# TRANSLATION EQUIVALENCE IN FREE GROUPS

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Abstract. Motivated by the work of Leininger on hyperbolic equivalence of homotopy classes of closed curves on surfaces, we investigate a similar phenomenon for free groups. Namely, we study the situation when two conjugacy classes $c, c'$ in a free group $F$ have the property that for every free isometric action of $F$ on an $\mathbb{R}$-tree $X$ the translation lengths of $c$ and $c'$ on $X$ are equal.

## 1. Introduction

Let $S$ be a closed oriented surface of negative Euler characteristic and let $\gamma$ be a free homotopy class of essential closed curves on $S$. If $\rho$ is a hyperbolic metric then $\gamma$ contains a unique curve $c$ of minimal $\rho$-length. We denote this length by $l_\rho(\gamma)$. The curve $c$ is a closed geodesic on $(S,\rho)$ and $l_\rho(\gamma)$ is the translation length of any representative of $\gamma$ in $G = \pi_1(S)$ for the action of $G$ on $\mathbb{H}^2 = \tilde{S}$ corresponding to $\rho$. There is an obvious identification between the set of conjugacy classes $C$ of $G$ and the set of free homotopy classes of closed curves on $S$ and we shall not distinguish between the two.

Thus each hyperbolic structure $\rho$ on $S$ defines a so-called *marked length spectrum* $l_\rho : C \to \mathbb{R}$. It is well-known and easy to see that if $l_{\rho_1} = l_{\rho_2}$ then $\rho_1$ and $\rho_2$ essentially coincide (that is, they define the same point of the Teichmuller space of $S$). However, the dual situation is different. For $\gamma_1, \gamma_2 \in C$ we say that $\gamma_1$ is *hyperbolically equivalent to* $\gamma_2$, denoted $\gamma_1 \equiv_h \gamma_2$, if for every hyperbolic structure $\rho$ on $S$ we have $l_\rho(\gamma_1) = l_\rho(\gamma_2)$. In more algebraic terms, for two conjugacy classes $\gamma_1, \gamma_2 \in C$ we have $\gamma_1 \equiv_h \gamma_2$ if for every discrete and co-compact isometric action of $G$ on $\mathbb{H}^2$ the translation lengths of $\gamma_1$ and $\gamma_2$ are equal. It can happen that $\gamma_1 \neq \gamma_2^{\pm 1}$ and yet $\gamma_1 \equiv_h \gamma_2$. The main source of hyperbolic equivalence comes from the so-called "trace identities" in $SL(2,\mathbb{C})$. A number of interesting new results(some of which we shall discuss later) about hyperbolic equivalence were recently obtained by Chris Leininger [8].

We are interested in investigating a similar phenomenon for free groups. In this context the Teichmuller space has to be replaced by the Culler-Vogtmann outer space [4], so that instead of actions on $\mathbb{H}^2$ we consider

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free and discrete actions on $\mathbb{R}$-trees. Recall that if $G$ is a group acting by isometries on an $\mathbb{R}$-tree $X$, and $g \in G$ is an element of $G$, the *translation length* $l_X(g)$ is defined as:

$$l_X(g) = \inf_{x \in X} d(x, gx).$$

It is easy to see that $l_X(g)$ only depends on the conjugacy class of $g$ and that in the above definition infimum can be replaced by a minimum. Thus $l_X(g) = 0$ if and only if $g$ fixes a point of $X$. The above discussion naturally leads us to the following definition:

**Definition 1.1.** Let $F$ be a finitely generated free group and let $g, h \in F$ be elements of $F$. We say that $g$ and $h$ are *translation equivalent in $F$*, denoted $g \equiv_{tr} h$, if for every free and discrete isometric action of $F$ on an $\mathbb{R}$-tree $X$ we have $l_X(g) = l_X(h)$.

Clearly, if $g \equiv_{tr} h$ in $F$ and $\phi : F \to F_1$ is an injective homomorphism to a free group $F_1$ then $\phi(g) \equiv_{tr} \phi(h)$ in $F_1$. Indeed, suppose $F_1$ acts freely and discretely by isometries on an $\mathbb{R}$-tree $X$. Then, by restriction, we get a free and discrete action of $\phi(F)$ on $X$, and via a twist by $\phi$, we also get a free and discrete action of $F$ itself on $X$. Namely, $f \cdot p := \phi(f)p$, where $p \in X, f \in F$. Since $u$ and $v$ are translation equivalent in $F$, it follows that $l_X(u) = l_X(v)$, that is, $l_X(\phi(u)) = l_X(\phi(v))$.

A phenomenon related to but different from translation equivalence was studied by Smillie and Vogtmann [12] who, given an arbitrary finite set of conjugacy classes in a free group, constructed multi-parametric families of free discrete actions where the translation length of each conjugacy class from this set remains constant through the family. Results similar in spirit to those of [12] were also obtained by Cohen, Lustig and Steiner [3].

The first natural problem is to demonstrate that there are some nontrivial instances of translation equivalence in free groups. Unlike in the surface group case, one cannot rely on trace identities here. We provide two different sources of translation equivalence: one based on two-variable palindromic reversion and another based on power redistribution for some products of translation equivalent elements. Both methods can be iterated and used to produce arbitrarily large finite collections of distinct conjugacy classes in $F$ that are pairwise translation equivalent. It turns out that both of these sources can also be used to produce hyperbolic equivalence in the context of $\mathbb{H}^2$-actions, although some distinctions do arise (see Example 7.5).

Another natural question is to give a more combinatorial and algebraic (and possibly algorithmic) characterization of translation equivalence.

Recall that an isometric action of a group $G$ on an $\mathbb{R}$-tree $X$ is *very small* [2] if the following conditions hold:

1. The action is *small*, that is, arc stabilizers do not contain free subgroups of rank two.
2. Stabilizers of tripods are trivial.

(3) For any $g \in G$ and for each $n \neq 0$ the fixed sets of $g$ and $g^n$ are equal.

In particular, every free action and, more generally, an action with trivial arc stabilizers, is very small. Results of Cohen-Lustig [2] and Bestvina-Feighn [1] imply that an action of $F_n$ on an $\mathbb{R}$-tree is very small if and only if this action represents a point in the standard compactification of the outer space. Thus a very small action can always be approximated (in the sense of length functions) by a sequence of free simplicial actions.

If $A$ is a basis of a free group $F$ and if $g \in F$, we denote by $||g||_A$ the cyclically reduced length of $g$ with respect to $A$ (that is $||g||_A = l_X(g)$ where $X$ is the Cayley graph of $F$ with respect to $A$). A *cyclic word* in $A^{\pm 1}$ is the set of cyclic permutations of a word. A cyclic word is *reduced* if every word in it is freely reduced. There is a canonical identification between the set of reduced cyclic words in $A^{\pm 1}$ and the set of conjugacy classes in $F$. If $w$ is a cyclic word and $x, y \in A^{\pm 1}$, we denote by $n(w; x, y)$ the total number of occurrences of subwords $xy$ and $y^{-1}x^{-1}$ in $w$. Thus $n(w; x, y) = n(w; y^{-1}, x^{-1}) = n(w^{-1}; x, y)$.

**Definition 1.2** (Whitehead graph)**.** Let $w$ be a nontrivial cyclic word in $F(A)$. The *Whitehead graph* $\Gamma_A(w)$ *of* $w$ *with respect to* $A$ is a labelled directed graph defined as follows. The vertex set of $\Gamma_A(w)$ is $A^{\pm 1}$. If $x, y \in A^{\pm 1}$ are such that $x \neq y$, there is a directed edge in $\Gamma_A(w)$ from $x$ to $y$ with label $n(w; x, y^{-1})$.

If $[g]$ is a nontrivial conjugacy class in $F$, then $[g]$ is represented by a unique reduced cyclic word $w$ in $F(A)$. Then the *Whitehead graph* $\Gamma_A([g])$ *of* $[g]$ *with respect to* $A$ is defined as $\Gamma_A(w)$.

Note that by definition for any nontrivial reduced cyclic word $w$ we have $\Gamma_A(w) = \Gamma_A(w^{-1})$.

