $n = k = 2$ any non-degenerate quadric is equivalent to either the *hyperbolic*, *elliptic*, or *parabolic* quadric. Among these three only hyperbolic and elliptic quadrics are in general position. In particular, if the Levi form at some point is hyperbolic or elliptic, it will remain the same in a neighbourhood.
The linearization problem for hyperbolic manifolds was resolved in [33]. It was proved that the only manifolds with non-linearizable isotropy are Cartesian products of a hyperquadric in $\mathbb{C}^2$ and a hypersurface in $\mathbb{C}^2$. Thus, the phenomenon of non-linearizability is strongly related to quadrics.
Application of similar techniques to elliptic manifolds [20, 21] showed that non-linearizability is an extremely strong condition that seemed that only quadrics could comply with.
In this article we adopt a different approach based on the results in [7, 34, 35] that proves this expectation wrong. In fact, we discover a family of non-quadratic *torsion-free* elliptic manifolds with (even individually) non-linearizable automorphisms. This family appears through a correspondence between torsion-free elliptic manifolds and second order ODE as studied in the 19th century by S. Lie [26, 27] and A. Tresse [41]. E. Cartan established the correspondence of second order ODE and projective connections on $T\mathbb{C}^2$ and solved the flatness problem for such projective connections. The analogous problem in higher dimensions is related to Schlesinger's equation and was studied by N. Hitchin [24]. A more general correspondence between elliptic CR-manifolds with vanishing curvature components and projective connections is described by A. Čap [6].
The article is organized as follows: In Section 1 we recall some basic properties of elliptic CR-manifolds including the existence of two distinguished direction fields that carry the full information of the CR structure. Then, in Section 2, we introduce

canonical coordinates for manifolds with two distinguished direction fields. In Section 3 we establish the correspondence between torsion-free elliptic manifolds and 2^{nd}order ODE. This is elaborated for the flat models in Section 4. Section 5 is devoted to ODE with a special *shear* symmetry that corresponds to a non-linearizable quadric automorphism. We prove that non-linearizability of the isotropy group cannot appear at an isolated point but automatically on some holomorphic curve through this point. In Sections 6-8 we give a classification of ODE's with shear symmetry and reveal the only ODE with more than one shear.
Appendices A and B contain Maple programs used for the calculations in Sections 6 and 7.

1. Elliptic CR-manifolds

A CR-manifold M of CR-dimension 2 and codimension 2 is called elliptic if the Levi form at any point consists of two components which do not admit a degenerate real linear combination. For any point $p \in M$ there are coordinates (z_1, z_2, w_1, w_2) centered in 0 such that the equation of M takes the form

$$\frac{w_1 - \bar{w}_2}{2\,\mathrm{i}} = z_1 \bar{z}_2 + O(3).$$

The CR-subspace $T_0^{CR}M \subset T_0 M$ is defined by $w_1 = w_2 = 0$. There are two distinguished directions $z_1 = 0$ and $z_2 = 0$, the isotropic directions of the Levi form. These distinguished directions form two distinguished complex direction fields that can be represented by two sections Z_1 and Z_2 of $T^{(1,0)}M$.
The direction fields $Z_1, \bar{Z}_2, [Z_1, \bar{Z}_2]$ define an almost complex structure on M (cf. [34, 7, 35]). For our purposes we need to consider only a special class of so-called torsion-free elliptic manifolds (cf. [34, 7, 35]).
We recall that an elliptic CR-manifold is called *torsion-free* if the the distributions $(Z_1, \bar{Z}_1)$ and $(Z_2, \bar{Z}_2)$ are involutive and the almost complex structure, as defined above, is integrable.

Lemma 1. *If M is a real-analytic, elliptic CR-manifold in $\mathbb{C}^4$ that can be written in the form*

$$\frac{w_1 - \bar{w}_2}{2\,\mathrm{i}} = z_1 \bar{z}_2 + N\left(z_1, \bar{z}_2, \frac{w_1 + \bar{w}_2}{2}\right) \tag{2}$$

then M is torsion-free.

It is convenient to use the following notations of *twisted* real and imaginary parts

$$\eta = \frac{w_1 + \bar{w}_2}{2}, \quad V = \frac{w_1 - \bar{w}_2}{2\,\mathrm{i}},$$

respectively, so that (2) becomes

$$V = z_1 \bar{z}_2 + N(z_1, \bar{z}_2, \eta). \tag{3}$$

Though, in general, it is difficult to explicitly determine Z_1 and Z_2, we can easily find these expressions for manifolds (3):

$$Z_1 = \frac{\partial}{\partial z_1} + 2\,\mathrm{i}\,\frac{\bar{z}_2 + \frac{\partial N}{\partial z_1}}{1 - \mathrm{i}\,\frac{\partial N}{\partial \eta}}\,\frac{\partial}{\partial w_1} \qquad Z_2 = \frac{\partial}{\partial z_2} + 2\,\mathrm{i}\,\frac{\bar{z}_1 + \frac{\partial \bar{N}}{\partial z_2}}{1 - \mathrm{i}\,\frac{\partial \bar{N}}{\partial \bar{\eta}}}\,\frac{\partial}{\partial w_2}$$

It is straightforward to verify that these complex fields are tangent to M and are isotropic for the Levi form.
Proof. Since Z_1 depends only on $z_1, \bar{z}_2, w_1, \bar{w}_2$ and $\bar{Z}_1$ depends only on $\bar{z_1}, z_2, \bar{w}_1, w_2$. It is clear that $[Z_1, \bar{Z}_1] = 0$. The same way one proves that $[Z_2, \bar{Z}_2] = 0$.
Notice that the fields $Z_1, \bar{Z}_2, [Z_1, \bar{Z}_2]$ are holomorphic with respect to the coordinates $z_1, \zeta_2 = \bar{z}_2, w_1, \omega_2 = \bar{w}_2$. Thus, they determine a complex structure on M, which is the same as the one induced from the ambient space with coordinates $(z_1, \zeta_2, w_1, \omega_2)$. □

2. Distinguished direction fields on complex 3-folds

Torsion-free elliptic CR-manifolds can be considered as complex 3-folds with two distinguished holomorphic direction fields $\mathfrak{z}_1 = Z_1$, $\mathfrak{z}_2 = \bar{Z}_2$ that span a non-involutive distribution. The initial CR-structure can be recovered as the distribution spanned by $\mathfrak{z}_1$, and $\bar{\mathfrak{z}}_2$.

Definition 1. *Let $\mathfrak{z}_1$ and $\mathfrak{z}_2$ be two direction fields in a neighbourhood of the origin in $\mathbb{C}^3$ that span a non-involutive distribution. A local system of coordinates (x, y, ζ) will be called canonical if $\mathfrak{z}_1$ and $\mathfrak{z}_2$ are represented by the vector fields*

$$\mathfrak{z}_1 = \frac{\partial}{\partial \zeta} \qquad \mathfrak{z}_2 = \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial y} + B(x, y, \zeta) \frac{\partial}{\partial \zeta}$$

Proposition 1. *Let $\mathfrak{z}_1$ and $\mathfrak{z}_2$ be two direction fields in a neighbourhood of the origin in $\mathbb{C}^3$ that span a non-involutive distribution. Then there exist canonical coordinates x, y, ζ. Canonical coordinates are determined up to a coordinate change in the x, y plane and a choice of the order of $\mathfrak{z}_1$ and $\mathfrak{z}_2$.*

Proof. We start with the existence part. Let $\mathfrak{z}_1$ be a vector field which represents one of the two direction fields. We choose coordinates with $\mathfrak{z}_1 = \frac{\partial}{\partial \zeta}$. Now, a vector field $\mathfrak{z}_2$ which represents the other direction field can be rescaled in such a way that

$$\mathfrak{z}_2 = \frac{\partial}{\partial x} + A \frac{\partial}{\partial y} + B \frac{\partial}{\partial \zeta}.$$

