Quasiconformal immersions of Riemannian manifolds and a Picard-type theorem

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Abstract

We study singularities of quasiconformal immersions and we show that for dimension $n \geq 3$ a phenomenon of compulsory continuation of the mapping holds. It leads, in particular, to a stronger version of the Picard-type theorem — to one without omitted values.

1 Main results

The aim of the present paper is to prove a theorem on the continuation of a germ of a locally inverse mapping that can serve as an instrument for study of singularities of quasiconformal immersions of Riemannian manifolds in dimension $n \geq 3$.

We consider manifolds of the same dimension, and by immersion, we mean a locally injective (locally invertible) mapping. Quasiconformality of a mapping means the boundedness of its quasiconformality coefficient (at least in the domains, where the mapping is injective).

Theorem 1 (Main Theorem — alternative) *Let M^n and N^n be Riemannian manifolds of finite dimension n, $\dot{U} = U \setminus \{o\}$ punctured neighborhood of a point $o \in M^n$, and $f : \dot{U}^n \to N^n$ quasiconformal immersion. If $n \geq 3$, then the following alternative holds:*

(a) *either the mapping f is injective in some punctured neighborhood of the point o;*

(b) *or each germ f_p^{-1} of the locally inverse mapping can be continued along every path in N^n starting at $p \in N^n$ and not intersecting the image $f(\partial U)$ of the boundary of the neighborhood U.*

Here $f(\partial U)$ can be considered as a limit set of f in N^n at the boundary ∂U. Of course, one can diminish the neighborhood U, so, without loss of generality, one can suppose that f is defined on ∂U or even in the neighborhood of ∂U.

We prove Theorem 1 in section 3.

Below (see section 5) we discuss some applications of Theorem 1. Among them the global homeomorphism theorem (asserting that for $n \geq 3$ every quasiconformal immersion $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally invertible), and Gromov's geometric version of the latter theorem.

As a byproduct we obtain the following Picard-type theorem.

Theorem 2 (Picard-type theorem for quasiconformal immersions) *Quasiconformal immersion with an isolated singular point of one Riemannian manifold to another one of the same dimension $n \geq 3$*

(a) *is either injective in some punctured neighborhood of the isolated singular point;*

(b) *or the mapping takes every such punctured neighborhood onto the whole target manifold.*

The mapping $z \longmapsto \exp \frac{1}{z}$ shows that the restriction $n \geq 3$ is essential.

On the other hand each of the two cases (a) and (b) of the alternative is realizable in every dimension $n \geq 2$. The following Hopf's mapping can serve as an example. Define $f : \dot{\mathbb{R}}^n = \mathbb{R}^n \setminus \{o\} \to S^{n-1} \times S^1$ by the correspondence $\dot{\mathbb{R}}^n \ni x \longmapsto \{2^k x \mid k \in Z\} \in \dot{\mathbb{R}}^n / G$, where G is the group of homoteties $g : \dot{\mathbb{R}}^n \to \dot{\mathbb{R}}^n$ generated by the mapping $x \longmapsto 2x$.

The proof of the main Theorem 1 is based on several lemmas. The key observation is the following connection between conformal capacity and geometry of the visible part of the boundary of a star-like domain.

Lemma 1 (Star Lemma) *Let $D \subset \mathbb{R}^n$ $(n \geq 2)$ be a star-like domain relative to a point $o \in \mathbb{R}^n$, i.e., let D be formed by rays or by their intervals originating from o. The finite intervals end at points of those part Δ of the boundary ∂D of D which is "visible" from o.*

Let Γ be the family of all curves in D that join Δ with ε-neighborhood $B(o, \varepsilon)$ of o.

