CHERN CLASSES OF SCHUBERT CELLS AND VARIETIES

PAOLO ALUFFI AND LEONARDO CONSTANTIN MIHALCEA

Abstract. We give explicit formulas for the Chern-Schwartz-MacPherson classes of all Schubert varieties in the Grassmannian of d-planes in a vector space, and conjecture that these classes are effective. We prove this is the case for (very) small values of d.

1. Introduction

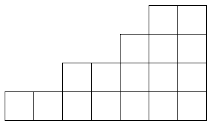
The classical *Schubert varieties* in the Grassmannian $G_d(V)$ parametrize subspaces of dimension d ('d-planes') of an ambient vector space V, satisfying prescribed incidence conditions with a flag of subspaces. Schubert varieties are among the most studied objects in algebraic geometry, and it may come as a surprise that something is *not* known about them. Yet, to our knowledge the *Chern classes* of Schubert varieties have not been available in the literature. This paper is devoted to their computation.

Most Schubert varieties are singular; in fact, the only nonsingular Schubert varieties are themselves (isomorphic to) Grassmannians, and computing their Chern classes is a standard exercise. For singular varieties, there is a good notion of Chern classes in 'homology' (i.e., in the Chow group), that is, the so-called *Chern-Schwartz-MacPherson* ('CSM') classes. They were defined independently by Marie-Hélène Schwartz ([Sch65a], [Sch65b]) and Robert MacPherson ([Mac74]), they agree with the total homology Chern class of the tangent bundle for nonsingular varieties, and they satisfy good functoriality properties (summarized, for example, in [Ful84], Example 19.1.7).

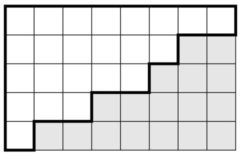
It is a consequence of these good functoriality properties classes that a Chern-Schwartz-MacPherson class can in fact be assigned to all Schubert *cells:* these are varieties isomorphic to affine spaces, parametrizing d-planes which satisfy prescribed incidence conditions *strictly.* Every Schubert *variety* is stratified by the Schubert *cells* contained in it, and as a consequence the CSM class of a Schubert variety may be written as a sum of CSM classes of Schubert cells contained in it.

With this understood, our task amounts to the computation of the CSM class of a Schubert cell, as an element of the Chow group of a Schubert variety containing it.

In order to state the result more precisely, we have to bring in some notation. We consider the 'abstract' Schubert variety $\mathbb{S}(\underline{\alpha})$ determined by a partition $\underline{\alpha} = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d \geq 0)$; we depict $\underline{\alpha}$ by the corresponding (upside-down) Young diagram $\underline{\alpha}$. For example, $\mathbb{S}(7 \geq 5 \geq 3 \geq 2)$ is the Schubert variety associated to the diagram



Embedding this Schubert variety in (for example) the Grassmannian $G_5(V)$ of 5-planes in a 13-dimensional vector space V realizes it as the subvariety parametrizing subspaces intersecting a fixed flag of subspaces of dimensions 1, 4, 6, 9, 12 in dimension $\geq 1, 2, 3, 4, 5$ respectively. It is perhaps more common to associate this latter realization with the 'complementary' Young diagram,



whose Schur function determines the class of $\mathbb{S}(\underline{\alpha})$ in the *cohomology* (or Chow ring) of $G_5(V)$. However, since CSM classes live in *homology,* we are naturally led to take the dual viewpoint. This also has the advantage of allowing us to deal with Schubert varieties as abstract varieties, independently of a (standard) embedding into a Grassmannian. The dimension of $\mathbb{S}(\underline{\alpha})$ equals the number of boxes in the corresponding 'homological' diagram.

We write $\underline{\beta} \leq \underline{\alpha}$ to denote $\beta_i \leq \alpha_i\ \forall i$; that is, the diagram corresponding to $\underline{\beta}$ is contained in the diagram corresponding to $\underline{\alpha}$. For $\underline{\beta} \leq \underline{\alpha}$ there are closed embeddings $\mathbb{S}(\underline{\beta}) \subset \mathbb{S}(\underline{\alpha})$; the Schubert *cell* $\mathbb{S}(\underline{\alpha})^\circ$ is the complement of $\cup_{\beta<\alpha}\mathbb{S}(\underline{\beta})$ in $\mathbb{S}(\alpha)$. As mentioned above, Schubert cells are isomorphic to affine spaces, and there is a decomposition

$$\mathbb{S}(\underline{\alpha}) = \coprod_{\beta\leq\alpha} \mathbb{S}(\underline{\beta})^\circ \quad .$$

It also follows that the Chow group $A_*\mathbb{S}(\underline{\alpha})$ is freely generated by the classes $[\mathbb{S}(\underline{\beta})]$ for all $\underline{\beta} \leq \underline{\alpha}$. For $\underline{\beta} \leq \underline{\alpha}$, the CSM class

$$c_{\mathrm{SM}}(\mathbb{S}(\underline{\beta})^\circ) \in A_*\mathbb{S}(\underline{\alpha})$$

of the corresponding Schubert cell is defined as the image via MacPherson's natural correspondence (cf. [Mac74]) of the constructible function $1\!\!1_{\mathbb{S}(\underline{\beta})^\circ}$ whose value is 1 on $\mathbb{S}(\underline{\beta})^\circ$, and 0 on its complement in $\mathbb{S}(\underline{\alpha})$. Since $1\!\!1_{\mathbb{S}(\underline{\alpha})} = \sum_{\beta\leq\alpha} 1\!\!1_{\mathbb{S}(\underline{\beta})^\circ}$, the basic covariance property of CSM classes implies that

$$c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha}) = \sum_{\underline{\beta}\leq\underline{\alpha}} c_{\mathrm{SM}}(\mathbb{S}(\underline{\beta})^\circ) \quad ,$$

reducing the problem of computing CSM classes of Schubert *varieties* to that of computing CSM classes of Schubert *cells.* Now,

$$c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ) = \sum_{\underline{\beta}\leq\underline{\alpha}} \gamma_{\underline{\alpha},\underline{\beta}}[\mathbb{S}(\underline{\beta})]$$

for uniquely determined coefficients $\gamma_{\underline{\alpha},\underline{\beta}} \in \mathbb{Z}$. Our main result consists of the computation of these coefficients, and can be stated in several different forms. For example:

Theorem 1.1. *Let* $\underline{\alpha} = (\alpha_1 \geq \cdots \geq \alpha_d)$, $\underline{\beta} = (\beta_1 \geq \cdots \geq \beta_d)$. *The integer* $\gamma_{\underline{\alpha},\underline{\beta}}$ *equals the coefficient of*

$$t_1^{\alpha_1} \cdots t_d^{\alpha_d} \cdot u_1^{\beta_1} \cdots u_d^{\beta_d}$$

in the expansion of the rational function

$$\Gamma_d(\underline{t},\underline{u}) = \frac{1}{(t_1^d\cdots t_d^1)(u_1^d\cdots u_d^1)}\cdot\prod_{1\le i<j\le d}\frac{(t_i-t_j)(u_i-u_j)}{1-2t_j+t_it_j}\cdot\prod_{1\le i,j\le d}\frac{1-t_i}{1-t_i(1+u_j)}$$

as a Laurent polynomial in $\mathbb{Z}[[\underline{t},\underline{u}]]$.

(Even) more explicitly, the coefficient $\gamma_{\underline{\alpha},\underline{\beta}}$ equals

$$\sum \det\left[\binom{\alpha_i-\ell^i_{i+1}-\cdots-\ell^i_d}{\beta_j+(i-j)+\ell^1_i+\cdots+\ell^{i-1}_i-\ell^i_{i+1}-\cdots-\ell^i_d}\right]_{1\le i,j\le d}$$

where the summation is over the $\binom{d}{2}$ integers ℓ^k_i, $1\le k<i\le d$, subject to the conditions

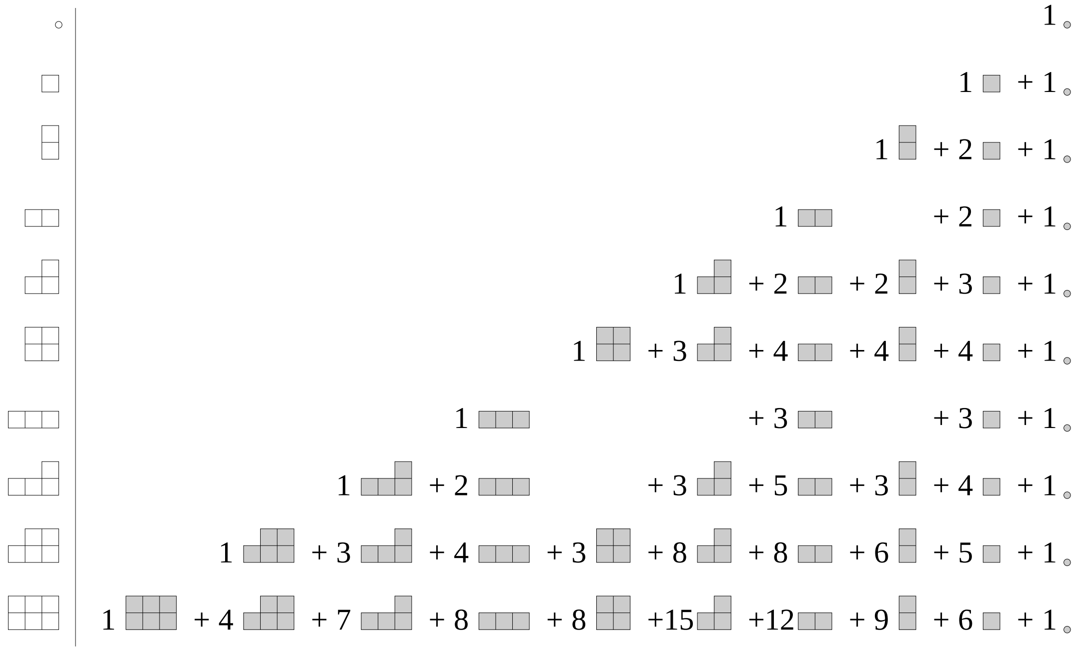
$$0\le \ell^k_{k+1}+\cdots+\ell^k_d\le \alpha_{k+1}\quad .$$

Other expressions for the coefficients $\gamma_{\underline{\alpha},\underline{\beta}}$ are given in §3. If $\beta_i=0$ for $i\ge 2$, that is, $\underline{\beta}=(r)$ corresponds to a one-row diagram, then $\gamma_{\underline{\alpha},\underline{\beta}}$ equals the coefficient of u^r in

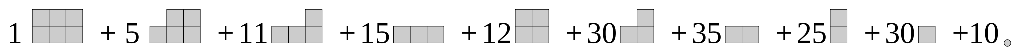
$$\prod_{i\ge 1}(1+iu)^{\alpha_i-\alpha_{i+1}}$$

(see Corollary 3.11 below). For 2-row diagrams $\underline{\alpha}$, a formula for $\gamma_{\underline{\alpha},\underline{\beta}}$ in terms of non-intersecting lattice paths is given in Theorem 4.5.

Example 1.2. The Grassmannian $G_2(\mathbb{C}^5)$ parametrizing lines in $\mathbb{P}^4$ is isomorphic to $\mathbb{S}(\underline{\alpha})$ for a 3×2 rectangle $\underline{\alpha}$. We list here the CSM classes of its Schubert cells, denoting by a darkened diagram $\underline{\beta}$ the corresponding class $[\mathbb{S}(\underline{\beta})]\in A_*\mathbb{S}(\underline{\alpha})=A_*G_2(\mathbb{C}^5)$, and by a dot the 'empty' diagram (corresponding to the class of a point):



For example, the Chern-Schwartz-MacPherson class of $G_2(\mathbb{C}^5)$ may be obtained by adding up all these classes:



As $G_2(\mathbb{C}^5)$ is nonsingular, $c_{\rm SM}(G_2(\mathbb{C}^5))$ agrees with the total Chern class of its tangent bundle, which may be computed from the standard tautological sequence on the Grassmannian.

