Gromov-Witten Classes, Quantum Cohomology and Enumerative Geometry

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ABSTRACT

The paper is devoted to the mathematical aspects of topological quantum field theory and its applications to enumerative problems of algebraic geometry. In particular, it contains an axiomatic treatment of Gromov–Witten classes, and a discussion of their properties for Fano varieties. Cohomological Field Theories are defined, and it is proved that tree level theories are determined by their correlation functions. Application to counting rational curves on del Pezzo surfaces and projective spaces are given.

§1. Introduction

Let V be a projective algebraic manifold.

Methods of quantum field theory recently led to a prediction of some numerical characteristics of the space of algebraic curves in V, especially of genus zero, eventually endowed with a parametrization and marked points. It turned out that an appropriate generating function Φ whose coefficients are these numbers has a physical meaning ("potential", or "free energy"), and its analytical properties can be guessed with such a precision that it becomes uniquely defined. In particular, when V is a Calabi–Yau manifold, Φ conjecturally describes a variation of Hodge structure of the mirror dual manifold in special coordinates (see contributions in [Y], [Ko], [Ma2]) which identifies Φ as a specific combination of hypergeometric functions.

In this paper, we use a different tool, the so called "associativity" relations, or WDVV–equations (see [W], [D]), in order to show that for Fano manifolds these equations tend to be so strong that they can define Φ uniquely up to a choice of a finite number of constants. (For Calabi–Yau varieties these equations hold as well, but they do not constraint Φ to such extent).

Mathematically, this formalism is based upon the theory of *the Gromov–Witten classes.* In our setup, they form a collection of linear maps $I^V_{g,n,\beta}: H^*(V,\mathbf{Q})^{\otimes n} \to H^*(\overline{M}_{g,n},\mathbf{Q})$ that ought to be defined for all integers $g \geq 0, n+2g-3 \geq 0$, and homology classes $\beta \in H_2(V,\mathbf{Z})$ and are expected to satisfy a series of formal properties as well as geometric ones. (Here $\overline{M}_{g,n}$ is the coarse moduli space of stable curves of genus g with n marked points).

In §2 of this paper, we compile a list of these formal properties, or "axioms" (see subsections 2.2.0–2.2.8), and explain the geometric intuition behind them (2.3.0–2.3.8). This is an elaboration of Witten's treatment [W].

Unfortunately, the geometric construction of these classes to our knowledge has not been given even for $V = \mathbf{P}^1$. The most advanced results were obtained for $g = 0$

by the techniques of symplectic geometry going back to M. Gromov (see [R], [RT]), but they fall short of the complete picture. In 2.4 we sketch an algebro–geometric approach to this problem based upon a new notion of *stable map* due to one of us (M. K.)

The axiomatic treatment of §2 in principle opens a way to prove this existence formally, at least for some Fano varieties V and $g = 0$. This is the content of §3 and the Reconstruction Theorem 3.1, which basically says that Gromov–Witten classes in certain situations can be recursively calculated. However the equations determining these classes form a grossly overdetermined family, so that checking compatibility at each step presents considerable algebraic difficulties. The Second Reconstruction Theorem 8.8 shows that it suffices to check this compatibility for codimension zero classes. This allows one to extend the construction of [RT] from codimension zero to all tree level GW–classes.

The subject matter of §4 is the beautiful geometric picture encoded in the potential function Φ constructed with the help of zero–codimensional Gromov–Witten classes of genus zero. Namely, over a convergence subdomain $M \subset H^*(V)$ (the cohomology space being considered as a linear supermanifold) Φ induces the following structures:

a). A structure of the (super)commutative associative algebra with identity on the (fibers of the) tangent bundle $\mathcal{T}H^*(V)$ depending on the point $\gamma \in H^*(V)$. The fibers $\mathcal{T}_\gamma H^*(V)$ were called by Vafa "quantum cohomology rings" of V.

b). A flat connection on $\mathcal{T}H^*(V)$ which was used by B. Dubrovin [D] in order to show that the associativity equations constitute a completely integrable system.

c). An extended connection on $\mathcal{T}H^*(V)$ lifted to $H^*(V) \times \mathbf{P}^1$ and its partial Fourier transform which may define a variation of Hodge structure.

We show that the axioms for the Gromov–Witten classes imply all the properties of Φ postulated in [D].

Together, §2 and §4 can be considered as a pedagogical attempt to present the formalism of correlation functions of topological sigma–models in a form acceptable for mathematicians with algebro–geometric background.

A more ambitious goal of our treatment is to define a framework for the conjectural interpretation of $H^*(V)$ as an extended moduli space (see [Ko] and Witten's contribution to [Y]).

In §5 we discuss examples. Since from the enumerative geometry viewpoint the logic of this discussion is somewhat convoluted, we try to describe it here.

Assuming the existence of the relevant Gromov–Witten classes we calculate the potential Φ and give the recursive formulas for its coefficients whenever feasible. *Assuming* in addition that these classes can be constructed and/or interpreted along the lines of §2, we state the geometric meaning of these numbers.

On the other hand, the potential Φ can be directly *defined* by using the (numerical version of the) Reconstruction Theorem. Then the redundancy of the associated equations translates into a family of strange number–theoretical identities. In principle, they can be also checked directly, without recourse to the geometric context in which they arose. Until this is done, they remain conjectural. We discuss del Pezzo surfaces from this angle (cf. also [I]).

The last three sections are devoted to a description of a less constrained structure of Cohomological Field Theory. Roughly speaking, we forget about the dependence of our theory on the target manifold V, and retain only its part dealing with moduli spaces. In §6, we give two definitions of a CohFT and prove their equivalence. One is modelled upon the axiomatics of Gromov–Witten classes, another is based upon (a version of) operads.

This formalism is used in §7 for a description of the cohomology of moduli spaces of genus zero. Keel in [Ke] described its ring structure in terms of generators, the classes of boundary divisors, and relations between them. We need more detailed understanding of linear relations between homology classes of boundary strata of any codimension, and derive from Keel's result a complete system of such relations. (E. Getzler informed us that he and R. Dijkgraaf obtained similar results).

Finally, in §8 we prove the second Reconstruction Theorem, which allows us, in particular, to classify Cohomology Field Theories via solutions of WDVV-equations, and to formally prove the existence of GW–classes e.g., for projective spaces. This theorem can be viewed as an instance of a general principle that a quantum field theory can be completely recovered from the collection of its Green functions.

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§2. Gromov–Witten classes

2.1. Setup. Let V be a projective algebraic manifold over $\mathbf{C}$ with canonical class K_V.

Denote by $B \subset H_2(V,\mathbf{Z})$ the semigroup consisting of homology classes β such $(L.\beta) \geq 0$ for all Kähler L.

In what follows, we will often consider cohomology classes as represented by differential forms, and then write e.g. $\int_C c_1(T(V))$ instead of $(-K_V.C)$. Cup product is denoted $\wedge$.

2.2. Definition. A *system* (resp. *tree level system*) *of Gromov–Witten (GW) classes* for V is a family of linear maps

$$I_{g,n,\beta}^V : \; H^*(V,\mathbf{Q})^{\otimes n} \to H^*(\overline{M}_{g,n},\mathbf{Q}) \tag{2.1}$$

defined for all $g \geq 0, n \geq 0, n+2g-3 \geq 0$ (resp. $g=0, n\geq 3$) and satisfying the following axioms.

2.2.0. Effectivity. $I_{g,n,\beta}^V = 0$ for $\beta \notin B$.

2.2.1. S_n–covariance. The symmetric group S_n acts upon $H^*(V,\mathbf{Q})^{\otimes n}$ (considered as superspace via $\mathbf{Z}$ mod 2 grading) and upon $\overline{M}_{g,n}$ via renumbering of marked points. The maps $I_{g,n,\beta}^V$ must be compatible with this action.

2.2.2. Grading. For $\gamma \in H^i$, put $|\gamma| = i$. The map $I_{g,n,\beta}^V$ must be homogeneous of degree $2(K_V.\beta) + (2g-2)\dim_{\mathbf{C}} V$, that is

$$|I_{g,n,\beta}^V(\gamma_1 \otimes \cdots \otimes \gamma_n)| = \sum_{i=1}^n |\gamma_i| + 2(K_V.\beta) + (2g-2)\dim_{\mathbf{C}} V. \tag{2.2}$$

Before stating the remaining axioms, let us introduce the following terminology. Call a GW-class *basic* if it corresponds to the least admissible values of (n,β) that is, belongs to the following list:

$$I_{0,3,\beta}^V(\gamma_1 \otimes \gamma_2 \otimes \gamma_3);\; I_{1,1,\beta}^V(\gamma);\; I_{g,0,\beta}^V(1),\; g \geq 2, \tag{2.3}$$

where 1 is the canonical generator of $H^*(V,\mathbf{Q})^{\otimes 0} = \mathbf{Q}$. Call a class *new* if it is not basic, and if among its (homogeneous) arguments γ_i there are none with $|\gamma| = 0$ or 2. Finally, call the number

$$2(3g-3+n) - |I_{g,n,\beta}^V(\gamma_1 \otimes \cdots \otimes \gamma_n)|$$

the *codimension* of the class (recall that $\dim_{\mathbf{C}} \overline{M}_{g,n} = 3g-3+n$). The classes of codimension zero are especially important and are expected to express the number of solutions of some counting problems (see 2.3 below). Instead of such a class $I_{g,n,\beta}^V$ we will often consider the corresponding number $\langle I_{g,n,\beta}^V \rangle$ defined by

$$\langle I_{g,n,\beta}^V \rangle(\gamma_1 \otimes \cdots \otimes \gamma_n) = \int_{\overline{M}_{g,n}} I_{g,n,\beta}^V(\gamma_1 \otimes \cdots \otimes \gamma_n). \tag{2.4}$$

Notice the following facts:

a). For $g=0$, all basic classes have codimension zero, because $\dim_{\mathbf{C}}\overline{M}_{0,3}=0$.

b). For a non-vanishing class, we have from (2.2):

$$(-K_V.\beta)-n-(g-1)\dim_{\mathbf{C}}V \le \frac{1}{2}\sum_{i=1}^{n}(|\gamma_i|-2)\le$$

$$(-K_V.\beta)-(g-1)(\dim_{\mathbf{C}}V-3), \tag{2.5}$$

with the second equality sign for codimension zero classes. If the class is new, and $H^1(V)=0$, the middle term of (2.5) must be non-negative, and even $\geq n$, if $H^3(V)=0$. Hence for $K_V=0, \dim_{\mathbf{C}}V=3$, (e.g., Calabi–Yau threefolds) there are no non-vanishing new classes. For Fano varieties, this inequality bounds n if β is fixed, and β if n is fixed.

The next two axioms partially reduce the calculation of GW-classes to that of basic ones and new ones.

2.2.3. Fundamental class. Let $e_V^0\in H^0(V)$ be the identity in the cohomology ring (i.e. its dual homology class is the fundamental class $[V]$). If the following class is not basic, we have

$$I^V_{g,n,\beta}(\gamma_1\otimes\cdots\otimes\gamma_{n-1}\otimes e_V^0)=\pi_n^* I^V_{g,n-1,\beta}(\gamma_1\otimes\cdots\otimes\gamma_{n-1}) \tag{2.6}$$

where $\pi_n:\ \overline{M}_{g,n}\to\overline{M}_{g,n-1}$ is the map forgetting the n–th section. In particular, (2.6) cannot be of codimension zero unless it vanishes.

In addition, for basic classes with argument e_V^0 we have

$$\langle I^V_{0,3,\beta}\rangle(\gamma_1\otimes\gamma_2\otimes e_V^0)=\begin{cases}0, \text{ if } \beta\ne 0;\\ \int_V\gamma_1\wedge\gamma_2, \text{ if } \beta=0.\end{cases} \tag{2.7}$$

2.2.4. Divisor. If $I^V_{g,n,\beta}(\gamma_1\otimes\cdots\otimes\gamma_n)$ is a non-basic codimension zero class with the last argument γ_n, $|\gamma_n|=2$, then

$$\langle I^V_{g,n,\beta}\rangle(\gamma_1\otimes\cdots\otimes\gamma_n)=\int_\beta\gamma_n\cdot\langle I^V_{g,n-1,\beta}\rangle(\gamma_1\otimes\cdots\otimes\gamma_{n-1}).$$

More generally, for non-necessarily codimension zero classes,

$$\pi_{n*}(I^V_{g,n,\beta}(\gamma_1\otimes\cdots\otimes\gamma_n))=\int_\beta\gamma_n\cdot I^V_{g,n-1,\beta}(\gamma_1\otimes\cdots\otimes\gamma_{n-1}).$$

2.2.5. Mapping to point. By this catchword we describe the situation when $\beta=0$.

For $g=0$, the situation is simple: the only non-vanishing classes must have $\sum_{i=1}^n|\gamma_i|=2\dim_{\mathbf{C}}V$, and

$$I^V_{0,n,0}(\gamma_1\otimes\cdots\otimes\gamma_n)=(\int_V\gamma_1\wedge\cdots\wedge\gamma_n)e^0_{\overline{M}_{0,n}}, \tag{2.8}$$

where $e^0_{\overline{M}_{0,n}}$ is the identity in $H^*(\overline{M}_{0,n})$.

For $g = 1$, the non-vanishing classes have $\sum_{i=1}^n |\gamma_i| = 0$ or 2 and therefore are lifts of the following two basic ones:

$$I^V_{1,1,0}(e^0_V) = \chi(V) e^0_{\overline{M}_{1,1}}, \tag{2.9}$$

$$|\gamma| = 2: \; I^V_{1,1,0}(\gamma) = (\int_V \mathcal{X}(V) \wedge \gamma) e^2_{\overline{M}_{1,1}}, \tag{2.10}$$

where $\chi(V) = \sum_i (-1)^i h^{i,0}(V)$, $e^2_{\overline{M}_{1,1}} = c_1(\mathcal{O}(1))$ (recall that $\overline{M}_{1,1} \simeq \mathbf{P}^1$), and $\mathcal{X}(V)$ is a certain characteristic class of the tangent bundle of V whose explicit description we postpone to 2.4.2.

Finally if $g \geq 2$, then $I^V_{g,n,0} = 0$ unless $\dim_{\mathbf{C}} V \leq 3$, and non-vanishing classes again can be described explicitly.

2.2.6. Splitting. Fix g_1, g_2 and n_1, n_2 such that $g = g_1 + g_2$, $n = n_1 + n_2$, $n_i + 2g_i - 2 \geq 0$. Fix also two complementary subsets $S = S_1, S_2$ of $\{1, \ldots, n\}$, $|S_i| = n_i$. Denote by $\varphi_S : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n}$ the canonical map which assigns to marked curves $(C_i; x^{(i)}_1, \ldots x^{(i)}_{n_i+1}), i = 1, 2$, their union $C_1 \cup C_2$, with $x^{(1)}_{n_1+1}$ identified to $x^{(2)}_1$. The remaining points are then renumbered by $\{1, \ldots n\}$ in such a way that their relative order is kept intact, and points on C_i are numbered by S_i.

Finally, choose a homogeneous basis $\{\Delta_a | a = 1, \ldots, D\}$ of $H^*(V, \mathbf{Q})$ and put $g_{ab} = \int_V \Delta_a \wedge \Delta_b, (g^{ab}) = (g_{ab})^{-1}$.

The Splitting Axiom now reads:

$$\varphi^*_S(I^V_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)) =$$

$$\varepsilon(S) \sum_{\beta = \beta_1 + \beta_2} \sum_{a,b} I^V_{g_1,n_1+1,\beta_1}((\otimes_{j \in S_1} \gamma_j) \otimes \Delta_a) g^{ab} \otimes I^V_{g_2,n_2+1,\beta_2}(\Delta_b \otimes (\otimes_{j \in S_2} \gamma_j)), \tag{2.11}$$

where $\varepsilon(S)$ is the sign of the permutation induced by S on $\{\gamma_j\}$ with odd $|\gamma|$.