Thus

$$[\mathfrak{z}_1, \mathfrak{z}_2] = \frac{\partial A}{\partial \zeta} \frac{\partial}{\partial y} + \frac{\partial B}{\partial \zeta} \frac{\partial}{\partial \zeta}$$

with $\frac{\partial A}{\partial \zeta} \neq 0$. After the coordinate change

$$\begin{aligned} x' &= x \\ y' &= y \\ \zeta' &= A \end{aligned}$$

we have

$$\mathfrak{z}_1 = \frac{\partial A}{\partial \zeta} \frac{\partial}{\partial \zeta'},$$

i.e. $\mathfrak{z}_1' = \frac{\partial}{\partial \zeta'}$ defines the same direction field as $\mathfrak{z}_1$.
Furthermore,

$$\mathfrak{Z}_2 = \frac{\partial}{\partial x'} + A\frac{\partial}{\partial y'} + \#\frac{\partial}{\partial \zeta'} = \frac{\partial}{\partial x'} + \zeta'\frac{\partial}{\partial y'} + B'(x', y', \zeta')\frac{\partial}{\partial \zeta'}.$$

(Here and further symbol # means a factor (constant or functional), whose value is not important.) For the proof of the uniqueness statement we observe that after having chosen the order of the two direction fields a coordinate change which preserves $\mathfrak{Z}_1 = \frac{\partial}{\partial \zeta}$ up to scale has the form

$$\begin{aligned} x_1 &= \alpha(x, y) \\ y_1 &= \beta(x, y) \\ \zeta_1 &= \gamma(x, y, \zeta) \end{aligned}$$

This coordinate change preserves $\mathfrak{Z}_2$ up to scale if $\frac{\partial \alpha}{\partial x} \neq 0$ and

$$\zeta_1 = \frac{\frac{\partial \beta}{\partial x} + \zeta \frac{\partial \beta}{\partial y}}{\frac{\partial \alpha}{\partial x} + \zeta \frac{\partial \alpha}{\partial y}}$$

□

Let $\Xi = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial \zeta}$
be an infinitesimal automorphism of the structure $(\mathbb{C}^3, \mathfrak{Z}_1, \mathfrak{Z}_2)$, i.e.,

$$[\Xi, \mathfrak{Z}_1] = \lambda \mathfrak{Z}_1, \tag{4}$$

$$[\Xi, \mathfrak{Z}_2] = \mu \mathfrak{Z}_2. \tag{5}$$

It follows that (4) is satisfied if and only if ξ and η depend only on x and y. Then (5) yields

$$\mu = -\frac{\partial \xi}{\partial x} - \zeta\frac{\partial \xi}{\partial y}, \qquad \phi = \frac{\partial \eta}{\partial x} + \zeta\left(\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x}\right) - \zeta^2\frac{\partial \xi}{\partial y}$$

and

$$\begin{aligned} &\xi\frac{\partial B}{\partial x} + \eta\frac{\partial B}{\partial y} + \phi\frac{\partial B}{\partial \zeta} + \left(2\frac{\partial \xi}{\partial x} + 3\zeta\frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial y}\right)B - \\ &\quad - \frac{\partial^2 \eta}{(\partial x)^2} + \zeta\left(\frac{\partial^2 \xi}{(\partial x)^2} - 2\frac{\partial^2 \eta}{\partial x \partial y}\right) + \zeta^2\left(2\frac{\partial^2 \xi}{\partial x \partial y} - \frac{\partial^2 \eta}{(\partial y)^2}\right) + \zeta^3\frac{\partial^2 \xi}{(\partial y)^2} = 0. \end{aligned} \tag{6}$$

3. Second order ODE

Let $B(x, y, \zeta)$ be a holomorphic function in some neighbourhood of the origin in $\mathbb{C}^3$ and let

$$y'' = B(x, y, y') \tag{7}$$

be a 2^{nd}order ODE on the holomorphic function $y(x)$. For any point (x_0, y_0, y_0') sufficiently close to the origin there is a unique solution $y(x)$ of (7) with $y(x_0) = y_0$ and $y'(x_0) = y_0'$. These curves lift to $(x, y = y(x), \zeta = y'(x))$ in $\mathbb{C}^3$ and form a foliation which is generated by the vector field

$$\frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial y} + B(x, y, \zeta) \frac{\partial}{\partial \zeta}.$$

The canonical projection $(x, y, \zeta) \to (x, y)$ defines another foliation into vertical lines generated by $\frac{\partial}{\partial \zeta}$. Thus, we obtain the same structure as before, namely $\mathbb{C}^3$ with two distinguished direction fields which span a non-involutive distribution.
Any transformation of the (x, y)-plane lifts to a transformation of the (x, y, ζ) space

$$\begin{aligned} x_1 &= \alpha(x, y) \\ y_1 &= \beta(x, y) \\ \zeta_1 &= \frac{dy_1}{dx_1} = \frac{\frac{\partial \beta}{\partial x} + \zeta \frac{\partial \beta}{\partial y}}{\frac{\partial \alpha}{\partial x} + \zeta \frac{\partial \alpha}{\partial y}}. \end{aligned}$$

These are the same transformations as in Proposition 1

4. The flat model

Let Q be the elliptic quadric

$$V = z_1 \bar{z}_2, \tag{8}$$

where $V = \frac{w_1 - \bar{w}_2}{2\,\mathrm{i}}$. The CR structure of Q is determined by two complex vector fields that span the CR distribution and annihilate the Levi form:

$$\mathfrak{z}_1 = \frac{\partial}{\partial z_1} + \mathrm{i}\, \bar{z}_2 \frac{\partial}{\partial \eta} \qquad \mathfrak{z}_2 = \frac{\partial}{\partial \bar{z}_2} - \mathrm{i}\, z_1 \frac{\partial}{\partial \eta}$$

Here we used intrinsic coordinates $z_1, z_2, \eta = \frac{w_1 + \bar{w}_2}{2}$. The coordinates

$$\begin{aligned} x &= \bar{z}_2 \\ y &= \frac{\eta}{2\,\mathrm{i}} + \frac{z_1 \bar{z}_2}{2} \\ \zeta &= z_1 \end{aligned}$$

are then canonical and the distinguished vector fields turn into

$$\mathfrak{z}_1 = \frac{\partial}{\partial \zeta} \qquad \mathfrak{z}_2 = \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial y}$$

These vector fields correspond to the ODE: $y'' = 0$.
The infinitesimal automorphisms of the flat structure can be easily determined from (6) for $B \equiv 0$:

$$-\frac{\partial^2 \eta}{(\partial x)^2} + \zeta \left(\frac{\partial^2 \xi}{(\partial x)^2} - 2 \frac{\partial^2 \eta}{\partial x \partial y} \right) + \zeta^2 \left(2 \frac{\partial^2 \xi}{\partial x \partial y} - \frac{\partial^2 \eta}{(\partial y)^2} \right) + \zeta^3 \frac{\partial^2 \xi}{(\partial y)^2} = 0.$$

It follows

$$\frac{\partial^2\eta}{(\partial x)^2} = 0 \tag{9}$$

$$\frac{\partial^2\xi}{(\partial x)^2} - 2\frac{\partial^2\eta}{\partial x\partial y} = 0 \tag{10}$$

$$2\frac{\partial^2\xi}{\partial x\partial y} - \frac{\partial^2\eta}{(\partial y)^2} = 0 \tag{11}$$

$$\frac{\partial^2\xi}{(\partial y)^2} = 0 \tag{12}$$

From (9) we obtain

$$\eta = \eta_0 + \eta_1 x$$

where η_0 and η_1 depend only on y. Then (10) implies

$$\xi = \xi_0 + \xi_1 x + \xi_2 x^2$$

where ξ_0, ξ_1, ξ_2 depend only on y and $\xi_2 = \eta_1'$, and (11) yields

$$\eta_0'' = 2\xi_1', \quad \eta_1'' = 4\xi_2'.$$

Finally, from (12) follows that $\xi_j'' = 0$ for $j = 1, 2, 3$.
Therefore,

$$\begin{aligned}
\xi &= \alpha_0 + \alpha_1 y + \alpha_2 x + \alpha_3 xy + \beta_3 x^2 \\
\eta &= \beta_0 + \beta_1 y + \alpha_3 y^2 + \beta_2 x + \beta_3 xy \\
\phi &= \beta_2 + \beta_3 y + (\beta_1 - \alpha_2 - \beta_3 x)\zeta - (\alpha_1 - 2\alpha_3 x)\zeta^2.
\end{aligned}$$