A number of features of CSM classes of Schubert cells are clear for 'geometric' reasons. For example:

- The contribution of the class of a point to the CSM class of a Schubert cell is 1; this is necessarily the case, since each cell is an affine space and the degree of the CSM class of a variety always equals its Euler characteristic (by functoriality);
- *Mutatis mutandis,* the CSM class for a diagram $\underline{\alpha}$ must agree with the CSM class for the transposed diagram, obtained from $\underline{\alpha}$ by interchanging rows and columns;
- The CSM class for a sequence $\underline{\alpha} = (\alpha_1 \geq \alpha_2 \geq \cdots)$ only depends on the nonzero terms in the sequence.

These facts correspond to identities involving the coefficients $\gamma_{\underline{\alpha},\underline{\beta}}$ computed in Theorem 1.1, which may appear more mysterious from the 'algebraic' point of view.

Less immediate features include adjunction-type formulas, and interpretations of the coefficients $\gamma_{\underline{\alpha},\underline{\beta}}$ as computing certain nonintersecting lattice paths in the plane; these are discussed in §4, and follow from the algebraic expressions obtained in §3.

There is one feature which is experimentally manifest, and for which we do not know a general proof from either the algebraic or the geometric viewpoint: all coefficients of the CSM classes of all Schubert cells appear to be *positive.* We give two proofs of this fact for diagrams with ≤ 2 rows in §4, as consequences of the facts mentioned in the previous paragraph. A proof of positivity for diagrams with 3 rows will appear elsewhere. Positivity of Chern classes is a well-explored theme in intersection theory; the case of Schubert cells hints that there may be an interesting, and as yet unknown, principle of positivity for Chern-Schwartz-MacPherson classes of singular or noncomplete varieties under suitable hypotheses.

The proof of Theorem 1.1 relies on explicit nonsingular birational models $\mathbb{V}(\underline{\alpha})$ of Schubert varieties $\mathbb{S}(\underline{\alpha})$—the well-known *Bott-Samelson resolutions* (cf. [Dem74], [Vak]). In §2 we give a self-contained description of these varieties, independent of the embedding of $\mathbb{S}(\underline{\alpha})$ in a Grassmannian. For the application to CSM classes it is necessary to compute explicitly the push-forward at the level of Chow groups

$$A_*(\mathbb{V}(\underline{\alpha})) \to A_*(\mathbb{S}(\underline{\alpha})) \quad ;$$

this is accomplished in Proposition 2.12, and may be of independent interest. The varieties $\mathbb{V}(\underline{\alpha})$ realize the splitting principle for the tautological bundle $\mathscr{S}$ on $\mathbb{S}(\underline{\alpha})$ (obtained as pull-back of the tautological subbundle from a Grassmannian), so that the Chern roots of $\mathscr{S}$ are realized as Chern classes $-\xi_i$ of line bundles $\mathscr{L}_i$ on $\mathbb{V}(\alpha)$. We explicitly compute the push-forward to $\mathbb{S}(\underline{\alpha})$ of a monomial in the ξ_i's, as the class (up to sign) of a smaller Schubert variety $\mathbb{S}(\underline{\beta})$. The reader will check that the classical Pieri's formula is an immediate consequence of this observation.

The variety $\mathbb{V}(\underline{\alpha})$ is a nonsingular compactification of the Schubert cell $\mathbb{S}(\underline{\alpha})^\circ$, with complement a divisor with simple normal crossing. The CSM class of $\mathbb{S}(\underline{\alpha})^\circ$ may then be obtained by computing the (ordinary) Chern class of a bundle of logarithmic tangent fields on $\mathbb{V}(\underline{\alpha})$. This is done in §3, leading to explicit expressions for $c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ)$, of which Theorem 1.1 is a sample.

This approach to the computation of Chern-Schwartz-MacPherson classes goes back to an observation (Proposition 15.3 in [GP02], Theorem 1 in [Alu99]) and has proven useful for other explicit computations. For example, it gives a one-sentence proof of Fritz Ehlers' formula for the CSM class of toric varieties ([Alu06]).

Chern-Schwartz-MacPherson classes of (possibly) singular varieties have been the object of intense study, but they have been computed explicitly in relatively few cases. To our knowledge, the largest class of varieties for which CSM classes are known consists of *degeneracy loci* of morphisms of vector bundles, satisfying a mild generality hypotheses, which are treated by Adam Parusiński and Piotr Pragacz in [PP95]. In fact, there is some overlap of the results in [PP95] and in the present paper: Schubert varieties indexed by $\underline{\alpha} = (\alpha_1 = \alpha_2 = \cdots = \alpha_{d-1} \geq \alpha_d)$ (that is, by a 'cohomological' one-row diagram) may be realized as degeneracy loci. Comparing the formulas obtained here with those in *loc. cit.* is a natural project.

Chern-Schwartz-MacPherson classes are arguably the most natural replacement for $c(TX) \cap [X]$ when the tangent bundle is not available, and computing them for varieties as well-known as Schubert varieties is a natural task. It would also be a natural task to compute other classes satisfying the same basic normalization requirement, such as Mather's Chern class ([Ful84], Example 4.2.9) or Fulton's Chern class ([Ful84], Example 4.2.6). Comparisons between these different classes for Schubert varieties would lead to the computation of other important invariants of their singularities, such as the local Euler obstruction.

Acknowledgements. We are grateful to the Max-Planck-Institut für Mathematik, Bonn, for hospitality and support in the Summer of 2006. We thank Don Zagier for many enlightening discussions on generating functions, and especially for a series of hints which catalyzed the proof of Corollary 3.11.

2. Schubert and Bott-Samelson varieties

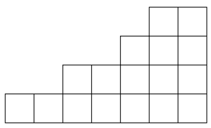
2.1. **Schubert varieties.** We work over an algebraically closed field k. In this section we recall some material concerning classical Schubert varieties, mainly for the purpose of setting notations. Proofs may be found in any standard reference, such as [Ful84], Chapter 14.

We denote by $\underline{\alpha}$ a *partition,* that is, a nonincreasing sequence of nonnegative integers α_i, such that $\alpha_i = 0$ for $i \gg 0$:

$$\underline{\alpha} : (\underline{\alpha}_1 \geq \underline{\alpha}_2 \geq \cdots \geq 0 \geq \dots) \quad .$$

If we write $\underline{\alpha} = (\alpha_1 \geq \cdots \geq \alpha_d)$, it is understood that $\alpha_i = 0$ for $i > d$.

Pictorially, every partition is associated to a Young diagram



with α_i boxes in the i-th row, counting from the bottom; with this in mind, we sometime call $\underline{\alpha}$ a *diagram.*

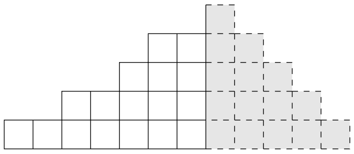
We can associate to $\underline{\alpha}$ a *Schubert variety* $\mathbb{S}(\underline{\alpha})$, as follows. For $\underline{\alpha} = (0)$ (the 'empty diagram'), $\mathbb{S}(\underline{\alpha})$ is a point. For nonzero $\underline{\alpha}$, let N, d be integers such that $N \geq \alpha_1$ and $\alpha_{d+1} = 0$. Let V be a vector space of dimension $N + d$, and fix a complete flag $F_\bullet$ in V:

$$F_0 = \{0\} \subset F_1 \subset \cdots \subset F_{N+d} = V \quad ,$$

where $\dim F_r = r$. The sequence $\underline{\alpha}$ selects d elements in the flag:

$$F_{\alpha_d+1} \subset F_{\alpha_{d-1}+2} \subset \cdots \subset F_{\alpha_1+d} \quad ;$$

the dimensions of the spaces in this subflag are the lengths of the rows of the diagram obtained adjoining to $\underline{\alpha}$ a 'd-step ladder':



Definition 2.1. The *Schubert variety* $\mathbb{S}(\underline{\alpha})$ denotes the variety of d-planes S in the Grassmannian $G_d(V)$ such that

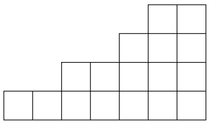
$$\dim(S \cap F_{\alpha_{d+1-i}+i}) \geq i \quad .$$

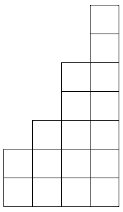
Note that $\mathbb{S}(\underline{\alpha})$ is not defined unless $\underline{\alpha}$ is a nonincreasing sequence of integers, as above.

Lemma 2.2. *Up to isomorphism, $\mathbb{S}(\underline{\alpha})$ only depends on $\underline{\alpha}$.*

Indeed, the definition of $\mathbb{S}(\underline{\alpha})$ makes sense as soon as $N = \alpha_1$ and d is the number of nonzero elements in $\underline{\alpha}$; increasing N does not affect the constraints, while increasing d amounts to direct-summing a fixed subspace to all subspaces parametrized by the variety, producing an isomorphic variety $\mathbb{S}(\underline{\alpha})$.

It is also the case that $\mathbb{S}(\underline{\alpha})$ and $\mathbb{S}(\underline{\alpha}^t)$ are isomorphic, for α, α^t 'transposed' diagrams, obtained from each other by interchanging rows and columns.





This is a consequence of the correspondence between subspaces of dual vector spaces.

2.2. **Schubert cells.** We assume that a sufficiently large vector space V, a complete flag $F_\bullet$ in V, and a sufficiently large d have been chosen.

We denote by $\underline{\beta} \leq \underline{\alpha}$ the ordering $\beta_i \leq \alpha_i \ \forall i$. If $\underline{\beta} \leq \underline{\alpha}$, there are embeddings

$$\mathbb{S}(\underline{\beta}) \subset \mathbb{S}(\underline{\alpha}) \quad :$$

indeed, the conditions defining $\mathbb{S}(\underline{\beta})$ are a strengthening of those defining $\underline{\alpha}$.

Definition 2.3. The Schubert *cell* corresponding to $\underline{\alpha}$ is the open dense subset

$$\mathbb{S}(\underline{\alpha})^\circ = \mathbb{S}(\underline{\alpha}) \setminus \cup_{\beta<\alpha}\mathbb{S}(\beta)$$

of the Schubert *variety* $\mathbb{S}(\underline{\alpha})$.

Equivalently, a d-plane $S \in \mathbb{S}(\underline{\alpha})$ is in the cell $\mathbb{S}(\underline{\alpha})^\circ$ if and only if

$$\dim(S \cap F_{\alpha_{d+1-i}+i}) = i \quad \text{and} \quad \dim(S \cap F_{\alpha_{d+1-i}+i-1}) = i-1$$

for $i = 1, \dots, d$.

An elementary coordinate computation proves the following:

Lemma 2.4. *The Schubert cell $\mathbb{S}(\alpha)^\circ$ is isomorphic to an affine space $\mathbb{A}^{\sum_i \alpha_i}$ of dimension $\sum_i \alpha_i$.*

It follows that $\dim \mathbb{S}(\underline{\alpha}) = \sum_i \alpha_i$; further, this shows that the Schubert varieties have a cellular decomposition (cf. [Ful84], Example 1.9.1) and in particular that $A_*(\mathbb{S}(\underline{\alpha}))$ is freely generated by the *Schubert classes* $[\mathbb{S}(\beta)]$ for $\beta \leq \alpha$. As stated in the introduction, our task is to express the Chern-Schwartz-MacPherson class of the Schubert *cell* $\mathbb{S}(\underline{\alpha})^\circ$, as a combination of Schubert classes:

$$c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ) = \sum_{\beta \leq \alpha} \gamma_{\underline{\alpha},\underline{\beta}}\, [\mathbb{S}(\underline{\beta})] \quad \in A_*(\mathbb{S}(\underline{\alpha})] \quad .$$

2.3. **Bott-Samelson varieties.** In the embedded setting, there is a natural way to produce a nonsingular model of the Schubert variety $\mathbb{S}(\underline{\alpha}) \subset G_d(V)$. If $\mathbb{S}(\underline{\alpha})$ is defined with respect to the complete flag

$$F_0 = \{0\} \subset F_1 \subset \cdots \subset F_{N+d} = V \quad ,$$

as in §2.1, let $\mathbb{V}(\underline{\alpha})$ be the corresponding *Bott-Samelson variety* of flags

$$\mathbb{V}(\underline{\alpha}) = \{(S^1 \subset S^2 \subset \cdots \subset S^d) \mid \dim S^i = i \text{ and } S^i \subset F_{\alpha_{d+1-i}+i}\} \quad .$$

The top space in the flag, S^d, satisfies the conditions defining $\mathbb{S}(\underline{\alpha})$; therefore, there is a natural map

$$\pi_\alpha : \mathbb{V}(\underline{\alpha}) \to \mathbb{S}(\underline{\alpha}) \quad .$$

It is clear from this description that $\pi_{\underline{\alpha}}$ has a section over the Schubert cell $\mathbb{S}(\underline{\alpha})^\circ$, defined by sending sending $S \in \mathbb{S}(\underline{\alpha})^\circ$ to the flag

$$(S \cap F_{\alpha_d+1} \subset S \cap F_{\alpha_{d-1}+2} \subset \cdots \subset S = S \cap F_{\alpha_1+d}) \quad .$$

In particular, $\pi_{\underline{\alpha}}$ is a birational isomorphism.