Notice that $\sum_{a,b} \Delta_a \otimes \Delta_b g^{ab}$ is the class of the diagonal in $H^*(V \times V)$.

The sum in (2.11) is finite because of the Effectivity Axiom 2.2.0.

2.2.7. Genus reduction. Denote by $\psi : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ the map corresponding to glueing together the last two marked points. Then

$$\psi^*(I^V_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)) = \sum_{a,b} I^V_{g-1,n+2,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes \Delta_a \otimes \Delta_b) g^{ab}. \tag{2.12}$$

2.2.8. Motivic Axiom. The maps $I^V_{g,n,\beta}$ are induced by certain correspondences in the Chow rings:

$$C^V_{g,n,\beta} \in C^*(V^n \times \overline{M}_{g,n}). \tag{2.13}$$

We recall that $c \in C^*(V \times W)$ induces the map

$$H^*(V) \to H^*(W): \ \gamma \mapsto \pi_{W*}(\pi_V^*(\gamma) \wedge [c]), \tag{2.14}$$

where π_V, π_W are projections of $V \times W$ to V, W and $[c]$ is the cohomology class of c.

2.3. Comments to the Definition. When Gromov–Witten classes are considered in the literature in an algebraic context, it is usually assumed that V is either Fano ($-K_V$ ample), or Calabi-Yau ($K_V = 0$) so that $(-K_V.\beta) \geq 0$ is automatically satisfied for all algebraic homology classes. However, nothing in the formalism forces us to postulate it. Notice that for manifolds of general type (2.5) implies vanishing of all $I^V_{g,n,\beta}$ with large enough β.

The simplest example of a tree level system of GW–classes on V is:

$$I^V_{0,n,\beta}(\gamma_1 \dots \gamma_n) = \begin{cases} 0, & \text{if } \beta \neq 0, \\ (\int_V \gamma_1 \wedge \dots \wedge \gamma_n) e^0_{\overline{M}_{0,n}} & \text{otherwise.} \end{cases} \tag{2.15}$$

In addition, any system of GW–classes posesses an obvious scaling transformation (if we allow to extend coefficients $\mathbf{Q}$ to $\mathbf{C}$): $I^V_{0,n,\beta} \mapsto e(\beta) I^V_{0,n,\beta}$ where $e: \ B \to \mathbf{C}^*$ is a semigroup homomorphism. If we put $e(\beta) = \exp(-t(\omega.\beta))$ for a Kähler class ω, then the scaling of any initial tree level system tends to (2.15) as $t \to \infty$. In terms of quantum cohomology (see 4.5 below) (2.15) gives rise to the classical cohomology ring, whereas $I^V_{0,n,\beta}$ supply the instanton corrections.

Intuitively, one can imagine the geometry behind these corrections $I^V_{g,n,\beta}$ as follows. For $\gamma_i \in H^*(V, \mathbf{Z})$, choose some generic representatives Γ_i of dual homology classes in V. Consider an appropriate space of triples $\{f, C; x_1, \dots, x_n\}$ where C is a curve of genus g with n marked points x_i, and $f: \ C \to V$ is an algebraic map such that $f_*([C]) = \beta$ and $f(x_i) \in \Gamma_i$ for all i. The projection of this space to $\overline{M}_{g,n}$, under some genericity conditions, must be the cycle dual to the cohomology class $I^V_{g,n,\beta}$.

As we have already remarked, the most powerful known constructions of the classes $I^V_{g,n,\beta}$ leave the domain of algebraic or even complex geometry, in order to satisfy the necessary genericity assumptions. The whole subject seems to belong rather to symplectic topology: cf. [R]. To our knowledge, only a part of the picture of Def. 2.2 is at the moment rigorously established even in this wider context.

We will use the naive picture described above in order to motivate the expected properties of GW-classes.

2.3.0. Effectivity. The meaning of this axiom is obvious.

2.3.1. Grading. The condition (2.2) expresses the following genericity assumptions. Firstly, the space of maps $f: \ C \to V$ landing at β must have the complex dimension of its first order infinitesimal approximation at a point, that is $H^0(C, f^*(N_{C/V}))$ which in turn must be unobstructed and coincide with $\chi(C, f^*(N_{C/V}))$. By Riemann–Roch,

$$\chi(C, f^*(N_{C/V})) = (-K_V.\beta) + (1-g)\dim_{\mathbf{C}} V.$$

Secondly, when we constrain $f(C)$ to intersect all Γ_i's, this diminishes (real) dimension by $\sum_i \dim_{\mathbf{R}}\Gamma_i$.

From this discussion it is clear that zero-codimensional classes, or rather numbers $\langle I^V_{g,n,\beta}\rangle$ morally count curves constrained by incidence conditions to such a degree that only a finite numbers of such curves occur "generically". For instance, the number of curves of given degree d on $\mathbf{P}^2$ passing through $3d-1$ fixed points in general position and having additional $(d-1)(d-2)/2$ double points elsewhere must actually coincide with $\langle I^{\mathbf{P}^2}_{0,3d-1,d\beta_0}\rangle(e^{\otimes 3d-1})$ where β_0 is the homology class of a line, and e the dual cohomology class of a point. However, in more complex situations such naive counting may be totally misleading.

We nevertheless use this language, but it should not be taken too literally.

2.3.2. S_n–covariance. The meaning of this Axiom is obvious. Notice that codimension zero classes are simply S_n–symmetric.

2.3.3. Fundamental class. The dual cycle to e^0_V is V. Therefore, the l.h.s. of (2.6) imposes no constraints on the n–th point of C. The r.h.s. expresses this in terms of moduli space.

2.3.4. Divisor. If $I^V_{g,n,\beta}(\gamma_1\otimes\dots\gamma_n)$ is zero–codimensional class, the same is true for $I^V_{g,n-1,\beta}(\gamma_1\dots\gamma_{n-1})$. Hence the l.h.s. of (2.4) (resp. the integral at r.h.s.) counts the number of marked curves passing through $\Gamma_1,\dots,\Gamma_n$ (resp. $\Gamma_1,\dots,\Gamma_{n-1}$). But the two problems differ only by the additional presence of the n–th point in the l.h.s. which may be chosen among intersection points of $f(C)$ and Γ_n. Their number is $\int_\beta \gamma_n$.

2.3.5. Mapping to point. If $\beta = f_*([C]) = 0$, then f maps C to a point which is constrained to belong to $\Gamma_1\cap\dots\cap\Gamma_n$; otherwise the curve and the marked points on it are arbitrary. This justifies the axiom (2.8) for $g=0$.

Unfortunately, for $g\ge 1$ this reasoning is too naive, and dealing with this very degenerate situation requires some sophistication. In 2.4.4 below we sketch an argument giving simultaneously (2.8)–(2.10) and formulas for $g\ge 2$.

2.3.6. Splitting. In the picture described above, the l.h.s. of (2.11) can be represented by a cycle on $\overline{M}_{g_1,n_1+1}\times\overline{M}_{g_2,n_2+1}$ which is the sum of cycles corresponding to various splittings $\beta=\beta_1+\beta_2$. When β_1,β_2 are chosen, we must consider pairs of maps $f_1: C_1\to V\times V$ and $f_2: C_2\to V\times V$ such that $\pi_2\circ f_1(C_1)$ and $\pi_1\circ f_2(C_2)$ are points, $f_{i*}([C_i]) = \pi^*_{2-i}(\beta_i)$, with the incidence conditions described by $\pi^*_i(\Gamma_i)$, the partition S, and the additional relation $f_1(x^{(1)}_{n_1+1}) = f_2(x^{(2)}_1)$. On $V\times V$, the latter can be expressed by intersecting with diagonal. This leads to the r.h.s. of (2.11).

2.3.7. Genus reduction. A similar reasoning motivates (2.12).

2.3.8. Motivic Axiom. For $g=0$, and at least V with ample tangent sheaf $\mathcal{T}_V$, one can try to construct $C^V_{0,n,\beta}$ directly as follows. Consider the scheme of maps $Map_\beta(\mathbf{P}^1,V)$ such that $f_*([\mathbf{P}^1])=\beta$. Construct the morphism

$$a_n:\ (\mathbf{P}^1)^n_0\times Map_\beta(\mathbf{P}^1,V)\to V^n\times\overline{M}_{0,n},$$

$$a_n(x_1,\dots,x_n;f)=(f(x_1),\dots,f(x_n);(\mathbf{P}^1;x_1,\dots x_n))$$

(here $(\mathbf{P}^1)_0^n$ is $(\mathbf{P}^1)^n$ with deleted diagonals.) The closure of the image of a_n (counted with appropriate multiplicity) is a candidate for $C_{0,n,\beta}^V$.

Generally, this construction is inadequate; but see 2.4 below for a more refined version which hopefully works for certain V.

Anyway, if one takes the Motivic Axiom for granted, then all the other axioms can and should be stated directly in terms of $C_{g,n,\beta}^V$. We will show how to do it for the Splitting Axiom leaving the remaining ones to the reader.

For a product $\prod_{i=1}^n W_i$ and a subset $T\subset\{1,\dots,n\}$, denote by $\pi_T:\ \prod_{i=1}^n W_i\to\prod_{i\in T}W_i$ the projection. Keeping the notation of 2.2.6, consider the following correspondences ($\Delta=$ the diagonal class in $C^*(V\times V)$):

$$C_1=(id_{V^n}\times\varphi_S)^*C_{g,n,\beta}^V\in C^*(V^n\times\overline{M}_{g_1,n_1+1}\times\overline{M}_{g_2,n_2+1}),$$

$$C_2=\sum_{\beta_1+\beta_2=\beta}\pi^*_{\{1,\dots,n_1+2\}}(C^V_{g_1,n_1+1,\beta_1})\pi^*_{\{n_1+3,\dots,n+4\}}(C^V_{g_2,n_2+1,\beta_2})\pi^*_{\{n_1+1,n_1+3\}}(\Delta)\in$$

$$C^*(V^{n_1+1}\times\overline{M}_{g_1,n_1+1}\times V^{n_2+1}\times\overline{M}_{g_2,n_2+1}).$$

Then

$$C_1=\pi_{\{1,2\}*}t^*(C_2),$$

where $t\in S_{n+4}$ is the obvious reshuffling of factors.

We did not start with data (2.14) for two reasons. First, at present, when constructions of GW-classes rely upon Gromov's symplectic methods (which actually work for non-algebraic manifolds), (2.14) looks unnecessarily restrictive. Second, using axioms like the Splitting Axiom directly in terms of correspondences would entail a very clumsy notation, especially in the next section.

2.4. Construction project. Fix an algebraic manifold V as above.

2.4.1. Definition. *A stable map (to V) is a structure $(C;\ \{x_1,\dots,x_n\},f)$ consisting of the following data.*

a). $(C;x_1,\dots,x_n)$, or simply C, is a connected reduced curve with $n\geq 0$ pairwise distinct marked non-singular points and at most ordinary double singular points.

b). $f:\ C\to V$ is a map having no non-trivial first order infinitesimal automorphisms, identical on V and $(x_1,\dots,x_n)$. This means that every component of C of genus 0 (resp. 1) which is contracted by f must have at least 3 (resp. 1) special (i.e., marked or singular) points on its normalization.

For an algebraic cohomology class $\beta\in H_2(V,\mathbf{Z})$, consider the stack $\overline{M}_{g,n}(V,\beta)$ of stable maps to V of n–marked curves of arithmetical genus g such that $f_*([C])=\beta$ for any point $((C;x_1,\dots,x_n),f)$ of this stack. We expect that this stack is proper and separated.

If $n+2g-3\geq 0$, there is a map $\overline{M}_{g,n}(V,\beta)\to V^n\times\overline{M}_{g,n}$ consecutively contracting the non-stable components of C. For $g=0$, it is useful to extend this map putting $\overline{M}_{0,n}=$*point* for $n\leq 2$.

2.4.2. Definition. *V is called convex if for any stable map $f: C \to V$ of genus zero we have $H^1(C, f^*(\mathcal{T}_V)) = 0$.*

Examples. a). Generalized flag spaces G/P are convex.

b). More generally, if for some $n > 0$, $\mathcal{T}_V^{\otimes n}$ is generated by global sections, V is convex.

c). Although in general Fano manifolds are not convex (look at exceptional curves on del Pezzo surfaces), it is conceivable that indecomposable Fano manifolds of sufficiently large anticanonical degree are.

We expect that $\overline{M}_{0,n}(V,\beta)$ is a smooth stack ("an orbifold") whenever V is convex.

2.4.3. Construction. For a convex V, denote by $C^V_{0,n,\beta}$ the image of the fundamental class of $\overline{M}_{0,n}(V,\beta)$ in $C^*(V^n \times \overline{M}_{0,n})$.

2.4.4. Hope. *For convex V, $\{C^V_{0,n,\beta}\}$ is a tree level motivic system of GW-classes.*

The main property to be checked is the Splitting Axiom. It must follow from the natural structure at infinity of $\overline{M}_{0,n}(V,\beta)$ which is parallel to that of $\overline{M}_{0,n}$ (stratification according to the degeneration graph of a curve).

If $g \geq 1$ and/or V is not convex, the fundamental class of $\overline{M}_{g,n}(V,\beta)$ is "too big", and $C^V_{g,n,\beta}$ must be defined as image of a characteristic class of an appropriate "obstruction complex" on this moduli space. We illustrate the arising complications on the "mapping to point" example.

2.4.5. Mapping to point. By Definition 2.4.1, we have

$$\overline{M}_{g,n}(V,0) = \overline{M}_{g,n} \times V.$$

The complex dimension of this space is bigger than the expected one by $g\dim_{\mathbf{C}} V := G$. This is precisely the rank of the locally free obstruction sheaf $\mathcal{T}^{(1)}$ on $\overline{M}_{0,n}(V,\beta)$ whose geometric fiber at the point $[f]$, $f(C) = v \in V$, is

$$H^1(C, f^*(\mathcal{T}_V)) \cong H^1(C, \mathcal{O}_C) \otimes T_v V.$$

Denote by $\mathcal{E}_{g,n}$ the locally free sheaf $R^1\pi_*\mathcal{O}$ on $\overline{M}_{g,n}$ where π is the projection of the universal curve. We have $\mathcal{T}^{(1)} \cong \mathcal{E}_{g,n} \boxtimes \mathcal{T}_V$.

Notice that $\mathcal{E}_{g,n}$ is simply the pullback of one of the basic sheaves $\mathcal{E}_{0,3} \cong 0$, $\mathcal{E}_{1,1}$, $\mathcal{E}_{g,0}$, for $g = 0, 1, \geq 2$ respectively.

Consider now the Euler class $c_G(\mathcal{T}^{(1)})$ and denote by p_1, p_2 the two projections of $\overline{M}_{g,n} \times V$. An intuitive argument appealing to our desire to pass to an unobstructed situation suggests the following definition:

$$I^V_{g,n,0}(\gamma_1 \otimes \cdots \otimes \gamma_n) = p_{1*}(c_G(\mathcal{E} \boxtimes \mathcal{T}_V) \wedge p_2^*(\gamma_1 \wedge \cdots \wedge \gamma_n)).$$

We hope to develop this construction in a separate publication.

We will end our general discussion of GW-classes with the following two constructions.

2.5. Tensor product. Let $\{I^{V_i}_{g,n,\beta}\}$, $i = 1,2$, be two full (or only tree level) GW-systems for V_1, V_2, $H^1(V_i) = 0$. Put

$$I^{V_1\times V_2}_{g,n,\beta_1\oplus\beta_2}((\gamma_1^{(1)}\otimes\gamma_1^{(2)})\otimes\dots\otimes(\gamma_n^{(1)}\otimes\gamma_n^{(2)})) :=$$

$$\varepsilon I^{V_1}_{g,n,\beta_1}(\gamma_1^{(1)}\otimes\dots\otimes\gamma_n^{(1)})\wedge I^{V_2}_{g,n,\beta_2}(\gamma_1^{(2)}\otimes\dots\otimes\gamma_n^{(2)}), \tag{2.16}$$

where ε is the sign of the relevant permutation of odd-dimensional arguments.