We obtain the following result:

**Theorem A.** *Let $F$ be a finitely generated free group and let $g, h \in F$ be nontrivial elements.*

*Then the following statements are equivalent:*

(1) *For every very small action of $F$ on an $\mathbb{R}$-tree $X$ we have $l_X(g) = l_X(h)$.*
(2) *For every free action of $F$ on an $\mathbb{R}$-tree $X$ we have $l_X(g) = l_X(h)$.*
(3) *We have $g \equiv_{tr} h$ in $F$.*
(4) *For every free basis $A$ of $F$ we have $||g||_A = ||h||_A$.*
(5) *For every free basis $A$ of $F$ the conjugacy classes $[g]$ and $[h]$ have the same Whitehead graphs with respect to $A$, that is $\Gamma_A([g]) = \Gamma_A([h])$.*

Theorem A immediately yields a more combinatorial version of translation equivalence:

**Corollary 1.3.** *Let $F$ be a finitely generated free group with a free basis $A$ and let $g, h \in F$. Then the following conditions are equivalent:*

(1) *We have* $g \equiv_{tr} h$ *in* $F$.
(2) *For every automorphism* $\phi$ *of* $F$ *we have* $||\phi(g)||_A = ||\phi(h)||_A$.
(3) *For every injective endomorphism* $\phi$ *of* $F$ *we have* $||\phi(g)||_A = ||\phi(h)||_A$.
(4) *For every free group* $F_1$ *with a free basis* $B$ *and for every injective homomorphism* $\phi : F \to F_1$ *we have* $||\phi(g)||_B = ||\phi(h)||_B$.

The a priori weakest condition in Theorem A is condition (4) which deserves further comment. Let $S$ be a closed surface as before. If $\gamma, \delta$ are free homotopy classes of closed curves on $S$, we denote by $i(\gamma, \delta)$ the *geometric intersection number* of $\gamma$ and $\delta$. We will say that free homotopy classes $\gamma_1, \gamma_2$ of closed curves on $S$ are *simple intersection equivalent* if for every essential *simple closed curve* $c$ on $S$ we have $i(\gamma_1, [c]) = i(\gamma_2, [c])$. It is easy to see that hyperbolic equivalence implies simple intersection equivalence. However, surprisingly, the converse is not true as was recently proved by Chris Leininger [8].

It is natural to ask what the analogue of simple intersection equivalence is for free groups. If $G = \pi_1(S)$ and $c$ is an essential simple closed curve on $S$, then $c$ defines a splitting of $G$ as either an amalgamated product or as an HNN-extension (depending on whether $c$ is separating or not) over a cyclic subgroup. Let $X$ be the Bass-Serre tree corresponding to that splitting. It is easy to see that for any $g \in G$ we have $i([g], [c]) = l_X(g)$. The difficulty is that for a free group $F$, no single element of $F$ (not even a primitive element) defines a splitting of $F$. However, a free basis $A$ of $F$ does define such a splitting, namely the splitting of $F$ as the multiple HNN-extension of the trivial group with the stable letters corresponding to the elements of $A$. The Bass-Serre tree $X$ of this splitting is precisely the Cayley graph of $F$ with respect to $A$. Then for any $g \in F$ we have $l_X(g) = ||g||_A$, the cyclically reduced length of $g$ with respect to $A$. Therefore for a free basis $A$ of $F$ and an element $g \in F$ we can define the *intersection number* $i(g, A) := ||g||_A$. By analogy with the surface group case we will say that $g_1, g_2 \in F$ are *simple intersection equivalent* in $F$ if for every free basis $A$ of $F$ we have $i(g_1, A) = i(g_2, A)$. Theorem A says, in particular, that, unlike for the surface group case, in free groups simple intersection equivalence is the same as translation equivalence.

We can now state the main sources of translation equivalence in free groups that we have discovered so far.

Let $w(x, y) \in F(x, y)$ be a freely reduced word. We will denote by $w^R(x, y)$ the word $w(x, y)$ read backwards, but without inverting the letters. Thus $w^R(x, y) = (w(x^{-1}, y^{-1}))^{-1}$. We prove:

**Theorem B.** *Let* $F$ *be a free group of rank* $k \geq 2$ *and let* $w(x, y) \in F(x, y)$ *be a freely reduced word. Then for any* $g, h \in F$ *we have*

$$w(g, h) \equiv_{tr} w^R(g, h) \quad \text{in } F.$$

**Theorem C.** *Let* $F$ *be a free group of rank* $k \geq 2$ *and let* $g, h \in F$ *be such that* $g \equiv_{tr} h$ *(for example, if* $g$ *is conjugate to* $h$*) and that* $g \neq h^{-1}$*. Then* *for any positive integers $p, q, i, j$ such that $p + q = i + j$ we have*

$$g^p h^q \equiv_{tr} g^i h^j \quad \text{in } F.$$

The following is an analogue of a result of Randol [14] about hyperbolic surfaces.

**Corollary 1.4.** *Let $F$ be a free group of rank $k \geq 2$. Then for any integer $M \geq 1$ there exist elements $g_1, \dots, g_M \in F$ such that $g_i \equiv_{tr} g_j$ in $F$ and such that for $i \neq j$ $g_i$ is not conjugate to $g_j^{\pm 1}$.*

*Proof.* Let $M \geq 1$ be an integer and let $(a, b, \dots)$ be a free basis of $F$. Consider $g = a$ and $h = ba^{-1}b^{-1}$. Put $g_i = g^i h^{2M-i} = a^i ba^{i-2M} b^{-1}$ for $i = 1, \dots, M$. Then by Theorem C $g_i \equiv_{tr} g_j$ in $F$. On the other hand $g_i$ is not conjugate to $g_j^{\pm 1}$ unless $i = j$. This can be seen, for example, by observing that $g_i$ and $g_j^{\pm 1}$ have distinct images in the abelianization of $F$ for $i \neq j$. □

**Remark 1.5.** Theorem B can be iterated to produce other examples with the same properties as in Corollary 1.4. Namely, let $\phi_i : F(x, y) \to F(x, y)$ be injective endomorphisms of $F(x, y)$ for $i = 1, \dots, N$. Let $\phi := \phi_N \circ \cdots \circ \phi_1$. For $i = 1, \dots, N-1$ put $\psi_i := \phi_N \circ \cdots \circ \phi_{i+1}$ and $\theta_i = \phi_i \circ \cdots \circ \phi_1$. Then $\phi = \psi_i \circ \theta_i$. Let $\psi_i(x) = u_i(x, y)$, $\psi_i(y) = v(x, y)$, $\theta_i(x) = t_i(x, y)$ and $\theta_i(y) = s_i(x, y)$. Denote $w(x, y) = \phi(x)$. Then $w = \psi_i(\theta_i(x)) = \psi_i(t_i(x, y)) = t_i(u_i(x, y), v_i(x, y))$. Denote $w_i = t_i^R(u_i(x, y), v_i(x, y))$. Theorem B implies that, $w(g, h) \equiv_{tr} w_i(g, h)$ for any $g, h \in F$ for $i = 1, \dots, N-1$. It is possible, for each $N \geq 1$, to choose the endomorphisms $\phi_i$ and then elements $g, h \in F$ so that $w_1, \dots, w_{N-1}$ are pairwise non-conjugate in $F$.

Moreover, as we observe later in Corollary 4.6, it is possible to generalize the proof of Corollary 1.4 to many non-free groups.

We show, using the $SL_2$ trace identities, that Theorem A and a version of Theorem B also hold for the standard hyperbolic equivalence. Using a limiting argument that we conclude that these statements also apply to the tree actions that occur in the Thurston boundary of the Teichmuller space of a closed hyperbolic surface (both orientable and non-orientable). Recall that each point $\mu$ in the Thurston boundary of the Teichmuller space is a measured lamination. There is an $\mathbb{R}$-tree $X_\mu$ dual to (the lift of) this lamination that comes equipped with a small isometric action of the fundamental group of the surface. Note that, in the case of a non-orientable surface, not all such actions are very small.