The varieties $\mathbb{V}(\underline{\alpha})$ are easily seen to be nonsingular, as they may be realized as a tower of projective bundles over a point.

For a thorough treatment of Bott-Samelson varieties (in a more general context) we refer the reader to [Vak]. We provide here a self-contained construction of these varieties, adapted to our application. In particular, we stress the independence of the definition of $\mathbb{V}(\underline{\alpha})$ from the ambient Grassmannian; we verify that the complement of $\mathbb{S}(\underline{\alpha})^\circ$ in $\mathbb{V}(\underline{\alpha})$ is a divisor with simple normal crossings; and we compute the push-forward $\pi_{\underline{\alpha}*}$ explicitly.

2.4. $\mathbb{V}(\underline{\alpha})$ **as a tower of projective bundles.** The partition $\underline{\alpha}$ determines the following inductive construction of $\mathbb{V}(\underline{\alpha})$. Each $\mathbb{V}(\underline{\alpha})$ will be a nonsingular projective variety, of dimension $\sum_{i\geq 1} \alpha_i$, endowed with bundles $\mathscr{L}_i$, $\mathscr{Q}_i$ of rank resp. 1, α_i. It will map birationally to the corresponding Schubert variety $\mathbb{S}(\underline{\alpha})$.

Here is the construction:

- For $\underline{\alpha} = (0)$, $\mathbb{V}(\underline{\alpha})$ is a point; for all $i \geq 0$, $\mathscr{Q}_i = 0$ and $\mathscr{L}_i = \mathscr{O}$;

- Given any $\underline{\alpha} = (\alpha_1 \geq \alpha_2 \geq \dots)$, denote by $\underline{\alpha}'$ the (shifted) 'truncation' $(\alpha_2 \geq \alpha_3 \geq \dots)$, and by $\mathscr{L}'_i$, $\mathscr{Q}'_i$ the corresponding bundles. Then $\mathbb{V}(\underline{\alpha})$ is the following projective bundle over $\mathbb{V}(\underline{\alpha}')$:

$$\begin{array}{c} \mathbb{V}(\underline{\alpha}) := \mathbb{P}(\mathscr{Q}'_1 \oplus \mathscr{O}^{\oplus(\alpha_1-\alpha_2+1)}) \\ \big\downarrow \rho^1_{\underline{\alpha}} \\ \mathbb{V}(\underline{\alpha}') \end{array}$$

- For $i \geq 2$, the distinguished bundles on $\mathbb{V}(\underline{\alpha})$ are

$$\mathscr{L}_i := (\rho^1_{\underline{\alpha}})^*(\mathscr{L}'_{i-1}) \quad , \quad \mathscr{Q}_i := (\rho^1_{\underline{\alpha}})^*(\mathscr{Q}'_{i-1}) \quad .$$

  The bundles $\mathscr{L}_1$, $\mathscr{Q}_1$ are defined by the tautological sequence

$$0 \longrightarrow \mathscr{L}_1 = \mathscr{O}(-1) \longrightarrow (\rho^1_{\underline{\alpha}})^*(\mathscr{Q}'_1 \oplus \mathscr{O}^{\oplus(\alpha_1-\alpha_2+1)}) = \mathscr{Q}_2 \oplus \mathscr{O}^{\oplus(\alpha_1-\alpha_2+1)} \longrightarrow \mathscr{Q}_1 \longrightarrow 0$$

2.5. We let

$$\xi_i := -c_1(\mathscr{L}_i) \quad .$$

By construction, if $\alpha_j = 0$ then $\mathscr{L}_j \cong \mathscr{O}$, and hence $\xi_j = 0$. We record the following immediate consequence of the construction:

Lemma 2.5. *Let d be such that $\alpha_j = 0$ for $j > d$. Then the Chow ring of $\mathbb{V}(\underline{\alpha})$ is generated by the classes ξ_i, $i = 1, \dots, d$; thus, every element of $A_*\mathbb{V}(\underline{\alpha})$ may be expressed as*

$$\xi_1^{r_1} \cdots \xi_d^{r_d} \cap [\mathbb{V}(\underline{\alpha})] \quad ,$$

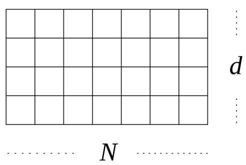
for $r_1, \dots, r_d \geq 0$.

Relations among the monomials $\xi_1^{r_1} \cdots \xi_d^{r_d}$ follow from the standard description of the Chow group of a projective bundle (cf. [Ful84], §3.3). The following observation will be crucial for our application, as it will be key to the computation of push-forwards (§2.10), and is less immediate. We postpone its proof to §2.11, at the end of this section.

We denote by (N^d) the sequence

$$(\underbrace{N \geq \cdots \geq N}_{d} \geq 0 \geq \cdots) \quad ,$$

corresponding to an $N \times d$ rectangle:



(The corresponding Schubert variety $\mathbb{S}((N^d))$ is then isomorphic to $G_d(k^{N+d})$.) Also, we denote by $\int$ the *degree* of a rational equivalence class.

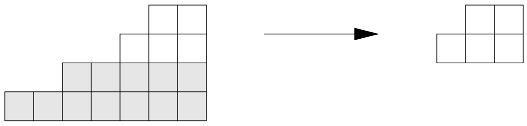
Lemma 2.6. *Let $d \geq 2$, and let $r_1, \dots, r_d$ be nonnegative integers. Then for all $1 \leq i < d$ such that $r_i > 0$:*

$$\int (\prod_{j=1}^{d} \xi_j^{r_j}) \cap [\mathbb{V}((N^d))] = -\int (\xi_1^{r_1} \cdots \xi_i^{r_{i+1}+1} \xi_{i+1}^{r_i-1} \cdots \xi_d^{r_d}) \cap [\mathbb{V}((N^d))] \quad .$$

2.6. By definition, if $\underline{\beta}$ is obtained from $\underline{\alpha}$ by deleting the first several entries, then there is a fibration

$$\rho : \mathbb{V}(\underline{\alpha}) \twoheadrightarrow \mathbb{V}(\underline{\beta}) \quad ,$$

which is in fact a composition of projective bundles; the distinguished bundles on $\mathbb{V}(\underline{\beta})$ pull back to distinguished bundles with shifted indices on $\mathbb{V}(\underline{\alpha})$.



If $\underline{\beta} \le \underline{\alpha}$, then there are closed embeddings

$$\iota : \mathbb{V}(\underline{\beta}) \hookrightarrow \mathbb{V}(\underline{\alpha}) \quad ,$$

such that (with evident notation)

$$\iota^* \mathscr{L}_{\underline{\alpha},i} = \mathscr{L}_{\underline{\beta},i} \quad , \quad \iota^* \mathscr{Q}_{\underline{\alpha},i} = \mathscr{Q}_{\underline{\beta},i} \oplus \mathscr{O}^{\alpha_i - \beta_i} \quad .$$

Indeed, assuming ι has been constructed for the truncated sequences:

$$\iota : \mathbb{V}(\underline{\beta}') \hookrightarrow \mathbb{V}(\underline{\alpha}') \quad ,$$

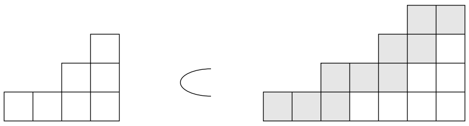
then the identification

$$\mathscr{Q}'_{\underline{\beta},1} \oplus \mathscr{O}^{\alpha_2 - \beta_2} = \iota^*(\mathscr{Q}'_{\underline{\alpha},1})$$

gives an inclusion

$$\mathscr{Q}'_{\underline{\beta},1} \oplus \mathscr{O}^{\beta_1 - \beta_2 + 1} \subset \mathscr{Q}'_{\underline{\beta},1} \oplus \mathscr{O}^{\alpha_1 - \beta_2 + 1} = \mathscr{Q}'_{\underline{\beta},1} \oplus \mathscr{O}^{\alpha_2 - \beta_2} \oplus \mathscr{O}^{\alpha_1 - \alpha_2 + 1} = \iota^*(\mathscr{Q}'_{\underline{\alpha},1} \oplus \mathscr{O}^{\alpha_1 - \alpha_2 + 1})$$

projectivizing which induces the needed embedding $\mathbb{V}(\underline{\beta}) \subset \mathbb{V}(\underline{\alpha})$.



Lemma 2.7. $[\mathbb{V}(\underline{\beta})] = \xi_1^{\alpha_1 - \beta_1} \cdots \xi_d^{\alpha_d - \beta_d} \cap [\mathbb{V}(\underline{\alpha})] \in A_*(\mathbb{V}(\underline{\alpha}))$.

Proof. This follows from a simple inductive argument, comparing the sequences of relative tangent bundles for $\mathbb{V}(\underline{\alpha})$, $\mathbb{V}(\underline{\beta})$ over the varieties $\mathbb{V}(\underline{\alpha}')$, $\mathbb{V}(\underline{\beta}')$ corresponding to the truncations (cf. [Ful84], §B.5.8). □

Note that Lemma 2.7 gives an explicit geometric realization of monomials $\xi_1^{r_1} \cdots \xi_d^{r_d}$, for exponents r_i such that the sequence $\alpha_1 - r_1, \dots, \alpha_d - r_d$ is a nonincreasing sequence of nonnegative integers.

2.7. The varieties $\mathbb{V}(\underline{\alpha})$ constructed above have natural maps to Grassmannians. As in §2.1, choose d and N so that

$$\begin{cases} \alpha_i = 0 \text{ for } i > d \\ N \ge \alpha_1 \end{cases} ;$$

and fix a vector space V of dimension $N + d$, and a complete flag $F_\bullet$ in V. These choices determine a map

$$\mathbb{V}(\underline{\alpha}) \to G_d(V)$$

to the Grassmannian of d-planes in V, as follows.

Denote by $\mathscr{F}_r$ the trivial bundle with fiber F_r.

Lemma 2.8. *For $i = 1, \dots, d$ there are compatible epimorphisms*

$$\mathscr{F}_{\alpha_i+(d+1-i)} \twoheadrightarrow \mathscr{Q}_i$$

of bundles over $\mathbb{V}(\underline{\alpha})$.

Proof. If $\underline{\alpha}$ is $0 \geq 0 \geq \dots$ there is nothing to show, for all d. For arbitrary $\underline{\alpha}$, assume that the epimorphisms have been constructed on the truncation $\mathbb{V}(\underline{\alpha}')$; then the corresponding epimorphisms for $i \geq 2$ are obtained on $\mathbb{V}(\underline{\alpha})$ by pulling back via $\rho^1_{\underline{\alpha}}$. In particular we have an epimorphism

$$\mathscr{F}_{\alpha_2+(d-1)} \twoheadrightarrow \mathscr{Q}_2 \quad ;$$

to obtain it for $i = 1$:

$$\mathscr{F}_{\alpha_1+d} = \mathscr{F}_{\alpha_2+(d-1)} \oplus \mathscr{O}^{\oplus(\alpha_1-\alpha_2+1)} \twoheadrightarrow \mathscr{Q}_2 \oplus \mathscr{O}^{\oplus(\alpha_1-\alpha_2+1)} \twoheadrightarrow \mathscr{Q}_1 \quad ,$$

where the rightmost epimorphism comes from the universal sequence defining $\mathscr{Q}_1$ on $\mathbb{V}(\underline{\alpha})$. $\square$

Let $\mathscr{S}_i$ denote the kernel of the epimorphisms obtained in Lemma 2.8:

$$(*) \qquad 0 \longrightarrow \mathscr{S}_i \longrightarrow \mathscr{F}_{\alpha_i+(d+1-i)} \longrightarrow \mathscr{Q}_i \longrightarrow 0 \quad ;$$

the rank of $\mathscr{S}_i$ is $d+1-i$ (that is, the length of the i row of the 'ladder'). By the universal property of Grassmannians, we obtain maps

$$\pi^i_{\underline{\alpha}} : \mathbb{V}(\underline{\alpha}) \to G_{d+1-i}\big(F_{\alpha_i+(d+1-i)}\big)$$

such that (*) is the pull-back of the tautological sequence over $G_{d+1-i}\big(F_{\alpha_i+(d+1-i)}\big)$.