2.5.1. Claim. (2.15) is a full (resp. tree level) system of GW-clases for $V_1\times V_2$ which is called *the tensor product* of given systems.

In fact, one easily checks all axioms, including the refined version of the Mapping to Point Axiom.

Notice that even if one is interested only in codimension zero classes of the tensor product, one has to know all classes of the factors. In the tree level setting, they can be in turn be calculated from the codimension zero classes of the factors, but in a highly non-trivial way. In fact, we have:

2.5.2. Proposition. *Let a tree level system of GW-classes $I^V_{0,n,\beta}$ be given for V. Then it can be uniquely reconstructed from its codimension zero subsystem.*

Proof. In fact, consider a class of codimension ≥ 1 with $n\geq 4$. The Splitting Axiom (2.11) allows one to calculate its restrictions to all boundary components of the moduli space (corresponding to lesser values of n). It remains to show that $\cap_S\mathrm{Ker}\ \varphi_S^* = H^{2n-6}(\overline{M}_{0,n})$. In fact, let $d_S\in H^2(\overline{M}_{0,n})$ be the dual class of the boundary component corresponding to the partition S. The whole cohomology ring is generated by these classes (see [Ke]). On the other hand, $\varphi_{S*}\varphi_S^*(\alpha) = \alpha\wedge d_S$ for any class α. And if $\alpha\neq 0$, $|\alpha| < 2(n-3)$, then by Poincaré duality there exists some non-constant monomial d in d_S such that $\int_{\overline{M}_{0,n}}\alpha\wedge d\neq 0$. Hence $\alpha\notin\cap_S\mathrm{Ker}\ \varphi_S^*$.

Summarizing, we may say that the tensor product can be defined on the codimension zero subsystems, but there are no simple formulas for doing it.

2.5.3. Cusp classes. Generally, let us call *cusp classes* those elements of $H^*(\overline{M}_{g,n})$ which vanish on all boundary divisors of this moduli space. For $g\geq 1$, non-trivial cusp classes may exist (e.g. the Ramanujan class for $g=1$, $n=13$). It would be interesting to have examples of GW-classes with non-trivial cusp components.

2.6. Restricted GW-systems. Let C^* be an intersection theory such that a given GW-system $\{I^V_{g,n,\beta}\}$ can be represented by C–correspondences. We have $C^*(V^n)\subset H^*(V)^{\otimes n}$. We will say that the maps $I^V_{g,n,\beta}$ restricted to $C^*(V^n)$ form *the restricted GW-system.* Slightly elaborating the discussion of the Splitting Axiom in 2.2.8, one can convince oneself that all the axioms restricted to $\oplus_n C^*(V^n)$ make sense and can be stated entirely in terms of this restriction, without appealing to extra cohomology classes like Δ_a in (2.11) and (2.12).

This is useful for those enumerative problems where we want to consider incidence conditions stated in terms of algebraic cycles only.

§3. First Reconstruction Theorem

3.1. Theorem. *Let V be a manifold for which a tree level system of GW-classes $\{I^V_{0,n,\beta}\}$ exists.*

If $H^*(V)$ as a ring is generated by $H^2(V)$, then $\{I^V_{0,n,\beta}\}$ can be uniquely reconstructed starting with the following system of codimension zero basic classes:

$$\{I^V_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes \gamma_3) \mid (-K_V.\beta) \le 2\mathrm{dim}_{\mathbf{C}} V + 1;$$

$$\sum_{i=1}^{3} |\gamma_i| = 2(-K_V.\beta) + 2\mathrm{dim}_{\mathbf{C}} V;\ |\gamma_3| = 2\}. \tag{3.1}$$

3.1.1. Comments. We may and usually will choose $|\gamma_i|$ from the elements of a fixed basis of $H^*(V)$. Then, if V is Fano, (3.1) is a finite set because the degree of β is also bounded. For instance, if $V = \mathbf{P}^n$, (3.1) is satisfied only by $\beta = 0$ and $\beta =$ class of a line. The $\beta = 0$ case is settled by (2.8). For the line, (3.1) gives $\{\gamma_1, \gamma_2, \gamma_3\} = \{e^{2n}, e^{2n}, e^2\}$, where $e^{2i} = c_1(\mathcal{O}(1))^i$. Since e^{2n} is the dual class of a point, one can imaginatively say that all enumerative problems about rational curves in $\mathbf{P}^n$ eventually reduce to counting the number of lines passing through two points.

On the other hand, for Calabi–Yau varieties with $K_V = 0$ (3.1) does not restrict β at all. Besides, $H^2(V)$ almost never generates $H^*(V)$. Nevertheless, Theorem 3.1 does say something about this case as well.

a). The algebraic (or Hodge) part of cohomology may be generated by H^2, and the corresponding restricted GW-system can be reconstructed from (3.1).

b). Then Theorem 3.1 says that all tree level classes with algebraic arguments can be reconstructed if one knows all β-contributions to the triple quantum intersection indices. This information is conjecturally supplied by the Mirror family.

3.2. Proof. It will consist of several reduction steps.

3.2.1. Step 1. *Every class $I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)$ of codimension ≥ 1 with $n \ge 4$ can be reconstructed from classes with lesser values of n.*

The proof of the Proposition 2.5.2 shows this.

It remains to deal with codimension zero classes, that is, numbers

$$\langle I^V_{0,n,\beta}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_n),\ \sum_{i=1}^{n} |\gamma_i| = 2(-K_V.\beta) + 2\mathrm{dim}_{\mathbf{C}} V + 2(n-3). \tag{3.2}$$

We start with some preliminaries.

3.2.2. Quadratic relations. Fix some $\{\beta; \gamma_1, \dots \gamma_N\}$, $N \ge 4$, and four pairwise distinct indices $\{i, j, k, l\} \subset \{1, \dots, N\}$. Assume that $I^V_{0,N,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_N)$ has topological codimension two. Take the cup product of this class with the following linear relation between the generators d_S established in [Ke]:

$$\sum_{\{ijSkl\}} d_S = \sum_{\{ikTjl\}} d_T,$$

where $\sum_{\{ijSkl\}}$ means that we sum over all partitions S such that $i,j \in S_1,\ k,l \in S_2$, or vice versa.

Calculate the degrees of all summands using (2.11). We get the following fundamental system of quadratic relations among codimension zero classes:

$$\sum_{\{ijSkl\}} \sum_{\beta_1+\beta_2=\beta} \sum_{a,b} \varepsilon(S)\langle I^V_{0,|S_1|+1,\beta_1}\rangle((\otimes_{r\in S_1})\gamma_r)\otimes\Delta_a)g^{ab}\langle I^V_{0,|S_2|+1,\beta_2}\rangle(\Delta_b\otimes(\otimes_{s\in S_2}\gamma_s)) =$$

$$\sum_{\{ikTjl\}} \sum_{\beta_1+\beta_2=\beta} \sum_{a,b} \varepsilon(T)\langle I^V_{0,|T_1|+1,\beta_1}\rangle((\otimes_{r\in T_1})\gamma_r)\otimes\Delta_a)g^{ab}\langle I^V_{0,|T_2|+1,\beta_2}\rangle(\Delta_b\otimes(\otimes_{s\in T_2}\gamma_s)). \tag{3.3}$$

Now, define a partial order on pairs $(\beta,n),\ \beta\in B,\ n\geq 3$, by setting $(\beta,n) > (\beta',n')$, iff either $\beta=\beta'+\beta'',\ \beta',\beta''\in B,\ \beta''\neq 0$, or $\beta=\beta',\ n>n'$.

Observe that the highest order terms enter in (3.3) linearly. In fact, for these terms we have either $\beta_1=\beta$ or $\beta_2=\beta$. The complementary class, with $\beta_2=0$ (resp. $\beta_1=0$), can be non-zero only if $|S_2|=2$ or $|T_2|=2$ (resp. $|S_1|=2$ or $|T_1|=2$): see (2.8). Hence there are four possibilities: $S_1=\{i,j\}; S_2=\{k,l\}; T_1=\{i,k\}; T_2=\{j,l\}$.

Let us look, say, at the first group of highest terms:

$$\varepsilon(S)\sum_{a,b}\langle I^V_{0,3,0}\rangle(\gamma_i\otimes\gamma_j\otimes\Delta_a)g^{ab}\langle I^V_{0,n-1,\beta}\rangle(\Delta_b\otimes(\otimes_{s\neq i,j}\gamma_s)). \tag{3.4}$$

We have by (2.8):

$$\langle I^V_{0,3,0}\rangle(\gamma_i\otimes\gamma_j\otimes\Delta_a)=\int_V\gamma_i\wedge\gamma_j\wedge\Delta_a.$$

Since $\langle I^V_{0,N-1,\beta}\rangle$ is (poly)linear, we can rewrite (3.4) as

$$\varepsilon(S)\langle I^V_{0,N-1,\beta}\rangle(\sum_{a,b}(\int_V\gamma_i\wedge\gamma_j\wedge\Delta_a)g^{ab}\Delta_b\otimes(\otimes_{s\neq i,j}\gamma_s))=$$

$$\varepsilon(S)\langle I^V_{0,N-1,\beta}\rangle(\gamma_i\wedge\gamma_j\otimes(\otimes_{s\neq i,j}\gamma_s)). \tag{3.5}$$

Using analogs of (3.5) for all four groups of highest order terms we can finally write (3.3) as

$$\pm\langle I^V_{0,N-1,\beta}\rangle(\gamma_i \wedge \gamma_j \otimes (\otimes_{s\neq i,j}\gamma_s)) \pm \langle I^V_{0,N-1,\beta}\rangle(\gamma_k \wedge \gamma_l \otimes (\otimes_{s\neq k,l}\gamma_s))$$

$$\pm\langle I^V_{0,N-1,\beta}\rangle(\gamma_i \wedge \gamma_k \otimes (\otimes_{s\neq i,k}\gamma_s)) \pm \langle I^V_{0,N-1,\beta}\rangle(\gamma_j \wedge \gamma_l \otimes (\otimes_{s\neq j,l}\gamma_s)) =$$

$$\text{a quadratic combination of lower order terms.} \tag{3.6}$$

3.2.3. Step 2. *Every class $I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)$ of codimension zero with $n \geq 4$ can be reconstructed from basic classes (with $n = 3$).*

In fact, it suffices to calculate numbers $\langle I^V_{0,n,\beta}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_n)$ for $n \geq 4$ and

$$2\dim_{\mathbf{C}}(V) \geq |\gamma_1| \geq \cdots \geq |\gamma_n| \geq 4$$

(if $|\gamma_n| = 2$, we can apply the Divisor Axiom to reduce n). We will now for the first time use the assumption that $H^*(V)$ is generated by $H^2(V)$ and write $\gamma_n = \sum_i \delta_i \wedge \delta'_i$ for some δ_i, δ'_i with $|\delta'_i| = 2$. Clearly, it suffices to treat the case $\gamma_n = \delta \wedge \delta', |\delta'| = 2$. Apply the construction of 3.2.2 to the codimension two class

$$I^V_{0,n+1,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta \otimes \delta')$$

and indices $\{i,j,k,l\} = \{1,2,n,n+1\}$. Relation (3.6) becomes

$$\pm\langle I^V_{0,n,\beta}(\gamma_1 \wedge \gamma_2 \otimes \gamma_3 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta \otimes \delta') \pm \langle I^V_{0,n,\beta}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta \wedge \delta')$$

$$\pm\langle I^V_{0,n,\beta}\rangle(\gamma_1 \wedge \delta \otimes \gamma_2 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta') \pm \langle I^V_{0,n,\beta}\rangle(\gamma_1 \otimes \gamma_2 \wedge \delta' \otimes \gamma_3 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta) =$$

$$\text{a quadratic combination of lower terms.} \tag{3.7}$$

Now, the second summand in (3.7) is our initial class $\langle I^V_{0,n,\beta}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes \gamma_n)$. The first and the third summands are lifted from $\overline{M}_{0,n-1}$because of the Divisor Axiom. Finally, in the fourth summand the last argument is of lesser dimension than in the initial class: $|\delta| < |\gamma_n|$. If $|\delta| = 2$, it is lifted from $\overline{M}_{0,n-1}$; if $|\delta| > 2$, we can repeat the same trick applying it to this summand. In a finite number of iterations, we will reduce n.

3.2.4. Step 3. *Every basic class $I^V_{0,3,\beta}$ can be calculated via those with restrictions (3.1).*

In fact, if $|\gamma_3| \geq 4$ in $I^V_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes \gamma_3)$, $\beta \neq 0$, (the $\beta = 0$ case is given by (2.8)), then we can apply to this class the reduction procedure described above and diminish $|\gamma_3|$. The remaining conditions follow from the Grading Axiom.

§4. Potential, associativity relations, and quantum cohomology

4.1. Setup. let M be a supermanifold endowed with a tensor g of rank two and a tensor A of rank three. To fix our sign conventions, it is convenient to choose a (local) coordinate system $\{x^a\}$ which defines the basis $\partial_a = \partial/\partial x^a$ of vector fields and the basis dx^a of 1-forms. Our tensors then have components g_{ab} and A_{ab}^c.

Generally, $\tilde{x}$ denotes the $\mathbf{Z}/2\mathbf{Z}$–degree of x. To simplify notation, in superscripts we replace $\tilde{x}_a$ by a so that e.g. $(-1)^{\tilde{x}_a\tilde{x}_b+\tilde{x}_c\tilde{x}_d}$ becomes $(-1)^{ab+cd}$. Hopefully, this will not lead to a confusion.

We want g and A to be even, i.e. $\tilde{g}_{ab} = (-1)^{a+b}$, $\tilde{A}_{ab}^c = (-1)^{a+b+c}$. The sign conventions about the de Rham complex are: d is odd, and Ω_M^* is the symmetric algebra of Ω_M^1.

4.2. Pairing, multiplication and connection (Dubrovin's formalism). We use g_{ab} in order to define an even $\mathcal{O}_M$–pairing on the tangent sheaf $\mathcal{T}_M$:

$$\langle \partial_a, \partial_b \rangle := g_{ab}. \tag{4.1}$$

We will always assume it to be symmetric ($g_{ba} = (-1)^{ab} g_{ab}$) and non-degenerate, so that the inverse matrix $(g^{ab}) = (g_{ab})^{-1}$ exists. In addition, we will assume that g_{ab} are constant, so that it defines a flat metric in a flat coordinate system (for non–flat coordinates, cf. [D]).

We use A_{ab}^c in order to construct, firstly, a structure of $\mathcal{O}_M$–algebra on $\mathcal{T}_M$ with multiplication $\circ$:

$$\partial_a \circ \partial_b := \sum_c A_{ab}^c \partial_c \tag{4.2}$$

and secondly, a family of connections on $\mathcal{T}_M$ depending on a even parameter λ and defined by the covariant differential

$$\nabla_\lambda(\partial_b) := \lambda \sum_{a,c} dx^a A_{ab}^c \otimes \partial_c \tag{4.3}$$

or equivalently, by covariant derivatives

$$\nabla_{\lambda,\partial_a}(\partial_b) := \lambda \sum_c A_{ab}^c \partial_c = \lambda(\partial_a \circ \partial_b). \tag{4.4}$$

We will now consecutively impose some relations upon A, g, and interpret them both in terms of multiplications and connections.