If conformal modulus of the family Γ is equal to zero, then

- *projection of Δ along these rays to the sphere $\partial B(o, \varepsilon)$ does not contain connected components different from a point;*
- *∂D does not locally separate the space provided $n > 2$.*

Recall that conformal modulus of the family Γ of curves in $\mathbb{R}^n$ is defined as follows:

$$\operatorname{mod} \Gamma := \inf \int_{\mathbb{R}^n} \rho^n \, dv \qquad ,$$

where the infimum is taken over all Borel measurable nonnegative functions $\rho : \mathbb{R}^n \to \mathbb{R}_+$ such that $\int_\gamma \rho \geq 1$ for each curve $\gamma \in \Gamma$.

2 Auxiliary statements

Proposition 1 (on asymptotic values and the covering) *Let $f : \dot{U} \to N^n$ be immersion (or even an open mapping) of a punctured neighborhood $\dot{U}$ of the point o of the manifold M^n to connected manifold N^n of the same dimension n.*

If this immersion has no asymptotic values at the singular point o, then for every punctured neighborhood $\dot{V} \subset \dot{U}$ of the point o the equality $f(\dot{V}) = N^n$ holds.

By an asymptotic value we mean a limit (finite or infinite) of f along a path that leads to the singular point. A finite limit is a point of the manifold N^n. The limit is said to be infinite if the limit set of f along the path is empty (for finite dimensional manifolds it is equivalent to the condition that the image of the point finally leaves any compact set on N^n; in other words, the image goes to absolute, or to infinity of N^n, as the point of the path tends to the singular point o on M^n).

Proposition 1 formulated above is very general and very easy to prove. The proof consists of the following two observations concerning immersions in general.

Remark 1 *Let $f : M^n \to N^n$ be an immersion. Suppose that pulling-back a path from N^n to M^n we meet an obstruction, i.e. the procedure (the path) cannot be continued over some point $p \in N^n$. Then the preimage of the lifted part of the path cannot have a finite limit or even nonempty limit set in M^n.*

In other words, as the point of the path in N^n tends to p the preimage of the point on the covering path in M^n finally leaves any compact subset of M^n. For finite n one can say that the preimage goes to the absolute or to infinity of M^n.

We develop Remark 1 by one more general observation related to specific properties of immersions.

Remark 2 *Let $f : M^n \to N^n$ be an immersion. Suppose f_D^{-1} is a univalent branch of the inverse mapping in the domain $D \subset N^n$. Consider two sequences of points in D converging to the same boundary point p of D. Suppose the corresponding points of these two sequences can be join by a sequence of arcs in D collapsing to p. Under these conditions, if there exists the limit of f_D^{-1} along one of the sequences under consideration, then there exists the same limit of f_D^{-1} along the second one.*

In particular, if D is locally connected at the boundary point p, for instance, if the boundary ∂D does not separate the space near the point p at all (as a ray in $\mathbb{R}^3$, but not in $\mathbb{R}^2$), then the described conditions are fulfilled automatically.

Note that in the latter case there even must exist a limit (finite or infinite) of f_D^{-1} at $p \in \partial D$.

All assertions of Remark 1 and Remark 2 are based on the fact that any immersion is locally injective and cannot send a nondegenerate connected set to a point. This observation proves Remark 2.

To prove Remark 1, it is sufficient to consider now a sequence of punctured neighborhoods of the singular point $o \in M^n$ and the intersection of its images in N^n.

If the intersection does not coincide with N^n, then it is either empty, or it has a boundary point p in N^n.

The first assumption contradicts to nonexistence of "infinite" asymptotic values.

Consider the second one. The mapping f is open, so by diminishing $\dot{U}$ if necessary we may assume that $p \notin f(\partial U)$. Then there are no points of $f(\partial U)$ in some small neighborhood of p, but there are points of $f(\dot{U})$ there. Take a path in N^n that starts at a point of $f(\dot{U})$, ends at p, and does not intersect $f(\partial U)$. There are no obstructions for lifting such path to M^n, for otherwise we obtain a finite asymptotic value of our mapping f at the singular point $o \in M^n$. Thus f is a covering mapping over a neighborhood of p in N^n not intersecting $f(\partial U)$. The preimage of this neighborhood would be a sequence of domains in $\dot{U}$ homeomorphic by f to the neighborhood under consideration. They must accumulate and collapse to the singular point $o \in M^n$. But then p is not a boundary point of the initially considered intersection.

