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Report for the Arbeitstagung

“Restriction maps in cohomology of arithmetic groups”

We give a criterion to determine when a Hecke translate of a cohomology class on a compact Shimura variety $S(\Gamma)$ restricts non-trivially to a smaller Shimura variety.

In details, let G, H be connected, semi-simple simply connected algebraic groups over $\mathbb{Q}$, such that all their $\mathbb{Q}$-simple factors are noncompact at infinity (this is to ensure that $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_f)$, $\mathbb{A}_f$ = the ring of finite adeles over $\mathbb{Q}$). Let $f : H \to G$ be a morphism of $\mathbb{Q}$-algebraic groups, with finite kernel. Fix a maximal compact subgroup $K_\infty \subset G(\mathbb{R})$ so that $K_\infty^H := f^{-1}(K_\infty)$ is a maximal compact subgroup of $H(\mathbb{R})$. One has the associated map $f : Y := H(\mathbb{R})/K_\infty^H \to G(\mathbb{R})/K_\infty := X$; of symmetric spaces. For each congruence arithmetic group $\Gamma \subset G(\mathbb{Q})$, write $S(\Gamma) = \Gamma\backslash X$, $S_H(\Gamma) = \Gamma \cap H\backslash Y$, and $j = j(\Gamma) : S_H(\Gamma) \to S(\Gamma)$. If $g \in G(\mathbb{Q})$, one has an isomorphism $S(g^+\Gamma g) \xrightarrow{g} S(\Gamma)$ induced by $x \mapsto g(z)$ on X, and $j_g : S_H(g^{-1}\Gamma g) \xrightarrow{j} S(\Gamma) \xrightarrow{g} S(\Gamma)$ is the composite. Consider the restriction map

$$\mathrm{Res} = \prod j_g^* : H^\circ(S(\Gamma)) \to \prod_{g\in G(\mathbb{Q})} (S_H(g^{-1}\Gamma g))$$

<u>Theorem 1</u>: Suppose G, H are as in the foregoing, so that both Y, X are Hermitian symmetric and $f : Y \to X$ is holomorphic. Suppose $S_H(\Gamma)$ is a divisor on $S(\Gamma)$. Then

$$\mathrm{Res} = H^m(S(\Gamma), \mathbb{C}) \to \prod H^m(S_H(\Gamma), \mathbb{C})$$

is injective for $m \le \dim S_H(\Gamma)$ $(= \dim S(\Gamma) - 1)$.

<u>Corollary (of the proof)</u>: Suppose G, H are as in Theorem 1. Suppose that up to compact factors, $G(\mathbb{R})$ and $H(\mathbb{R})$ are of the form
(1) $G(\mathbb{R}) = SU(n,1)$, $H(\mathbb{R}) = SU(m,1)$ included in the natural imbedding in $SU(n,1)$
(2) $G(\mathbb{R}) = \mathrm{Spin}(n,2)$, $H(\mathbb{R}) = \mathrm{Spin}(m,2)$
(3) $G(\mathbb{R}) = \mathrm{Sp}_2$, $H(\mathbb{R}) = \mathrm{Sp}_1 \times \mathrm{Sp}_1$.
Then $\mathrm{Res} : H^i(S(\Gamma)) \to \prod H^i(S_H(\Gamma))$ $(i \le \dim Y)$ is injective.

Let g_0, k_0 be the real Lie algebras of $G(\mathbb{R})$ and K_∞, and $g_0 = k_0 \oplus p_0$ the associated Cartan decomposition. Let g, k, p be the complexifications of g_0, k_0 and p_0.

We view g_0 and $g_u = k_0 \oplus i\, p_0$ as real subalgebras of g. Let $G_{\mathbb{C}}$ be the complexifications of G and $G_u \subset G_{\mathbb{C}}$ the analytic subgroup with Lie algebra g_u. Let $\hat{X} = G_u/K_\infty$ be the "compact dual" of X. One has the Matsushima formula $H^\circ(S(\Gamma)) = \mathrm{Hom}_{K_\infty}(\overset{\circ}{\Lambda}_p, \mathcal{C}^\infty(\Gamma\backslash G)(0))$. Thus

$$H^\circ(\hat{X}) = \mathrm{Hom}_{K_\infty}(\Lambda_p, \mathbb{C}) \hookrightarrow H^\circ(S(\Gamma)).$$

Here Λ_p is the exterior algebra of p and $\mathcal{C}^\infty(\Gamma\backslash G)(0)$ the space of smooth functions on $\Gamma\backslash G$ killed by the Casimir of g. Let $\hat{Y} \hookrightarrow \hat{X}$ be the compact dual of Y, and $[\hat{Y}]$ the Poincaré dual of the cycle $\hat{Y}$ in $\hat{X}$. Thus $[\hat{Y}] \in H^\circ(\hat{X}) \subset H^\circ(S(\Gamma))$. Let $[S_H(\Gamma)]$ be the Poincaré dual of $S_H(\Gamma)$ in $H^\circ(S(\Gamma))$.