4.2.1. Commutativity/vanishing torsion.

$$\forall a, b, c, \qquad A_{ba}^c = (-1)^{ab} A_{ab}^c. \tag{4.5}$$

In view of (4.2), this means supercommutativity of $(\mathcal{T}_M, \circ)$. From (4.4) it follows that

$$\forall a, b, \qquad \nabla_{\lambda,\partial_a}(\partial_b) = (-1)^{ab} \nabla_{\lambda,\partial_b}(\partial_a). \tag{4.6}$$

4.2.2. Associativity/flatness.

$$\forall a,b,c,d, \qquad \sum_e A^e_{ab}A^d_{ec} = (-1)^{a(b+c)}\sum_e A^e_{bc}A^d_{ea}, \tag{4.7a}$$

$$\forall a,b,c,d, \qquad \partial_d A^c_{ab} = (-1)^{ad}\partial_a A^c_{db}. \tag{4.7b}$$

Using (4.2) one checks that (4.7a) is equivalent to the associativity relations $(\partial_a \circ \partial_b)\circ\partial_c = \partial_a \circ (\partial_b \circ \partial_c)$.

Using (4.3) one checks that (4.7a) and (4.7b) *together* are equivalent to $\nabla_\lambda^2 = 0$. More precisely, if one puts symbolically $\nabla_\lambda = \nabla_0 + \lambda A$, then (4.7b) is $\nabla_0(A) = 0$, and (4.7a) is equivalent to $[A,A]=0$ if one assumes (4.5) or (4.6).

4.2.3. Frobenius algebra/covariantly constant pairing. Put $A_{abc} := \sum_a A^e_{ab}g_{ec}$. The next relation we impose is:

$$\forall a,b,c, \qquad A_{abc} = (-1)^{a(b+c)}A_{bca}. \tag{4.8}$$

Together with (4.6), this means that A_{abc} is S_3–invarinat (in the sense of superalgebra).

In terms of multiplication, (4.8) reads

$$\langle \partial_a \circ \partial_b, \partial_c\rangle = \langle \partial_a, \partial_b \circ \partial_c\rangle,$$

that is, the scalar product is invariant wrt multiplication. In terms of connection, (4.8) reads (use (4.4) and (4.1)):

$$\langle \nabla_{\lambda,\partial_a}(\partial_b), \partial_c\rangle = (-1)^{ab}\langle \partial_b, \nabla_{\lambda,\partial_a}(\partial_c)\rangle \tag{4.9}$$

Since g_{ab} are constant, (4.9) means that g is covariantly constant wrt ∇_λ. This assumption will be valid in our main application to GW-classes.

4.2.4. Identity. Assume that the coordinate vector field ∂_0 is even, and

$$\forall b,c, \qquad A^c_{0b} = \delta_{bc} \tag{4.10a}$$

or equivalently,

$$A_{0bc} = g_{bc}. \tag{4.10b}$$

According to (4.2), this means that ∂_0 is identity in $(\mathcal{T}_M, \circ)$. According to (4.3), this is equivalent to

$$\nabla_\lambda(\partial_0) = \lambda\sum_a dx^a\partial_a$$

or more suggestively,

$$\nabla_\lambda(\partial_0) = \lambda d.$$

This also means that

$$\forall b, \qquad \nabla_{\lambda,\partial_0}(\partial_b) = \lambda\partial_b. \tag{4.11}$$

4.2.5. Potential. An even (local) function Φ on M is called a *potential* for the (A,g)–structure, if

$$\forall a,b,c, \qquad A_{abc} = \partial_a \partial_b \partial_c \Phi. \tag{4.12}$$

Such a potential always exists locally. On the other hand, for any function Φ and the tensor of its derivatives A_{abc}, the S_3–invariance, in particular (4.8), is automatic. If we then define A_{ab}^c by $\sum_e A_{abe} g^{ec}$, then (4.5) is also automatic. If in addition g^{ec} are constant, (4.7b) follows.

The crucial associativity relations (4.7a) then become a remarkable system of quadratic differential equations called WDVV–equations in [D]:

$$\forall a,b,c,d, \qquad \sum_{ef} \partial_a \partial_b \partial_e \Phi \cdot g^{ef} \partial_f \partial_c \partial_d \Phi = (-1)^{a(b+c)} \sum_{ef} \partial_b \partial_c \partial_e \Phi \cdot g^{ef} \partial_f \partial_a \partial_d \Phi. \tag{4.13}$$

We will now show how to derive a potential (A,g)–structure from a tree level system of GW-classes.

4.3. GW-potential. Let V be a manifold equipped with a system of tree level GW-classes. We will actually use only the numbers $\langle I_{0,n,\beta}^V \rangle (\gamma_1 \otimes \cdots \otimes \gamma_n)$ with properties postulated in §2.

We will consider $H^*(V,\mathbf{C})$ as a linear superspace $\mathbf{Z}/2\mathbf{Z}$–graded by $\tilde{\gamma} := |\gamma| \bmod 2$, and as a supermanifold which we then denote H^V. Our potential Φ_ω will depend on a choice of a class $\omega \in H^2(V,\mathbf{C})$ whose real part lies in the Kähler cone. We first define Φ_ω as a formal sum depending on a variable point $\gamma \in H^V$:

$$\Phi_\omega(\gamma) := \sum_{n\geq 3} \sum_{\beta} e^{-\int_\beta \omega} \frac{1}{n!} \langle I_{0,n,\beta}^V \rangle (\gamma^{\otimes n}). \tag{4.14}$$

To make sense of this expression, choose a basis $\{\Delta_a\}$ of $H^*(V,\mathbf{C})$, write the generic point as $\gamma = \sum_{a=0}^D x^a \Delta_a$, $\tilde{x}^a = \tilde{\Delta}_a$, and define the metric by Poincaré duality:

$$\langle \partial_a, \partial_b \rangle = g_{ab} := \int_V \Delta_a \wedge \Delta_b. \tag{4.15}$$

Since $\langle I_{0,n,\beta}^V \rangle (\gamma_1 \otimes \cdots \otimes \gamma_n)$ is simply S_n–invariant when all γ_i are even, we have

$$\frac{1}{n!} \langle I_{0,n,\beta}^V \rangle (\gamma^{\otimes n}) =$$

$$\sum_{n_0+\cdots+n_D=n\geq 3} \frac{\varepsilon(n_0,\ldots,n_D)}{n_1!\ldots n_D!} \langle I_{0,n,\beta}^V \rangle (\Delta_0^{\otimes n_0} \otimes \cdots \otimes \Delta_D^{\otimes n_D}) x_0^{n_0} \ldots x_D^{n_D}, \tag{4.16}$$

where $\varepsilon(n_0,\ldots,n_D) = \varepsilon = \pm 1$ is the sign acquired in the supercommutative algebra $S(H^*(V))[x^0,\ldots,x^D]$ after reshuffling $\prod_{a=0}^D (x^a \Delta_a)^{n_a} = \varepsilon \prod_{a=0}^D \Delta_a^{n_a} \prod_{a=0}^D (x^a)^{n_a}$.

There are several natural convergence assumptions that can be made about (4.14).

A. $\forall n \geq 3$, *there exists only finitely many effective β satisfying the grading condition (3.2) for the zero–codimensional classes.*

This is the case of Fano manifolds. If this hypothesis is satisfied, (4.14) can be interpreted as a formal series in x^a

$$\Phi_\omega(\gamma) =$$

$$\sum_{n_0+\dots+n_D\geq 3} \sum_{\beta} \frac{\varepsilon(n_0,\dots,n_D)e^{-\int_\beta \omega}}{n_0!\dots n_D!} \langle I^V_{0,n,\beta}\rangle(\Delta_0^{\otimes n_0}\otimes\dots\otimes\Delta_D^{\otimes n_D})(x^0)^{n_0}\dots(x^D)^{n_D}, \tag{4.17}$$

because each interior sum $\sum_\beta$ is effectively finite.

B. *The previous condition is not satisfied, but each $\sum_\beta$ in the r.h.s. of (4.17) converges for each n, at least when ω has a sufficiently large real Kähler part.*

(Conjecturally, this is the case for Calabi–Yau manifolds).

Then again, (4.17) is a well-defined formal series.

C. A *or* B *is satisfied, and in addition $\Phi_\omega(\gamma)$ converges in a subdomain M of H^V (which may depend on ω) as a function of $\{x^a\}$.*

We expect this to be generally true, because of the conjectural exponential growth estimates for $\frac{1}{n!}\langle I^V_{0,n,\beta}\rangle$.

The following formal calculations can be easily justified in each of these contexts.

4.4. Proposition. *Let $\gamma = \gamma' + \gamma^2 + \gamma^0$, where $|\gamma^2| = 2$, $|\gamma^0| = 0$, and γ' = the sum of components of dimension $\neq 0, 2$. Then*

a). $\Phi_\omega(\gamma'+\gamma^2+\gamma^0) = \Phi_\omega(\gamma'+\gamma^2)+$ a function quadratic in $\gamma'+\gamma^2$.

b). If x^0 is the coefficient of γ at $\Delta_0 = e^0_V$ (identity in $H^*(V)$) so that $\gamma^0 = x^0\Delta_0$, we have

$$\partial_0\partial_b\partial_c\Phi_\omega(\gamma) = g_{bc}. \tag{4.18}$$

c). We have

$$\Phi_\omega(\gamma'+\gamma^2) = \Phi_{\omega-\gamma^2}(\gamma') +$$
$$\text{a function quadratic in } \gamma,\ \gamma'. \tag{4.19}$$

Proof. a). $\gamma^{\otimes n} = (\gamma'+\gamma^2)^{\otimes n} + \sum_{i+j=n-1}(\gamma'+\gamma^2)^i\otimes\gamma^0\otimes(\gamma'+\gamma^2)^j$ plus terms containing γ^0 at least twice. From the Fundamental Class Axiom it follows that $\langle I^V_{0,n,\beta}\rangle = 0$ for $n\geq 4$, if e^0_V is among the arguments. Hence the contribution of γ^0 to $\Phi_\omega(\gamma)$ is restricted to the terms $n=3$ in (4.17), and they are (no more than) quadratic in $\gamma'+\gamma^2$.

b). We can now calculate $\partial_0\partial_b\partial_c\Phi_\omega$ taking into account only the $n=3$ terms:

$$\partial_0\partial_b\partial_c\langle I^V_{0,3,\beta}\rangle(\gamma\otimes\gamma\otimes\gamma) = 6\langle I^V_{0,3,\beta}\rangle(e^0\otimes\Delta_b\otimes\Delta_c).$$

This vanishes if $\beta\neq 0$ and is g_{bc} otherwise: see (2.7).

c). We have

$$\Phi_{\omega-\gamma^2}(\gamma') = \sum_{p\geq 3}\sum_\beta e^{-\int_\beta(\omega-\gamma^2)}\frac{1}{p!}\langle I^V_{0,p,\beta}\rangle(\gamma'^{\otimes p}) =$$

$$\sum_{p\geq 3, q\geq 0}\sum_{\beta} e^{-\int_\beta \omega}\frac{1}{p!q!}\langle I^V_{0,p,\beta}\rangle(\gamma'^{\otimes p})(\int_\beta \gamma^2)^q;$$

$$\Phi_\omega(\gamma'+\gamma^2)=\sum_{n\geq 3}\sum_{\beta} e^{-\int_\beta \omega}\frac{1}{n!}\langle I^V_{0,n,\beta}\rangle((\gamma'+\gamma^2)^{\otimes n})=$$

$$\sum_{p+q\geq 3}\sum_{\beta} e^{-\int_\beta \omega}\frac{1}{p!q!}\langle I^V_{0,n,\beta}\rangle(\gamma'^{\otimes p}\otimes(\gamma^2)^{\otimes q}).$$

Finally, because of the Divisor Axiom, for $p\geq 3,\ q\geq 0$, we have

$$\langle I^V_{0,p+q,\beta}\rangle(\gamma'^{\otimes p}\otimes(\gamma^2)^{\otimes q})=\langle I^V_{0,p,\beta}\rangle(\gamma'^{\otimes p})(\int_\beta \gamma^2)^q.$$

4.5. Theorem-Definition. *The tensors g_{ab} and $\partial_a\partial_b\partial_c\Phi_\omega$ define a potential Dubrovin structure satisfying all the properties (4.1)–(4.11), with ∂_0 as identity.*

The fibers of $\mathcal{T}_M$ endowed with multiplication $\circ$ are called the quantum cohomology rings of V associated with the tree level system of GW-classes I^V.

Proof. It remains only to check the relations (4.13). Let us calculate the l.h.s. of (4.13) using (4.17). The terms with fixed e,f are:

$$\sum_{n_1\geq 3,\beta_1} e^{-\int_{\beta_1}\omega}\frac{1}{(n_1-3)!}\langle I^V_{0,n_1,\beta_1}\rangle(\gamma^{\otimes(n_1-3)}\otimes\Delta_a\otimes\Delta_b\otimes\Delta_e)g^{ef}\times$$

$$\times\sum_{n_2\geq 3,\beta_2} e^{-\int_{\beta_2}\omega}\frac{1}{(n_2-3)!}\langle I^V_{0,n_2,\beta_2}\rangle(\Delta_f\otimes\Delta_c\otimes\Delta_d\otimes\gamma^{\otimes(n_2-3)})=$$

$$\sum_{n\geq 6,\beta}\frac{1}{(n-6)!}e^{-\int_\beta\omega}\sum_{\beta_1+\beta_2=\beta}\sum_{n_1+n_2=n}\binom{n-6}{n_1-3}\langle I^V_{0,n_1,\beta_1}\rangle(\gamma^{\otimes(n_1-3)}\otimes\Delta_a\otimes\Delta_b\otimes\Delta_e)g^{ef}\times$$

$$\times\langle I^V_{0,n_2,\beta_2}\rangle(\Delta_f\otimes\Delta_c\otimes\Delta_d\otimes\gamma^{\otimes(n_2-3)}).$$

Rewriting similarly the r.h.s., we see that (4.13) is equivalent to the family of identities

$$\sum_{n_1+n_2=n}\sum_{\beta_1+\beta_2}\sum_{e,f}\binom{n-6}{n_1-3}\langle I^V_{0,n_1,\beta_1}\rangle(\gamma^{\otimes(n_1-3)}\otimes\Delta_a\otimes\Delta_b\otimes\Delta_e)g^{ef}\times$$

$$\times\langle I^V_{0,n_2,\beta_2}\rangle(\Delta_f\otimes\Delta_c\otimes\Delta_d\otimes\gamma^{\otimes(n_2-3)})=$$

$$(-1)^{a(b+c)}\sum_{n_1+n_2=n}\sum_{\beta_1+\beta_2=\beta}\sum_{e,f}\binom{n-6}{n_1-3}\langle I^V_{0,n_1,\beta_1}\rangle(\gamma^{\otimes(n_1-3)}\otimes\Delta_b\otimes\Delta_c\otimes\Delta_e)g^{ef}\times$$

$$\times\langle I^V_{0,n_2,\beta_2}\rangle(\Delta_f\otimes\Delta_a\otimes\Delta_d\otimes\gamma^{\otimes(n_2-3)}).$$

Obviously, this is a particular case of (3.3).

Remark. A polarization argument shows that, vice versa, (4.13) implies all the quadratic relations (3.3).

We will show now that the grading conditions imply an additional symmetry of the (A,g)–structure with potential Φ_ω.

4.6. Scaling. If $\{\gamma_1,\dots\gamma_n\} = \{\Delta_a$ with multiplicity n_a, $a=0,\dots,D\}$, then the grading condition (3.2) for non-vanishing summands of Φ_ω becomes

$$\sum_{a=0}^{D} n_a(|\Delta_a|-2) - 2\int_\beta c_1(V) = 2(\dim_{\mathbf{C}} V - 3), \tag{4.20}$$

This leads to the following flows on our geometric data.

4.6.1. Scaling transformation of the potential.

$$x^a \mapsto e^{(|\Delta_a|-2)t}x^a,$$

$$\omega \mapsto \omega + 2c_1(T(V))t,$$

$$\Phi \mapsto e^{2(\dim_{\mathbf{C}} V-3)t}\Phi.$$

(See (4.17).)