These form the 8-dimensional Lie algebra of projective transformations of $\mathbb{CP}^2$ (with affine part $\mathbb{C}^2_{x,y}$): parameters α_0, and β_0 correspond to infinitesimal shifts of the origin, $\alpha_1, \alpha_2, \beta_1, \beta_2$ to linear transformations in the affine part and α_3, β_3 to the 1-parametric families of non-linear projective transformations

$$(x, y) \to \left(\frac{x}{1 - \alpha_3 ty}, \frac{y}{1 - \alpha_3 ty}\right), \quad (x, y) \to \left(\frac{x}{1 - \beta_3 tx}, \frac{y}{1 - \beta_3 ty}\right),$$

respectively. Since the solutions of the flat ODE are straight lines, it is natural to view them as globally defined curves in the projective space $\mathbb{CP}^2$. Their extension to the projectivized tangent bundle $\mathbb{P}(T\mathbb{CP}^2)$ has a nice geometric interpretation: to any point at some line we assign the direction of the line, which is represented by the line itself. Thus, a point of $\mathbb{P}(T\mathbb{CP}^2)$ can be identified with a $(0, 1)$-flag in $\mathbb{CP}^2$. The extended action of $\mathrm{PSL}(2, \mathbb{C})$ on $\mathbb{P}(T\mathbb{CP}^2)$ coincides with the induced action on the flag manifold.
The following quadric automorphisms

$$\begin{aligned}
z_1^* &= \frac{z_1 + a_1 w_1}{1 - 2\,\mathrm{i}\,\bar a_2 z_1 - (r + \mathrm{i}\, a_1 \bar a_2) w_1} & \qquad w_1^* &= \frac{w_1}{1 - 2\,\mathrm{i}\,\bar a_2 z_1 - (r + \mathrm{i}\, a_1 \bar a_2) w_1} \\
z_2^* &= \frac{z_2 + a_2 w_2}{1 - 2\,\mathrm{i}\,\bar a_1 z_2 - (\bar r + \mathrm{i}\, a_2 \bar a_1) w_2} & \qquad w_2^* &= \frac{w_2}{1 - 2\,\mathrm{i}\,\bar a_1 z_2 - (\bar r + \mathrm{i}\, a_2 \bar a_1) w_2}
\end{aligned} \tag{13}$$

are non-linearizable.
We will be interested in automorphisms of real-analytic CR-manifolds which coincide up to terms of higher weight with an automorphism (13) with $a_1 = r = 0$. Such quadric automorphisms belong to a one-parametric family, which is generated by the infinitesimal automorphism

$$\chi_Q = w_1 \frac{\partial}{\partial z_1} + 2\,\mathrm{i}\, z_2^2 \frac{\partial}{\partial z_2} + 2\,\mathrm{i}\, z_1 w_1 \frac{\partial}{\partial w_1}. \tag{14}$$

Set $2\,\mathrm{i}\,\bar{a}_2 = 1$. In (x, y, ζ) coordinates this one-parametric family is

$$x^* = x + ty \qquad y^* = y \qquad \zeta^* = \frac{\zeta}{1 - t\zeta}.$$

The infinitesimal automorphism takes the form

$$\Theta = y \frac{\partial}{\partial x} - \zeta^2 \frac{\partial}{\partial \zeta}.$$

We note that this automorphism is the extension of the infinitesimal *shear automorphism*

$$y \frac{\partial}{\partial x}$$

on the (x, y)-plane that generates the 1-parametric family of automorphisms

$$(x, y) \mapsto (x + ty, y).$$

5. Non-linearizable automorphisms of elliptic CR-manifolds

It can be proved by the methods in [20, 21] that the isotropy group of an elliptic CR-manifold M is linearizable if M is not torsion-free. Moreover, we may assume that a non-linearizable infinitesimal automorphism χ of the torsion-free manifold

$$V = z_1 \bar{z}_2 + N(z_1, \bar{z}_2, \eta)$$

coincides with χ_Q up to terms of higher order.
According to the results in [37], in Q-normalized coordinates χ equals χ_Q. In particular, there must be a curve of fixed points passing through the origin. We use this to prove that for any torsion-free CR-manifold, which allows an infinitesimal automorphism χ, there are canonical coordinates (x, y, ζ) such that $\chi = \Theta$. We need the following *Darboux*-type lemma.

Lemma 2. *Let $\theta = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ be a vector field in a neighbourhood of the origin in $\mathbb{C}^2$ such that $\frac{\partial \xi}{\partial y}(0) \neq 0$ and θ vanishes on $\{\xi = 0\}$. Then there exist coordinates (x_1, y_1) such that $\theta = y_1 \frac{\partial}{\partial x_1}$.*

Proof. Without loss of generality we may assume that $\xi = y$. Since θ vanishes at $\{\xi = 0\}$, we get

$$\theta = y \left(\frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y} \right).$$

Then

$$y_1\frac{\partial}{\partial x_1} = y\left(\frac{\partial}{\partial x} + \eta_1\frac{\partial}{\partial y}\right),$$

if $x_1 = \phi(x,y)$, $y_1 = \psi(x,y) = y\chi(x,y)$ with

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= -y\eta_1\frac{\partial\chi}{\partial x} + \chi\\ \frac{\partial\phi}{\partial y} &= -y\eta_1\frac{\partial\chi}{\partial y} - \eta_1\chi\end{aligned}\tag{15}$$

This system is solvable if

$$\left(y\frac{\partial\eta_1}{\partial y}\right)\frac{\partial\chi}{\partial x} - \left(1 + y\frac{\partial\eta_1}{\partial x}\right)\frac{\partial\chi}{\partial y} - \frac{\partial\eta_1}{\partial x}\chi = 0.$$

Choose $\chi(x,y)$ (with $\chi(0) = 1$) that satisfies the above equation and then solve (15) for ϕ. □

Our aim is to find CR manifolds that allow the infinitesimal transformation

$$\Theta = y\frac{\partial}{\partial x} - \zeta^2\frac{\partial}{\partial\zeta}.$$

We always have

$$\left[\Theta, \frac{\partial}{\partial\zeta}\right] = 2\zeta\frac{\partial}{\partial\zeta}.$$

Thus the only condition is

$$\left[\Theta, \frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial y} + B\frac{\partial}{\partial\zeta}\right] = \mu\left(\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial y} + B\frac{\partial}{\partial\zeta}\right)$$

From

$$[\Theta, \mathfrak{z}_2] = -\zeta\frac{\partial}{\partial x} - \zeta^2\frac{\partial}{\partial y} + \left(2\zeta B + y\frac{\partial B}{\partial x} - \zeta^2\frac{\partial B}{\partial\zeta}\right)\frac{\partial}{\partial\zeta}$$

we get $\mu = -\zeta$ and

$$y\frac{\partial B}{\partial x} - \zeta^2\frac{\partial B}{\partial\zeta} + 3\zeta B = 0.$$

It follows

$$B(x,y,\zeta) = F\left(x - \frac{y}{\zeta}, y\right)\zeta^3.$$

For analytic B we have

$$B = f_0(y)(y - x\zeta)^3 + f_1(y)(y - x\zeta)^2\zeta + f_2(y)(y - x\zeta)\zeta^2 + f_3(y)\zeta^3.\tag{16}$$

We summarize the above reasoning as

Theorem 1. *A real-analytic elliptic CR-manifold M has a non-linearizable isotropy group at some point $p \in M$ if and only if it is torsion-free and corresponds to an ODE of the form (16).*