Therefore the intersection coincides with the entire manifold N^n, and Remark 2 is proved.

The following assertion is the main step in the proof of Theorem 1. It uses both the quasiconformality of f and the condition $n \geq 3$ on the dimension of manifolds.

Proposition 2 (on local quasiconformal continuation) *Let M^n, N^n be connected Riemannian manifolds of dimension n. Suppose $\dot{U} = U \setminus \{O\}$ is punctured neighborhood of the set $O \subset M^n$ of conformal capacity zero (for instance, a point may serve as $\{O\}$); and $f : \dot{U} \to N^n$ is quasiconformal immersion. If $n \geq 3$, then for each point $a \in f(\dot{U}) \setminus f(\partial U)$ there are the neighborhood $W \subset N^n$ and the set S (possibly empty) of conformal capacity zero, such that every germ f_a^{-1} of the locally inverse mapping at the point $a \in N^n$ can be continued along any path in $W \setminus S$ starting at a. A branch of the inverse mapping thus obtained is univalent in $W \setminus S$.*

At the moment we prove Proposition 2 using the Star Lemma formulated in section 1 above. We prove the Star Lemma in section 4 below.

The assertion of Proposition 2 is actually related with the local continuation of the germ f_a^{-1} in a neighborhood of the point $a \in N^n$. Hence, without loss of generality, one can assume that $N^n = \mathbb{R}^n$ and that a is the origin of $\mathbb{R}^n$.

Let W be a ball centered at a such that $W \subset f(\dot{U}) \setminus f(\partial U)$. Extend the germ f_a^{-1} in W along the rays from a as far as possible. We get a star-like domain $D \subset W$ and a univalent branch f_D^{-1} of f^{-1} obtained by described continuation of the germ f_a^{-1}. The part Δ of the boundary ∂D of D visible from a consists of singular points in the sense that f_D^{-1} can not be extended over the points of Δ. Thus (see the proof of Proposition 1) the limit set of f_D^{-1} is empty at every point of Δ.

Since $W \subset f(\dot{U}) \setminus f(\partial U)$, we conclude that f_D^{-1} takes any path in D ending in Δ to a path in $\dot{U}$ ending in the set $O = U \backslash \dot{U}$ (and not in ∂U).

By assumption, conformal capacity of the set O is equal to zero. Therefore conformal modulus of all curves in $\mathbb{R}^n$ with the closure intersecting the set O is also equal to zero.

The mapping f_D^{-1} is quasiconformal. Hence modulus of all curves in D ending in Δ is equal to zero.

By the Star Lemma, the set $\partial D \cap W$ locally does not separate the space. Hence, by extending f_D^{-1} to the points of $\partial D \cap W$, where the extension is still possible, we still obtain univalent quasiconformal mapping $f_{\tilde{D}}^{-1}$ in an extended domain $\tilde{D} \cap W$ (see Remark 2). According to the construction, the mapping $f_{\tilde{D}}^{-1}$ can not be extended to any point of the set $S = \partial \tilde{D} \cap W$.

By Remark 2, each point of S is singular for $f_{\tilde{D}}^{-1}$, and we can repeat the above reasoning to conclude that conformal modulus of all curves in $\tilde{D}$ ending in S is equal to zero. Therefore the conformal capacity of S is equal to zero.

There is at most countable set of different germs f_a^{-1} at the point a. But countable union of sets S of conformal capacity zero is still a set of conformal capacity zero. Proposition 2 is proved.

It is well known that the set of conformal capacity zero is very thin: its metric dimension is equal to zero. In a sense the set S constructed above is a quasiconformal image of the singular set O (or part of O) and obtained property of S ($\text{cap}(S) = 0$) is therefore very natural.