These maps are clearly compatible with the projections and embeddings defined in §2.6. In fact, a simple chase shows that the sequences (*) fit in diagrams with exact rows and columns:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathscr{L}_i \arrow[r] \arrow[u] & \big(\mathscr{Q}_{i+1} \oplus \mathscr{O}^{\oplus(\alpha_i-\alpha_{i+1}+1)}\big) \arrow[r] \arrow[u] & \mathscr{Q}_i \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathscr{S}_i \arrow[r] \arrow[u, two heads] & \mathscr{F}_{\alpha_i+(d+1-i)} \arrow[r] \arrow[u, two heads] & \mathscr{Q}_i \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & \mathscr{S}_{i+1} \arrow[r, equal] \arrow[u] & \mathscr{S}_{i+1} \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

for all $i = 1, \dots, d$.

2.8. In particular, for $i = 1$ we obtain a map

$$\pi_{\underline{\alpha}} : \mathbb{V}(\underline{\alpha}) \to G_d(F_{\alpha_1+d}) \subset G_d(V) \quad ,$$

where the last inclusion is induced by the inclusion $F_{\alpha_1+d} \subset V$.

Proposition 2.9. *The map $\pi_{\underline{\alpha}}$ is a birational isomorphism onto $\mathbb{S}(\underline{\alpha})$.*

Of course this is nothing but the map $\pi_{\underline{\alpha}}$ mentioned in §2.3: the fibers $\mathscr{S}_i|_v$ over $v \in \mathbb{V}(\underline{\alpha})$ give a flag

$$\begin{tikzcd}
\mathscr{S}_d|_v \arrow[r, hook] \arrow[d, hook] & \mathscr{S}_{d-1}|_v \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & \mathscr{S}_1|_v \arrow[d, hook] \\
F_{\alpha_d+1} \arrow[r, hook] & F_{\alpha_{d-1}+2} \arrow[r, hook] & \cdots \arrow[r, hook] & F_{\alpha_1+d}
\end{tikzcd}$$

as prescribed in the description of $\mathbb{V}(\underline{\alpha})$ given in §2.3, and $\pi_{\underline{\alpha}}(v)$ consists of the fiber $\mathscr{S}_1|_v$ viewed as a subspace of V. That is, $\pi_{\underline{\alpha}}(v)$ is a d-plane satisfying the incidence conditions defining $\mathbb{S}(\underline{\alpha})$. The image $\pi_{\underline{\alpha}}(v)$ is in the cell $\mathbb{S}(\underline{\alpha})^\circ$ when

$$\dim(\mathscr{S}_1|_v \cap F_{\alpha_i+(d+1-i)-1}) < i \quad ;$$

that is, when

$$\mathscr{S}_i|_v \not\subset F_{\alpha_i+(d+1-i)-1}$$

for all $i = 1, \dots, d$. The following proposition formalizes the discussion given in §2.3.

Proposition 2.10. *The map $\pi_{\underline{\alpha}}$ admits a section over $\mathbb{S}(\underline{\alpha})^\circ$. The complement of the image of $\mathbb{S}(\underline{\alpha})^\circ$ in $\mathbb{V}(\underline{\alpha})$ is a simple normal crossing divisor, whose components have class ξ_i, $i \geq 1$.*

Proof. The section $i : \mathbb{S}(\underline{\alpha})^\circ \to \mathbb{V}(\underline{\alpha})$ may be defined inductively. For $\underline{\alpha} = (0)$, both $\mathbb{S}(\underline{\alpha})^\circ$ and $\mathbb{V}(\underline{\alpha})$ are points. For arbitrary $\underline{\alpha}$, let $\underline{\alpha}' = (\alpha_2 \geq \alpha_3 \geq \cdots)$ as usual, and assume

$$i' : \mathbb{S}(\underline{\alpha}')^\circ \hookrightarrow \mathbb{V}(\underline{\alpha}')$$

has been constructed. For $S \in \mathbb{S}(\underline{\alpha})^\circ$, the intersection of the corresponding d-plane with $F_{\alpha_2+(d-1)}$ has dimension exactly $d-1$, hence it determines a point $S' \in \mathbb{S}(\underline{\alpha}')^\circ$.

$$\begin{tikzcd}
\mathbb{S}(\underline{\alpha})^\circ \arrow[r, hook, dotted, "i"] \arrow[d] & \mathbb{V}(\underline{\alpha}) \arrow[d, "\rho"] \\
\mathbb{S}(\underline{\alpha}')^\circ \arrow[r, hook, "i'"] & \mathbb{V}(\underline{\alpha}')
\end{tikzcd}$$

By construction, S' is naturally identified with the fiber $\mathscr{S}_2|_{v'}$ of $\mathscr{S}_2$ over $v' = i'(S')$. The one dimensional quotient S/S' determines a one-dimensional subspace of

$$\frac{F_{\alpha_1+d}}{S'} = \left(\frac{\mathscr{F}_{\alpha_1+d}}{\mathscr{S}_2}\right)|_{v'} = (\mathscr{Q}_2 \oplus \mathscr{O}^{\oplus \alpha_1-\alpha_2+1})|_{v'} \quad ,$$

that is, a point v of

$$\mathbb{P}(\mathscr{Q}_2 \oplus \mathscr{O}^{\oplus \alpha_1-\alpha_2+1}) = \mathbb{V}(\underline{\alpha})$$

lying over v'. Setting $i(S) = v$ lifts i', as needed.

The statement on the complement of $\mathbb{S}(\underline{\alpha})^\circ$ in $\mathbb{V}(\underline{\alpha})$ may also be verified inductively. The complement D' of $\mathbb{S}(\underline{\alpha}')^\circ$ in $\mathbb{V}(\underline{\alpha}')$ may be assumed to be a simple normal crossing divisor, with components of class ξ_i, $i \geq 2$. For S over $S' \in \mathbb{S}(\underline{\alpha}')^\circ$, the condition $S \in \mathbb{S}(\underline{\alpha})^\circ$ is equivalent to $S \not\subset F_{\alpha_1+d-1}$, that is (with notation as above) to

$$v \notin \mathbb{P}(\mathscr{Q}_2 \oplus \mathscr{O}^{\oplus \alpha_1 - \alpha_2}) \quad .$$

Therefore, the complement of $\mathbb{S}(\underline{\alpha})^\circ$ in $\mathbb{V}(\underline{\alpha})$ consists of $D = E \cup \rho^{-1}(D')$, where $E := \mathbb{P}(\mathscr{Q}_2 \oplus \mathscr{O}^{\oplus \alpha_1 - \alpha_2})$ is a hypersurface of $\mathbb{V}(\underline{\alpha})$ of class ξ_1. It remains to verify that D is a divisor with simple normal crossings, and this is an instance of the following more general situation: let $\pi : Y \to X$ be a smooth morphism of varieties, $E \subset Y$ an irreducible divisor, smooth over X, and D' a simple normal crossing divisor on X. Then $\pi^{-1}(D') \cup E$ is a simple normal crossing divisor on Y, as needed. □

2.9. **Pull-back.** The left column in the diagram displayed at the end of §2.7 is the exact sequence

$$0 \longrightarrow \mathscr{S}_{i+1} \longrightarrow \mathscr{S}_i \longrightarrow \mathscr{L}_i \longrightarrow 0$$

($i = 1, \dots, d$). The existence of these sequences amounts to the fact that the variety $\mathbb{V}(\underline{\alpha})$ is a concrete realization of the splitting construction for the restriction of the tautological bundles to the Schubert varieties $\mathbb{S}(\alpha_i \geq \alpha_{i+1} \geq \dots)$: for each $i = 1, \dots, d$ we have constructed a map

$$\pi^i_{\underline{\alpha}} : \mathbb{V}(\underline{\alpha}) \twoheadrightarrow \mathbb{S}(\alpha_i \geq \alpha_{i+1} \geq \dots) \subset G_{d+1-i}(V)$$

such that $\mathscr{S}_i$ is the pull-back of the tautological subbundle.

Lemma 2.11. *With notation as above:*

- $c(\mathscr{S}_i) = (1+\xi_i)\cdots(1+\xi_d)$*;*
- *the image of the pull-back map*

$$\pi^{i\,*}_{\underline{\alpha}} : A^*(G_{d+1-i}(V)) \to A^*(\mathbb{V}(\underline{\alpha}))$$

  consists of the symmetric functions in $\xi_i, \dots, \xi_d$.

Proof. The first statement is an immediate consequence of the exact sequences recalled above. The second follows from the fact that the Chow ring of the Grassmannian is generated by the Chern classes of the tautological subbundle, and the considerations preceding the statement. □

2.10. **Push-forward.** Lemmas 2.6 and 2.11 allow us to determine the push-forward

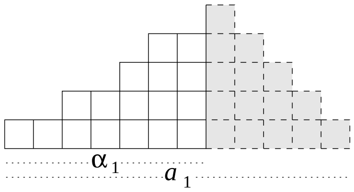
$$\pi_{\underline{\alpha}*} : A_*(\mathbb{V}(\underline{\alpha})) \to A_*(G_d(V)) \quad ,$$

and this will be necessary for our application.

We need another piece of notation. Let $a_1, \dots, a_d$ denote integers; if $a_1 > \dots > a_d > 0$, we let

$$\Omega(a_d, \dots, a_1) = \mathbb{S}(a_1 - d \geq a_2 - (d-1) \geq \dots \geq a_d - 1 \geq 0 \geq \dots) \quad .$$

The notation is chosen in order to match standard terminology (cf. [Ful84], §14.7). The number a_i denotes the number of boxes in the i-th row (from the bottom) of the usual diagram, *with* a d-ladder adjoined to the right:



For arbitrary $a_1, \dots, a_d$, define the class $[\Omega(a_d, \dots, a_1)]$ in the Chow group of a Schubert variety to be

$$[\Omega(a_d, \dots, a_1)] = (-1)^\sigma \, [\Omega(a_{\sigma(d)}, \dots, a_{\sigma(1)})]$$

if $a_1, \dots, a_d$ are positive and distinct, and $\sigma \in S_d$ is the permutation such that $a_{\sigma(d)} < \cdots < a_{\sigma(1)}$; and 0 if the integers $a_1, \dots, a_d$ are not positive and distinct.