**Theorem D.** *Let $S$ be a closed surface (possibly non-orientable) of negative Euler characteristic and let $G = \pi_1(S)$. Then the following hold:*

(1) *For any $g, h \in G$ and for any $w(x, y) \in F(x, y)$ and for any tree action $\mu$ of $G$ in the Thurston boundary of the Teichmuller space of $S$ we have*
$$l_{X_\mu}(w(g, h)) = l_{X_\mu}(w^R(g, h)).$$

(2) *For any conjugate elements $g, h \in G$, for any $p, q > 0$ and for any tree action $\mu$ of $G$ in the Thurston boundary of the Teichmuller space of $S$ we have*
$$l_{X_\mu}(g^p h^q) = l_{X_\mu}(g^q h^p).$$
(3) *If $S$ is orientable then for any conjugate elements $g, h \in G$, for each point $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$ and for any positive integers $p, q, i, j$ such that $p + q = i + j$ we have*
$$l_{X_\mu}(g^p h^q) = l_{X_\mu}(g^i h^j).$$

The following interesting question remains open:

**Problem 1.6.** Is there an algorithm that, given two elements in a finitely generated free group, decides whether or not they are translation equivalent?

It can be deduced from the results of Leininger [8] that hyperbolic equivalence in surface groups is algorithmically decidable using standard commutative algebra techniques applied to the representation variety of the surface group. We would also like to find some other sources of translation equivalence, apart from those discussed in this paper.

## 2. Whitehead graphs and a characterization of translation equivalence

Let $F = F(a_1, \dots, a_k)$ where $k \geq 2$. Denote $A = \{a_1, \dots, a_k\}$. Thus $F = F(A)$.

If $w$ is a cyclic word and $x \in A^{\pm 1}$, we denote by $n(w; x)$ to be the total number of occurrences of $x^{\pm 1}$ in $w$. Thus $n(w; x) = n(w, x^{-1})$.

Let $x, y \in \{a_1, \dots, a_k\}^{\pm 1}$ such that $x \neq y^{\pm 1}$. Denote by $\phi_{x,y}$ the automorphism of $F$ that sends $x$ to $xy$ and fixes each generator $a_i \neq x^{\pm 1}$. For $i \neq j, 1 \leq i, j \leq k$ denote $\phi_{i,j} := \phi_{a_i, a_j}$.

The following lemma is obvious:

**Lemma 2.1.** *For any cyclic word $w$ we have and any $x, y \in \{a_1, \dots, a_k\}^{\pm 1}$ such that $x \neq y^{\pm 1}$ we have*
$$||\phi_{x,y}(w)|| - ||w|| = n(w; x) - 2n(w; x, y^{-1}).$$

**Lemma 2.2.** *Let $w$ be a cyclic word. Then:*
$$n(w; a_i) = \frac{1}{k}\big(||w|| + \sum_{j \neq i}(||\phi_{i,j}(w)|| - ||\phi_{j,i}(w)||)\big).$$

*Proof.* Note that $n(w; x, y^{-1}) = n(w; y, x^{-1})$. Therefore

$$||\phi_{x,y}(w)|| - ||w|| = n(w; x) - 2n(w; x, y^{-1})$$
$$||\phi_{y,x}(w)|| - ||w|| = n(w; y) - 2n(w; y, x^{-1}) = n(w; y) - 2n(w; x, y^{-1}),$$
and so
$$||\phi_{x,y}(w)|| - ||\phi_{y,x}(w)|| = n(w; x) - n(w; y).$$

Let $x = a_i$ and $y$ vary over $a_j, j \neq i$. Then summing up the instances of the above equality for $x = a_i, y = a_j$ we get

$$(k-1)n(w;a_i) - \sum_{j\neq i} n(w,a_j) = \sum_{j\neq i}(||\phi_{i,j}(w)|| - ||\phi_{j,i}(w)||)$$

On the other hand

$$n(w;a_i) + \sum_{j\neq i} n(w,a_j) = ||w||.$$

Adding the above formulas we get

$$k\, n(w;a_i) = ||w|| + \sum_{j\neq i}^{n}(||\phi_{i,j}(w)|| - ||\phi_{j,i}(w)||),$$

which yields the statement of the lemma. □

**Proposition 2.3.** *Let $F$ be a finitely generated free group and let $u, v \in F$ be nontrivial elements such that for every free basis $A$ of $F$ we have $||u||_A = ||v||_A$. Then for every free basis $A$ of $F$ the Whitehead graphs of $[u]$ and $[v]$ with respect to $A$ are equal.*

*Proof.* Let $A$ be a free basis of $F$. We consider the conjugacy classes $[u]$, $[v]$ as cyclic words over $A$. The assumptions of the proposition imply that for every automorphism $\phi$ of $F$

$$||\phi(u)||_A = ||\phi(v)||_A.$$

By Lemma 2.2 it follows that for each $x \in A$ we have $n([u];x) = n([v],x)$. Therefore by Lemma 2.1 for any $x, y \in A^{\pm 1}$ such that $x \neq y^{\pm 1}$ we have

$$n([u];x,y) = n([v];x,y).$$

For a fixed $x \in A$ and an arbitrary cyclic word $w$

$$n(w,x) = n(w;x,x) + \sum_{y\neq x^{\pm 1}} n(w;x,y).$$

Since $n([u];x) = n([v];x)$ and $n([u];x,y) = n([v],x,y)$ for any $y \neq x^{\pm 1}$, $y \in A^{\pm 1}$, it follows that $n([u];x,x) = n([v];x,x)$.

Thus we have shown that for any $x, y \in A^{\pm 1}$ such that $x \neq y^{-1}$ we have $n([u];x,y) = n([v],x,y)$. This means that $[u]$ and $[v]$ have equal Whitehead graphs with respect to $A$, as claimed. □

We can now establish Theorem A from the Introduction:

**Theorem 2.4.** *Let $F$ be a finitely generated free group and let $g, h \in F$ be nontrivial elements.*

*Then the following statements are equivalent:*

1. *For every very small action of $F$ on an $\mathbb{R}$-tree $X$ we have $l_X(g) = l_X(h)$.*
2. *For every free action of $F$ on an $\mathbb{R}$-tree $X$ we have $l_X(g) = l_X(h)$.*
3. *We have $g \equiv_{tr} h$ in $F$.*

(4) *For every free basis $A$ of $F$ we have $||g||_A = ||h||_A$.*
(5) *For every free basis $A$ of $F$ the conjugacy classes $[g]$ and $[h]$ have the same Whitehead graphs with respect to $A$, that is $\Gamma_A([g]) = \Gamma_A([h])$.*

*Proof.* Let $k$ be the rank of $F$. We may assume that $k \geq 2$ since for $k = 1$ the statement of the theorem is obvious.

The implications $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are obvious. Moreover, (3) implies (1) since by the results of [2] every very small action is the limit (in the sense of length functions) of free discrete actions. Thus $(1), (2), (3)$ are equivalent. The implication $(4) \Rightarrow (5)$ is Proposition 2.3.

We will now show that (5) implies (3). Indeed, suppose that $F$ is acting freely dicretely and isometrically on an $\mathbb{R}$-tree $X$. Let $Y = X/F$ be the quotient graph of $X$. Since $X$ is a metric tree, the edges $e$ of $Y$ come equipped with the lengths $l(e)$. Thus every edge-path in $Y$ has a *length* which is the sum of the lengths of the edges of this path. There is an obvious canonical identification between $F$ and $\pi_1(Y, y)$ where $y$ is a vertex of $Y$. Choose an orientation $EY = E^+Y \sqcup E^-Y$ on $Y$.

Given a conjugacy class $g$, represent it by an immersed loop $\gamma$ in $Y$. Then $l_X(g) = \sum_{e \in E^+Y} n(e, \gamma) l(e)$, where $n(e, \gamma)$ is the number of times that $\gamma$ traverses $e$ (in either direction). We prove that (5) implies (3) by showing that, given $e \in E^+Y$, the number $n(e, \gamma)$ is completely determined by the Whitehead graphs of $g$ with respect to free bases of $F$.