Of course, we are only interested in such manifolds that are not equivalent to the flat one. This can be sorted out by three different methods:
1. Lie derived a criterion for a 2^{nd}order ODE to be flat in [26]. Notice that the ODE with B as above satisfy Lie's linear necessary flatness conditions.
2. Since the flat ODE is the only one with 8-dimensional algebra of infinitesimal automorphisms we can estimate the dimension of the algebra, i.e. we solve (6). This approach leads to many interesting examples and will be partially developed in Section 7 but, like the first approach, it does not answer the question when two ODE of the given form are equivalent.
The complete answer to this question will be given by a third method, namely a construction of a normal form for shear invariant ODE. In the next section we prove that an ODE of the form $y'' = B(x, y, y')$ with B as described in (16) can be reduced by a coordinate change to $y'' = \tilde{B}(x, y, y')$, where

$$\tilde{B} = \tilde{f}_0(y)(y - xy')^3 + \tilde{f}_1(y)(y - xy')^2 y'. \tag{17}$$

The question about the available freedom in the choice of coordinates such that the ODE takes the reduced form (17) will be answered in in Section 8. The answer depends on whether the ODE has only one 1-parametric family of shear automorphisms or more. Therefore we will study shear invariant ODE with additional symmetries in Section 7.
We conclude this section by an immediate corollary of Theorem 1 that disproves the intuitive perception of non-linearizability as an isolated phenomenon.

Theorem 2. *If M is an elliptic CR-manifold, which has a non-linearizable isotropy group at some point $p \in M$ then there is a holomorphic (in sense of the canonic complex structure on M) curve γ through p such that for any $q \in \gamma$ the isotropy $\operatorname{Aut}_q M$ is non-linearizable.*

Proof. According to Theorem 1, M corresponds to an ODE of the form (16). The translation $(x, y) \to (x + t, y)$ preserves the form (16) of the ODE and therefore the existence of a shear transformation. Thus, any point at $y = \zeta = 0$ has a non-linearizable isotropy group. $\square$

6. Normal form of shear invariant ODE. Existence

In this section we will construct the reduced form for equations $y'' = B(x, y, y')$ with B of the form (16) mentioned above, by choosing special local coordinates in the (x, y)-plane.

Proposition 2. *For any shear invariant ODE*

$$y'' = \sum_{k=0}^{3} f_k(y)(y - xy')^{3-k}(y')^k$$

there is a local coordinate change which eliminates the summands that contain f_2 and f_3, i.e. transforms the ODE into

$$y'' = \sum_{k=0}^{1} \tilde{f}_k(y)(y - xy')^{3-k}(y')^k. \tag{18}$$

Proof. We choose new coordinates (x_1, y_1) such that $x = \Xi(x_1, y_1)$, $y = \mathrm{H}(x_1, y_1)$ and the shear transformation $y\frac{\partial}{\partial x}$ after the coordinate change is again a shear transformation $y_1\frac{\partial}{\partial x_1}$. Thus

$$\mathrm{H}(x_1, y_1)\frac{\partial}{\partial x} = y_1\frac{\partial \Xi}{\partial x_1}\frac{\partial}{\partial x} + y_1\frac{\partial\,\mathrm{H}}{\partial x_1}\frac{\partial}{\partial y}.$$

It follows $\mathrm{H}(x_1, y_1) = y_1\phi(y_1)$ and $\Xi(x_1, y_1) = \psi(y_1) + \phi(y_1)x_1$.
Then

$$\begin{aligned}\tilde{f}_2 &= -\phi''\phi + (\phi, \phi', \psi, \psi', f_0, \ldots, f_3)\\ \tilde{f}_3 &= -\psi''(\phi + y_1\phi') + y_1\phi''\psi' + 2\phi'\psi' + (\phi, \phi', \psi, \psi', f_0, \ldots, f_3),\end{aligned}$$

where the parentheses mean some expression that involves the indicated functions. Thus, $\tilde{f}_2 = \tilde{f}_3 = 0$ can be achieved by solving the corresponding system of second order ODE. □

In particular, the equations

$$y'' = f_2(y)(y - xy')(y')^2 + f_3(y)(y')^3$$

are equivalent to $y'' = 0$. For

$$y'' = f_3(y)(y')^3$$

this equivalence can be established by the simple transformation

$$\begin{aligned}x &= \psi(y_1) + x_1\\ y &= y_1\end{aligned}$$

with $\psi'' = -f_3$.
The uniqueness issue will be postponed to Section 8. Here we just observe that if $y\frac{\partial}{\partial x}$ is a unique (up to scale) shear symmetry for an ODE then *any* mapping that takes the ODE into

$$y'' = f_0(y)(y - xy')^3 + f_1(y)(y - xy')^2 y'$$

has the form

$$x_1 = \psi(y) + c_1\phi(y)x, \quad y_1 = c_2 y\phi(y), \tag{19}$$

where ϕ, ψ satisfy the differential equations from Proposition 2 and the initial conditions $\psi(0) = 0$, $\phi(0) = 1$. Thus, the coordinates are determined up to 4 parameters: $\psi'(0), c_1 \neq 0, c_2 \neq 0, \phi'(0)$. Two of them, namely c_1, c_2 are related to the linear transformation $(x, y) \to (c_1x, c_2y)$. Parameter $\psi'(0) = 0$ is related to the shears $(x, y) \to (x + ty, y)$ (that are automorphisms by construction).

7. Shear invariant ODE with additional symmetries

It would be interesting to compute the complete symmetry algebras for all shear invariant ODE's. This can be done by solving equation (6) for ξ, η. Let $B = \sum b_k(x, y)\zeta^k$. By separating powers $0, 1, 2, 3, k > 3$ of ζ we get

$$\xi\frac{\partial b_0}{\partial x}+\eta\frac{\partial b_0}{\partial y}+\frac{\partial\eta}{\partial x}b_1+\left(2\frac{\partial\xi}{\partial x}-\frac{\partial\eta}{\partial y}\right)b_0-\frac{\partial^2\eta}{(\partial x)^2}=0 \tag{20}$$

$$\xi\frac{\partial b_1}{\partial x}+\eta\frac{\partial b_1}{\partial y}+2\frac{\partial\eta}{\partial x}b_2+\frac{\partial\xi}{\partial x}b_1+3\frac{\partial\xi}{\partial y}b_0+\frac{\partial^2\xi}{(\partial x)^2}-2\frac{\partial^2\eta}{\partial x\partial y}=0 \tag{21}$$

$$\xi\frac{\partial b_2}{\partial x}+\eta\frac{\partial b_2}{\partial y}+3\frac{\partial\eta}{\partial x}b_3+\frac{\partial\eta}{\partial y}b_2+2\frac{\partial\xi}{\partial y}b_1-\frac{\partial^2\eta}{(\partial y)^2}+2\frac{\partial^2\xi}{\partial x\partial y}=0 \tag{22}$$

$$\xi\frac{\partial b_3}{\partial x}+\eta\frac{\partial b_3}{\partial y}+4\frac{\partial\eta}{\partial x}b_4+\left(2\frac{\partial\eta}{\partial y}-\frac{\partial\xi}{\partial x}\right)b_3+\frac{\partial\xi}{\partial y}b_2+\frac{\partial^2\xi}{(\partial y)^2}=0 \tag{23}$$

and

$$\begin{aligned}\xi\frac{\partial b_k}{\partial x}+\eta\frac{\partial b_k}{\partial y}+(k+1)\frac{\partial\eta}{\partial x}b_{k+1}+&\\ +\left((k-1)\frac{\partial\eta}{\partial y}-(k-2)\frac{\partial\xi}{\partial x}\right)b_k-(k-4)\frac{\partial\xi}{\partial y}b_{k-1}&=0.\end{aligned} \tag{24}$$