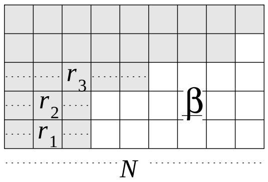
Proposition 2.12. *Let $r_1, \dots, r_d$ be nonnegative integers. Then*

$$\pi_{\underline{\alpha}*}\left(\xi_1^{r_1} \cdots \xi_d^{r_d} \cap [\mathbb{V}(\underline{\alpha})]\right) = [\Omega(\alpha_d - r_d + 1, \dots, \alpha_1 - r_1 + d)] \quad .$$

Proof. The formula is immediate if $d = 1$, so we assume $d \geq 2$. By compatibility with embeddings (§2.6), we can assume that $\underline{\alpha} = (N^d)$, that is, $\alpha_1 = \cdots = \alpha_d = N$, $\alpha_{d+1} = 0$. If $0 \leq r_1 \leq \cdots \leq r_d \leq N$ then

$$\xi_1^{r_1} \cdots \xi_d^{r_d} \cap [\mathbb{V}((N^d))] = [\mathbb{V}(\underline{\beta})]$$

for $\underline{\beta} = (N - r_1 \geq \cdots N - r_d \geq 0 \cdots)$,



by Lemma 2.7. Thus

$$\pi_{\underline{\alpha}*}\left(\xi_1^{r_1} \cdots \xi_d^{r_d} \cap [\mathbb{V}((N^d))]\right) = \pi_{\underline{\alpha}*}\left([\mathbb{V}(\underline{\beta})]\right) = [\mathbb{S}(\underline{\beta})] = [\Omega(N - r_d + 1, \dots, N - r_1 + d)]$$

if $0 \leq r_1 \leq \cdots \leq r_d \leq N$, that is, if

$$0 < N - r_d + 1 < \cdots < N - r_1 + d \quad .$$

In order to prove the formula in the general case, it suffices to show that both sides behave in the same way after permutations of this list of integers; that is, it suffices to show that

$$\pi_{\underline{\alpha}*}\left(\xi_1^{s_1} \cdots \xi_d^{s_d} \cap [\mathbb{V}((N^d))]\right) = (-1)^\sigma \pi_{\underline{\alpha}*}\left(\xi_1^{r_1} \cdots \xi_d^{r_d} \cap [\mathbb{V}((N^d))]\right)$$

if the list

$$N - s_d + 1 \quad , \quad \dots \quad , \quad N - s_1 + d$$

is obtained by applying the permutation σ to the list

$$N - r_d + 1 \quad , \quad \dots \quad , \quad N - r_1 + d \quad .$$

As transpositions generate the group of permutations, it suffices to prove that

$$\pi_{\underline{\alpha}*}\left(\xi_1^{r_1} \cdots \xi_i^{r_{i+1}+1} \xi_{i+1}^{r_i - 1} \cdots \xi_d^{r_d} \cap [\mathbb{V}((N^d))]\right) = -\pi_{\underline{\alpha}*}\left(\xi_1^{r_1} \cdots \xi_d^{r_d} \cap [\mathbb{V}((N^d))]\right)$$

for all nonnegative integer $r_1, \ldots, r_d$, assuming $r_i > 0$. By Poincaré duality in the Grassmannian and the projection formula, it suffices to prove that

$$\int \xi_1^{r_1} \cdots \xi_i^{r_{i+1}+1} \xi_{i+1}^{r_i-1} \cdots \xi_d^{r_d} \cdot \pi_{\underline{\alpha}}^*(C) \cap [\mathbb{V}((N^d))] = -\int \xi_1^{r_1} \cdots \xi_d^{r_d} \cdot \pi_{\underline{\alpha}}^*(C) \cap [\mathbb{V}((N^d))]$$

for all classes C of codimension $dN - \sum r_j$ in the Grassmannian.

By Lemma 2.11,

$$\pi_{\underline{\alpha}}^*(C) = P(\xi_1, \ldots, \xi_d)$$

for a homogeneous symmetric polynomial $P(\xi_1, \ldots, \xi_d)$ of degree $dN - \sum r_j$; we have to prove that

$$\xi_1^{r_1} \cdots \xi_d^{r_d} \cdot P(\xi_1, \ldots, \xi_d)$$

and

$$-\xi_1^{r_1} \cdots \xi_i^{r_{i+1}+1} \xi_{i+1}^{r_i-1} \cdots \xi_d^{r_d} \cdot P(\xi_1, \ldots, \xi_d)$$

agree in $[\mathbb{V}((N^d))]$. Now, there is a one-to-one correspondence between monomials in these expressions: each monomial

$$(\prod_{j=1}^{d} \xi_j^{r_j}) \cdots \xi_i^a \xi_{i+1}^b \cdots$$

in the first one corresponds to exactly one monomial

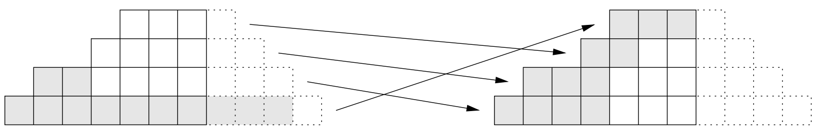
$$-(\xi_1^{r_1} \cdots \xi_i^{r_{i+1}+1} \xi_{i+1}^{r_i-1} \cdots \xi_d^{r_d}) \cdots \xi_i^b \xi_{i+1}^a \cdots$$

in the second one; it suffices to show that these two terms match. After absorbing all exponents into the r_i's, the stated equality is then reduced to the statement of Lemma 2.6 (proved in §2.11 below), and we are done. □

The push-forward operation may be visualized as follows: adjoin the d-ladder to $\underline{\alpha}$; take away boxes from the rows as dictated by the given monomial in $\xi_1, \ldots, \xi_d$; rearrange the remaining rows in *strictly* increasing order, keeping track of the sign of the needed permutation; the push-forward is then read from the complement of the ladder.

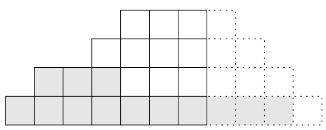
Example 2.13. Let $\underline{\alpha} = (7 \geq 6 \geq 4 \geq 3)$; then

$$\pi_{\underline{\alpha}*}\left(\xi_1^{10}\xi_2^2 \cap [\mathbb{V}(\underline{\alpha})]\right) = -[\mathbb{S}(3 \geq 3 \geq 2)] \quad :$$



The push-forward is 0 if these operations cannot be performed; for example, if the numbers of 'white' boxes in the rows are not distinct:

$$\pi_{\underline{\alpha}*}\left(\xi_1^{10}\xi_2^3 \cap [\mathbb{V}(\underline{\alpha})]\right) = 0 \quad .$$



2.11. **Proof of Lemma 2.6.** Let

$$\rho : \mathbb{V}((N^d)) \to \mathbb{V}(0) = \text{point}$$

be the projection; Lemma 2.6 is equivalent to

$$\rho_*\left(\xi_1^{r_1}\cdots\xi_d^{r_d}\cap[\mathbb{V}((N^d))]\right) = -\rho_*\left(\xi_1^{r_1}\cdots\xi_i^{r_{i+1}+1}\xi_{i+1}^{r_i-1}\cdots\xi_d^{r_d}\cap[\mathbb{V}((N^d))]\right) \quad .$$

This reduces the lemma to a particular case of the following template situation.

Let X be any scheme, and let $X_2 = \mathbb{P}(\mathscr{E})$ be a projective bundle over X, with $\operatorname{rk}\mathscr{E} = N+1$. The universal sequence over X_2:

$$0 \longrightarrow \mathscr{O}(-1) \longrightarrow \mathscr{E} \longrightarrow \mathscr{Q} \longrightarrow 0$$

determines a quotient bundle $\mathscr{Q}$, of rank N. Let $X_1 = \mathbb{P}(\mathscr{Q}\oplus\mathscr{O})$, projecting to X:

$$\rho : X_1 \xrightarrow{\rho_1} X_2 \xrightarrow{\rho_2} X \quad .$$

Lemma 2.6 is an immediate consequence of the following explicit computation.

Lemma 2.14. *Let* $\xi_2 = -c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-1))$ *on* X_2 *(and its pull-back to* X_1*); and let* $\xi_1 = -c_1(\mathscr{O}_{\mathbb{P}(\mathscr{Q}\oplus\mathscr{O})})(-1))$ *on* X_1*. Then*

$$\rho_*(\xi_1^a\xi_2^b\cap[X_1]) = -\rho_*(\xi_1^{b+1}\xi_2^{a-1}\cap[X_1]) \quad .$$

Proof. By the very definition of Segre class,

$$\rho_{1*}(\sum_j \xi_1^j\cap[X_1]) = s(\mathscr{Q}\oplus\mathscr{O})\cap[X_2] = s(\mathscr{E})c(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-1))\cap[X_2] = s(\mathscr{E})(1-\xi_2)\cap[X_2] \quad .$$

Therefore

$$\rho_{1*}(\xi_1^a\cap[X_1]) = (s_{a-N}(\mathscr{E}) - s_{a-1-N}(\mathscr{E})\,\xi_2)\cap[X_2] \quad .$$

By the same token,

$$\rho_{2*}(\sum_j \xi_2^j\cap[X_2]) = s(\mathscr{E})\cap[X] \quad ,$$

and it follows that

$$\rho_*(\xi_1^a\xi_2^b\cap[X_1]) = (s_{a-N}(\mathscr{E})s_{b-N}(\mathscr{E}) - s_{a-1-N}(\mathscr{E})s_{b+1-N}(\mathscr{E}))\cap[X] \quad .$$

The substitution $a\mapsto b+1$, $b\mapsto a-1$ switches the summands in the right-hand-side, proving the statement. □

3. Chern-Schwartz-MacPherson classes of Schubert cells

3.1. **Chern classes of singular varieties.** We now assume that the ground field k has characteristic 0. As recalled in the introduction, there is a good theory of Chern classes for singular varieties: $c_{\mathrm{SM}}(X)\in A_*(X)$ will denote the *Chern-Schwartz-MacPherson* class of the (possibly) singular variety X. There are a number of different approaches to the definition of these classes; the following is best suited to our purposes. A more thorough discussion may be found in [Alu] or [Alu06].

Let X be a complete variety over an algebraically closed field of characteristic zero. We denote by $F(X)$ the group of *constructible functions* on X, that is, the free abelian group on characteristic functions of subvarieties of X. Thus, every constructible function may be written (uniquely) as $\sum m_Z\,1\!\!1_Z$, where the sum ranges over finitely many closed subvarieties $Z \subset X$, $m_Z \in \mathbb{Z}$, and $1\!\!1_Z$ denotes the function with value 1 on Z and 0 on its complement.

Constructible functions may also be written as finite sums $\sum m_W 1\!\!1_W$, for $W \subset X$ *locally* closed subvarieties of X; of course these expressions are no longer unique. For a given $\varphi \in F(X)$, we consider any decomposition

$$X = \bigcup_{j \in J} W_j$$

of X into finitely many, locally closed, *nonsingular* subvarieties W_j, such that

$$\varphi = \sum_{j \in J} m_j \, 1\!\!1_{W_j} \quad ,$$

for $m_j \in \mathbb{Z}$. We use such a decomposition to associate to φ a class

$$c_*(\varphi) \in A_* X \quad ,$$

as follows.

By resolution of singularities, if $W \subset X$ is nonsingular then there exists a nonsingular completion $\overline{W}$ of W, and a proper morphism

$$\omega : \overline{W} \to X$$

such that the complement $\overline{W} \setminus W$ is a divisor D with normal crossings and nonsingular components D_i, $i \in I$.

Definition 3.1. For $W \subset X$ a nonsingular locally closed subvariety, we set

$$c_*(1\!\!1_W) := \omega_* \left(c(T\overline{W}(-\log D)) \cap [\overline{W}] \right) \in A_*(X) \quad .$$

Here $T\overline{W}(-\log D)$ denotes the bundle of vector fields with logarithmic zeros along the components of D; as is well known,

$$c(T\overline{W}(-\log D)) = \frac{c(T\overline{W})}{\prod_{i \in I}(1 + D_i)} \quad .$$

One can verify that the class $c_*(1\!\!1_W)$ is independent of the chosen completion $\overline{W}$.

The class $c_*(\varphi)$ is defined by linearity: for a decomposition $X = \cup_{j \in J} W_j$ as above, we set

$$c_*(\varphi) := \sum_{j \in J} m_j \, c_*(1\!\!1_{W_j}) \quad .$$

One can verify that the class $c_*(\varphi)$ only depends on the constructible function φ, not on the chosen decomposition. In fact (cf. [Alu06], Théorème 3.3), the homomorphism $c_* : F(X) \to A_*(X)$ agrees with the one induced by MacPherson's natural transformation from the functor F (with covariance defined by Euler characteristic of fibers, cf. [Mac74] or [Ful84], Example 19.1.7) to the Chow group functor A_*. In particular, for $\varphi = 1\!\!1_X$ one obtains the *Chern-Schwartz-MacPherson* class of X:

$$c_{\mathrm{SM}}(X) := c_*(1\!\!1_X) \in A_*(X) \quad ;$$

and the preceding discussion shows that if

$$X = \coprod_{j \in J} W_j$$

is a decomposition of X into disjoint nonsingular locally closed subvarieties, then

$$c_{\mathrm{SM}}(X)=\sum_{j\in J} c_*(1\!\!1_{W_j}) \quad .$$

We denote $c_*(1\!\!1_{W_j})$ by $c_{\mathrm{SM}}(W_j)$, for notational consistency; this abuse of language is harmless in context, provided that the reader keeps in mind that $c_{\mathrm{SM}}(W_j)$ is a class in the Chow group of the ambient variety X, not of W_j.

3.2. We are essentially ready to compute $c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ)$, by applying the formula given in Definition 3.1; the only missing ingredient is the Chern class of the tangent bundle of $\mathbb{V}(\underline{\alpha})$.