4.6.2. Scaling transformation of the Dubrovin structure. Since A_{abc} is the tensor of the third derivatives of Φ_ω, the Proposition 4.4 allows one to replace the flow shift of ω by the reverse shift of $\gamma^2 = \sum_{|\Delta_b|=2} x^b\Delta_b$ by $-2c_1(T(V))t$. Defining the numbers $\{\xi^b \mid |\Delta_b|=2\}$ by

$$c_1(T(V)) = \sum_{|\Delta_b|=2} \xi^b\Delta_b,$$

we obtain the following flow: $\omega\mapsto\omega$, and

$$x^a \mapsto e^{(|\Delta_a|-2)t}x^a, \qquad |\Delta_a| \neq 2;$$

$$x^b \mapsto x^b - 2\xi^b t, \qquad |\Delta_b| = 2; \tag{4.21}$$

$$A_{abc} \mapsto e^{(2\dim_{\mathbf{C}} V-|\Delta_a|-|\Delta_b|-|\Delta_c|)t}A_{abc}. \tag{4.22}$$

Since the Poincaré pairing is invariant, and $g_{ab}\neq 0$ only for $|\Delta_a|+|\Delta_b| = 2\dim_{\mathbf{C}} V$, from these basic formulas we get furthermore

$$\partial_a \mapsto e^{(2-|\Delta_a|)t}\partial_a,\ dx^a \mapsto e^{(|\Delta_a|-2)t}dx^a;$$

$$A_{ab}^c \mapsto e^{(|\Delta_c|-|\Delta_a|-|\Delta_b|)t}A_{ab}^c, \tag{4.23a}$$

and finally, from (4.3),

$$\nabla_\lambda \mapsto \nabla_0 + e^{2t}(\nabla_\lambda - \nabla_0). \tag{4.23b}$$

We now introduce an extended supermanifold $\widehat{H}^V = H^V \times \mathbf{P}^1$ where $\mathbf{P}^1$ is the completion of the affine line with coordinate λ, and extend the flow (4.21) to $\widehat{H}^V$ by

$$\lambda \mapsto e^{2t}\lambda \tag{4.24}$$

which is another form of (4.23b).

From now on, ω can and will be assumed fixed.

4.6.3. Infinitesimal form. The flow (4.21) can be written as $\exp(tX)$ where X is the following even vector field on H^V :

$$X = \sum_a (|\Delta_a| - 2)x^a \partial_a - 2 \sum_{|\Delta_b|=2} \xi^b \partial_b. \tag{4.25}$$

In view of (4.24), its natural extension to $\widehat{H}^V$ is

$$Y = 2\lambda \frac{\partial}{\partial \lambda} + X. \tag{4.26}$$

Let $\pi : \widehat{H}^V \to H^V$ be the projection, and $\widehat{\mathcal{T}} = \pi^*(\mathcal{T}_{H^V})$. For a local section ∂ of $\mathcal{T}_{H^V}$, we denote by $\widehat{\partial}$ its lift to $\widehat{\mathcal{T}}$. We will now extend ∇_λ to a connection on $\widehat{\mathcal{T}}$.

4.7. Proposition. *Put*

$$\widehat{\nabla}_{\partial_b}(\widehat{\partial}_a) := (\nabla_{\lambda,\partial_b}(\partial_a))\widehat{\ } = \lambda(\partial_b \circ \partial_a)\widehat{\ } = \lambda \sum_c A^c_{ba} \widehat{\partial}_c,$$

$$\widehat{\nabla}_Y(\widehat{\partial}_a) := (2 - |\Delta_a|)\widehat{\partial}_a. \tag{4.27}$$

Then $\widehat{\nabla}$ is a flat connection on $\widehat{\mathcal{T}}$.

Proof. Clearly, (∂_b, ∂_c)–components of the curvature vanish because of (4.7a). It remains to check that

$$\widehat{\nabla}_{[\partial_b, Y]}(\widehat{\partial}_a) = [\widehat{\nabla}_{\partial_b}, \widehat{\nabla}_Y](\widehat{\partial}_a). \tag{4.28}$$

We have

$$[\partial_b, Y]x^c = (|\Delta_c| - 2)\delta_{bc}, \ [\partial_b, Y]\lambda = 0,$$

so that

$$[\partial_b, Y] = (|\Delta_b| - 2)\partial_b,$$

and

$$\widehat{\nabla}_{[\partial_b, Y]}(\widehat{\partial}_a) = \lambda(|\Delta_b| - 2)\sum_c A^c_{ba}\widehat{\partial}_c. \tag{4.29}$$

On the other hand, in view of (4.27), (4.23),

$$\widehat{\nabla}_{\partial_b}\widehat{\nabla}_Y(\widehat{\partial}_a) = \lambda(2 - |\Delta_a|)\sum_c A^c_{ba}\widehat{\partial}_c,$$

$$\widehat{\nabla}_Y \widehat{\nabla}_{\partial_b}(\partial_a) = \widehat{\nabla}_Y(\lambda \sum_c A^c_{ba}\widehat{\partial}_c) =$$

$$2\lambda \sum_c A^c_{ba}\widehat{\partial}_c + \lambda \sum_c X(A^c_{ba})\widehat{\partial}_c + \lambda \sum_c A^c_{ba}(2 - |\Delta_c|)\widehat{\partial}_c =$$

$$2\lambda \sum_c A^c_{ba}\widehat{\partial}_c + \lambda \sum_c (|\Delta_c| - |\Delta_b| - |\Delta_c|) A^c_{ba}\widehat{\partial}_c + \lambda \sum_c A^c_{ba}(2 - |\Delta_c|)\widehat{\partial}_c =$$

$$(4 - |\Delta_b| - |\Delta_a|)\lambda \sum_c A^c_{ba}\widehat{\partial}_c$$

so that

$$[\widehat{\nabla}_{\partial_b}, \widehat{\nabla}_Y](\widehat{\partial}_a) = (|\Delta_b| - 2)\lambda \sum_c A^c_{ba}\widehat{\partial}_c,$$

which agrees with (4.29).

4.8. Horizontal sections. Consider an even local section $\psi = \sum_a \psi^a \widehat{\partial}_a$ of $\widehat{\mathcal{T}}$. From (4.27) we find

$$\widehat{\nabla}_{\partial_b}(\psi) = \sum_a [(\partial_b \psi^a)\widehat{\partial}_a + (-1)^{ab}\lambda \sum_c \psi^a A^c_{ba}\widehat{\partial}_c],$$

$$\widehat{\nabla}_Y(\psi) = \sum_a [Y\psi^a + (2 - |\Delta_a|)\psi^a]\widehat{\partial}_a.$$

Hence ψ is horizontal iff

$$\forall b, a, \qquad \partial_b \psi^a = -\lambda \sum_c \psi^c A^a_{cb}, \tag{4.30}$$

$$\forall a, \qquad Y\psi^a + (2 - |\Delta_a|)\psi^a = 0. \tag{4.31}$$

But

$$Y\psi^a = 2\lambda \frac{\partial \psi^a}{\partial \lambda} + \sum_c (|\Delta_c| - 2)x^c \partial_c \psi^a - 2 \sum_{|\Delta_b|=2} \xi^b \partial_b \psi^a.$$

Replacing here the ∂-derivatives of ψ by the r.h.s. of (4.30), we get finally the equation governing the λ-dependence of the horizontal sections:

$$2\lambda \frac{\partial \psi^a}{\partial \lambda} - \lambda \sum_c (|\Delta_c| - 2)x^c \sum_e \psi^e A^a_{ec} +$$

$$2\lambda \sum_{|\Delta_b|=2} \xi^b \sum_e \psi^e A^a_{eb} + (2 - |\Delta_a|)\psi^a = 0,$$

or else

$$2\lambda \frac{\partial \psi^a}{\partial \lambda} - \lambda \sum_e \psi^e \Big(\sum_c (|\Delta_c| - 2)x^c A^a_{ce} - 2 \sum_{|\Delta_b|=2} \xi^b A^a_{be}\Big) + (2 - |\Delta_a|)\psi^a = 0. \tag{4.32}$$

The equation (4.32) has two singular points of which $\lambda = 0$ is regular, but $\lambda = \infty$ is irregular one.

Therefore we make a formal Fourier transform (of $\hat{\mathcal{T}}$ as a relative $\mathcal{D}$-module in λ-direction)

$$\lambda \mapsto \frac{\partial}{\partial \mu}, \ \frac{\partial}{\partial \lambda} \mapsto -\mu, \ \psi^a \mapsto \varphi^a.$$

4.9. Proposition. *The formal Fourier transform of (4.32) is an isomonodromy deformation of a holonomic $\mathcal{D}$-module on $\mathbf{P}^1$ with regular singular points (in which H^V is the space of deformation parameters).*

§5. Examples

5.1. The structure of Φ. In this section, we discuss in more detail some classes of manifolds V. We always tacitly assume that at least tree level GW-classes for V exist, and that the potential Φ constructed from them satisfies one of the convergence hypotheses of 4.3.

We start with (4.17) and take into account the Proposition 4.4 in order to drop the redundant terms, in particular those that are no more than quadratic in γ. Our conventions are:

a). $\omega = 0$ (because this can be achieved by shifting γ).

b). $D = r+1$, $\Delta_0 = e_V^0$, $\Delta_{r+1} = e_V^{2\dim_{\mathbf{C}} V}$ = the dual class of a point.

We also assume that $H^1(V) = 0$ so that $2 \leq |\Delta_a| \leq 2\dim_{\mathbf{C}} V - 2$ for $a = 1, \dots, r$.

The coordinates are renamed: $\gamma = x\Delta_0 + \sum_{a=1}^r y^a \Delta_a + z\Delta_{r+1}$.

From the proof of Proposition 4.4 one knows that only the ($\beta = 0, n = 3$)–term in (4.17) depends on x, and is $\frac{1}{6}\int_V \gamma \wedge \gamma \wedge \gamma := \frac{1}{6}(\gamma^3)$ (see (2.8)). The ($\beta = 0, n > 3$)–terms all vanish because of the grading condition (4.20) combined with (2.8). So we start with

$$\Phi^V(\gamma) = \frac{1}{6}(\gamma^3) +$$

$$\sum_{n_1+\dots+n_{r+1}=n\geq 3}\ \sum_{\beta\neq 0} \frac{\varepsilon(n_1, \dots, n_{r+1})}{n_1! \dots n_{r+1}!} \langle I_{0,n,\beta}^V \rangle (\Delta_1^{\otimes n_1} \otimes \dots \otimes \Delta_{r+1}^{\otimes n_{r+1}})(y^1)^{n_1} \dots (y^r)^{n_r} z^{n_{r+1}} \tag{5.1}$$

and the grading condition for non-vanishing terms

$$\sum_{a=1}^{r+1} n_a(|\Delta_a| - 2) = 2(-K_V.\beta) + 2(\dim_{\mathbf{C}} V - 3). \tag{5.2}$$

Dimension 1. Let $V = \mathbf{P}^1$, $r = 0$, $\beta = d[\mathbf{P}^1]$, $d \geq 1$. From (5.2) it follows that ($d \geq 2$)–terms vanish. In view of the Divisor Axiom,

$$\langle I_{0,n,[\mathbf{P}^1]}^{\mathbf{P}^1} \rangle (\Delta_1^{\otimes n}) = \langle I_{0,3,[\mathbf{P}^1]}^{\mathbf{P}^1} \rangle (\Delta_1^{\otimes 3}) \tag{5.3}$$

for all $n \geq 3$, and this must be 1, which is the number of the automorphisms of $\mathbf{P}^1$ fixing three points. So finally we find a (conditional) answer:

5.1.1. Proposition.

$$\Phi^{\mathbf{P}^1}(xe^0 + ze^2) = \frac{1}{2}x^2 z + e^z - 1 - z - \frac{z^2}{2}. \tag{5.4}$$

In the following calculations, we will be omitting quadratic, linear, and constant terms of Φ without changing its notation.

Dimension 2. Here $\{\Delta_1, \dots, \Delta_r\}$ form a basis of $H^2(V)$; (5.2) is equivalent to

$$n_{r+1} = (-K_V.\beta) - 1 := k(\beta) - 1, \tag{5.5}$$

so that we must sum only over β with $k(\beta) \geq 1$.

5.1.2. Proposition. *For surfaces V we have up to terms no more than quadratic in γ*

$$\Phi^V(\gamma) = \frac{1}{6}(\gamma^3) + \sum_\beta N(\beta)\frac{z^{k(\beta)-1}}{(k(\beta)-1)!}e^{(\beta.\gamma)}. \tag{5.6}$$

Here for $k(\beta) \geq 4$ we put

$$N(\beta) = \langle I^V_{0,k(\beta)-1,\beta}\rangle(\Delta_{r+1}^{\otimes(k(\beta)-1)}). \tag{5.7}$$

For $k(\beta) \leq 3$, the definition of $N(\beta)$ is given below: see (5.9).

Proof. If $(-K_V.\beta) \geq 4$, then $n_{r+1} \geq 3$ (see (5.5)) so that the contribution of β to (5.1) in view of the Divisor Axiom takes form

$$\frac{z^{k(\beta)-1}}{(k(\beta)-1)!}\sum_{n_i\geq 0}\frac{1}{n_1!\dots n_r!}\langle I^V_{0,n,\beta}\rangle(\Delta_1^{\otimes n_1}\otimes\dots\otimes\Delta_r^{n_r}\otimes\Delta_{r+1}^{\otimes(k(\beta)-1)}(y^1)^{n_1}\dots(y^r)^{n_r} =$$

$$\frac{z^{k(\beta)-1}}{(k(\beta)-1)!}\sum_{n_i\geq 0}\frac{1}{n_1!\dots n_r!}\langle I^V_{0,k(\beta)-1,\beta}\rangle(\Delta_{r+1}^{\otimes(k(\beta)-1)})((\beta.\Delta_1)y^1)^{n_1}\dots((\beta.\Delta_r)y^r)^{n_r} =$$

$$N(\beta)\frac{z^{k(\beta)-1}}{(k(\beta)-1)!}e^{(\beta.\gamma)}. \tag{5.8}$$

For $1 \leq k(\beta) \leq 3$ the calculation is only slightly longer. The actual contribution of β is given by the same formula as the first expression in (5.8), but this time with summation taken over $n_1 + \dots + n_r \geq 4 - k(\beta)$. First, this sum lacks the terms of total degree ≤ 2 in y^i, z, but they are negligible. Second, in order to represent this sum as the last expression in (5.8), we are bound to put

$$N(\beta) := \frac{\langle I^V_{0,n,\beta}\rangle(\Delta_1^{\otimes n_1}\otimes\dots\otimes\Delta_r^{\otimes n_r}\otimes\Delta_{r+1}^{\otimes(k(\beta)-1)})}{(\beta.\Delta_1)^{n_1}\dots(\beta.\Delta_r)^{n_r}}. \tag{5.9}$$

For fixed n_i, the r.h.s. is well defined if $(\beta.\Delta_i) \neq 0$ for all i. One can secure this by choosing an appropriate basis (eventually depending on β). The result does not depend on n_i. In fact, one can reach any point $(n_1, \dots n_r)$ in the set $\sum n_i \geq 4 - k(\beta), n_i \geq 0$, from any other point, without ever leaving this set, by adding and subtracting 1's from coordinates. In view of the Divisor Axiom, these steps multiply the numerator and the denominator of (5.9) by the same amount.

We expect that $N(\beta)$ counts the number of rational curves in the homology class β passing through $k(\beta) - 1$ points, at least in unobstructed problems.

Dimension 3. In this dimension, Calabi–Yau manifolds make their first appearance, and we consider their potentials. Since $K_V = 0$, (5.2) shows that $n_a \neq 0$ only for $|\Delta_a| = 2$. Therefore, we may and will disregard the other elements of the basis of $H^*(V)$, and in this subsection denote by $\{\Delta_1, \dots, \Delta_r\}$ a basis of $H^2(V)$.