Note that if the initial set O reduces to a single singular point, then the set S in W is either empty, or is a single point as well.

In this case the mapping $f_{\tilde{D}}^{-1}$ can be extended to this single point and the initial mapping f is injective in the punctured neighborhood of the point O and f admit quasiconformal extension to O.

As a direct corollary of the arguments above we obtain the following assertion.

Proposition 3 (alternative on asymptotic values and the removability of singularities of quasiconformal immersions)
Let M^n, N^n be Riemannian manifolds of dimension n, and $f : \dot{U} \to N^n$ be quasiconformal immersion of a punctured neighborhood $\dot{U} = U \setminus \{o\}$ of a point $o \in M^n$.

If $n \geq 3$, then the following alternative holds:

- *(a) either o is a removable singularity of the immersion f,*
- *(b) or the mapping f has no finite asymptotic values at the point o.*

Recall that in the definition of asymptotic value we allow both finite and infinite limits.

3 Proof and generalization of the Main Theorem

In the proof of Proposition 1 it was shown that if the lifting of a path $\gamma \subset N^n$ to $\dot{U} \subset M^n$ has an obstruction at $p \in N^n$, then the lifted part of γ either ends at the singular point $o \in U$, or goes to the boundary ∂U of the neighborhood $U \subset M^n$.

But if $\gamma \cap f(\partial U) = \emptyset$, then only the first of the two mentioned cases is possible. Thus, p becomes an asymptotic value of f at the singular point $o \in M^n$. Hence, by Proposition 3, f must be injective in some punctured neighborhood of o, which proves Theorem 1.

Note that the cases (a) and (b) of Theorem 1 and of Proposition 3 may appear simultaneously. (For instance, take the identity map of a disk to the punctured disk and, of course, suppose that the mapping is defined outside of a punctured point o.) But if (a) and (b) appear simultaneously, then f has an "infinite" asymptotic value. Hence the manifold N^n can be completed by a point o^* and the mapping f can be extended by the relation $f(o) = o^*$. A small neighborhood of o^* in N^n will be homeomorphic to the punctured n-dimensional ball. Since $n \geq 3$, it is a simply connected domain.

This Proposition can be completed by the following

Remark 3 *Under the conditions of Proposition 3, the mapping under consideration is either injective in some punctured neighborhood of the singular point, or it has no (finite or infinite) asymptotic values at the singular point.*

Note that what is essential for the proof of Theorem 1 (besides the condition $n \geq 3$) is that the family of all curves ending at the singular point o is of conformal capacity zero and that its image under the mapping f is a family of conformal capacity zero as well.

This observation allows to extend the class of immersions f under consideration. For instance, one can allow f to be quasiconformal only in the mean. Besides, this observation leads to the following generalization of Theorem 1.

Theorem 3 (on quasiconformal immersions of manifolds of conformally parabolic type)

Let U^n be n-dimensional Riemannian manifold with a compact (possibly empty) boundary ∂U^n. Let U^n be of conformally parabolic type and let $f : U^n \to N^n$ be quasiconformal immersion of U^n to n-dimensional Riemannian manifold N^n.

If $n \geq 3$, then each germ f_p^{-1} of locally inverse mapping can be continued along every path that starts at the point p in N^n and does not intersect neither the image $f(\partial U^n)$ of the boundary nor some (possibly empty) singular set S of conformal capacity (and of metric dimension) zero.

Conformal parabolicity of a manifold implies (see [ZK]) that the conformal capacity of the manifold at infinity is equal to zero. The latter property is equivalent to the following one: conformal capacity of the family of all curves in U^n going to infinity (i.e. eventually leaving any compact set in U^n) is equal to zero.

This allows us to apply Star Lemma. Using the same method as in the proof of the Main Theorem, we construct a univalent branch of the inverse mapping. This branch is defined in a simply connected domain V everywhere outside of a singular set S_V of conformal capacity zero.