Recall that $\mathscr{Q}_i$ is trivial for $i\gg 0$; thus $c(\mathscr{L}_i^\vee\otimes\mathscr{Q}_i)=1$ for $i\gg 0$, and only finitely many terms contribute nontrivially to the formula in the following statement.

Proposition 3.2.

$$c(T\mathbb{V}(\underline{\alpha}))=\prod_{i\ge 1} c(\mathscr{L}_i^\vee\otimes\mathscr{Q}_i) \quad .$$

Proof. As $\mathbb{V}(\underline{\alpha})$ is defined inductively, we have to check that this is correct for $\underline{\alpha}=(0)$, which is trivially the case, and that the classes have identical behavior when going from the truncation $\underline{\alpha}'=(\alpha_2\ge\dots)$ to $\underline{\alpha}=(\alpha_1\ge\alpha_2\ge\dots)$.

Now, by definition,

$$\mathbb{V}(\underline{\alpha})=\mathbb{P}(\mathscr{Q}_2\oplus\mathscr{O}^{\oplus\alpha_1-\alpha_2+1})$$

over $\mathbb{V}(\underline{\alpha}')$, with relative tangent bundle $\mathscr{L}_1^\vee\otimes\mathscr{Q}_1$: this is computed by tensoring the universal sequence

$$0\longrightarrow\mathscr{L}_1\longrightarrow\mathscr{Q}_2\oplus\mathscr{O}^{\oplus\alpha_1-\alpha_2+1}\longrightarrow\mathscr{Q}_1\longrightarrow 0$$

by $\mathscr{L}_1^\vee$ (cf. [Ful84], §B.5.8.) Then the statement follows immediately. ☐

The main results of this paper will be obtained from the following consequence.

Corollary 3.3. *Let $\pi_{\underline{\alpha}}:\mathbb{V}(\underline{\alpha})\to\mathbb{S}(\underline{\alpha})$ be the birational isomorphism defined in §2.8. Then the Chern-Schwartz-MacPherson class of the Schubert cell $\mathbb{S}(\underline{\alpha})^\circ$ is*

$$\begin{aligned}c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ)&=\pi_{\underline{\alpha}*}\left(\frac{c(T\mathbb{V}(\underline{\alpha}))}{\prod_{i\ge1}c(\mathscr{L}_i^\vee)}\cap[\mathbb{V}(\underline{\alpha})]\right)\\&=\pi_{\underline{\alpha}*}\left(\prod_{i\ge1}c(\mathscr{L}_i^\vee)^{\alpha_i-\alpha_{i+1}}\,c(\mathscr{L}_i^\vee\otimes\mathscr{Q}_{i+1})\cap[\mathbb{V}(\underline{\alpha})]\right) \quad .\end{aligned}$$

Proof. By Proposition 2.10, we can identify $\mathbb{S}(\underline{\alpha})^\circ$ with an open dense subset of $\mathbb{V}(\underline{\alpha})$, and the complement of $\mathbb{S}(\underline{\alpha})^\circ$ in $\mathbb{V}(\underline{\alpha})$ is a divisor with normal crossings and nonsingular components, with class $\xi_i=c_1(\mathscr{L}_i^\vee)$. According to Definition 3.1,

$$c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ)=\pi_{\underline{\alpha}*}\left(\frac{c(T\mathbb{V}(\underline{\alpha}))}{\prod_{i\ge1}c(\mathscr{L}_i^\vee)}\cap[\mathbb{V}(\underline{\alpha})]\right)$$

(since $\mathscr{L}_i=\mathscr{O}$ for $i\gg0$, only finitely many terms contribute to the denominator).

Now tensor the sequences

$$0\longrightarrow\mathscr{L}_i\longrightarrow\mathscr{Q}_{i+1}\oplus\mathscr{O}^{\oplus\alpha_i-\alpha_{i+1}+1}\longrightarrow\mathscr{Q}_i\longrightarrow 0$$

by $\mathscr{L}_i^\vee$, obtaining

$$c(\mathscr{L}_i^\vee\otimes\mathscr{Q}_i)=c(\mathscr{L}_i^\vee\otimes\mathscr{Q}_{i+1})\,c(\mathscr{L}_i^\vee)^{\alpha_i-\alpha_{i+1}+1}\quad;$$

applying Proposition 3.2 gives

$$c_{\rm SM}(\mathbb{S}(\underline{\alpha})^\circ)=\pi_{\underline{\alpha}*}\left(\prod_{i\ge 1}\frac{c(\mathscr{L}_i^\vee\otimes\mathscr{Q}_{i+1})\,c(\mathscr{L}_i^\vee)^{\alpha_i-\alpha_{i+1}+1}}{c(\mathscr{L}_i^\vee)}\cap[\mathbb{V}(\underline{\alpha})]\right)$$

with the stated result. □

The next task is to obtain explicit formulas from the statement of Corollary 3.3. We offer several versions, all easily amenable to computer implementation. Since $c_{\rm SM}(\mathbb{S}(\underline{\alpha})^\circ)\in A_*\mathbb{S}(\underline{\alpha})$,

$$c_{\rm SM}(\mathbb{S}(\underline{\alpha})^\circ)=\sum_{\underline{\beta}\le\underline{\alpha}}\gamma_{\underline{\alpha},\underline{\beta}}[\mathbb{S}(\underline{\beta})]\quad,$$

for uniquely determined coefficients $\gamma_{\underline{\alpha},\underline{\beta}}\in\mathbb{Z}$; we give formulas computing these coefficients.

The reader should note that the formulas will often appear to depend on the choice of an integer d such that $\alpha_{d+1}=0$; the results of the computations must be independent of this choice, since so is the class $c_{\rm SM}(\mathbb{S}(\underline{\alpha})^\circ)$.

3.3. **Explicit computations of $\gamma_{\underline{\alpha},\underline{\beta}}$.** We denote by

$$h_a(y_1,\dots,y_n)=\sum_{i_1+\cdots+i_n=a}y_1^{i_1}\cdots y_n^{i_n}$$

the *complete symmetric polynomial* of degree a in the variables $y_1,\dots,y_n$.

Theorem 3.4. *Let $\underline{\alpha}=(\alpha_1\ge\cdots\ge\alpha_d)$, $\underline{\beta}=(\beta_1\ge\cdots\ge\beta_d)$ be partitions.*

For $b_1,\dots,b_d$ positive integers, let $C_{\underline{\alpha}}(b_1,\dots,b_d)$ be the coefficient of

$$x_1^{\alpha_1+d-b_1}\cdots x_d^{\alpha_d+1-b_d}$$

in the polynomial

$$\prod_{i=1}^d(1+x_i)^{\alpha_i-\alpha_{i+1}}\cdot h_{\alpha_i+1}(1+x_i,x_{i+1},\dots,x_d)\quad.$$

Then

$$\gamma_{\underline{\alpha},\underline{\beta}}=\sum_{\sigma\in S_d}(-1)^\sigma C_{\underline{\alpha}}(b_{\sigma(1)},\dots,b_{\sigma(d)})\quad,$$

where $b_i=\beta_i+(d+1-i)$.

Proof. By Corollary 3.3,

$$c_{\rm SM}(\mathbb{S}(\underline{\alpha})^\circ)=\pi_{\underline{\alpha}*}\left(\prod_{i=1}^d c(\mathscr{L}_i^\vee)^{\alpha_i-\alpha_{i+1}}\,c(\mathscr{L}_i^\vee\otimes\mathscr{Q}_{i+1})\cap[\mathbb{V}(\underline{\alpha})]\right)\quad,$$

where d is such that $\alpha_{d+1}=0$ (so that $\mathscr{L}_i=\mathscr{O}$ and $\mathscr{Q}_i=0$ for $i>d$).

Recall that for all i there are exact sequences ((*) in §2.7)

$$0\longrightarrow\mathscr{S}_i\longrightarrow\mathscr{F}_{\alpha_i+(d+1-i)}\longrightarrow\mathscr{Q}_i\longrightarrow 0\quad;$$

and in particular

$$c(\mathscr{Q}_{i+1}) = \frac{1}{c(\mathscr{S}_{i+1})} = \frac{1}{(1-\xi_{i+1})\cdots(1-\xi_d)} = \sum_{j=1}^{\alpha_{i+1}} h_j(\xi_{i+1},\dots,\xi_d) \quad .$$

Therefore ([Ful84], Example 3.2.2)

$$c(\mathscr{L}_i^\vee \otimes \mathscr{Q}_{i+1}) = h_{\alpha_{i+1}}(1+\xi_i,\xi_{i+1},\dots,\xi_d)$$

and hence

$$\prod_{i=1}^d c(\mathscr{L}_i^\vee)^{\alpha_i-\alpha_{i+1}}\, c(\mathscr{L}_i^\vee \otimes \mathscr{Q}_{i+1}) = \prod_{i=1}^d (1+\xi_i)^{\alpha_i-\alpha_{i+1}}\, h_{\alpha_{i+1}}(1+\xi_i,\xi_{i+1},\dots,\xi_d) \quad .$$

Now let $x_1,\dots,x_d$ be variables, and write

$$\prod_{i=1}^d (1+x_i)^{\alpha_i-\alpha_{i+1}}\, h_{\alpha_{i+1}}(1+x_i,x_{i+1},\dots,x_d) = \sum_{r_1,\dots,r_d\ge 0} e_{r_1,\dots,r_d}\, x_1^{r_1}\cdots x_d^{r_d} \quad ;$$

by Proposition 2.12,

$$\begin{aligned} c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ) &= \pi_{\underline{\alpha}*}\left(\sum_{r_1,\dots,r_d\ge 0} e_{r_1,\dots,r_d}\, \xi_1^{r_1}\cdots\xi_d^{r_d} \cap [\mathbb{V}(\underline{\alpha})]\right) \\ &= \sum_{r_1,\dots,r_d\ge 0} e_{r_1,\dots,r_d}\, [\Omega(\alpha_d-r_d+1,\dots,\alpha_1-r_1+d)] \quad , \end{aligned}$$

with notation as in §2.10. By definition, $\gamma_{\underline{\alpha},\underline{\beta}}$ is the coefficient of $[\mathbb{S}(\beta)]$ in this expression. Let then $b_1=\beta_1+d$, …, $b_d=\beta_d+1$; thus $b_1>\cdots>b_d$ are positive integers, $[\mathbb{S}(\underline{\beta})]=[\Omega(b_d,\dots,b_1)]$, and

$$[\Omega(b_{\sigma(d)},\dots,b_{\sigma(1)})] = (-1)^\sigma [\mathbb{S}(\underline{\beta})] \quad .$$

With this notation, the coefficient of $[\mathbb{S}(\underline{\beta})]$ is given by

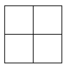
$$\sum_{\sigma\in S_d} (-1)^\sigma e_{\alpha_1+d-b_{\sigma(1)},\dots,\alpha_d+1-b_{\sigma(d)}} = \sum_{\sigma\in S_d} (-1)^\sigma C_{\underline{\alpha}}(b_{\sigma(1)},\dots,b_{\sigma(d)}) \quad ,$$

as stated. □

Example 3.5. For two-row diagrams, i.e., $\underline{\alpha}=(\alpha_1\ge\alpha_2)$, the CSM class of the corresponding Schubert cell may be read off the expansion of

$$(1+x_1)^{\alpha_1-\alpha_2}(1+x_2)^{\alpha_2}\sum_{i=0}^{\alpha_2}(1+x_1)^i x_2^{\alpha_2-i} \quad .$$

For the open cell in $G_2(\mathbb{C}^4)$ (=the Grassmannian of projective lines in $\mathbb{P}^3$):



this is

$$1+3\,x_2+4\,x_2^2+5\,x_1x_2+x_1^2+4\,x_1x_2^2+x_1^2x_2^2+\left(2\,x_1+2\,x_1^2x_2+3\,x_2^3+x_1x_2^3+x_2^4\right) \quad .$$

The coefficient of the Schubert class corresponding to

$$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$$

that is, of $[\mathbb{S}(1 \geq 1)]$, is

$$\gamma_{(2\geq 2),(1\geq 1)} = C_{2\geq 2}(3,2) - C_{2\geq 2}(2,3) = e_{1,1} - e_{2,0} = 5 - 1 = 4 \quad .$$

The terms in parentheses do not correspond to any subdiagram of $\underline{\alpha}$; they do correspond to classes in $\mathbb{V}(\underline{\alpha})$ (setting $x_i = \xi_i$), but they all push-forward to 0 in $A_*(G_2(\mathbb{C}^4))$, as may be checked by applying Proposition 2.12.