5.1.3. Proposition. *For a threefold V with $K_V = 0$, we have, up to terms of degree ≤ 2 in γ,*

$$\Phi^V = \frac{1}{6}(\gamma^3) + \sum_{\beta\neq 0} \tilde{N}(\beta)e^{(\beta.\gamma)}, \tag{5.10}$$

where

$$\tilde{N}(\beta) = \frac{\langle I^V_{0,n,\beta}\rangle(\Delta_1^{\otimes n_1} \otimes \cdots \otimes \Delta_r^{\otimes n_r})}{(\beta.\Delta_1)^{n_1}\dots(\beta.\Delta_r)^{n_r}} \tag{5.11}$$

for any $n = n_1 + \cdots + n_r \geq 3$ and any basis of $H^2(V)$ such that $(\beta.\Delta_i) \neq 0$ for all i.

The proof does not differ much from the previous one.

Let us guess the geometric meaning of $\tilde{N}(\beta)$ restricting ourselves to the case $r = \dim H^2(V) = \mathrm{rk}\, Pic(V) = 1$. Let Δ_1 be the ample generator of $H^2(V)$, β_0 the effective generator of $H_2(V, \mathbf{Z})$ with $(\beta_0.\Delta_1) = 1$, and $\beta = d\beta_0$. If $N(d)$ is the "geometric" number of unparametrized rational curves in the class $d\beta_0$, then the number of parametrized curves with three marked points landing in β and incident to three fixed cycles dual to Δ_1 must be $d^3N(d)$.

According to [AM], the parametrizations of degree m must be counted with multiplicity m^{-3}. Hence we expect that

$$\tilde{N}(d\beta_0) = \sum_{k/d} \frac{1}{(d/k)^3}N(k)d^3 = \sum_{k/d} N(k)k^3 \tag{5.12}$$

which can also be taken as a formal definition of numbers $N(k)$ via GW-classes. Rewriting (5.11) in this situation, we get for $y = y^1$

$$\Phi^V(\gamma) = \frac{1}{6}(\gamma^3) + \sum_{k\geq 1} N(k)Li_3(e^{ky}) \tag{5.13}$$

where $Li_3(z) = \sum_{m\geq 1} \frac{z^m}{m^3}$.

Projective spaces. Let $V = \mathbf{P}^{r+1}$, $\Delta_i = c_1(\mathcal{O}(1))^i, |\Delta_i| = 2i$, $r \geq 1$. Put for $\sum_{a=2}^{r+1} n_a(a-1) = (r+2)d + r - 2$ (this is (5.2)),

$$N(d; n_2, \dots, n_{r+1}) = \frac{\langle I^V_{0,n,d[\mathbf{P}^1]}\rangle(\Delta_1^{\otimes n_1} \otimes \cdots \otimes \Delta_{r+1}^{n_{r+1}})}{d^{n_1}}, \tag{5.14}$$

where $n = n_1 + \cdots + n_{r+1} \geq 3$. The r.h.s. of (5.13) being independent on n_1, for $n_2 + \cdots + n_{r+1} \geq 3$ one can take $n_1 = 0$. Again, (5.15) must be the number of rational curves of degree d in $\mathbf{P}^{r+1}$ intersecting n_a hyperplanes of codimension a, $a = 2, \dots, r+1$.

A version of previous calculations now gives:

5.1.4. Proposition.

$$\Phi^{\mathbf{P}^{r+1}}(\gamma)=\frac{1}{6}(\gamma^3)+\sum_{d,n_a}N(d;n_2,\dots,n_{r+1})\frac{(y^2)^{n_2}\dots(y^{r+1})^{n_{r+1}}}{n_2!\dots n_{r+1}!}e^{dy_1}. \qquad (5.15)$$

5.2. Enumerative predictions. We start with projective plane: (5.6) for $r=1$. Put

$$\varphi(y,z)=\Phi^{\mathbf{P}^2}(\gamma)-\frac{1}{6}(\gamma^3)=\sum_{d=1}^{\infty}N(d)\frac{z^{3d-1}}{(3d-1)!}e^{dy},\ N(1)=1.$$

We have the following simple

5.2.1. Claim. *The associativity condition for the potential $\Phi^{\mathbf{P}^2}$ is equivalent to the single equation*

$$\varphi_{zzz}=\varphi_{yyz}^2-\varphi_{yyy}\varphi_{yzz} \qquad (5.16)$$

which in turn is equivalent to the recursive relation

$$N(d)=\sum_{k+l=d}N(k)N(l)k^2l\left[l\binom{3d-4}{3k-2}-k\binom{3d-4}{3k-1}\right],\ d\geq 2, \qquad (5.17)$$

uniquely defining $N(d)$ and φ.

This discovery made by M. Kontsevich was the starting point for this paper. The first values of $N(d)$, starting with $d=2$, are 1,12,620,87304,26312976,14616808192.

From 3.1 and 3.1.1 it follows that a similar uniqueness result holds for any projective space: in the notation of (5.14), (5.15), we have

5.2.2. Claim. *The associativity relations together with the initial condition $N(1;0,\dots,0,2)=1$ uniquely define all $N(d;n_2,\dots,n_{r+1})$ and the potential $\Phi^{\mathbf{P}^{r+1}}$.*

Here, however, the compatibility of the associativity relations must be established either geometrically (via a construction of GW-classes), or by number theoretic and combinatorial means. We will now look at some of the identities implied by associativity for del Pezzo surfaces.

Del Pezzo surfaces. Let $V=V_r$ be a del Pezzo surface that can be obtained by blowing up $0\leq r\leq 8$ points (in sufficiently general position) of $\mathbf{P}^2$. A choice of such a representation $\pi: V_r\to\mathbf{P}^2$ allows one to identify $Pic(V_r)$ with $\mathbf{Z}^{r+1}$ via

$$L=a\Lambda-b_1l_1-\dots-b_rl_r\mapsto(a,b_1,\dots b_r)$$

where $\Lambda=\pi^*(c_1(\mathcal{O}(1)))$ and $l_i=$ inverse image of the i–th blown point. Under this identification, the intersection index becomes $((a,b_i).(a',b_i'))=aa'-\sum b_ib_i'$, and $-K_V=(3;1,\dots,1)$ so that $(-K_V.L)=3a-\sum b_i$. The cone of effective classes B is generated by its indecomposable elements Λ for $r=0$, $\Lambda-l_1$ and l_1 for $r=1$, and all exceptional classes for $r\geq 2$. (Recall that l is exceptional iff $(l^2)=-1$ and $(-K_V.\beta)=1$; for more details see [Ma1]).

This allows us to rewrite (5.6) as an explicit sum over B.

Writing $\gamma=xe^0+y\Lambda+ze^4-\sum_{i=1}^r y^il_i$, $\Phi^{V_r}(\gamma)=\frac{1}{6}(\gamma^3)+\varphi(\gamma)$, we can easily check the following generalization of 5.2.1:

5.2.3. Claim. *a). One of the associativity relations reads*

$$\varphi_{zzz} = \varphi_{yyz}^2 - \sum_{i=1}^{r} \varphi_{yy^iz}^2 - \varphi_{yyy}\varphi_{yzz} + \sum_{i=1}^{r} \varphi_{yyy^i}\varphi_{y^izz}.$$

b). This relation is equivalent to the following recursive formula for the coefficients $N(\beta)$ (see (5.6)):

$$N(\beta) =$$

$$\sum_{\beta_1+\beta_2=\beta} N(\beta_1)N(\beta_2)(\beta_1.\beta_2)(\Lambda.\beta_1)\left[(\Lambda.\beta_2)\binom{(-K_V.\beta)-4}{(-K_V.\beta_1)-2} - (\Lambda.\beta_1)\binom{(-K_V.\beta)-4}{(-K_V.\beta_1)-1}\right]. \tag{5.18}$$

The initial conditions for (5.18) consist of the list of values of $N(\beta)$ for all indecomposable elements of B. It is expected that all these values are 1.

The redundancy of the associativity relations is reflected here in the presence of Λ which depends on the choice of $\pi : V_r \to \mathbf{P}^2$. The number c_r of such choices for $r = 1, \dots, 8$ is respectively $1, 1, 2, 5, 2^4, 2^3.3^2, 2^6.3^2, 2^7.3^3.5$. In fact, the symmetry group W_r of the configuration of exceptional classes acts upon the set of associativity relations, and $c_r = |W_r|/r!$, the denominator corresponding to the renumberings of blown points.

Question. Is it true that the linear span of all relations (5.18) for various choices of π contains all the associativity relations, at least for larger values of r?

5.2.4. Quadric. The quadric $V = \mathbf{P}^1 \times \mathbf{P}^1$ is the last del Pezzo surface. Here $Pic(V) = \mathbf{P}^2$, $-K_V = (2,2)$, and all associativity relations were written explicitly in [I]. In self-explanatory notation

$$\varphi(\gamma) = \sum_{a+b\geq 1} N(a,b) \frac{z^{2a+2b-1}}{(2a+2b-1)!} e^{ay^1+by^2},$$

and the associativity relations together with initial conditions $N(0,1) = N(1,0) = 1$ imply the following recursive definition of $N(a,b)$ in the effective cone $a \geq 0, b \geq 0$:

$$N(a,b) =$$

$$\sum_{\substack{a_1+a_2=a \\ b_1+b_2=b}} N(a_1,b_1)N(a_2,b_2)(a_1b_2+a_2b_1)b_2 \left[a_1\binom{2a+2b-4}{2a_1+2b_1-2} - a_2\binom{2a+2b-4}{2a_1+2b_1-1}\right]. \tag{5.19}$$

The remaining relations are:

$$N(a,b) = N(b,a), \tag{5.20}$$

$$2abN(a,b) = \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b}} N(a_1,b_1)N(a_2,b_2)\binom{2a+2b-3}{2a_1+2b_1-1} a_1^2(a_1b_2-a_2b_1)b_2^2, \tag{5.21}$$

$$aN(a,b) = \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b}} N(a_1,b_1)N(a_2,b_2)\binom{2a+2b-3}{2a_1+2b_1-1} a_1(a_1^2b_2^2 - b_1^2a_2^2), \tag{5.22}$$

$$0 = \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b}} N(a_1,b_1)N(a_2,b_2)a_1^2[(a_2+b_2-1)(b_1a_2+a_1b_1) - (2a_1+2b_1-1)a_2b_2]. \tag{5.23}$$

Question. Can one deduce (5.20)-(5.23) directly from (5.19)?

5.2.6. Nonsingular rational curves. Consider an effective class β with $p_a(\beta) := (\beta.\beta + K_V)/2 + 1 = 0$, i.e.

$$(-K_V.\beta) = d \geq 0, \ (\beta.\beta) = d - 2.$$

Any irreducible curve in this class must be nonsingular rational, so that passing through points imposes only linear conditions. Thus we may expect that $N(\beta) = 1$ for such a class. This was observed numerically on cubic surfaces V_6 by C. Itzykson.

Question. Can one deduce directly from (5.19) that $N(\beta) = 1$ whenever $p_a(\beta) = 0$?

Notice that there are infinitely many such classes on any V_r with $r \geq 1$. The simplest family is: $r = 1, \beta = n\Lambda - (n-1)l_1$ projecting into rational curves of degree n with one $(n-1)$–ple point on $\mathbf{P}^2$.

§6. Cohomological Field Theory

6.1. Definition. A two-dimensional cohomological field theory (CohFT) with coefficient field K consists of the following data:

a). A K-linear superspace (of fields) A, endowed with an even non-degenerate pairing.

b). A family of even linear maps (correlators)

$$I_{g,n}:\ A^{\otimes n} \to H^*(\overline{M}_{g,n}, K),$$

defined for all $g \geq 0$ and $n + 2g - 3 \geq 0$.

These data must satisfy the following axioms:

6.1.1. S_n**–covariance.**

6.1.2. Splitting. In order to state this axiom, we retain all notation of 2.2.6 with the following minimal changes: $\{\Delta_a\}$ denotes a basis of A, $\Delta = \sum g^{ab}\Delta_a \otimes \Delta_b$ is the Casimir element of the pairing, and all mention of V and β's is omitted. Then the axiom reads:

$$\varphi_S^*(I_{g,n}(\gamma_1 \otimes \cdots \otimes \gamma_n)) =$$

$$\varepsilon(S)\sum_{a,b} I_{g_1,n_1+1}(\otimes_{j\in S_1}\gamma_j \otimes \Delta_a)g^{ab} \otimes I_{g_2,n_2+1}(\Delta_b \otimes (\otimes_{j\in S_2}\gamma_j)). \tag{6.1}$$

6.1.3. Genus reduction. This is (2.12), with V and β omitted.

6.2. Remarks. a). We will mostly assume A finite-dimensional. However, A can also be graded with finite-dimensional components, or an object of a K-linear tensor category, etc.

b). We will be mostly concerned with *tree level* CohFT. Correlators for such a theory must be given only for $g = 0,\ n \geq 3$, and the Genus Reduction Axiom is irrelevant.

6.3. Example: GW-theories. Any system of GW-classes for a manifold V satisfying appropriate convergence assumptions gives rise to the following cohomological field theory depending on a class ω as in 4.3: $A = H^*(V, \mathbf{C})$ with Poincaré pairing,

$$I_{g,n} := \sum_{\beta} I^V_{g,n,\beta}\, e^{-\int_\beta \omega}. \tag{6.2}$$

The series in β can also be treated as a formal one. Equivalently, we can put $A = \oplus_{\beta\in B} H^*(V, K)$ and work with B-graded objects.

6.4. Operations on field theories. a). Tensor product of two cohomological field theories can be defined as in 2.5.

b). If a group G acts upon the space of fields A of a tree level CohFT preserving scalar product and $I_{g,n}$, a new tree level CohFT can be obtained by restricting all maps to the space of invariants A^G.

c). Following Witten ([W]), one can define an infinite-dimensional family of deformations of any CohFT. The parameter space will be the formal neighborhood of zero in the vector superspace $A \otimes \mathbf{C}[[x]]$ where x is an even formal variable. The space of fields A and its scalar product are kept undeformed.

For a point $\alpha = \sum_0^\infty \alpha_i x^i$, $\alpha_i \in A$, of the parameter space, define the new correlators by

$$\widetilde{I}_{g,n}(\gamma_1 \otimes \cdots \otimes \gamma_n) =$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1,\dots,i_k} \pi_{k*}\left(I_{g,n+k}(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes \alpha_{i_1} \otimes \cdots \otimes \alpha_{i_k}) \wedge (\wedge_{j=1}^{k} c_1(\mathcal{T}^*_{x_{n+j}}C)^{i_j})\right).$$

Here x_i denotes the i–th marked point of the universal curve C, π_k is the projection $\overline{M}_{g,n+k} \to \overline{M}_{g,n}$ forgetting the last k points, π_* is the direct image in cohomology for the proper map π of smooth stacks (orbifolds).

6.5. Mirror Symmetry. Physicists believe that to each Calabi–Yau manifold V of sufficiently big Kähler volume one can associate *two* different cohomological field theories, called A– and B– models in [W].

A–model depends only on the cohomology class $[\omega]$ of the symplectic form and remains invariant when one deforms the complex structure of V.

In the infinite volume limit A-model should approximate the GW-model from 6.3, taking its existence for granted. One expects (see [BCOV]) that the difference between any correlators in A– and GW–models is a cohomology class of a moduli space Poincaré dual to a homology class supported on the boundary.

B–model should depend only on the complex structure of V via the universal infinitesimal variation of its Hodge structure and, possibly, on some additional data. The space of fields in the B–model must be $\oplus_{p,q} H^q(\wedge^p \mathcal{T}_V)$ with grading $(p+q)$ mod 2. The quantum multiplication in the computed examples is given by the symbol of iterated canonical connection on the cohomology space whose definition requires a choice of the global volume form on V.

Mirror symmetry ought interchange A– and B– models of dual varieties.

It would be important to have a treatment of A– and B– models axiomatizing their dependence on the geometry of V.

In the remaining part of this section, we introduce an operadic firmalism for description of tree level CohFT's. Our framework is similar to that of [V], [BG], [GM].

6.6. Trees. We will formally introduce trees describing combinatorial structure of a marked stable curve of arithmetical genus 0. Their vertices correspond to components, and edges to special points.