The metric (Hausdorff) dimension of such set S_V is equal to zero (see [Fu]). In contrast with the case of $n = 2$, in the case of dimension $n \geq 3$ we can remove the set S_V from V without affecting the fundamental group. Thus V S_V is simply connected (provided V was).

A countable union of the zero capacity sets is a set of capacity zero as well. One can cover N^n (or being more precise, $N^n \setminus f(\partial U)$) with a countable collection of sets of the type V. The union of corresponding singular sets S_V form the set S of conformal capacity zero.

By construction, each path in N^n starting at a point $p \in f(V) \subset N^n$ and not intersecting neither $f(\partial U)$ nor S can be covered by means of f by a path in U^n starting at any point $f^{-1}(p) \in U^n$.

4 Proof of the Star Lemma

We need the following result from [Z1] concerning conformal capacity. In [Z1] it was formulated and proved for the case of dimension $n = 3$ but both the formulation and the proof hold for any finite dimension $n \geq 2$.

Lemma 2 (on an estimate of conformal modulus) *Let $U_{x \in I} D(x, r(x))$ be a union of $(n-1)$-dimensional disks $D(x, r(x))$ (x center, $r(x)$ radius) situated on the parallel hyperplanes orthogonal to the interval I in the space $\mathbb{R}^n$ $(n \geq 2)$.*

On the $(n-2)$-dimensional boundary of each disk $D(x, r(x))$ of finite radius fix a point $x^* = x^*(x)$ and consider the family $\Gamma(x)$ of all curves that join x and $x^*(x)$ in $D(x, r(x))$. Let $\Gamma = \cup_{x \in I} \Gamma(x)$. Then

$$\mathrm{mod}\ \Gamma \geq c(n)\ \frac{mesE_r}{r}\ ,$$

where $E_r := \{x \in I \mid r(x) < r\}$, $mesE_r$ is the length (inner 1-measure) of $E_r \subset I$, and $c(n)$ is a positive constant depending only on n.

Now we can prove the Star Lemma formulated in section 1.

Assume the converse, i.e., suppose that the image of Δ under described projection to the sphere $S = \partial B(o, \varepsilon)$ contains a connected component δ different from a point. Then one can choose an interval $I \subset \mathbb{R}^n$ centered at $o \in \mathbb{R}^n$ and such that the image of δ under the orthogonal projection to the straight line containing the interval I covers I. Consider $(n-1)$-dimensional cones with the vertex o and with direction I as the axis. Every such conic hypersurface is completely determined by the angle between the axis and an element (generatrix) of the cone. By construction, there is an interval $[\alpha, \beta]$, $0 < \alpha < \beta < \pi/2$ such that each conic hypersurface corresponding to an angle from the interval $[\alpha, \beta]$ intersects with the set $\delta \subset S$. Consider the following domain $G \subset \mathbb{R}^n$ located outside of the ball $B(o, \varepsilon)$: G is bounded by the conic surfaces corresponding to angles α and β, and by the part

S_α^β of the sphere S cut out by these two conic surfaces. The domain G allows an explicit axially symmetric quasiconformal mapping that transforms the conic surfaces to parallel hyperplanes and S_α^β to a cylindrical surface Z with generatrix parallel to I. Conformal modulus of a family of curves is a quasiconformal quasiinvariant. Hence, if mod $\Gamma = 0$ for the initial family Γ of curves that join S and Δ, then the same is true for the subfamily $\Gamma_G \subset \Gamma$ of Γ in G, and for the image $\tilde{\Gamma}_G$ of Γ_G under the quasiconformal mapping described above. However, we shall show now that if δ is not a one point set, then mod $\tilde{\Gamma}_G > 0$.