For $B=f_0(y)(y-x\zeta)^3+f_1(y)(y-x\zeta)^2\zeta$ equations (20-23) take the form

$$-\frac{\partial^2\eta}{(\partial x)^2}+(y^2f_1-3xy^2f_0)\frac{\partial\eta}{\partial x}+(y^3f_0'+3y^2f_0)\eta-y^3f_0\eta'+2y^3f_0\frac{\partial\xi}{\partial x}=0$$

$$\begin{aligned}\frac{\partial^2\xi}{(\partial x)^2}+(y^2f_1-3xy^2f_0)\frac{\partial\xi}{\partial x}-3y^2f_0\xi+3y^3f_0\xi'-2\frac{\partial\eta'}{\partial x}-3xy^2f_0'\eta-&\\ -6xyf_0\eta+2yf_1\eta+y^2f_1'\eta+6x^2yf_0\frac{\partial\eta}{\partial x}-4xyf_1\frac{\partial\eta}{\partial x}&=0\end{aligned}$$

$$\begin{aligned}-\eta''+(3x^2yf_0-2xyf_1)\eta'+3x^2(f_0+yf_0')\eta-2x(f_1+yf_1')\eta+&\\ +3x^2(f_1-xf_0)\frac{\partial\eta}{\partial x}+2\frac{\partial\xi'}{\partial x}+2y^2(f_1-3xf_0)\xi'+(6xyf_0-2yf_1)\xi&=0\end{aligned}$$

$$\begin{aligned}\xi''+x^2(xf_0-f_1)\frac{\partial\xi}{\partial x}+(3x^2yf_0-2xyf_1)\xi'+(2xf_1-3x^2f_0)\xi+&\\ +2x^2(f_1-xf_0)\eta'+x^2(f_1'-xf_0')\eta&=0,\end{aligned}$$

where $'$ stands for the derivative with respect to y. Now, we separate the components according to powers of x. Let $\xi=\sum\xi_k(y)x^k$ and $\eta=\sum\eta_k(y)x^k$. Then

$$\begin{aligned}-(k+2)(k+1)\eta_{k+2}-y^3f_0\eta_k'-y^2(3(k-1)f_0-yf_0')\eta_k+&\\ +2y^3(k+1)f_0\xi_{k+1}+(k+1)y^3f_1\eta_{k+1}&=0\end{aligned} \tag{25}$$

$$\begin{aligned}(k+2)(k+1)\xi_{k+2}-3(k+1)y^2f_0\xi_k+3y^3f_0\xi_k'-2(k+1)\eta_{k+1}'-&\\ -y[(4k-2)f_1-yf_1']\eta_k+3y[2(k-2)f_0-yf_0']\eta_{k-1}&=0\end{aligned} \tag{26}$$

$$\begin{aligned}-\eta_k''+3yf_0\eta_{k-2}'-3((k-3)f_0-yf_0')\eta_{k-2}+2(k+1)\xi_{k+1}'-&\\ -6y^2f_0\xi_{k-1}'+6yf_0\xi_{k-1}+[(3k-5)f_1-2yf_1']\eta_{k-1}-2yf_1\eta_{k-1}'-&\\ -2yf_1\xi_k+2y^2f_1\xi_k'&=0\end{aligned} \tag{27}$$

$$\begin{aligned}(28)\quad & \xi_k'' + (k-5)f_0\xi_{k-2} + 3yf_0\xi_{k-2}' - 2f_0\eta_{k-3}' - f_0'\eta_{k-3} - \\ & - (k-3)f_1\xi_{k-1} - 2yf_1\xi_{k-1}' + f_1'\eta_{k-2} + 2f_1\eta_{k-2}' = 0.\end{aligned}$$

These equations are satisfied for all $k \geq 0$. Components ξ_k and η_k with negative indices have to be replaced by 0.
Using equations (27) and (28) one can determine $\xi_0, \xi_1, \eta_0, \eta_1$ up to eight complex parameters $\alpha_0, \dots, \alpha_3, \beta_0, \dots, \beta_3$. All ξ_k, η_k with $k \geq 2$ can be calculated recursively from (25) and (26). Equations (27) and (28) provide then additional conditions on f_0, f_1. A general implementation of this strategy comes across some tedious algebraic calculations. Since our aim is to identify ODE's with more than one shear we will simplify the system by stipulating $\alpha_0 = \xi_0(0) = 0$ and $\beta_0 = \eta_0(0) = 0$. Since the shear $y\frac{\partial}{\partial x}$ is always a symmetry we assume $\alpha_1 = 0$. Then

$$\begin{aligned}(29)\quad & \xi_0 = 0 \\ & \xi_1 = \alpha_2 + \alpha_3 y \\ & \eta_0 = \beta_1 y + \alpha_3 y^2 \\ & \eta_1 = \beta_2 + \beta_3 y + 2\alpha_3 y^2\Phi_1 - 2\alpha_2 y\Phi_1 - 2\beta_1 y^2\Phi_1' - 2\alpha_3 y^3\Phi_1'.\end{aligned}$$

Another simplification of (25)-(28), that leads to interesting examples, including the only non-flat ODE with two shears, is to consider the particular case $f_1 \equiv 0$ (even with α_0, β_0 not necessarily vanishing).

7.1. **The case** $B = f_0(y)(y - x\zeta)^3$**.** For $B = f(y)(y - x\zeta)^3$ equations (25)-(28) take the form

$$(30)\quad -(k+2)(k+1)\eta_{k+2} - y^3f\eta_k' - y^2(3(k-1)f - yf')\eta_k + 2y^3(k+1)f\xi_{k+1} = 0$$

$$\begin{aligned}(31)\quad & (k+2)(k+1)\xi_{k+2} - 3(k+1)y^2f\xi_k + 3y^3f\xi_k' - 2(k+1)\eta_{k+1}' + \\ & +3y[2(k-2)f - yf']\eta_{k-1} = 0\end{aligned}$$

$$\begin{aligned}(32)\quad & -\eta_k'' + 3yf\eta_{k-2}' - 3((k-3)f - yf')\eta_{k-2} + 2(k+1)\xi_{k+1}' - 6y^2f\xi_{k-1}' + \\ & +6yf\xi_{k-1} = 0\end{aligned}$$

$$(33)\quad \xi_k'' + (k-5)f\xi_{k-2} + 3yf\xi_{k-2}' - 2f\eta_{k-3}' - f'\eta_{k-3} = 0.$$

(For simplicity we omitted the subscript of f_0).
It follows immediately from (33) that $\xi_0 = \alpha_0 + \alpha_1 y$ and $\xi_1 = \alpha_2 + \alpha_3 y$ are linear functions, and

$$\xi_2'' = 3f\xi_0 - 3yf\xi_0' = 3\alpha_0 f.$$

We find

$$(34)\quad \xi_2 = 3\alpha_0\Phi + \alpha_4 + \alpha_5 y,$$

where $\Phi'' = f$ with $\Phi(0) = \Phi'(0) = 0$. Then (32) implies

$$\begin{aligned}\eta_0'' &= 2\xi_1' = 2\alpha_3 \\ \eta_1'' &= 4(3\alpha_0\Phi') + 4\alpha_5 + 6\alpha_0 yf\end{aligned}$$

and therefore,

$$\eta_0 = \alpha_3 y^2 + \beta_0 + \beta_1 y$$
$$\eta_1 = 6\alpha_0 y\Phi + \beta_2 + \beta_3 y + 2\alpha_5 y^2$$

From (31) one obtains

$$\xi_2 = \frac{3\alpha_0}{2} y^2 \Phi'' + 6\alpha_0 y\Phi' + 6\alpha_0 \Phi + \beta_3 + 4\alpha_5 y,$$

which must be equal to the expression found in (34). Thus, $\alpha_5 = 0$, $\alpha_4 = \beta_3$ and

$$3\alpha_0\Phi + 6\alpha_0 y\Phi' + \frac{3\alpha_0}{2} y^2\Phi'' + \beta_3 = 0.$$

It follows

$$\alpha_0(k^2 + 3k + 2)\Phi_k = 0 \text{ for } k \geq 0$$

Since $f = \Phi''$ does not vanish identically, we have $\alpha_0 = 0$. Comparison of ξ_4'' obtained from (31) and ξ_4' obtained from (33) yields

$$(y^2 f''' + 10yf'' + 20f')\beta_2 + (y^3 f''' + 15y^2 f'' + 15yf' + 15f)\beta_3 = 0.$$