There are two cases.

First, suppose that $e$ does not separate $Y$. Choose a maximal tree not containing $e$, and consider the associated free basis $A$ of $F = \pi_1(Y, y)$. Then $n(e, \gamma) = n(g, a_e)$, where $a_e \in A$ is the generator corresponding to $e$.

If $e$ separates $Y$, so that $Y = Y_1 \cup e \cup Y_2$, choose a basis $A$ of $F$ associated to any maximal tree. It is partitioned as $A = A_1 \sqcup A_2$, with $a \in A_i$ if and only if it corresponds to an edge in $Y_i$. Then $n(e, \gamma) = \displaystyle\sum_{a \in A_1, b \in A_2} n(g; a, b)$. $\square$

## 3. Palindromic sources of translation equivalence

Let $F$ be a free group of rank $k \geq 2$ and let $A = \{a_1, \dots, a_k\}$ be a free basis of $F$. Let $u = u(a_1, \dots, a_k) \in F$ be a freely reduced word over $A$. We define the *palindromic reverse of $u$ with respect to $A$*, denoted $u^R$, as:

$$u^R := u(a_1^{-1}, \dots, a_k^{-1})^{-1}.$$

Thus $u^R$ is the word $u$ read backwards without inverting the letters.

Similarly, we define the palindromic reverse $w^R$ of a cyclic word $w$ over $A$. Thus $w^R$ is again a cyclic word. Namely, if $w$ is represented by a cyclically reduced word $u$ then $w^R$ is represented by the cyclically reduced word $u^R$.

**Proposition 3.1.** *Let $F = F(a, b)$ be free of rank two. Then for any cyclic word $w$ in $F$ and for any $\phi \in Aut(F)$ we have:*

$$(\phi(w))^R = \phi(w^R).$$

*Proof.* For any $\psi \in Aut(F)$ denote by $\overline{\psi}$ the image of $\psi$ in $Out(F)$. For a free basis $A = (a, b)$ of $F$ denote by $\tau_A$ the automorphism of $F$ defined as $\tau_A(a) = a^{-1}, \tau_A(b) = b^{-1}$. It is easy to see that in $Out(F)$ the element $\overline{\tau_A}$ commutes with all the elementary Nielsen automorphisms with respect to $A$ and hence $\overline{\phi_A}$ is central in $Out(F)$. Therefore for any other free basis $B$ of $F$ we have $\overline{\tau_A} = \overline{\tau_B}$.

Suppose now that $w(a, b)$ is a cyclic word in $F(a, b)$ and $\phi \in Aut(F)$. Let $u(a, b) = \phi(w)$. Then by the above observation $\phi(w(a^{-1}, b^{-1})) = u(a^{-1}, b^{-1})$. Since $w^R = (w(a^{-1}, b^{-1}))^{-1}$ and $u^R = (u(a^{-1}, b^{-1}))^{-1}$, this implies that $(\phi(w))^R = \phi(w^R)$. □

**Remark 3.2.** It is well-known that $Out(F_2) \cong GL(2, \mathbb{Z})$ and that the center of $GL(2, \mathbb{Z})$ is cyclic of order two. Thus in fact the outer automorphism $\overline{\tau_A}$ is the only nontrivial element of the center of $Out(F_2)$, although we did not need this fact in the above proof.

We can now prove Theorem B from the Introduction:

**Theorem 3.3.** *Let $F$ be a free group and let $g, h \in F$. Then for any freely reduced word $w(x, y) \in F(x, y)$ the elements $w(g, h)$ and $w^R(g, h)$ are translation equivalent in $F$.*

*Proof.* If $g$ and $h$ commute in $F$ then the statement is obvious. Suppose now that $g$ and $h$ do not commute and hence $F_1 = \langle g, h \rangle$ is a free group of rank two. Let $F$ act freely and discretely by isometries on an $\mathbb{R}$-tree $X$. Let $X_1$ be the minimal $F_1$-invariant subtree and let $Y = X_1/F_1$ be the quotient graph. Then topologically $Y$ is either a wedge of two circles or is a $\theta$-graph or $Y$ consists of two disjoint circles joined by an edge. In each case $Y$ possesses an involution isometry $\sigma$ that leaves a maximal subtree $T$ of $Y$ invariant and takes every edge $e$ outside of a maximal tree to $\bar{e}$. This is shown in Figure 1.

Thus if $c_1, c_2$ is the basis of $\pi_1(Y, y)$ corresponding to $T$, then for any cyclic word $u(c_1, c_2)$ we have $\sigma_\# u(c_1, c_2) = u(c_1^{-1}, c_2^{-1})$. Since $\sigma$ is an isometry of $Y$, this means that $l_X(u(c_1, c_2)) = l_X(u(c_1^{-1}, c_2^{-1}))$. On the other hand $u(c_1^{-1}, c_2^{-1}) = (u^R(c_1, c_2))^{-1}$ and so

$$l_X(u(c_1, c_2)) = l_X(u^R(c_1, c_2)).$$

The pair $(c_1, c_2)$ is a free basis of $F_1 = F(g, h)$. Write $w(g, h) = u(c_1, c_2)$ and $w^R(g, h) = u'(c_1, c_2)$. By Proposition 3.1 $u' = u^R$. Hence

$$l_X(w(g, h)) = l_X(u(c_1, c_2)) = l_X(u^R(c_1, c_2)) = l_X(w^R(g, h)),$$

as required. □

**Corollary 3.4.** *Let $F$ be a finitely generated free group. Then for any $w(x, y) \in F(x, y)$ and for any conjugate elements $g, h$ in $F$ we have*

$$w(g, h) \equiv_{tr} w(h, g) \text{ in } F.$$

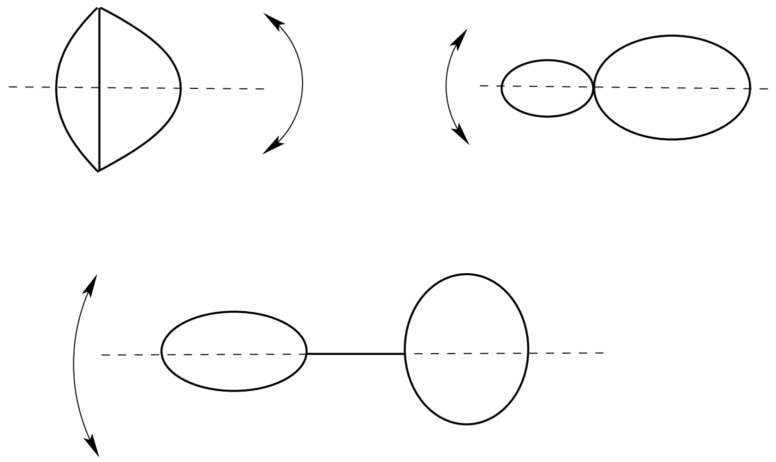

FIGURE 1. "Hyper-elliptic involution"

*Proof.* Suppose $h = f^{-1}gf$. Then by Theorem 3.3 $w(g,h) \equiv_{tr} w(g^{-1}, h^{-1}) = w(g^{-1}, f^{-1}g^{-1}f)$. Again by Theorem 3.3

$$w(g^{-1}, f^{-1}g^{-1}f) \equiv_{tr} w(g, fgf^{-1}) = f^{-1}w(f^{-1}gf, g)f = f^{-1}w(h,g)f.$$

Hence $w(g,h) \equiv_{tr} w(h,g)$, as required. $\square$

## 4. Axis diagrams for free actions

Let $G$ be a group acting by isometries on an $\mathbb{R}$-tree $X$. Recall that $g$ is called *elliptic* if $l_X(g) = 0$ and $g$ is called *hyperbolic* if $l_X(g) > 0$. Thus $g$ is elliptic iff it fixes a point of $X$.