Indeed, consider the subfamily $\bar{\Gamma}$ of $\tilde{\Gamma}_G$ formed by the curves situated in the parallel hypersurfaces obtained above from the conic surfaces. Since $\bar{\Gamma} \subset \tilde{\Gamma}_G$, we have $\operatorname{mod}\bar{\Gamma} \le \operatorname{mod}\tilde{\Gamma}_G$.

On the other hand the family $\bar{\Gamma}$ can be considered as obtained by shortening of curves of the family Γ appeared in Lemma 2. Therefore mod $\bar{\Gamma} \ge$ mod Γ. But this relation together with the inequality of Lemma 2 lead to relation $\operatorname{mod}\bar{\Gamma} > 0$ under the assumption that δ is a nonempty set different from a point. The obtained contradiction completes the proof of Star Lemma.

5 Some corollaries

5.1 The global homeomorphism theorem

The theorem mentioned in the title of the subsection describes (see [Z1], [Z2]) the following specifically multidimensional phenomenon:

For $n \ge 3$ every locally homeomorphic quasiconformal mapping (immersion) $f : \mathbb{R}^n \to \mathbb{R}^n$ must be injective and $f(\mathbb{R}^n) = \mathbb{R}^n$.

We can now obtain this theorem as a corollary of the Main Theorem 1 proved above.

Indeed, up to a stereographic projection we may assume that we deal with an immersion $\tilde{f} : \dot{S}^n \to \dot{S}^n \subset S^n$ of the punctured n-dimensional sphere $\dot{S}^n$ to itself or to the sphere S^n. By Theorem 1, each germ of the inverse mapping can be continued along every path in S^n that does not pass through at most one distinguished point of S^n.

Thus, because of $n \ge 3$, the obtained branch of the inverse mapping is univalent outside this point.

Hence, the initial mapping $\tilde{f}$ is injective and $\tilde{f}(\dot{S}^n)$ is homeomorphic to the punctured sphere. Returning to the mapping f and using $f(\mathbb{R}^n) \subset \mathbb{R}^n$ we conclude that f is a global homeomorphism and $f(\mathbb{R}^n) = \mathbb{R}^n$.

5.2 Gromov's geometric version of the global homeomorphism theorem.

M. Gromov has generalized the global homeomorphism theorem (injectivity property) to quasiconformal immersions of Riemannian manifolds in the following form:

If $f : M^n \to N^n$ is a quasiconformal immersion of a complete Riemannian manifold M^n of finite volume to a simply connected one N^n of the same dimension n, then for $n \geq 3$ the mapping f is injective and $dim(N^n \setminus f(M^n)) = 0$.

Here $dim(N^n \setminus f(M^n))$ is the metric (Hausdorff) dimension of the omitted set and M^n, N^n are connected.

To prove this statement one can use the generalization of the Main Theorem described in section 4. Indeed, it can easily be checked that if $\mathrm{vol}\, M^n$ is finite, then the manifold M^n is of conformally parabolic type. Hence, by Theorem 3, each germ of the inverse mapping can be continued along every path in N^n that does not pass through a set S of conformal capacity equal to zero.

By assumption, N^n is a simply connected manifold, and since $n \geq 3$, $N^n \setminus S$ is simply connected as well. Therefore the obtained branch of the inverse mapping is univalent. Thus, the initial mapping f is injective.

By the construction, the set $N^n \setminus f(M^n)$ is a set of conformal capacity equal to zero. Thus, its metric dimension is equal to zero (a well-known general fact, see [Fu]).

5.3 A Picard-type theorem for quasiconformal immersions

Now we prove Theorem 2, a Picard-type alternative, related to the specific behavior of quasiconformal immersions at an isolated singular point provided that the dimension of the manifolds under consideration is greater than two.

Analyzing the proof of Theorem 1 we have made a useful Remark 3. It says that if the quasiconformal immersion f is not injective in any punctured neighborhood of the singular point o, then f has no asymptotic values at o.

To complete the proof of Theorem 2 it remains to quote Proposition 1 from section 2.

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