Suppose first that $f \not\equiv const$. Then the above equation is equivalent to

$$\beta_2 + \beta_3 \left(y + 5\frac{y^2 f'' + 8yf' + 12f}{y^2 f''' + 10yf'' + 20f'} \right) = 0.$$

Now, either $\beta_3 = \beta_2 = 0$ or $\beta_3 \neq 0$. Consider the second case. Then

$$\frac{-5}{y - \frac{1}{c}} = \frac{(y^2 f'' + 8yf' + 12f)'}{y^2 f'' + 8yf' + 12f}$$

with $c = -\beta_3/\beta_2 \neq 0$ and, consequently,

$$\frac{K}{(1-cy)^5} = -\frac{(y^4 f)''}{y^2}.$$

It follows

$$f(y) = \frac{K}{(1-cy)^3}.$$

This is equivalent to $f(y) = const$ (see (36)).
Suppose now $\beta_2 = \beta_3 = 0$. We equate now ξ_3' from (31) and from (32). It follows

$$(\alpha_3 y^2 + \beta_1 y + \beta_0) y^3 f''' + (15\alpha_3 y^2 + (12\beta_1 + 2\alpha_2)y + 11\beta_0) y^2 f'' +$$
$$+ (60\alpha_3 y^2 + (36\beta_1 + 16\alpha_2)y + 28\beta_0) yf' + (60\alpha_3 y^2 + 24(\beta_1 + \alpha_2)y + 12\beta_0) f = 0.$$

Let $f = \sum_{k=0}^{\infty} f_k y^k$. For $k \geq 0$ we have then

$$(k^3 + 8k^2 + 19k + 12) f_k \beta_0 + (k^3 + 6k^2 + 11k + 6) f_{k-1}\beta_1 +$$
$$(2k^2 + 10k + 12) f_{k-1}\alpha_2 + (s^3 + 6s^2 + 11s + 6) f_{k-2}\alpha_3 = 0$$

(with $f_{-1} = f_{-2} = 0$).
By recursion, $\beta_0 \neq 0$ would imply that all f_k must vanish, hence, $\beta_0 = 0$.
If $\alpha_2 = 0$, then, again by recursion, $\beta_1 = \alpha_3 = 0$ (or all f_k must vanish). In this case $y\frac{\partial}{\partial x}$ would be the only infinitesimal automorphism.

Suppose $\alpha_2 \neq 0$. Let f_k be the first non-vanishing term. Then

$$\beta_1 = \frac{-2}{k+2}\alpha_2$$
$$\alpha_3 = \frac{2}{(k+2)(k+3)}\frac{f_{k+1}}{f_k}\alpha_2$$

For all $\ell > k$ we now have

$$f_{\ell+1} = \frac{(k+2)(\ell+3)}{2(\ell-k+1)}\frac{\alpha_3 f_\ell}{\alpha_2} = \frac{(\ell+3)}{\ell-k+1}\frac{f_{k+1}}{(k+3)f_k}f_\ell.$$

Therefore,

$$f_{k+n} = \frac{(k+n+2)!}{(k+2)!\,n!}\left(\frac{f_{k+1}}{(k+3)f_k}\right)^n f_k.$$

The corresponding power series sums up to

$$f = Ky^k(1-cy)^{-k-3}$$

where $K = f_k \neq 0$ and $c = \frac{f_{k+1}}{(k+3)f_k}$ are free parameters.
If $k > 0$ we have only one symmetry in addition to the shear $y\frac{\partial}{\partial x}$, namely

$$[(k+2)x + 2cxy]\frac{\partial}{\partial x} - 2y(1-cy)\frac{\partial}{\partial y}.$$

In the remaining situation if $f = f_0 \neq 0$, that is equivalent to (7.1) we find three symmetries: two shears $y\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial y}$ and the hyperbolic rotation $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

7.2. **ODE with two shears.** If $\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$ is a second shear it has a zero at $x = y = 0$. The initial terms $\xi_0, \xi_1, \eta_0, \eta_1$ were earlier determined in (29). Using (25) and (26) we can find the complete series for ξ and η.
We compare now ξ_2' obtained from (25) and from (27). If $f_{1,k}$ is the first non-vanishing term in the power series of $f_1(y)$ then

$$(\alpha_2 + (k+2)\beta_1)f_{1,k}y^{k+1} = 0. \tag{35}$$

Thus, either $f_1 \equiv 0$ or α_2 is a multiple of β_1. Since the field we are looking for has a non-isolated zero at the origin, either $\beta_1 = \frac{\partial \eta}{\partial y}(0) = 0$ or $\alpha_2 = \frac{\partial \xi}{\partial x}(0) = 0$. But then (35) implies $\beta_1 = \alpha_2 = 0$.
Now, comparison of ξ_2' calculated in both different ways gives

$$\alpha_3 f_{1,k}y^{k+2} = 0.$$

Again, either $f_1 \equiv 0$ or $\alpha_3 = 0$.
Next, we compare ξ_3' calculated from (25) and from (27). It follows that

$$\beta_2(3-k-k^2)f_{1,k} = 0$$

and again, either $f_1 \equiv 0$ or $\beta_2 = 0$. The same way one shows that $\beta_3 = 0$ (if $f_1 \not\equiv 0$).
Thus, a second shear symmetry can appear only if $f_1 \equiv 0$. According to the results of subsection 7.1 this can only happen if $f_0 = K/(1-cy)^3$. For $K = 0$ this is the flat ODE. If $K \neq 0$ it can be reduced to

$$y'' = (y - xy')^3 \tag{36}$$

by $(x, y) \to (x/(1 - cy), y/\sqrt{K}(1 - cy))$.
Its symmetries are the two shears $y\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial y}$ and $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ that generate a copy of $\mathfrak{sl}(2, \mathbb{C})$.
The ODE (36) is equivalent to

$$y'' = \frac{1}{y^3}.$$

Incidently, it appears in Lie's list of ODE with three symmetries [28]. Notice that our representation of this ODE is regular at the origin and has a linear symmetry group, namely $\mathrm{SL}(2, \mathbb{C})$ with its standard action on $\mathbb{C}^2_{x,y}$. It follows that the ODE (36) is, in fact, invariant with respect to any shear automorphism.
In order to embed the corresponding abstract CR-manifold into $\mathbb{C}^4$ we need to find four coordinate functions that are annihilated by

$$\bar{Z}_1 = \frac{\partial}{\partial \bar{\zeta}} \quad \text{and} \quad \bar{Z}_2 = \frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial y} + (y - x\zeta)^3\frac{\partial}{\partial \zeta}.$$

Such functions are

$$z_1 = \frac{\zeta - xy(y - x\zeta)^2}{1 - x^2(y - x\zeta)^2}, \qquad z_2 = \bar{x}$$
$$w_1 = \frac{(y - x\zeta)^2}{1 - x^2(y - x\zeta)^2}, \qquad w_2 = \bar{y}.$$

The coordinates z_1, w_1 have a nice geometric interpretation, namely, they parameterize the family of integral curves of the initial ODE:

$$(y - z_1x)^2 - w_1^2x^2 = w_1$$

The coordinate functions $z_1(x, y, y'), w_1(x, y, y')$ assign to any point $(x, y, y' = \zeta)$ the parameters of the corresponding integral curve.
The embedded CR-manifold can be described by the quartic equation

$$w_1 + w_1^2\bar{z}_2^2 - (\bar{w}_2 - z_1\bar{z}_2)^2 = 0.$$

The foliations that correspond to the distinguished vector fields are nothing but the projections to the (z_1, w_1) and the (z_2, w_2)-planes, respectively.
The action of $\mathrm{SL}(2, \mathbb{C})$ extends to an action on $\mathbb{C}^4$ with the three following generators:

$$(z_1, z_2, w_1, w_2) \to (z_1 + t,\ z_2,\ w_1,\ w_2 + \bar{t}z_2)$$
$$(z_1, z_2, w_1, w_2) \to \left(\frac{z_1}{r^2},\ \bar{r}z_2,\ \frac{w_1}{r^2},\ \frac{w_2}{\bar{r}}\right)$$
$$(z_1, z_2, w_1, w_2) \to \left(\frac{z_1 + s(z_1^2 - w_1^2)}{(1 + sz_1)^2 - s^2w_1^2},\ z_2 + \bar{s}w_2,\ \frac{w_1}{(1 + sz_1)^2 - s^2w_1^2},\ w_2\right).$$