For a hyperbolic $g \in G$ put

$$L_g = \{x \in X : d(x, gx) = l_X(g)\}.$$

Then $L_g$ is the smallest $g$-invariant subtree of $X$ which is isometric to a line and on which $g$ acts by a translation of magnitude $l_X(g)$. The set $L_g$ is called the *axis* of $g$. In this section if an $\mathbb{R}$-tree $X$ is fixed, we will omit the subscript and denote the translation length of an isometry $g$ of $X$ by $l(g)$.

The following simple proposition enumerates all the possibilities for the configuration of $L_{gh}$ with respect to $Lg, L_h$ for two hyperbolic isometries $g$ and $h$. The following is essentially a restatement of Proposition 1.6 of Paulin [13].

**Proposition 4.1.** *Let $X$ be an $\mathbb{R}$-tree and let $g, h \in Isom(X)$ be two hyperbolic isometries of $X$. Then the following hold:*

(1) *Suppose that $|L_g \cap L_h| \leq 1$ and let $D = d(L_g, L_h)$. Then*

$$l(gh) = l(g) + l(h) + 2D.$$

(2) *Suppose that $L_g \cap L_h$ is a nondegenerate segment $[x, y]$.*

(a) *If the translation directions of $g$ and $h$ on $[x, y]$ coincide then*

$$l(gh) = l(g) + l(h).$$

(b) *If the translation directions of $g$ and $h$ on $[x, y]$ are opposite then*

$$l(gh) = \begin{cases} l(g) + l(h) - 2d(x, y) & \text{if } l(g) \geq d(x, y), l(h) \geq d(x, y) \\ |l(g) - l(h)| & \text{otherwise.} \end{cases}$$

(3) *If $L_g$ and $L_h$ are equal or intersect in a ray, then*

$$l(gh) = \begin{cases} l(g) + l(h) & \text{if } g \text{ and } h \text{ translate in the same direction} \\ |l(g) - l(h)| & \text{otherwise.} \end{cases}$$

**Remark 4.2.** The case where $L_g \cap L_h$ consists of a single point is omitted in Proposition 1.6 of [13]. Only the cases where $L_g \cap L_h$ is empty or contains a nondegenerate segment are explicitly covered there. However the proofs for the cases where $L_g \cap L_h$ is empty and where $L_g \cap L_h$ is a single point are completely analogous.

We need another simple fact (see Proposition 1.8 of Paulin [13])

**Proposition 4.3.** *Let $g, h$ be two elliptic isometries of an $\mathbb{R}$-tree $X$. Then*

$$l_X(gh) = 2 \min\{d(x, y) | gx = x, hy = y\}.$$

**Theorem 4.4.** *Let $G$ be a group acting isometrically with trivial arc stabilizers on an $\mathbb{R}$-tree $X$. Let $g, h \in G$ be nontrivial elements of $G$ such that $g \neq h^{-1}$.*

(1) *Suppose that $l_X(g) = l_X(h) > 0$. Then for any positive integers $p, q, i, j$ such that $p + q = i + j$ we have*

$$l_X(g^p h^q) = l_X(g^i h^j).$$

(2) *Suppose that $l(g) = l(h) = 0$.*
*Then for any integers $p, q, i, j$ such that $g^p, h^q, g^i, h^j$ are nontrivial we have*

$$l_X(g^p h^q) = l_X(g^i h^j)$$

*Proof.* Part (2) follows directly from Proposition 4.3. Indeed, since the action of $G$ has trivial arc stabilizers, there are some $x, y \in X$ such that $Fix(g) = \{x\}$ and $Fix(h) = \{y\}$. Then for any $p, q, i, g$ such that $g^p, g^i, h^q, h^j$ are nontrivial, we have $Fix(g^p) = Fix(g^i) = \{x\}$ and $Fix(h^q) = Fix(h^j) = \{y\}$ and hence by Proposition 4.3 $l_X(g^p h^q) = d(x, y) = L_X(g^i h^j)$.

Suppose now that the assumptions of part (1) of Theorem 4.4 are satisfied.

Denote $a = l(g) = l(h)$. If $g = h$ then the statement is obvious. Suppose now that $g \neq h$, so that $g \neq h^{\pm 1}$. Let $p, q, i, j \geq 1$ be integers such that $p + q = i + j$.

Observe that $L_g = L_{g^n}$ and $L_h = L_{h^n}$ for any $n \geq 1$. Moreover, in this case $l(g^n) = l(h^n) = nl(h) = nl(g) = na$.

Suppose first that $L_g \cap L_h$ consists of at most one point. Put $D = d(L_g, L_h)$. Then by part (1) of Proposition 4.1

$$l(g^p h^q) = l(g^p)+l(h^q)+2D = pl(g)+ql(h)+2D = pa+qa+2D = (p+q)a+2D.$$

Thus we see that $l(g^p h^q)$ depends only on $p+q$ and hence $l(g^p h^q) = l(g^i h^j)$, as required.

If the intersection of $L_g$ and $L_h$ contains a ray then either $gh$ or $gh^{-1}$ fixes a segment of that ray. This is impossible since the action of $G$ on $X$ has trivial arc stabilizers.

Suppose now that $L_g \cap L_h = [x, y]$ and that $d(x, y) > 0$. If the translation directions of $g$ and $h$ on $[x, y]$ coincide then by part (2a) of Proposition 4.1 we have:

$$l(g^p h^q) = l(g^p) + l(h^q) = (p + q)a = (i + j)a = l(g^i h^j).$$

Assume now that $g$ and $h$ translate on $[x, y]$ in the opposite directions.

If $a < d(x, y)$ then $gh$ fixes an arc contained in $[x, y]$, yielding a contradiction. Hence $l(g) = l(h) = a \geq d(x, y)$. Then by part (2b) of Proposition 4.1 we have

$$l(g^p h^q) = l(g^p)+l(h^q)-2d(x, y) = (p+q)a-2d(x, y) = (i+j)a-2d(x, y) = l(g^i h^j).$$

□

The following lemma is an elementary exercise, but we provide a proof for completeness.

**Lemma 4.5.** *Let $A$ be a finite abelian group and let $g \in A$ be an element of order bigger than four. Then for any $u \in A$ there is an integer $i$ such that $g^i u$ has order bigger than four.*

*Proof.* Note that if $a = bc$ in $A$ then the order of any of these three elements divides the least common multiple of the orders of the other two.

If $u \in \langle g \rangle$, then $u = g^n$ for some $n$ and the conclusion of the lemma holds with $i = -n - 1$. Suppose now that $u$ is not a power of $g$ and that for every $i$ the order of $g^i u$ is at most four. Hence for each integer $i$ the order of $g^i u$ is two, three or four. If $i$ is an integer then by the observation above the orders of $g^i u$ and $g^{i+1} u$ cannot be both even or both equal to three. Indeed, in that case the order of $g$ would divide either three or four, contrary to the assumption that the order of $g$ is bigger than four. Choose $i$ such that the order of $g^i u$ is three. Then the order of $g^{i+2} u$ is also three and the orders of $g^{i+1} u, g^{i+3} u$ are both even. Hence the order of $g^2 = g^{i+2} u (g^i u)^{-1} = g^{i+3} u (g^{i+1} u)^{-1}$ divides both three and four, yielding a contradiction. □

The following is a generalization of Corollary 1.4 from the Introduction.

**Corollary 4.6.** *let $G$ be a group.*

(1) *Suppose that $G$ is nonabelian and that the abelianization of $G$ contains an element of infinite order. Then for every integer $M \geq 1$ there exist nontrivial elements $g_1, \dots, g_M \in G$ such that $g_i$ is not conjugate to $g_j^{\pm 1}$ for $i \neq j$ and such that for every action of $G$ on an $\mathbb{R}$-tree $X$ with trivial arc stabilizers we have $l_X(g_i) = l_X(g_j)$.*

(2) *Suppose that $G$ is nonabelian and that the abelianization of $G$ contains an element of order bigger than four. Then there exist nontrivial elements $g_1, g_2 \in G$ such that $g_1$ is not conjugate to $g_2^{\pm 1}$ and such that for every action of $G$ on an $\mathbb{R}$-tree $X$ with trivial arc stabilizers we have $l_X(g_1) = l_X(g_2)$.*

*Proof.* Let $\overline{G}$ be the abelianization of $G$. For an element $x \in G$ we denote by $\overline{x}$ the image of $x$ in $\overline{G}$.