Consider the direct limit $\varinjlim_{\Gamma} H^\circ(S(\Gamma))$ as Γ varies through congruence arithmetic subgroups of $G(\mathbb{Q})$. The density of $G(\mathbb{Q})$ in $G(\mathbb{A}_f)$ implies that

$$\varinjlim H^\circ(S(\Gamma)) = \mathrm{Hom}_{K_\infty}(\Lambda p, \mathcal{C}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))(0))$$

$$\overset{\mathrm{def}}{=} H^\circ(\mathcal{S}h(G)).$$

Now $G(\mathbb{A}_f)$ acts on $H^\circ(\mathcal{S}h(G))$ by right translations, and it is known that $H^\circ(\mathcal{S}h(G))$ may be written as a direct sum $H^\circ(\mathcal{S}h(G)) = \bigoplus m(\pi_f)\pi_f$ of irreducible representations π_f, each occurring with a finite multiplicity, $m(\pi_f)$.

Now $H^\circ(S(\Gamma)) = H^\circ(\mathcal{S}h(G))^K$, where K is the closure of $\Gamma(\subset G(\mathbb{Q}) \subset G(\mathbb{A}_f))$ in $G(\mathbb{A}_f)$. Hence $\xi_\Gamma = [S_H(\Gamma)]$ is a class in $H^\circ(\mathcal{S}h(G))$. Let V_Γ be the $\mathbb{C}$-span of $G(\mathbb{A}_f)$-translates of ξ_Γ.

<u>Theorem 2</u>:
1) The space $V_\Gamma^{G(\mathbb{A}_f)}$ of $G(\mathbb{A}_f)$-invariants in V_Γ, is spanned by $[\hat{Y}]$.
2) If $w \in H^\circ(\mathcal{S}h(G))$ such that $\mathrm{Res}(w) = 0$, then $w \wedge [\hat{Y}] = 0$.

We now indicate a proof that Theorem 2 $\Rightarrow$ Theorem 1. For simplicity assume that $G(\mathbb{R}) = SU(n,1)$, $H(\mathbb{R}) = SU(n-1,1)$ up to compact factors. Then $X =$ unit ball in $\mathbb{C}^n \hookleftarrow Y =$ unit ball in $\mathbb{C}^{n-1}$, $\hat{X} = \mathbb{P}^n(\mathbb{C})$, and $[\hat{Y}] = L \in H^2(\mathbb{P}^n(\mathbb{C}))$ is a generator. If $w \in H^m(\mathcal{S}h(G))$, with $m \leq n-1$, then $\mathrm{Res}(w) = 0 \Rightarrow w \wedge L = 0$ (by Th. 2). This implies $w = 0$ by Lefschetz's theorem or hyperplane sections.

From now on we assume $f : Y \to X$ is a holomorphic imbedding of Hermitian symmetric domains. Let $T \subset K_\infty$ be a maximal torus in K_∞ (also in G). Let t_0, t its real and complex Lie algebras, $\Phi^+(k,t) =$ roots of t in k which are 'positive'. Fix $X \in i\ \mathrm{Lie}(T)$, s.t. $\alpha(X) \geq 0$ for all $\alpha \in \Phi^+(k,t)$, set $q = g^x \bigoplus_{\alpha(X)>0} g_\alpha$, $U = \bigoplus_{\alpha(X)>0} g_\alpha$. Then $u \cap p = u \cap p^+ \oplus u \cap p^-$, put $m = \dim(u \cap p)$. Let $u^- = \bigoplus_{\alpha(x)<0} g_\alpha$, and consider $w(q) = u \cap p^+ \oplus u^- \cap p^+ \subset p^+$. $w(q)$ is also m-dimensional. Let $V^+(q)$ be the K_∞-module generated by $\overset{m}{\Lambda} W(q)$ in $\overset{m}{\Lambda} p^+$. Write $E(G,H,m) = K_\infty$-module generated by $\overset{m}{\Lambda} p_H^+$ in $\overset{m}{\Lambda} p^+$. Put $V(q) = K_\infty$-span of $\overset{m}{\Lambda} u \cap p$ in $\overset{m}{\Lambda}$.
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Theorem (Parthasarathy, Vogan-Zuckerman): Irreducible unitary representations π_∞ of $G(\mathbb{R})$ with $H^\circ(g, K_\alpha, \pi_\infty) \neq 0$ are parametrised by the q's as above. Denote the associated representation by A_q. Then

$$H^i(g, K_\infty, \pi_\infty) = \mathrm{Hom}_{K_\infty}(V(q)_i, \pi_\infty)$$

where $V(q)_i$ is the isotypic of $V(q)$ in $\overset{i}{\Lambda}\,\varphi$.

Definition: We say that $w \in H^i(\mathcal{S}h(G))$ is of type A_q if the image of w (w thought of as a homomorphism of $\overset{i}{\Lambda}\,p$ into $\mathcal{C}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$) generates a submodule isomorphic to A_q. If the degree i of w is the dimension of $u \cap p$, then we say that w is strongly primitive.

Theorem 3 Let $w \in H^m(\mathcal{S}h(G))$ be a strongly primitive class of type A_q. If $\mathrm{Res}(w) = 0$, then

$$V^+(q) \cap E(G, H, m) = 0.$$

Thus Theorem 3 gives a sufficient condition that a class w has nonzero restriction. By applying Theorem 3 to cup-products, we get:

Theorem 4: Let w, w' be strongly primitive classes of type A_q, $A_{q'}$ and of degrees m, m' on $\mathcal{S}h(G)$, with $m + m' \leq \dim(\mathcal{S}h(G))$. Then $g^*w \wedge w' = 0$ for all $g \in G(\mathbb{A}_f)$ only if

$$V^+(q) \wedge V^+(q') = 0.$$