There is a useful alternative to the expression given in Theorem 3.4:

Theorem 3.6. *Let $\underline{\alpha} = (\alpha_1 \geq \cdots \geq \alpha_d)$, $\underline{\beta} = (\beta_1 \geq \cdots \geq \beta_d)$ be partitions.*
For $b_1, \dots, b_d$ positive integers, let $C'_{\underline{\alpha}}(b_1, \dots, b_d)$ be the coefficient of

$$x_1^{\alpha_1+d-b_1} \cdots x_d^{\alpha_d+1-b_d}$$

in the expansion of the rational function

$$\frac{\prod_{i=1}^d (1+x_i)^{\alpha_i+(d-i)}}{\prod_{1\leq i<j\leq d}(1+x_i-x_j)}$$

at 0*. Then*

$$\gamma_{\underline{\alpha},\underline{\beta}} = \sum_{\sigma\in S_d} (-1)^\sigma C'_{\underline{\alpha}}(b_{\sigma(1)}, \dots, b_{\sigma(d)}) \quad ,$$

where $b_i = \beta_i + (d+1-i)$.

Proof. Use again the exact sequence

$$0 \longrightarrow \mathscr{S}_i \longrightarrow \mathscr{F}_{\alpha_i+(d+1-i)} \longrightarrow \mathscr{Q}_i \longrightarrow 0$$

to obtain

$$c(\mathscr{L}_i^\vee \otimes \mathscr{Q}_i) = \frac{c(\mathscr{L}_i^\vee)^{\alpha_i+(d+1-i)}}{c(\mathscr{L}_i^\vee \otimes \mathscr{S}_i)} = \frac{c(\mathscr{L}_i^\vee)^{\alpha_i+(d+1-i)}}{\prod_{i\leq j\leq d} c(\mathscr{L}_i^\vee \otimes \mathscr{L}_j)} \quad ,$$

and hence

$$\frac{c(T\mathbb{V}(\underline{\alpha}))}{\prod_{1\leq i\leq d} c(\mathscr{L}_i^\vee)} = \prod_{1\leq i\leq d} \frac{c(\mathscr{L}_i^\vee)^{\alpha_i+(d-i)}}{\prod_{i<j\leq d} c(\mathscr{L}_i^\vee \otimes \mathscr{L}_j)} = \frac{\prod_{1\leq i\leq d}(1+\xi_i)^{\alpha_i+(d-i)}}{\prod_{1\leq i<j\leq d}(1+\xi_i-\xi_j)} \quad .$$

Then argue as in the proof of Theorem 3.4. □

Example 3.7. The numbers $C_{\underline{\alpha}}(b_1, \dots, b_d)$ and $C'_{\underline{\alpha}}(b_1, \dots, b_d)$ are *not* equal in general. For example,

$$C_{(2\geq 1\geq 1)}(5,2,1) = 4 \quad , \quad C'_{(2\geq 1\geq 1)}(5,2,1) = 5 \quad .$$

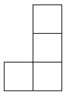
It is a consequence of Theorem 3.4 and 3.6 that, however, the 'antisymmetrization' of these coefficients must agree. Thus

$$C_{(2\geq 1\geq 1)}(5,2,1) - C_{(2\geq 1\geq 1)}(5,1,2) = 3 = C'_{(2\geq 1\geq 1)}(5,2,1) - C'_{(2\geq 1\geq 1)}(5,1,2)$$

both compute the coefficient $\gamma_{(2\geq 1\geq 1),(2)}$ of the Schubert class corresponding to

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$$

in the Chern-Schwartz-MacPherson class of the Schubert cell of



(the other permutations of the arguments $5, 2, 1$ all give vanishing contributions to the computation of $\gamma_{(2\geq 1\geq 1),(2\geq 0\geq 0)}$).

3.4. **Determinant form.** One can easily provide explicit determinantal expressions for the coefficients $\gamma_{\underline{\alpha},\underline{\beta}}$. The following form was stated in §1:

Theorem 3.8. *Let d be such that $\alpha_{d+1} = 0$. Then*

$$\gamma_{\underline{\alpha},\underline{\beta}} = \sum \det \left[\binom{\alpha_i - \ell^i_{i+1} - \cdots - \ell^i_d}{\beta_j + (i-j) + \ell^1_i + \cdots + \ell^{i-1}_i - \ell^i_{i+1} - \cdots - \ell^i_d} \right]_{1\leq i,j\leq d}$$

where the summation is over the $\binom{d}{2}$ integers ℓ^k_i, $1 \leq k < i \leq d$, subject to the conditions

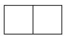
$$0 \leq \ell^k_{k+1} + \cdots + \ell^k_d \leq \alpha_{k+1} \quad .$$

Proof. This is obtained by computing explicitly the coefficients $C_{\underline{\alpha}}(b_1, \dots, b_d)$ appearing in Theorem 3.4, and interpreting $\sum_{\sigma\in S_d}(-1)^\sigma C_{\underline{\alpha}}(b_{\sigma(1)}, \dots, b_{\sigma(d)})$ as a determinant. $\square$

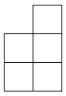
Example 3.9. For three-row diagrams, Theorem 3.8 computes $\gamma_{\underline{\alpha},\underline{\beta}}$ as

$$\sum_{0\leq \ell^1_2+\ell^1_3\leq\alpha_2} \; \sum_{0\leq\ell^2_3\leq\alpha_3} \det \begin{pmatrix} \binom{\alpha_1-\ell^1_2-\ell^1_3}{\beta_1-\ell^1_2-\ell^1_3} & \binom{\alpha_1-\ell^1_2-\ell^1_3}{\beta_2-1-\ell^1_2-\ell^1_3} & \binom{\alpha_1-\ell^1_2-\ell^1_3}{\beta_3-2-\ell^1_2-\ell^1_3} \\ \binom{\alpha_2-\ell^2_3}{\beta_1+1+\ell^1_2-\ell^2_3} & \binom{\alpha_2-\ell^2_3}{\beta_2+\ell^1_2-\ell^2_3} & \binom{\alpha_2-\ell^2_3}{\beta_3-1+\ell^1_2-\ell^2_3} \\ \binom{\alpha_3}{\beta_1+2+\ell^1_3+\ell^2_3} & \binom{\alpha_3}{\beta_2+1+\ell^1_3+\ell^2_3} & \binom{\alpha_3}{\beta_3+\ell^1_3+\ell^2_3} \end{pmatrix}$$

For example, the coefficient of the Schubert class corresponding to the diagram



in the Chern-Schwartz-MacPherson class of the Schubert cell of



is the sum of the determinants of the matrices

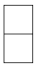
$$\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&2&1\\0&1&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&2\\0&1&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}$$

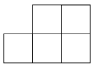
that is,

$$1 + 0 + 1 + 0 + 0 + 0 + 1 + 1 + 2 + 0 + (-1) + 1 = 6 \quad .$$

Duality implies that the same result may be obtained in a completely different way: this coefficient must equal the coefficient of the Schubert class corresponding to



in the Chern-Schwartz-MacPherson class of the Schubert cell of



that is, (again by Theorem 3.8) by the sum of binomial determinants

$$\sum_{\ell=0}^{\alpha_2} \det \begin{pmatrix} \binom{\alpha_1-\ell}{\beta_1-\ell} & \binom{\alpha_1-\ell}{\beta_2-1-\ell} \\ \binom{\alpha_2}{\beta_1+1+\ell} & \binom{\alpha_2}{\beta_2+\ell} \end{pmatrix} = \det \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} + \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \det \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 6 \quad .$$

In general, duality implies rather complicated combinatorial identities between sums of binomial determinants. It would be interesting to give a direct proof of these identities.

3.5. **Generating function.** It is also possible to present the numbers $\gamma_{\underline{\alpha},\beta}$ directly as coefficients of the series expansion of a rational function; this is the form given in Theorem 1.1.

Theorem 3.10. *Let d be such that $\alpha_{d+1} = 0$. Then $\gamma_{\underline{\alpha},\underline{\beta}}$ equals the coefficient of*

$$t_1^{\alpha_1} \cdots t_d^{\alpha_d} \cdot u_1^{\beta_1} \cdots u_d^{\beta_d}$$

in the expansion of the rational function

$$\Gamma_d(\underline{t},\underline{u}) = \frac{1}{(t_1^d \cdots t_d^1)(u_1^d \cdots u_d^1)} \cdot \prod_{1 \le i < j \le d} \frac{(t_i - t_j)(u_i - u_j)}{1 - 2t_j + t_i t_j} \cdot \prod_{1 \le i,j \le d} \frac{1 - t_i}{1 - t_i(1 + u_j)}$$

as a Laurent polynomial in $\mathbb{Z}[[t,u]]$.

This follows from a residue computation, based on the expression given in Theorem 3.6.

Proof. For the sake of notation we will give the argument for diagrams with at most $d = 2$ rows, that is,

$$\underline{\alpha} = (\alpha_1 \ge \alpha_2 \ge 0 \ge \cdots) \quad ;$$

the adaptations for larger d present no difficulties.

For $d = 2$, the expression in Theorem 3.6 is

$$\frac{(1+x_1)^{\alpha_1+1}(1+x_2)^{\alpha_2}}{1 + x_1 - x_2} \quad ;$$

the number $C'_{\underline{\alpha}}(b_1, b_2)$ is the coefficient of

$$x_1^{\alpha_1+2-b_1} x_2^{\alpha_2+1-b_2} \quad .$$

We can consider all $\underline{\alpha}$ at once as follows: $C'_{\underline{\alpha}}(b_1, b_2)$ is the coefficient of $x_1^{-b_1} x_2^{-b_2}$ in the Laurent expansion of

$$\frac{x_1^{-\alpha_1-2} x_2^{-\alpha_2-1}(1+x_1)^{\alpha_1+1}(1+x_2)^{\alpha_2}}{1+x_1-x_2} = \frac{x_1^{-2} x_2^{-1}(1+x_1)}{1+x_1-x_2}(1+x_1^{-1})^{\alpha_1}(1+x_2^{-1})^{\alpha_2} \quad ;$$

that is, in the coefficient of $t_1^{\alpha_1} t_2^{\alpha_2}$ in the expansion of

$$\frac{x_1^{-2} x_2^{-1}(1+x_1)}{1+x_1-x_2} \cdot \frac{1}{1 - t_1(1+x_1^{-1})} \cdot \frac{1}{1 - t_2(1+x_2^{-1})} \quad .$$

The coefficient of $x_1^{-b_1}$ in this expression may be viewed as the residue

$$\frac{x_2^{-1}}{1-t_2(1+x_2^{-1})}\frac{1}{2\pi i}\oint\frac{x_1^{b_1-2}(1+x_1)}{1+x_1-x_2}\cdot\frac{1}{1-t_1(1+x_1^{-1})}\cdot\frac{dx_1}{x_1}\quad,$$

or, after the change of variable $y=(1-t_1(1+x_1^{-1}))x_1$,

$$\frac{x_2^{-1}}{1-t_2(1+x_2^{-1})}\frac{1}{2\pi i}\oint\left(\frac{y+t_1}{1-t_1}\right)^{b_1-2}\frac{1+y}{1+y-x_2(1-t_1)}\cdot\frac{dy}{y(1-t_1)}\quad.$$

If $b_1\geq 2$, this is evaluated as

$$(*)\qquad\frac{x_2^{-1}}{1-t_2(1+x_2^{-1})}\cdot\frac{t_1^{b_1-2}}{(1-t_1)^{b_1-1}}\cdot\frac{1}{1-x_2(1-t_1)}\quad;$$

if $b_1=1$, an extra residue is picked up at $y=-t_1$, and is evaluated as

$$\frac{x_2^{-1}}{1-t_2(1+x_2^{-1})}\cdot\frac{-1}{t_1(1-x_2)}\quad;$$

but we can ignore this term, since we are only interested in the coefficients of $t_1^{\alpha_1}$ for $\alpha_1\geq 0$.