(1) Suppose first that $G$ is non-abelian and that $\overline{G}$ contains an element of infinite order. Then there exists a noncentral element $g$ of $G$ whose image has infinite order in $\overline{G}$. Indeed, suppose not. Then all noncentral elements of $G$ have finite order images in $\overline{G}$. Take $g_1 \in G$ such that $\overline{g_1}$ has infinite order in $\overline{G}$. Then by assumption $g_1$ is central in $G$. Since $G$ is non-abelian, there exists a non-central element $u \in G$. Again, by assumption, $\overline{u}$ has finite order. But then $g = g_1 u$ is noncentral and has infinite order in $\overline{G}$, yielding a contradiction.

Thus we can choose a noncentral element $g \in G$ such that $\overline{g}$ has infinite order. Then there is some $f \in G$ such that $h := f^{-1} g^{-1} f \neq g^{-1}$. Choose $M \geq 1$ and put $g_i := g^i h^{2M+1-i} = g^i f^{-1} g^{i-2M-1} f$ for $i = 1, \dots, M$. Then $\overline{g}_i = \overline{g}^{2i-2M-1} \neq 1$ for $1 \leq i \leq M$. Since $\overline{g}$ has infinite order in $G$, for $i \neq j$ we have $\overline{g_i} \neq \overline{g_j}^{\pm 1}$ and therefore $g_i$ is not conjugate to $g_j^{\pm 1}$ in $G$. Also, since $\overline{g_i} \neq 1$, it follows that $g_i \neq 1$ for $1 \leq i \leq M$. Theorem 4.4 implies that for every action of $G$ on an $\mathbb{R}$-tree $X$ with trivial arc stabilizers we have $l_X(g_i) = l_X(g_j)$. This establishes part (1) of the corollary.

(2) Suppose now that $G$ is nonabelian and that $\overline{G}$ has an element of order bigger than four.

We claim that in this case there exists a noncentral element $g$ of $G$ whose image has order bigger than four in $\overline{G}$. Indeed, suppose not. Then all noncentral elements of $G$ map to elements of order at most four in $\overline{G}$. Take an element $g \in G$ such that $\overline{g}$ has order bigger than four. Hence $g$ is central in $G$. By assumption $G$ is nonabelian. Hence there exists a noncentral element $u \in G$. Therefore $\overline{u}$ has order at most four in $\overline{G}$. If $\overline{g}$ has infinite order, then $g_1 = gu$ is noncentral and $\overline{g_1}$ has infinite order, yielding a contradiction. Suppose now that $\overline{g}$ has finite order, so that $\langle \overline{g}, \overline{u} \rangle$ is a finite abelian group. Hence by Lemma 4.5 there is an integer $i$ such that $\overline{g^i u}$ has order at least five. Then the element $g_1 := g^i u$ is noncentral in $G$ and $\overline{g_1}$ has order bigger than four, yielding a contradiction. This verifies the claim.

Thus let $g \in G$ be a noncentral element such that $\overline{g}$ has order bigger than four. Hence there exists $f \in F$ such that $h := f^{-1} g f \neq g^{-1}$.

Put $g_1 = g^4h = g^4f^{-1}g^{-1}f$, $g_2 = g^3h^2 = g^3f^{-1}g^{-2}f$. Then $\overline{g_1} = \overline{g^3}$ and $\overline{g_2} = \overline{g}$. Since $\overline{g}$ has order bigger than four, the elements $\overline{g}, \overline{g}^{-1}, \overline{g^3}$ are nontrivial and pairwise distinct in $\overline{G}$. Hence $g_2 \neq 1$ and $g_1$ is not conjugate to $g_2^{\pm 1}$ in $G$. Theorem 4.4 again implies that for every action of $G$ on an $\mathbb{R}$-tree $X$ with trivial arc stabilizers we have $l_X(g_1) = l_X(g_2)$. □

## 5. Palindromes and traces

The following statement is well-known and probably goes back to the work of Klein in late 19-th century. We present a proof for completeness (see [6, 9] for a more detailed account).

**Lemma 5.1.** *For any freely reduced word $w(a,b) \in F(a,b)$ there exist a polynomial $f_w \in \mathbb{Z}[x,y,z]$ such that for any field $\mathbb{K}$ and any matrices $A, B \in SL(2,\mathbb{K})$*

$$tr\, w(A,B) = f_w(tr\, A, tr\, B, tr\, AB).$$

*Proof.* Recall the so-called basic trace identity [6, 9]:

$$tr\, XY + tr\, XY^{-1} = tr\, X\, tr\, Y \text{ for any } X, Y \in SL(2,\mathbb{K}).$$

Recall also that $tr\, X = tr X^{-1}$ for any $X \in SL(2,\mathbb{K})$.

We will prove the lemma by induction on $|w|$. If $|w| \leq 1$, the statement is obvious. Suppose now that $|w| = n > 1$ and that the lemma has been established for all words of length $\leq n-1$. Since trace is preserved by conjugation, we may assume that $w$ is cyclically reduced.

Suppose there is $c \in \{a,b\}^{\pm 1}$ such that $c$ occurs at least twice in $w$. Without loss of generality we may assume that $c = a$. Then after a cyclic permutation we may assume that $w$ is of the form $w = u_1 a u_2 a$ and this product is freely reduced. Let $u$ be the freely reduced form of $u_1 u_2^{-1} \in F(a,b)$. Then by the basic trace identity:

$$tr\, w(A,B) = tr[u_1(A,B)A] tr[u_2(A,B)A] - tr[u_1(A,B)AA^{-1}u_2^{-1}(A,B)] =$$
$$(f_{u_1a} f_{u_2a})(tr\, A, tr\, B, tr\, AB) - f_u(tr\, A, tr\, B, tr\, AB).$$

Thus we can take $f_w = f_{u_1a} f_{u_2a} - f_u$.

Suppose now that no $c \in \{a,b\}^{\pm 1}$ occurs in $w$ more than once. This implies that $|w| \leq 4$.

If $0 < |w| \leq 3$ then, since $w$ is cyclically reduced, it follows that $w$ has length 1 and there is nothing to prove. Suppose that $|w| = 4$. Then up to a cyclic permutation and inversion, $w = aba^{-1}b^{-1}$. We can write $w = a \cdot ba^{-1}b^{-1}$. Then by the basic trace identity

$$tr\, ABA^{-1}B^{-1} = (tr A)^2 - tr ABAB^{-1}.$$

The word $abab^{-1}$ has two occurrences of $a$ and is thus covered by the previous case. Hence we can take $f_w = x^2 - f_{abab^{-1}}$. This completes the proof. □

**Proposition 5.2.** *Let $w(x, y) \in F(x, y)$ be a freely reduced word, let $\mathbb{K}$ be a field and let $A, B \in GL(2, \mathbb{K})$ be arbitrary matrices. Then*

$$tr\, w(A, B) = tr\, w^R(A, B) \tag{$\dagger$}$$

*Proof.* We first will prove (†) under the assumption that both $A, B \in SL(2, \mathbb{K})$.

Let $f_w(x, y, z)$ be the polynomial provided by Lemma 5.1. Let $A, B \in SL(2, \mathbb{K})$ be arbitrary matrices. Note that $tr(A) = tr(A^{-1})$, $tr(B) = tr(B^{-1})$, and $tr(AB) = tr(B^{-1}A^{-1}) = tr(A^{-1}B^{-1})$. Then by Lemma 5.1

$$tr(w(A, B)) = f_w(tr(A), tr(B), tr(AB)) =$$
$$f_w(tr(A^{-1}, tr(B^{-1}), tr(A^{-1}B^{-1}) = tr(w(A^{-1}, B^{-1})) = tr(w^R(A, B)),$$

where the last equality holds because $w^R(a, b) = [w(a^{-1}, b^{-1})]^{-1}$.