6.6.1. Definition. *A (stable) tree τ is a collection of finite sets V_τ (vertices), E_τ (interior edges), T_τ (exterior edges, or tails), and two boundary maps $b: T_\tau \to V_\tau$ (every tail has one end vertex), and $b: E_\tau \to$ {unordered pairs of distinct vertices} (every interior edge has exactly two vertices).*

The geometric realization of τ must be connected and simply connected. Every vertex must belong to at least three edges, exterior and/or interior (stability).

6.6.2. Definition. *A morphism of trees $f: \tau \to \sigma$ is a collection of three maps (notice arrow directions)*

$$f_v : V_\tau \to V_\sigma,\ f^t : T_\sigma \to T_\tau,\ f^e : E_\sigma \to E_\tau,$$

with the following properties:

a). f_v is surjective, f^t and f^e are injective.

b). If v_1, v_2 are ends of an edge e' of τ, then either $f_v(v_1) = f_v(v_2)$, or $f_v(v_i)$ are ends of an edge e'' of σ; we say that e' covers this edge, and we must then have $e' = f^e(e'')$.

c). If $v' \in V_\tau$ is such a vertex that $f_v(v')$ is the end of $t'' \in T_\tau$, then v' is the end of $f^t(t'')$.

In other words, f contracts interior edges from $E_\tau \setminus f^e(E_\sigma)$ and tails from $T_\tau \setminus f^t(T_\sigma)$, and is one–to–one on the remaining edges. We will denote by $f(e)$ the image of a non–contracted edge.

The composition of morphisms is the composition of maps. In this way, trees form a category.

6.6.3. Flags and dimension. A pair $\{$*edge, one end of it*$\}$ is called a flag. For a tree τ, we denote by F_τ the set of its flags, and by $F_\tau(v)$ the set of flags ending in vertex v. We have $|F_\tau| = 2|E_\tau| + |T_\tau|$.

The *dimension* of τ is defined by

$$\dim \tau := \sum_{v \in V_\tau} (|F_\tau| - 3) = 2|E_\tau| + |T_\tau| - 3|V_\tau|. \tag{6.3}$$

6.6.4. Glueing. Let (τ_i, t_i), $i = 1, 2$, be two pairs consisting each of a tree and its tail. Their glueing (t_1 to t_2) produces a pair (τ, e) consisting of a tree and its interior edge:

$$(\tau, e) := (\tau_1, t_1) * (\tau_2, t_2).$$

Formally:

$$V_\tau = V_{\tau_1} \coprod V_{\tau_2},\ E_\tau = E_{\tau_1} \coprod E_{\tau_2} \coprod \{e\},$$

$$T_\tau = (T_{\tau_1} \coprod T_{\tau_2}) \setminus \{t_1, t_2\},\ b(e) = \{b(t_1), b(t_2)\}.$$

This operation is functorial in the following sense: for two morphisms $f_i : \tau_i \to \sigma_i$ not contracting t_i, we have a selfexplanatory morphism

$$f_1 * f_2 : (\tau_1, t_1) * (\tau_2, t_2) \to (\sigma_1, f_1(t_1)) * (\sigma_2, f_2(t_2)).$$

Finally, $F_\tau = F_{\tau_1} \coprod F_{\tau_2}$.

6.7. Products of families. Assume that we work in a monoidal category where products of families of objects are associative and commutative in a functorial way. Then we can use a convenient formalism (spelled out e.g., by Deligne) of products of families indexed by *finite sets* and functorial with respect to the maps of such sets. In this sense, we will use notation like $\prod_{i\in F} A_i$, $\otimes_{i\in F}A_i$, $A^{\otimes F}$ (for a constant family $A_i = A$), etc.

In the same vein, if $|F| \geq 3$, we will denote by $\overline{M}_{0,F}$ the moduli space of stable curves of genus zero with $|F|$ marked points indexed by F.

6.8. From trees to moduli spaces. In this subsection, we define a functor

$$\mathcal{M} : \ \{trees\} \to \{algebraic\ manifolds\}$$

(ground field is arbitrary).

6.8.1. Objects. Put

$$\mathcal{M}(\tau) = \prod_{v\in V_\tau} \overline{M}_{0,F_\tau(v)}. \tag{6.4}$$

We have dim $\mathcal{M}(\tau) =$ dim τ.

This space parametrizes a family of (generally reducible) stable rational curves $C(\tau)$ with marked points indexed by T_τ. The dual graph of a generic (but not arbitrary) curve of this family is (canonically identified with) τ. To describe it, consider a point $x = (x_v) \in \mathcal{M}(\tau)$, $x_v \in \overline{M}_{0,F_\tau(v)}$, and let $C(x_v)$ be the fiber of a universal curve at this point. If v_1, v_2 bound an edge e of τ, $C(x_v)$ contains a point $y(v_i, e)$ marked by the flag (v_i, e). Identify $y(v_1, e)$ with $y(v_2, e)$ in the disjoint union $\coprod_{v\in F_\tau} C(x_v)$ for all e. This will be $C(\tau)(x)$.

Clearly, its remaining special points are marked by T_τ so that we have a canonical morphism (closed embedding) $\mathcal{M}(\tau) \to \overline{M}_{0,T_\tau}$. This is a special case of morphisms defined below.

6.8.2. Morphisms. Any morphism of trees $f : \ \tau \to \sigma$ contracting no tails induces a closed embedding $\mathcal{M}(\tau) \to \mathcal{M}(\sigma)$. To construct it, identify $T_\tau = T_\sigma = T$ by means of f^t, and denote by ρ the one–vertex tree with tails T. Clearly, $\mathcal{M}(\rho) = \overline{M}_{0,T}$, and by universality, we have embeddings of $\mathcal{M}(\sigma)$ and $\mathcal{M}(\tau)$ into $\mathcal{M}(\rho)$. In this embedding, $\mathcal{M}(\sigma) \subset \mathcal{M}(\tau)$ which is the seeked for morphism.

Any morphism of one–vertex trees contracting tails induces the forgetful morphism of the respective moduli spaces (see e.g. [Ke]).

The general construction of a moduli space morphism corresponding to a morphism of trees can be obtained by combining these two cases: embed $\mathcal{M}(\tau)$ into $\overline{M}_{0,T_\tau}$, $\mathcal{M}(\sigma)$ into $\overline{M}_{0,T_\sigma}$, and restrict the forgetful map onto $\mathcal{M}(\tau)$.

6.8.3. Glueing. If $(\tau, e) = (\tau_1, t_1) * (\tau_2, t_2)$, we have canonically ($H^*\mathcal{M}(\cdot) := H^*(\mathcal{M}(\cdot), K)$):

$$\mathcal{M}(\tau) = \mathcal{M}(\tau_1) \times \mathcal{M}(\tau_2),$$
$$H^*\mathcal{M}(\tau) = H^*\mathcal{M}(\tau_1) \otimes H^*\mathcal{M}(\tau_2). \tag{6.5}$$

6.9. From trees to tensors. Let A be a linear superspace over K with a Casimir element Δ, as in 6.1.

6.9.1. Objects. For a tree τ, put $\mathcal{A}(\tau) = A^{\otimes F_\tau}$. We will show that this construction is (contravariant) functorial with respect to *pure contractions* that is, morphisms of trees contracting no tails.

6.9.2. Morphisms. Let $f: \tau \to \sigma$ be a pure contraction, F_τ^s the set of non-contracted flags, and E_τ^c the set of contracted edges. Each such edge gives rise to a pair of contracted flags. Therefore, we have two identifications:

$$f^{-1}: \mathcal{A}(\sigma) = A^{\otimes F_\sigma} \to A^{\otimes F_\tau^s},$$

$$A^{\otimes F_\tau^s} \otimes (A \otimes A)^{E_\tau^c} \to A^{\otimes F_\tau},$$

the second being defined only up to switches in $A \otimes A$ factors. Since the Casimir element is invariant with respect to the switch, we can unambiguously set

$$\mathcal{A}(f)(\otimes_{i \in F_\sigma} \gamma_i) = \left(\otimes_{f^{-1}(i) \in F_\tau^s} \gamma_{f^{-1}(i)}\right) \otimes \Delta^{\otimes E_\tau^c}. \tag{6.6}$$

6.9.3. Glueing. Obviously, for (τ, e) as in 6.8.3, we have canonically

$$\mathcal{A}(\tau) = \mathcal{A}(\tau_1) \otimes \mathcal{A}(\tau_2). \tag{6.7}$$

We see that with respect to pure contractions and glueing, $\mathcal{A}$ has the same properties as $H^*\mathcal{M}$ from 6.8.

We can now state our new definition.

6.10. Definition. *An operadic tree level CohFT is a morphism of functors compatible with glueing*

$$I: \mathcal{A} \to H^*\mathcal{M}.$$

In other words, it consists of a family of maps indexed by trees

$$I(\tau): A^{\otimes F_\tau} \to H^*(\mathcal{M}(\tau), K)$$

such that for any pure contraction $f: \tau \to \sigma$ we have:

$$I(\tau) \circ \mathcal{A}(f) = H^*\mathcal{M}(f) \circ I(\sigma): A^{\otimes F_\sigma} \to H^*\mathcal{M}(\tau), \tag{6.8}$$

and for any τ glued from τ_1, τ_2,

$$I(\tau) = I(\tau_1) \otimes I(\tau_2): \mathcal{A}(\tau) = \mathcal{A}(\tau_1) \otimes \mathcal{A}(\tau_2) \to H^*\mathcal{M}(\tau_1) \otimes H^*\mathcal{M}(\tau_2). \tag{6.9}$$

6.11. Claim. *The two definitions of a tree level CohFT are equivalent.*

Proof (sketch). Notice that a two–vertex tree with tails $\{1, \ldots, n\}$ is the same as a partition $S = (S_1, S_2)$ of $\{1, \ldots, n\}$. Now, given an operadic tree level CohFT, restrict it to the following subclass of trees: $T_\tau = \{1, \ldots, n\}$ for some n, $|V_\tau| = 1$ or 2. One easily checks that morphisms φ_S from 2.2.6 are induced by non-trivial pure contractions in this subclass, and that (6.7) restricted to it becomes (6.1), and S_n–covariance corresponds to the functorality with respect to bijective maps of tails.

Conversely, given a tree level CohFT in the sense of 6.1, we first rewrite it as a fragment of an operadic CohFT as above, and then reconstruct the whole operadic CohFT using glueing and decomposition of pure contractions into products of morphisms contracting exactly one edge each.

§7. Homology of moduli spaces

7.1. Additive generators. If $T_\tau = \{1, \ldots, n\}$, we will call τ an n–tree. A morphism of n–trees $\tau \to \sigma$ identical on tails will be called n–morphism. If such a morphism exists, it is unique. Let ρ_n be a one–vertex n–tree. Then $\mathcal{M}(\rho_n) = \overline{M}_{0,n}$. For any n–tree τ, there exists a unique n–contraction $\tau \to \rho_n$. Let $d_\tau \in H_*(\overline{M}_{0,n})$ be the homology class of $\mathcal{M}(\tau)$ corresponding to this contraction. It depends only on the n–isomorphism class of τ. The manifolds $\mathcal{M}(\tau)$ embedded into each other in this way will be called strata.

Lemma 7.1.1. *d_τ span $H_*(\overline{M}_{0,n})$ (over any coefficient ring).*

Proof. Easy induction by n, as in the proof of 2.5.2.

7.2. Linear relations. Choose a system $R = (\tau, \{i, j, k, l\}, v)$, where τ is an n–tree, $1 \leq i, j, k, l \leq n$ are its pairwise distinct tails, and $v \in V_\tau$ is such a vertex that paths from v to i, j, k, l start with pairwise distinct edges e_i, e_j, e_k, e_l respectively (some of these edges may be tails themselves).

Consider all n–contractions $\tau' \to \tau$ which contract exactly one edge onto the vertex v and satisfy the following condition: lifts to τ' of e_i, e_j on the one hand, and e_k, e_l on the other, are incident to different vertices of the contracted edge. Below we will denote by $\{ij\tau'kl\}$ the summation over n–isomorphism classes of such contractions, R being fixed.

7.2.1. Lemma. *For any R, we have*

$$\sum_{\{ij\tau' kl\}} d_{\tau'} = \sum_{\{ik\tau'' jl\}} d_{\tau''} \qquad (7.1)_R$$

in $H_*(\overline{M}_{0,n})$.

Proof. Consider a morphism of τ contracting all edges and tails except of i, j, k, l. It induces the forgetful morphism $\mathcal{M}(\tau) \to \overline{M}_{0,\{ijkl\}} \cong \mathbf{P}^1$. Two fibers over boundary divisors of the latter moduli space are represented by the cycles $\sum_{\{ij\tau' kl\}} \mathcal{M}(\tau')$ and $\sum_{\{ik\tau'' jl\}} \mathcal{M}(\tau'')$ respectively.

7.3. Theorem. *Relations $(7.1)_R$ span the space of all linear relations between d_τ.*

7.3.1. Lemma ([Ke]). *As an algebra, $H^* := H^*(\overline{M}_{0,n})$ is generated by the boundary divisorial cohomology classes D_S indexed by unordered partitions S of $\{1, \ldots, n\}$ into two parts S_1, S_2 of cardinality ≥ 2 and satisfying the following generating relations:*

$$\sum_{\{ijSkl\}} D_S = \sum_{\{ikTjl\}} D_T \qquad (7.2)_{ijkl}$$

and

$$D_S D_T = 0 \qquad (7.3)$$

if four sets $S_i \cap T_j$ are pairwise distinct and non-empty. (In this case we will call S and T incompatible).

In 3.2.2, D_S were denoted d_S whereas here we reserve lower case letters for homology classes. Classes D_S are dual to the homology classes d_σ where σ run over n-trees with two vertices, and (7.2) is a consequence of (7.1).

Denote now by H_* the linear space generated by the symbols $[d_\sigma]$ subject to all relations $(7.1)_R$ where σ, τ run over all n-isomorphism classes of n-trees.

There is an obvious surjective map $a: \ H_* \to H^*$:

$$a([d_\sigma]) := \textit{ the cohomology class dual to } d_\sigma. \tag{7.4}$$

7.3.2. Main Lemma. *H_* can be endowed with a structure of cyclic H^*-module generated by $[d_{\rho_n}] := 1$ so that the map*

$$b: \ H^* \to H_*, \ b(h) = h \cdot 1 \tag{7.5}$$

is surjective.

Comparing (7.4) and (7.5) we see that dim $H_* = \dim H^*$ and both a and b are linear isomorphisms. Theorem 7.3 follows.

Proof of 7.3.2. (Sketch). Every interior edge e of an n-tree σ determines a partition $S(\tau, e)$ of its tails. Put

$$\mathcal{S}(\tau) = \{S \mid \exists e, S = S(\tau, e)\}.$$

The stratum $\mathcal{M}(\tau)$ is the transversal intersection of pairwise compatible boundary divisors in $\mathcal{M}(\rho_n)$:

$$\mathcal{M}(\tau) = \cap_{S \in \mathcal{S}(\tau)} \mathcal{M}(\tau_S).$$

Therefore we devise the action of H^* upon H_* in such a way that

$$\prod_{S \in \mathcal{S}(\tau)} D_S.1 = [d_\tau] \tag{7.6}$$

making obvious the surjectivity of (7.5). If we now want to define any product $D_S.[d_\tau]$, (7.6) forces us to consider three cases.