The infinitesimal automorphisms are

$$(\tau - 2\rho z_1 - \sigma(z_1^2 + w_1^2))\frac{\partial}{\partial z_1} + (\bar{\rho}z_2 + \bar{\sigma}w_2)\frac{\partial}{\partial z_2} - (2\rho w_1 + 2\sigma z_1w_1)\frac{\partial}{\partial w_1} + (\bar{\tau}z_2 - \bar{\rho}w_2)\frac{\partial}{\partial w_2}$$

8. Normal form of shear invariant ODE. Uniqueness

Theorem 3. *A reduced shear invariant ODE*

$$y'' = f_0(y)(y - xy')^3 + f_1(y)(y - xy')^2 y'$$

is equivalent

1. *to the flat equation if and only if $f_0 \equiv f_1 \equiv 0$,*
2. *to $y'' = (y - xy')^3$ if and only if $f_0 = K/(1 - cy)^3$ and $f_1 \equiv 0$,*
3. *to $y'' = \tilde{f}_0(y - xy')^3 + \tilde{f}_1(y - xy')^2 y'$ if and only if there is a coordinate change of the the form*
$$x_1 = \frac{c_1 x}{1 - cy}, \quad y_1 = \frac{c_2 y}{1 - cy}$$
that transforms one ODE to the other.

Proof. The flat equation is the only one in the considered family with an 8-dimensional group, the equations $y'' = K(y - xy')^3/(1 - cy)^3$ are the only non-flat ones with more than one shear. This proves the first and the second statement. The third statement follows immediately from the observation at the end of Section 6, the results of Section 7 and the fact that the given transformation preserves the reduced form. □

9. Appendix A

Here we calculate the reduction of a shear invariant ODE. The Maple8 work-sheet can be downloaded from
ftp://ftp.math.uni-bonn.de/people/schmalz/transformation.mws

Consider the second order shear invariant ODE: ODE=0 with ODE as defined below. Here x, y, zeta, omega are the independent variable, the dependent variable and the first and second order derivatives, respectively.

```
> ODE:=omega-sum(f[k](y)*(y-x*zeta)^(3-k)*zeta^k,k=0..3);
```

(x,y) are the old coordinates, (xi, eta) are the new coordinates. phi(0) not 0

```
> x:=psi(eta) + lambda*xi*phi(eta); y:=eta*phi(eta);
```

zeta is the derivative of y with respect to x; zeta1 is the derivative of eta with respect to xi.

```
> zeta:=(phi(eta)+eta*diff(phi(eta),eta))*zeta1/
> ((diff(psi(eta),eta)+xi*diff(phi(eta),eta))*zeta1+phi(eta));
```

omega is the second derivative of y with respect to x, omega1 is the second derivative of eta with respect to xi.

```
> omega:=((eta*diff(phi(eta),eta)+phi(eta))*phi(eta)*omega1
> +(((2*diff(phi(eta),eta)+eta*diff(phi(eta),eta$2))*phi(eta))
> -2*(diff(phi(eta),eta)*(eta*diff(phi(eta),eta)+phi(eta))))
> *zeta1^2+((2*diff(phi(eta),eta)+eta*diff(phi(eta),`$`(eta,2)))
> *(diff(psi(eta),`$`(eta,1))+xi*diff(phi(eta),`$`(eta,1)))
> -((diff(psi(eta),eta$2)+xi*diff(phi(eta),eta$2))*
> (phi(eta)+eta*diff(phi(eta),eta))))*zeta1^3)/(zeta1*
> diff(psi(eta),eta)+zeta1*xi*diff(phi(eta),eta)+phi(eta))^3;
```

We divide the ODE in new variables by a common non-vanishing factor. Then assign the numerator to ODE1. ODE1=0 is the new ODE.

```
> factor(ODE/(eta*(diff(phi(eta),eta)+phi(eta))*phi(eta))):
> ODE1:=numer(%);
```

The coefficient at the new omega1 in the new ODE

```
> coeff(ODE1,omega1);
```

The coefficients at (xi zeta1)^j for j=3,2,1,0. They give the new f[0]. In particular, the new f[0] vanishes if the old one did.

```
> coeff(ODE1,zeta1,3):  subs(xi=0,diff(%,xi$3))/3!;
> coeff(ODE1,zeta1,2):subs(xi=0,diff(%,xi$2))/2!;
> coeff(ODE1,zeta1,1):  subs(xi=0,diff(%,xi$1));
> coeff(ODE1,zeta1,0):  subs({omega1=0,xi=0},%);
```

The coefficients at zeta1 (xi zeta1)^j for j=2,1,0. They give the new f[1]. In particular, the new f[1] vanishes if the old f[0] and f[1] did. We also check here the correctness of the relations between these coefficients.

```
> coeff(ODE1,zeta1,3):  A32:=factor(subs(xi=0,diff(%,xi$2))/2!);
> coeff(ODE1,zeta1,2):  A21:=factor(subs(xi=0,diff(%,xi$1)));
> expand(A32*2*eta+A21);
> coeff(ODE1,zeta1,1):  A10:=factor(subs(xi=0,%));
> expand(A10-A32*eta^2);
```

The coefficients at zeta1^2 (xi zeta1)^j for j=1,0. They give the new f[2]. Vanishing of this new f[2] gives one of the ODE that determine psi and phi.

```
> coeff(ODE1,zeta1,2):  A20:=factor(subs(xi=0,%));
> coeff(ODE1,zeta1,3):  A31:=subs(xi=0,diff(%,xi$1));
> expand(A20+A31*eta);
```

The coefficient at zeta1^3. It gives the new f[3]. Vanishing of this new f[3] gives the second ODE that determines psi and phi.

```
> coeff(ODE1,zeta1,3):  A30:=factor(subs(xi=0,%));
```

10. Appendix B

Here we calculate the automorphisms of the family of shear invariant ODE with $f_0 = 0$. The Maple8 work-sheet can be downloaded from
ftp://ftp.math.uni-bonn.de/people/schmalz/Special3.mws

The right hand side of the ODE

```
> Bb:=f(y)*(y-x*zeta)^3;
> B[0]:=subs(zeta=0,Bb); for j1 from 1 to 5
> do
> B[j1]:=subs(zeta=0,diff(Bb,zeta$j1)/(j1!))od;
```