Consider now the general case of $GL(2, \mathbb{K})$. Since every field embeds in an algebraically closed field, it suffices to prove (†) for algebraically closed $\mathbb{K}$. Let now $\mathbb{K}$ be algebraically closed, $w \in F(x, y)$ be a freely reduced word and let $X, Y \in GL(2, \mathbb{K})$ be arbitrary. Since $\mathbb{K}$ is algebraically closed, there exist nonzero $r, q \in \mathbb{K}$ such that $r^2 = det(X)$ and $q^2 = det(Y)$. Put $X_1 := X/r$ and $Y_1 := Y/q$. Then $\det(X_1) = \det(Y_1) = 1$, so that $X_1, Y_1 \in SL(2, \mathbb{K})$. Therefore, by the already established fact, we have

$$tr\, w(X_1, Y_1) = tr\, w^R(X_1, Y_1).$$

Let $\sigma$ be the exponent sum on $x$ in $w$ (and hence in $w^R$) and let $\tau$ be the exponent sum on $y$ in $w$ (and hence in $w^R$). Then $w(X, Y) = w(rX_1, qY_1) = r^\sigma q^\tau w(X_1, Y_1)$ and, similarly $w^R(X, Y) = w^R(rX_1, qY_1) = r^\sigma q^\tau w^R(X_1, Y_1)$.

Therefore

$$tr\, w(X, Y) = r^\sigma q^\tau tr\, w(X_1, Y_1) = r^\sigma q^\tau tr\, w^R(X_1, Y_1) = tr\, w^R(X, Y),$$

as required.

□

**Proposition 5.3.** *For any field $\mathbb{K}$, for any integers $p, q$ and for any conjugate matrices $A, B \in GL(2, \mathbb{K})$ we have*

$$tr\, A^p B^q = tr\, A^q B^p.$$

*Proof.* Again, we may assume that $\mathbb{K}$ is algebraically closed. For any matrices $A, B \in SL(2, \mathbb{K})$ conjugate in $GL(2, \mathbb{K})$ we have $tr\, A = tr\, B$, and hence $tr\, A^p B^q = tr\, B^p A^p = tr\, A^q B^p$ by Lemma 5.1 applied to $w(x, y) = x^p y^q$.

Suppose now $A, B \in GL(2, \mathbb{K})$ are conjugate. Since $\mathbb{K}$ is algebraically closed, there exists $s \in \mathbb{K}$ such that $s^2 = \det(A) = \det(B)$. Then $A_1 = A/s$ and $B_1 = B/s$ have determinant 1 and are still conjugate in $GL(2, \mathbb{K})$. Therefore $tr\, A_1^p B_1^q = tr\, A_1^q B_1^p$. However $A^p B^q = s^{p+q} A_1^p B_1^q$ and $A^q B^p = s^{p+q} A_1^q B_1^p$ which implies that $tr\, A^p B^q = tr\, A^q B^p$, as required. □

## 6. Tree actions in the boundary of the Teichmuller space

Let $S$ be a closed surface (possibly non-orientable) of negative Euler characteristic and let $\mathcal{T}(S)$ be the Teichmuller space of $S$. We think of $\mathcal{T}(S)$ as the set of marked hyperbolic structures on $S$ or, equivalently, as the set of free discrete and cocompact isometric actions of $G = \pi_1(S)$ on $\mathbb{H}^2$. Let $\overline{\mathcal{T}(S)}$ be the Thurston compactification of $\mathcal{T}(S)$. The points of $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$ are measured laminations of $S$. Each such lamination $\mu$ defines a small action of $G$ on an $\mathbb{R}$-tree dual $X_\mu$ to the lift of this lamination to $\mathbb{H}^2$ [10, 11]. We refer the reader to [7] for a detailed discussion on this topic.

We next show that our results regarding two-variable palindromes also apply to the elements of $\overline{\mathcal{T}(S)}$.

**Theorem 6.1.** *Let $S$ be a closed surface (possibly non-orientable) of negative Euler characteristic and let $G = \pi_1(S)$. Then for any $g, h \in G$ and for any $w(x, y) \in F(x, y)$ we have:*

(1) *For each point $p \in \mathcal{T}(S)$, thought of as an action of $G$ on $\mathbb{H}^2$, the elements $w(g, h)$ and $w^R(g, h)$ have equal translation lengths.*
(2) *For each point $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$ we have*

$$l_{X_\mu}(w(g,h)) = l_{X_\mu}(w^R(g,h)).$$

*Proof.* Let $p \in \mathcal{T}(S)$ and consider the corresponding action $\phi : G \to Isom(\mathbb{H}^2)$ of $G$ by isometries on $\mathbb{H}^2$. Recall that, when $\mathbb{H}^2$ is considered in the upper-half space model, there is a canonical isometric action of $GL(2, \mathbb{R})$ on $\mathbb{H}^2$ whose image is the full isometry group of $\mathbb{H}^2$. Namely, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$. If $\det(A) > 0$ then $Az = \frac{az+b}{cz+d}$ for any $z \in \mathbb{H}^2$. If $\det(A) < 0$ then $Az = \frac{a\bar{z}+b}{c\bar{z}+d}$ for any $z \in \mathbb{H}^2$. The first case gives us an orientation-preserving isometry of $\mathbb{H}^2$ and the second case gives an orientation-reversing isometry. It is well-known that for $A \in GL(2, \mathbb{R})$ the trace $tr(A)$ and the determinant $\det(A)$ uniquely determine the translation length of $A$ as an isometry of the hyperbolic plane.

Let $w(x, y) \in F(x, y)$ and let $g, h \in G$. Let $A \in GL(2, \mathbb{R})$ represent $\phi(g)$ and let $B \in GL(2, \mathbb{R})$ represent $\phi(h)$. Then by Proposition 5.2 $tr\, w(A, B) = tr\, w^R(A, B)$. Moreover, obviously, $\det w(A, B) = \det w^R(A, B)$. Therefore we have $l_{\mathbb{H}^2}(w(g, h)) = l_{\mathbb{H}^2}(w^R(g, h))$, as claimed.

Thus part (1) of the theorem is established.

Now, part (1) immediately implies part (2). Indeed, recall that each element $p \in \mathcal{T}(S)$ determines a *marked length spectrum* $l_p : G \to \mathbb{R}$ where $l_p(g)$ is the translation length of $g$ for the isometric action of $G$ on $\mathbb{H}^2$ corresponding to $p$. Then it is well-known that for any $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$ the length-function $l_{X_\mu} : G \to \mathbb{R}$ is the limit of marked length spectra $l_{p_n}$ for some sequence of $p_n \in \mathcal{T}(S)$. That is, for every $f \in G$

$$l_{X_\mu}(f) = \lim_{n\to\infty} l_{p_n}(f).$$

Since for $f = w(g,h)$ and $f' = w^R(g,h)$ we know by (1) that $l_{p_n}(f) = l_{p_n}(f')$ for all $n$, it follows that $l_{X_\mu}(f) = l_{X_\mu}(f')$, as required. □

The following result immediately implies Theorem D from the Introduction:

**Theorem 6.2.** *Let $S$ be a closed surface (possibly non-orientable) of negative Euler characteristic and let $G = \pi_1(S)$. Then for any conjugate $g, h \in G$ and for any integers $p, q$ we have:*

(1) *For each point $p \in \mathcal{T}(S)$, thought of as an action of $G$ on $\mathbb{H}^2$, the elements $g^p h^q$ and $g^q h^p$ have equal translation lengths.*
(2) *For each point $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$ we have*
$$l_{X_\mu}(g^p h^q) = l_{X_\mu}(g^q h^p).$$
(3) *If $S$ is orientable then for any conjugate elements $g, h \in G$, for each point $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$ and for any positive integers $p, q, i, j$ such that $p + q = i + j$ we have*
$$l_{X_\mu}(g^p h^q) = l_{X_\mu}(g^i h^j).$$

*Proof.* Parts (1) and (2) are established exactly as Theorem 6.1, but using Proposition 5.3 instead of Proposition 5.2.