Case 1. S is incompatible with some $T \in \mathcal{S}(\sigma)$. Then we put

$$D_S.[d_\sigma] = 0. \tag{7.7}$$

Case 2. S is compatible with all $T \in \mathcal{S}(\sigma)$ but $S \notin \mathcal{S}(\sigma)$. Then we check that $\mathcal{S}(\sigma) \cup \{S\} = \mathcal{S}(\tau)$ for a unique τ, and put

$$D_S.[d_\sigma] = [d_\tau]. \tag{7.8}$$

Case 3. $S \in \mathcal{S}(\tau)$. This case concentrates all the difficulties. Let us start with a two–vertex σ, $\mathcal{S}(\sigma) = \{S\}$. Choose $i, j \in S_1$, $k, l \in S_2$ and apply D_S to the relation between homology classes dual to $(7.2)_{ijkl}$. We are forced to put

$$D_S.[d_\sigma] = -\sum_\tau [d_\tau] \tag{7.9}$$

where τ runs over all trees with $\mathcal{S}(\tau) = \{S, S'\}$, $S' \neq S$, $i, j \in S'_1$, $k, l \in S'_2$, and then to check that, modulo postulated relations, the r.h.s. of (7.8) does not depend on the choice of i, j, k, l.

Now, consider the r.h.s. of (7.9) as a divisorial cohomology class in $H^*(\mathcal{M}(\sigma)) \cong H^*(\overline{M}_{0,S_1}) \otimes H^*(\overline{M}_{0,S_2})$. It can be naturally represented as a sum of two components $D_1 \otimes 1 + 1 \otimes D_2$. If d_τ is represented by $\mathcal{M}(\tau) \subset \mathcal{M}(\sigma)$, we have $(\tau, e) = (\tau_1, t_1) * (\tau_2, t_2)$, where e is the lift of the unique edge of σ to τ. The action of D_i upon d_{τ_i} is assumed to be inductively defined, which determines the action of D_S upon d_τ.

It remains to check that these prescriptions are compatible with (7.1)–(7.3).

This is a tedious but straightforward verification which we omit.

§8. Second Reconstruction Theorem

8.1. Definition. An abstract tree level system of correlation functions (ACF) over a coefficient field K consists of a pair (A, Δ) as in Def. 6.1 and a family of even linear maps

$$Y_n : \ A^{\otimes n} \to K, \ n \geq 3 \tag{8.1}$$

satisfying the following axioms:

8.1.1. S_n**–invariance.**

8.1.2. Coherence. In notation of 6.1 and (3.3) it reads: for any pairwise distinct $1 \leq i, j, k, l \leq n$,

$$\sum_{\{ijSkl\}} \sum_{a,b} \varepsilon(S) Y_{|S_1|+1}(\otimes_{r\in S_1}\gamma_r \otimes \Delta_a) g^{ab} Y_{|S_2|+1}(\Delta_b \otimes (\otimes_{s\in S_2}\gamma_s)) =$$

$$\sum_{\{ikTjl\}} \sum_{a,b} \varepsilon(T) Y_{|T_1|+1}(\otimes_{r\in T_1}\gamma_r \otimes \Delta_a) g^{ab} Y_{|T_2|+1}(\Delta_b \otimes (\otimes_{s\in T_2}\gamma_s)). \tag{8.2}$$

8.2. Example. For a tree level CohFT $I = (A, \Delta, I_{0,n})$, put

$$\langle I_{0,n}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_n) := \int_{\overline{M}_{0,n}} I_{0,n}(\gamma_1 \otimes \cdots \otimes \gamma_n). \tag{8.3}$$

These polynomials are called correlation functions of I. Their S_n–symmetry is obvious, and (8.2) follows from the Splitting Axiom 6.1 in the same way as (3.3) in the context of GW–classes. The following main result of this section shows that these examples essentially exhaust ACFs.

8.3. Theorem. *Any tree level ACF consists of correlation functions of a unique tree level CohFT.*

8.3.1. Remark. Starting with any ACF, one can construct a potential

$$\Phi(\gamma) = \sum_{n\geq 3} \frac{1}{n!} Y_n(\gamma^{\otimes n})$$

and check that the differential equations (4.13) are equivalent to the coherence relations (8.2). In this sense, WDVV equations are equivalent to tree level CohFTs.

We start a proof of 8.3 with some preliminaries.

8.4. Correlation functions on trees. In this context, we will be considering tensors

$$\mathcal{B}(\tau) := A^{\otimes T_\tau} \tag{8.4}$$

rather than $\mathcal{A}(\tau) = A^{\otimes F_\tau}$. This construction is obviously functorial with respect to pure contractions. More important is its behaviour with respect to glueing and cutting.

If $(\tau, e) = (\tau_1, t_1) * (\tau_2, t_2)$, we say that (τ_i, t_i) are obtained by cutting τ across e. One can easily generalize this notion for any subset of edges $E \subset E_\tau$ instead of $\{e\}$. For instance, cutting τ across all edges results in a set of one–vertex trees, *stars* of vertices $v \in V_\tau$. Formally, a star $\rho(v, \tau)$ has v as its vertex and $F_\tau(v)$ as its tails.

Let $\tau|E$ be the set of trees obtained from τ by cutting it across all $e \in E$. Tails of any $\sigma \in \tau|E$ consist of some tails of τ and some "half–edges" of τ, each edge in E giving rise to two tails of the latter type. Therefore, we can construct a well defined map

$$\mathcal{B}(E): \ \mathcal{B}(\tau) = A^{\otimes T_\tau} \to \otimes_{\sigma \in \tau|E} \mathcal{B}(\sigma) = A^{\otimes(\coprod T_\sigma)} \cong A^{\otimes T_\tau} \otimes (A \otimes A)^E \tag{8.5}$$

which tensor multiplies any element of $\mathcal{B}(\tau)$ by $\Delta^{\otimes E} \in (A \otimes A)^E$. (Compare this to (6.6)).

8.4.1. Lemma. *For any system of S_n–symmetric polynomials $Y_n: \ A^{\otimes n} \to K$, there exists a unique extension to trees*

$$Y(\tau): \ \mathcal{B}(\tau) \to K$$

with the following properties:

a). If ρ_n is one–vertex tree with tails $\{1, \dots, n\}$, then $Y(\rho_n) = Y_n$.

b). For any τ and any $E \subset E_\tau$, we have

$$Y(\tau) = \left(\otimes_{\sigma \in \tau|E} Y(\sigma)\right) \circ \mathcal{B}(E). \tag{8.6}$$

c). $Y(\tau)$ are compatible with tree isomorphisms.

Proof. Put

$$Y(\tau) := \otimes_{\sigma \in \tau|E_\tau} Y(\sigma) \circ \mathcal{B}(E_\tau) = \otimes_{v \in V_\tau} Y(\rho(v, \tau)) \circ \mathcal{B}(E_\tau), \tag{8.7}$$

and $Y(\rho(v, \tau)) = Y_{|F_\tau(v)|}$. Then a) follows by definition from the S_n–symmetry of Y_n, and (8.6) becomes a corollary of the associativity of tensor products.

As an example of (8.6), let $\tau = \tau_S$ be a tree with vertices v_1, v_2 and tails S_i ending at i–th vertex. Put $Y_{(i)} = Y(\rho(v_i, \tau))$. Cutting τ across its edge we get

$$Y(\tau)(\otimes_{i \in S} \gamma_i) = (Y_{(1)} \otimes Y_{(2)})(\otimes_{r \in S_1} \gamma_r \otimes \Delta \otimes (\otimes_{s \in S_2} \gamma_s)) =$$

$$\sum_{a,b} Y_{(1)}(\otimes_{r \in S_1} \gamma_r \otimes \Delta_a) g^{ab} Y_{(2)}(\Delta_b \otimes (\otimes_{s \in S_2} \gamma_s)). \tag{8.8}$$

We turn now to correlation functions.

8.5. Lemma. *Let $\{Y_n\}$ be correlation functions of a CohFT I, $\{Y(\tau)\}$ their extension to trees which we will call operadic correlation functions.*

For a tree τ, denote by $f:\ \tau \to \rho$ a maximal pure contraction identical on tails, and by $\varphi : \mathcal{M}(\tau) \to \mathcal{M}(\rho)$ the corresponding embedding. Then

$$Y(\tau) = \int_{\mathcal{M}(\tau)} \varphi^*(I(\rho)). \tag{8.9}$$

Proof. From (6.4), we know that

$$\mathcal{M}(\tau) = \prod_{v \in V_\tau} \mathcal{M}(\rho(v,\tau)).$$

Write the relation (6.8) for f taking in account the following identifications:

$$\mathcal{B}(\tau) = \mathcal{A}(\rho) = A^{\otimes T_\tau},\ \mathcal{A}(f) = \mathcal{B}(E_\tau),$$

$$H^*\mathcal{M}(\tau) = \otimes_{v \in V_\tau} H^*\mathcal{M}(\rho(v,\tau)),\ H^*\mathcal{M}(f) = \varphi^*.$$

Thus, applying in addition (8.7), we see that the following two functions $A^{\otimes T_\tau} \to K$ coincide:

$$\int_{\mathcal{M}(\tau)} I(\tau) \circ \mathcal{A}(f) = \left(\otimes_{v \in V_\tau} \int_{\mathcal{M}(\rho(v,\tau))} I(\rho(v,\tau)) \right) \circ \mathcal{B}(E_\tau) = Y(\tau)$$

and

$$\int_{\mathcal{M}(\tau)} H^*\mathcal{M}(f) \circ I(\rho) = \int_{\mathcal{M}(\tau)} \varphi^*(I(\rho)).$$

8.5.1. Corollary. *Let $f_\alpha :\ \tau_\alpha \to \rho$ be a family of maximal pure contractions defining strata $\mathcal{M}_\alpha$ in $\mathcal{M}(\rho) \cong \overline{M}_{0,n}$ whose homology classes d_α satisfy a linear relation $R:\ \sum_\alpha a_\alpha d_\alpha = 0$.*

If $Y(\tau)$ are operadic correlation functions of a CohFT I, they satisfy the identity

$$\sum_\alpha a_\alpha Y(\tau_\alpha) = 0. \tag{8.10}$$

We will say that (8.10) is *correlated* with R.

For example, the Coherence Axiom 8.1.2 together with (8.6) means that ACF satisfy all equations correlated with Keel's linear relations between boundary divisors.

The central observation is that this implies the following stronger statement:

8.6. Lemma. *Any ACF satisfies all the equations correlated with linear relations between strata homology classes.*

Proof. Clearly, it suffices to treat relations $(7.1)_R$ where $R = (\tau, \{i,j,k,l\}, v)$ as in (7.2). Consider the star $\rho = \rho(v,\tau)$ and its four tails $\bar{i}, \bar{j}, \bar{k}, \bar{l}$ corresponding to e_i, e_j, e_k, e_l. Write the Keel relation in $H_*(\mathcal{M}(\rho(v,\tau))$ and the correlated identity (8.10):

$$\sum_{\{\bar{i}\bar{j}\rho'\bar{k}\bar{l}\}} Y(\rho') = \sum_{\{\bar{i}\bar{k}\rho''\bar{j}\bar{l}\}} Y(\rho'') \tag{8.11}$$

where the sums are taken over two–vertex trees with tails F_ρ.

There is a natural bijection between summands in (8.11) and $(7.1)_R$.

Denote by E the set of all interior edges of τ incident to v excepting e_i, e_j, e_k, e_l. Cut τ across these edges, and denote by T the set of resulting trees excepting the star of v. According to (8.6), for the terms of l.h.s in (8.11) and $(7.1)_R$ corresponding to each other, we have

$$Y(\tau') = Y(\rho') \otimes (\otimes_{\sigma \in T}) Y(\sigma) \circ \mathcal{B}(E),$$

and similarly for the r.h.s.

Hence from (8.11) it follows that

$$\sum_{\{ij\tau'kl\}} Y(\tau') = \sum_{\{ik\tau''jl\}} Y(\tau'').$$

This identity is correlated with $(7.1)_R$.

8.7. Proof of the Theorem 8.3. Start with an ACF Y_n. It suffices to reconstruct an "economy class" CohFT defined by 6.1 rather than the full fledged operadic one. Therefore, in this section, as in 7.1, we may and will consider only n–trees and n–contractions.

First, construct $I(\rho_n) = I_{0,n}$. From (8.9) we know the integrals of $I(\rho_n)$ over all tree strata, whose homology classes generate $H_*(\mathcal{M}(\rho_n))$, and by Lemma 8.6, these integrals extend to a linear functional on $H_*(\mathcal{M}(\rho_n))$. By Poincaré duality, this uniquely defines $I_{0,n}$.

Second, check (6.1). It suffices to verify that integrals of both sides over any stratum $\mathcal{M}(\tau)$ contained in the relevant boundary divisor $\mathcal{M}(\tau_S)$ coincide. But this is a particular case of (8.6). In fact, if $\tau = \tau_S$, this is exactly (8.8). Generally, one can assume that $(\tau, e) = (\tau_1, t_1) * (\tau_2, t_2)$, $T_{\tau_1} = S_1 \cup \{t_1\}$, $T_{\tau_2} = S_2 \cup \{t_2\}$, and apply (8.6) to this glueing.

In the context of GW–classes, we can now more or less formally deduce the following version of the Reconstruction Theorem 8.3.

8.8. Theorem. *Assume that for a manifold V, a system of maps*

$$Y^V_{n,\beta} : \ H^*(V, \mathbf{Q})^{\otimes n} \to \mathbf{Q}$$

is given satisfying all the conditions that are imposed by Axioms 2.2.0–2.2.6 on the tree level codimension zero GW-classes.

Then there exists a unique tree level system of GW–classes $I^V_{0,n,\beta}$ *such that* $\langle I^V_{0,n,\beta}\rangle = Y^V_{n,\beta}$.

To prove it, we first reconstruct the relevant CohFT, and then check the Axioms involving the geometry of V. We leave the details to an interested reader.

Thus starting with $V = \mathbf{P}^1$ (see (5.3) and (5.4)), and applying the tensor product construction, we can define GW–classes for $\mathbf{P}^1 \times \cdots \times \mathbf{P}^1$. In particular, identities in 5.2.4 are thereby established. Since $((\mathbf{P}^1)^n)^{S_n} \cong \mathbf{P}^n$, we get GW–classes for $\mathbf{P}^n$ as well.

Finally, results announced in [RT] give codimension zero tree level GW-classes and therefore all tree level GW-classes for semi–positive symplectic manifolds.

References

[AM] P. S. Aspinwall, D. R. Morrison. Topological field theory and rational curves. Preprint OUTP-91-32P, 1991.

[BCOV] M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa. Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes. Preprint HUTP-93/A025.

[BG] A. Beilinson, V. Ginzburg. Infinitesimal structure of moduli spaces of G–bundles. Int. Math. Res. Notices, Duke Math. Journ., 4 (1992), 63–74.

[D] B. Dubrovin. Integrable systems in topological field theory. Nucl. Phys. B 379 (1992), 627–689.

[GK] E. Getzler, M. M. Kapranov. Cyclic operads and cyclic homology. Preprint, 1994.

[GiK] V. A. Ginzburg, M. M. Kapranov. Koszul duality for operads. Preprint, 1993.

[I] C. Itzykson. Counting rational curves on rational surfaces. Preprint Saclay T94/001.

[Ke] S. Keel. Intersection theory of moduli spaces of stable n–pointed curves of genus zero. Trans. AMS, 330 (1992), 545–574.

[Ko] M. Kontsevich. A_∞–algebras in mirror symmetry. Bonn MPI Arbeitstagung talk, 1993.

[Ma1] Yu. Manin. Cubic forms: algebra, geometry, arithmetic. North Holland, 1974.

[Ma2] Yu. Manin. Problems on rational points and rational curves on algebraic varieties. (To be published in Surveys of Diff. Geometry).

[R] Y. Ruan. Topological sigma model and Donaldson type invariants in Gromov theory. Preprint MPI, 1992.

[RT] Y. Ruan, G. Tian. Mathematical theory of quantum cohomology. Preprint, 1993.

[V] A. Voronov. Topological field theories, string backgrounds, and homotopy algebras. Preprint, 1993.

[W] E. Witten, Two-dimensional gravity and intersection theory on moduli space. Surveys in Diff. Geom., 1 (1991), 243–310.

[Y] S.–T. Yau, ed. Essays on Mirror Manifolds. International Press Co., Hong Cong, 1992.

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