Xi(x,y) and Eta(x,y) are the coefficients of an infinitesimal automorphism. Here are the three equations according to the degrees 0,1,2,3 in zeta

```
> A[0]:=expand(-diff(Eta(x,y),x$2)-B[0]*diff(Eta(x,y),y)
> +2*B[0]*diff(Xi(x,y),x)+Xi*diff(B[0],x)+ Eta(x,y)*diff(B[0],y)
> +diff(Eta(x,y),x)*B[1]);
```

```
> A[1]:=expand(-2*diff(Eta(x,y),x,y)+diff(Xi(x,y),x$2)
> -B[1]*diff(Eta(x,y),y)+2*B[1]*diff(Xi(x,y),x)+
> 3*B[0]*diff(Xi(x,y),y)+Xi(x,y)*diff(B[1],x)+
> Eta(x,y)*diff(B[1],y)+2*diff(Eta(x,y),x)*B[2]+
> (diff(Eta(x,y),y)-diff(Xi(x,y),x))*B[1]);
```

```
> A[2]:=expand(-diff(Eta(x,y),y$2)+2*diff(Xi(x,y),x,y)-
> B[2]*diff(Eta(x,y),y)+2*B[2]*diff(Xi(x,y),x)+
> 3*B[1]*diff(Xi(x,y),y)+Xi(x,y)*diff(B[2],x)+
> Eta(x,y)*diff(B[2],y)+3*diff(Eta(x,y),x)*B[3]+
> 2*(diff(Eta(x,y),y)-diff(Xi(x,y),x))*
> B[2]-diff(Xi(x,y),y)*B[1]);
```

```
> A[3]:=expand(diff(Xi(x,y),y$2)-B[3]*diff(Eta(x,y),y)+
> 2*B[3]*diff(Xi(x,y),x)+3*B[2]*diff(Xi(x,y),y)
> +Xi(x,y)*diff(B[3],x)+ Eta(x,y)*diff(B[3],y)
> +4*diff(Eta(x,y),x)*B[4]+3*(diff(Eta(x,y),y)-
> diff(Xi(x,y),x))*B[3]-2*diff(Xi(x,y),y)*B[2]);
```

```
> A[4]:=-B[4]*diff(Eta(x,y),y)+2*B[4]*diff(Xi(x,y),x)+
> 3*B[3]*diff(Xi(x,y),y) +Xi(x,y)*diff(B[4],x)+
> Eta(x,y)*diff(B[4],y)+5*diff(Eta(x,y),x)*B[5]+
> 4*(diff(Eta(x,y),y)-diff(Xi(x,y),x))*B[4]-
> 3*diff(Xi(x,y),y)*B[3];
```

Let Xi= sum xi[k](y) x^k and Eta= sum eta[k](y) x^k. One derives from A[0]=0 ...A[3]=0 recursive formulas for eta[k] and xi[k] (for $k \geq 2$) and for the 1st and 2nd order derivatives xi1[k] and xi2[k] of xi[k] (for $k \geq 1$, resp. $k \geq 0$)

```
> eta[k](y):=(-y^3*f(y)*diff(eta[k-2](y),y)-
> y^2*(3*(k-3)*f(y) -y*diff(f(y),y))*eta[k-2](y)
> +2*y^3*(k-1)*f(y)*xi[k-1](y))/k/(k-1);
```

```
> xi[k](y):=(3*(k-1)*y^2* f(y)*xi[k-2](y)
> -3*y^3*f(y)*diff(xi[k-2](y),y) +2*(k-1)*diff(eta[k-1](y),y) -
> 3*y*(2*(k-4)*f(y)-y*diff(f(y),y))*eta[k-3](y))/k/(k-1);
```

```
> xi1[k](y):=(diff(eta[k-1](y),y$2)
> -3*y*f(y)*diff(eta[k-3](y),y)
> +3*((k-4)*f(y)-y*diff(f(y),y))*eta[k-3](y) +6*y^2*f(y)
> *diff(xi[k-2](y),y)-6*y*f(y)*xi[k-2](y))/2/k;
```

```
> xi2[k](y):=-(k-5)*f(y)*xi[k-2](y)
> -3*y*f(y)*diff(xi[k-2](y),y)
> +2*f(y)*diff(eta[k-3](y),y)+diff(f(y),y)*eta[k-3](y);
```

Using the formula for xi2[0] xi2[1] and xi2[2] we obtain

```
> xi[0](y):=a0 +a1*y; xi[1](y):=a2 +a3*y;
> xi[2](y):=3*a0*Phi(y)+a4+a5*y;
```

where Phi(y) is the second antiderivative of f(y) with Phi(0)=Phi'(0)=0.

```
> f(y):=diff(Phi(y),y$2);
```

Using the formula for xi1[1] and xi1[2] we find the initial terms

```
> eta[0](y):=a3*y^2 +b0 +b1*y;
> eta[1](y):=6*a0*y*Phi(y) + b2 +b3*y;
```

Now we can determine xi[k], eta[k] for $k \geq 2$ recursively from A[0]=0 and A[1]=0 and then compare their derivatives with the expressions from A[2]=0 and A[3]=0. By def1[k], resp. def2[k] we denote the difference of the first, resp. second derivatives of xi[k](y) calculated in both different ways. First, set formally

```
> eta[-1](y):=0;
```

```
> for k from 2 to 2 do
> eta[k](y):=expand((-y^3*f(y)*diff(eta[k-2](y),y)-
> y^2*(3*(k-3)*f(y) - y*diff(f(y),y))*eta[k-2](y)
> +2*y^3*(k-1)*f(y)*xi[k-1](y))/k/(k-1));
> xi[k](y):=expand((3*(k-1)*y^2*f(y)*xi[k-2](y)
> -3*y^3*f(y)*diff(xi[k-2](y),y) +2*(k-1)*diff(eta[k-1](y),y) -
> 3*y*(2*(k-4)*f(y)-y*diff(f(y),y))*eta[k-3](y))/k/(k-1));
> xi1[k](y):=expand(diff(eta[k-1](y),y$2)-3*y*f(y)*
> diff(eta[k-3](y),y)+3*((k-4)*f(y)-y*diff(f(y),y))*
> eta[k-3](y)+6*y^2*f(y) *diff(xi[k-2](y),y)-
> 6*y*f(y)*xi[k-2](y))/2/k;
> def1[k]:=expand(diff(xi[k](y),y)-xi1[k](y));
> xi2[k](y):=expand(-(k-5)*f(y)*xi[k-2](y)
> -3*y*f(y)*diff(xi[k-2](y),y)+2*f(y)*diff(eta[k-3](y),y)
> +diff(f(y),y)*eta[k-3](y));
> def2[k]:=expand(diff(xi[k](y),y$2)-xi2[k](y)) od;
```

From def1[2]=0 we find a0=0 (or else Phi"=f=0). From the expression of xi[2] from above we conclude also a4=b3 and a5=0.

```
> a0:=0; a4:=b3; a5:=0
```

(Before executing this, redefine xi[0], xi[1]... with a0=0, a4=b3 and a5=0.)

```
> for k from 3 to 4 do
> eta[k](y):=expand((-y^3*f(y)*diff(eta[k-2](y),y)-
> y^2*(3*(k-3)*f(y) - y*diff(f(y),y))*eta[k-2](y)
> +2*y^3*(k-1)*f(y)*xi[k-1](y))/k/(k-1));
> xi[k](y):=expand((3*(k-1)*y^2*f(y)*xi[k-2](y)
> -3*y^3*f(y)*diff(xi[k-2](y),y) +2*(k-1)* diff(eta[k-1](y),y)
> -3*y*(2*(k-4)*f(y)-y*diff(f(y),y))*eta[k-3](y))/k/(k-1));
> xi1[k](y):=expand(diff(eta[k-1](y),y$2)
> -3*y*f(y)*diff(eta[k-3](y),y)
> +3*((k-4)*f(y)-y*diff(f(y),y))*eta[k-3](y) +6*y^2*f(y)
> *diff(xi[k-2](y),y)-6*y*f(y)*xi[k-2](y))/2/k;
> def1[k]:=expand(diff(xi[k](y),y)-xi1[k](y));
> xi2[k](y):=expand(-(k-5)*f(y)*xi[k-2](y)
> -3*y*f(y)*diff(xi[k-2](y),y)+2*f(y)*diff(eta[k-3](y),y)
> +diff(f(y),y)*eta[k-3](y));
> def2[k]:=expand(diff(xi[k](y),y$2)-xi2[k](y)) od;
```

From def2[4]=0 we conclude b2=b3=0 or f satisfies an ODE which implies $f = k/(1-cy)^3$.

From def1[3]=0 we get an ODE on Phi"=f=$\sum_j^\infty$ f[n] y^n. The corresponding recursive formulas yield f[j+n]:=(j+n+2)!/(j+2)!/n!*a^n f[j] with a=f[j+1]/(j+3)/f[j].

```
> sum( (j+n+2)!/(j+2)!/n!*a^n*y^(j+n)*f[j],n=0..infinity);
```

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