To see that part (3) holds observe that, by well-known results, if $S$ is orientable then every orbit of the action of the mapping class group of $S$ on $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$ is dense in $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$. In particular, this applies to the orbit of a point corresponding to the stable foliation of a pseudo-Anosov homeomorphism of $S$. Therefore the set of those $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$, such that the action of $G$ on the tree $X_\mu$ is free, is dense in $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$. Since Theorem 4.4 applies to free actions, part (3) of Theorem 6.2 follows. □

**Remark 6.3.** The argument used in the proof of part (3) of Theorem 6.2 does not work in the case of non-orientable surfaces, as follows from the results of Danthony and Nogueira [5]. At the moment we do not know how to prove part (3) of Theorem 6.2 in the non-orientable case.

## 7. Examples

**Example 7.1.** The palindrome and the $g^p h^q$ phenomena described above no longer hold for the class of small actions (as opposed to free or very small actions).

Consider the graph of groups $\mathbb{A}$ shown in Figure 2. Let $v$ be the vertex of the graph incident to all three marked up edges $e, f, t$. Put $G = \pi_1(\mathbb{A}, v)$ and let $X = \widetilde{(\mathbb{A}, v)}$ be the Bass-Serre universal covering tree. It is easy to see that the action of $G$ on $X$ is minimal and that $G = F(x, y)$ is a free group of rank two with free basis $x = eae, y = tf$. Consider now the elements $g = xyx^2y^{-1}$ and $h = y^{-1}x^2yx$. Thus $h$ is the plaindromic reverse of $g$ in $F(x, y)$. Moreover, with $a = x$ and $b = yxy^{-1}$ we see that $g = ab^2$ and $h$ is conjugate to $a^2b$.

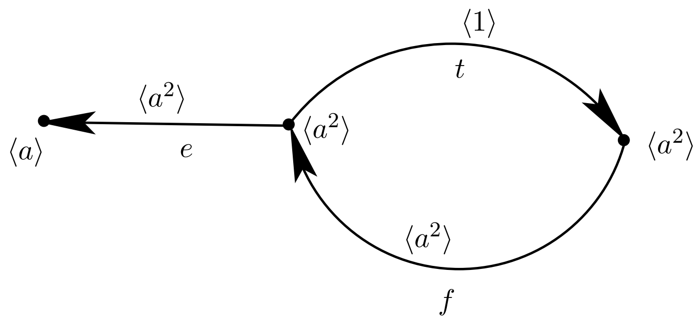


FIGURE 2. Couterexample

In $G$ we have:

$$g = xyx^2y^{-1} = eae^{-1}tfea^2e^{-1}f^{-1}t^{-1} = eae^{-1}tfa^2f^{-1}t^{-1} = eaeta^2t^{-1}$$
$$h = y^{-1}x^2yx = f^{-1}t^{-1}ea^2e^{-1}tfeae^{-1} = f^{-1}t^{-1}a^2tfeae^{-1}.$$

Both $eaeta^2t^{-1}$ and $f^{-1}t^{-1}a^2tfeae^{-1}$ are cyclically reduced closed paths in $\mathbb{A}$. Therefore $l_X(g) = 4$ while $l_X(h) = 6$.

**Remark 7.2.** We believe that the palindrome phenomenon also holds for small actions of a free group $F$ such that, if $g \in F$ is nontrivial and fixes more than one point, then $Fix(g)$ is a compact segment containing no branch point in its interior.

**Example 7.3.** We have remarked earlier that translation equivalence is preserved by injective homomorphisms, and hence by passing to bigger free groups, containing the given free group as a subgroup. However, passing to subgroups, even if they have finite index in $F$, generally no longer preserves translation equivalence.

Let $F = F(a, b)$ and let $H = \langle a, bab^{-1}, b^2\rangle \le F$. It is not hard to see that $H$ is a subgroup of index two in $F$ and that $x = a, y = bab^{-1}, z = b^2$ is a free basis of $H$, so that $H = F(x, y, z)$. Consider the elements $g = aba^2b^{-1}$ and $h = a^2bab^{-1}$ in $F$. Then $g \equiv_{tr} h$ in $F$ by, for example, Theorem 4.4. We also have $g, h \in H$ and $g = xy^2, h = x^2y$. Clearly, $g$ and $h$ have different Whitehead graphs in $F(x, y, z)$ and therefore $g \not\equiv_{tr} h$ in $H$.

On the other hand, it is easy to see that if $H$ is a free factor of $F$ and $g, h \in H$ then $g \equiv_{tr} h$ in $F$ iff $g \equiv_{tr} h$ in $H$.

**Example 7.4.** Theorem A states, in particular, that two elements of $F$ are translation equivalent in $F$ if and only if they have equal Whitehead graphs with respect to each free basis of $F$. This statement no longer holds if we consider "higher rank" $m \ge 2$ analogues of Whitehead graphs, where symmetrized numbers of occurrences of subwords of length $m$ are recorderd. Consider, again, $g = aba^2b^{-1}$ and $h = a^2bab^{-1}$ in $F(a, b)$. We already know that $g \equiv_{tr} h$ in $F$. However, consider the cyclic words $w$ and $u$ defined by $g$

and $h$ accordingly. The number of occurrences of $b^{-1}ab$ in $w$ is equal to 1, while neither $b^{-1}ab$ nor its inverse $b^{-1}a^{-1}b$ occur in $u$.

**Example 7.5.** Although translation equivalence in free groups seems to be related to the $SL_2$ trace identities, these phenomena are different. For example, we know from Theorem C that if $g$ and $h$ are conjugate in $F$ and $g \neq h^{-1}$ then $g^3h \equiv_{tr} g^2h^2$ in $F$. On the other hand, let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1/2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = CAC^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 5/2 \end{bmatrix}$ be matricies in $SL(2, \mathbb{R})$. A direct computation shows that $tr\ A^3B = -79/16$ while $tr\ A^2B^2 = -143/16$. Thus there is no precise analogue of Theorem C for the $SL_2$ trace identities.

## References


[1] M. Bestvina and M. Feighn, *Outer Limits*, preprint, 1993.

[2] M. Cohen and M. Lustig, *Very small group actions on R-trees and Dehn twist automorphisms.* Topology **34** (1995), no. 3, 575–617

[3] M. Cohen, M. Lustig and M. Steiner, *R-tree actions are not determined by the translation lengths of finitely many elements.* Arboreal group theory (Berkeley, CA, 1988), 183–187, Math. Sci. Res. Inst. Publ., **19**, Springer, New York, 1991

[4] M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free groups.* Invent. Math. **84** (1986), no. 1, 91–119

[5] C. Danthony and A. Nogueira, *Measured foliations on nonorientable surfaces.* Ann. Sci. cole Norm. Sup. (4) **23** (1990), no. 3, 469–494

[6] R. Horowitz, *Characters of free groups represented in the two-dimensional special linear group.* Comm. Pure Appl. Math. **25** (1972), 635–649

[7] M. Kapovich, *Hyperbolic manifolds and discrete groups.* Progress in Mathematics, 183. Birkhuser Boston, Inc., Boston, MA, 2001

[8] C. J. Leininger, *Equivalent curves in surfaces.* Geom. Dedicata **102** (2003), 151–177

[9] W. Magnus, *Rings of Fricke characters and automorphism groups of free groups.* Math. Z. **170** (1980), no. 1, 91–103.

[10] J. Morgan and S. Shalen, *Valuations, trees, and degenerations of hyperbolic structures. I.* Ann. of Math. (2) 120 (1984), no. 3, 401–476.

[11] J. Morgan and S. Shalen, *Degenerations of hyperbolic structures. III. Actions of 3-manifold groups on trees and Thurston's compactness theorem.* Ann. of Math. (2) **127** (1988), no. 3, 457–519

[12] J. Smillie and K. Vogtmann, *Length functions and outer space.* Michigan Math. J. **39** (1992), no. 3, 485–493

[13] F. Paulin, *The Gromov topology on R-trees.* Topology Appl. **32** (1989), no. 3, 197–221

[14] B. Randol, *The length spectrum of a Riemann surface is always of unbounded multiplicity.* Proc. Amer. Math. Soc. **78** (1980), no. 3, 455–456

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