An entirely analogous computation evaluates the coefficient of $x_2^{-b_2}$ in (*) as

$$\frac{t_1^{b_1-2}}{(1-t_1)^{b_1-1}}\cdot\frac{t_2^{b_2-1}}{(1-t_2)^{b_2-1}}\cdot\frac{1}{1-2t_2+t_1t_2}\quad.$$

Summarizing, $C'_{\underline{\alpha}}(b_1,b_2)$ equals the coefficient of $t_1^{\alpha_1}t_2^{\alpha_2}$ in the Laurent expansion of

$$\frac{1}{1-2t_2+t_1t_2}\cdot\frac{(1-t_1)(1-t_2)}{t_1^2t_2}\cdot\frac{t_1^{b_1}}{(1-t_1)^{b_1}}\cdot\frac{t_2^{b_2}}{(1-t_2)^{b_2}}\quad.$$

Multiplying by bookkeeping terms $u_1^{b_1}$, $u_2^{b_2}$ and adding over all $b_1\geq 0$, $b_2\geq 0$ shows that $C'_{\underline{\alpha}}(b_1,b_2)$ is the coefficient of $t_1^{\alpha_1}\,t_2^{\alpha_2}\,u_1^{b_1}\,u_2^{b_2}$ in the expansion of

$$\frac{1}{1-2t_2+t_1t_2}\cdot\frac{(1-t_1)^2(1-t_2)^2}{t_1^2t_2}\cdot\frac{1}{1-t_1(1+u_1)}\cdot\frac{1}{1-t_2(1+u_2)}\quad,$$

for all positive b_1, b_2. It follows that $\gamma_{\underline{\alpha},\underline{\beta}}$ equals the coefficient of $t_1^{\alpha_1}\,t_2^{\alpha_2}\,u_1^{b_1}\,u_2^{b_2}$ in the expansion of

$$\frac{1}{1-2t_2+t_1t_2}\cdot\frac{(1-t_1)^2(1-t_2)^2}{t_1^2t_2}\cdot\det\left[\frac{1}{1-t_i(1+u_j)}\right]_{1\leq i,j\leq 2}$$

for $b_1=\beta_1+2$, $b_2=\beta_2+1$. That is, the coefficient of $t_1^{\alpha_1}\,t_2^{\alpha_2}\,u_1^{\beta_1}\,u_2^{\beta_2}$ in the expansion of

$$\frac{1}{1-2t_2+t_1t_2}\cdot\frac{(1-t_1)^2(1-t_2)^2}{(t_1^2t_2)(u_1^2u_2)}\cdot\det\left[\frac{1}{1-t_i(1+u_j)}\right]_{1\leq i,j\leq 2}\quad.$$

The determinant may be evaluated directly; in general,

$$\det\left[\frac{1}{1-t_i(1+u_j)}\right]_{1\leq i,j\leq d}=\frac{\prod_{1\leq i<j\leq d}(t_i-t_j)(u_i-u_j)}{\prod_{1\leq i,j\leq d}(1-t_i(1+u_j))}\quad,$$

an application of *Cauchy's double alternant* (see for example [Kra99], (2.7)). This yields the formula given in the statement. □

3.6. **Contribution of one-row diagrams.** As an application of Theorem 3.10, we can give a rather compact evaluation of the contribution of Schubert classes corresponding to one-row diagrams to the CSM class of an arbitrary $\underline{\alpha}$; this formula was mentioned in §1.

Corollary 3.11. *For all partitions $\underline{\alpha}$,*

$$\sum_{r\ge 0} \gamma_{\underline{\alpha},(r)}\, u^r = \prod_{i\ge 1} (1+iu)^{\alpha_i-\alpha_{i+1}} \quad .$$

Proof. Assume $\alpha_{d+1}=0$. By Theorem 3.10, $\gamma_{\underline{\alpha},(r)}$ equals the coefficient of $t_1^{\alpha_1}\cdots t_d^{\alpha_d}\cdot u_1^r$ in the expansion of $\Gamma_d(\underline{t},\underline{u})$. Standard manipulations show that this equals the coefficient of

$$t_1^{\alpha_1+d-1}\cdots t_{d-1}^{\alpha_{d-1}+1} t_d^{\alpha_d}\cdot u^r$$

in the expansion of

$$F(\underline{t}) = \prod_{1\le i<j\le d} \frac{t_i-t_j}{1-2t_j+t_it_j}\cdot \prod_{1\le i\le d}\frac{1}{1-t_i(1+u)} \quad .$$

Now set $a_i=\alpha_i+d-i$, and note that $a_1>a_2>\cdots>a_d\ge 0$. We say that two rational functions of $t_1,\dots,t_d$ *agree on the good cone* if the coefficients of $t_1^{a_1}\cdots t_d^{a_d}$ agree whenever $a_1>\cdots>a_d\ge 0$. We proceed to find a simpler rational function agreeing with F on the good cone.

A partial fraction decomposition of $F(\underline{t})$ as a function of t_1 gives

$$F(\underline{t}) = \frac{C_1}{1-t_1(1+u)} + \sum_{j=2}^{d} \frac{C_j}{1-2t_j+t_1t_j}$$

with C_j rational functions in $t_2,\dots,t_d,u$, expanding to elements of $\mathbb{Z}[[t_2,\dots,t_d,u]]$. It follows that $F(\underline{t})$ agrees with

$$\frac{C_1}{1-t_1(1+u)}$$

on the good cone, since the exponent of t_1 in every other summand is no larger than the exponent of t_j, for some $j>1$. Evaluating C_1 shows that $F(\underline{t})$ and

$$\frac{1}{1-t_1(1+u)}\cdot \prod_{2\le i<j\le d}\frac{t_i-t_j}{1-2t_j+t_it_j}\cdot\prod_{2\le i\le d}\frac{1}{1+u-t_i(1+2u)}$$

agree on the good cone.

Applying the same procedure to C_1, viewed as a function of t_2, shows that $F(\underline{t})$ agrees with

$$\frac{1}{(1-t_1(1+u))(1-t_2(1+2u))}\cdot \prod_{3\le i<j\le d}\frac{t_i-t_j}{1-2t_j+t_it_j}\cdot\prod_{3\le i\le d}\frac{1}{1+2u-t_i(1+3u)}$$

on the good cone. Repeating again, at the d-th stage one obtains that $F(\underline{t})$ and

$$G(\underline{t}) := \prod_{i=1}^{d} \frac{1}{1+(i-1)u-t_i(1+iu)}$$

agree on the good cone.

Therefore, $\gamma_{\underline{\alpha},(r)}$ equals the coefficient of $t_1^{\alpha_1+d-1}\cdots t_d^{\alpha_d}\cdot u^r$ in the expansion of $G(\underline{t})$. This can be evaluated easily, and yields the statement. $\square$

The formula given in Corollary 3.11 implies the following pretty expression for the contribution of $[\mathbb{S}((r))]$ to the total Chern class of a Grassmannian $G_d(V)$: this equals the coefficient of u^r in

$$\frac{1}{d!\,u^d}\sum_{i=0}^{d}\binom{d}{i}(-1)^{d-i}(1+iu)^{\dim V}\quad .$$

For example, the coefficients of one-row Schubert classes in the total Chern class of $G_2(\mathbb{C}^5)$ may be read off

$$\frac{1}{2u^2}\left(1-2(1+u)^5+(1+2u)^5\right)=10+30u+35u^2+15u^3\quad ,$$

cf. Example 1.2.

4. Positivity and related issues

4.1. Substantial computer experimentation suggests the following:

Conjecture 1. *For all $\underline{\alpha}$, $c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ)\in A_*(\mathbb{S}(\underline{\alpha}))$ is represented by an effective cycle.*

That is, we conjecture that the coefficients $\gamma_{\underline{\alpha},\underline{\beta}}$ are nonnegative, for all $\underline{\beta}\le\underline{\alpha}$.

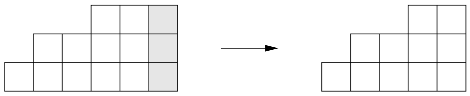
As a consequence, the Chern-Schwartz-MacPherson class of every Schubert *variety* would be positive in the sense of [Ful84], Chapter 12. This is well-known for the Grassmannian (that is, for $\underline{\alpha}=(N^d)$): indeed, the tangent bundle of a Grassmannian is generated by its sections ([Ful84], Examples 12.2.1 and 12.1.7). Such standard positivity arguments do not seem to apply directly to Chern classes of singular varieties. Conjecture 1, if true, may reflect a new positivity principle for such Chern classes.

Proving positivity from the explicit formulas obtained in §3 is an interesting challenge. Note that some of the terms in the expression given in Theorem 3.4 may give a negative contribution to a coefficient (cf. Example 3.5: x_1^2 contributes $-e_{2,0}=-1$ to $\gamma_{(2\ge 2),(1\ge 1)}$); and some of the individual determinants in Theorem 3.8 may be negative (cf. Example 3.9). Conjecture 1 amounts to the statement that these negative contributions are canceled by corresponding positive contributions, and proving that such cancellations always occur appears to be difficult in general.

On the other hand, Corollary 3.11 implies that all coefficients $\gamma_{\underline{\alpha},(r)}$ of one-row Schubert classes in the CSM class of any Schubert cell are positive; this is some evidence for Conjecture 1. In particular, Conjecture 1 is immediate for Schubert cells corresponding to one-row diagrams. It is also true for diagrams with 2 rows: in this section we give two different proofs of this easy result, which highlight two different interesting facts on Chern-Schwartz-MacPherson classes of Schubert cells.

These facts are a partial recursion for the coefficients $\gamma_{\underline{\alpha},\underline{\beta}}$, and a concrete interpretation of these coefficients as counting certain lattice paths between sets of points in the plane. We are able to prove positivity for all diagrams with 3 rows, by refining this latter counting argument. However, the proof in this case is substantially more technical, and we will report on it elsewhere.

4.2. **Partial recursion.** Given a non-empty diagram $\underline{\alpha}$, we denote by $\underline{\alpha}^-$ the diagram obtained by removing the rightmost column from $\underline{\alpha}$:



That is, $\underline{\alpha}^- = \underline{\alpha} - (1^d)$, where d is the largest integer for which $\alpha_d \neq 0$.

We have the inclusion of Schubert *varieties*

$$\mathbb{S}(\underline{\alpha}^-) \subset \mathbb{S}(\underline{\alpha}) \quad .$$

Proposition 4.1. *Let $\underline{\alpha}$ be a diagram with exactly d rows, and embed $\mathbb{S}(\underline{\alpha})$ in a Grassmannian $G_d(V)$. Let $\mathscr{S}$ be the universal subbundle on $G_d(V)$. Then*

$$c_d(\mathscr{S}^\vee) \cap c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})) = c(\mathscr{S}^\vee) \cap c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha}^-)) \quad ,$$

and the same relation holds for CSM classes of cells.

This statement may be interpreted as an 'adjunction formula', in the sense that the given identity is the formula that would follow from the exact sequence of the normal bundle if $\mathbb{S}(\underline{\alpha}^-) \subset \mathbb{S}(\underline{\alpha})$ were nonsingular, with normal bundle the restriction of $\mathscr{S}^\vee$. Simple examples show that such formulas do not hold in general for CSM classes; the fact that that they do hold for the embeddings $\mathbb{S}(\underline{\alpha}^-) \subset \mathbb{S}(\underline{\alpha})$ calls for a more geometric explanation.

Proof. By additivity, it suffices to prove the corresponding formula at the level of CSM classes of Schubert cells:

$$c_d(\mathscr{S}^\vee) \cap c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ) = c(\mathscr{S}^\vee) \cap c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha}^-)^\circ) \quad .$$

Let $\pi : \mathbb{V}(\underline{\alpha}) \to \mathbb{S}(\underline{\alpha})$, $\pi^- : \mathbb{V}(\underline{\alpha}^-) \to \mathbb{S}(\underline{\alpha}^-)$ be the birational isomorphisms constructed in §2.8. Recall that $\mathbb{V}(\underline{\alpha}^-)$ embeds in $\mathbb{V}(\underline{\alpha})$, compatibly with these maps; and $[\mathbb{V}(\underline{\alpha}^-)]$ has class $\xi_1 \cdots \xi_d$ in $A_*(\mathbb{V}(\underline{\alpha}))$ by Lemma 2.7.

By the very constructions of π,

$$\mathscr{S}_1 = \pi^* \mathscr{S} \quad ,$$

so that

$$c(\pi^* \mathscr{S}) = \prod_{i=1}^{d} c(\mathscr{L}_i^\vee)$$

(cf. §2.9). By Corollary 3.3 and the projection formula,

$$\begin{aligned} c_d(\mathscr{S}^\vee) \cap c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ) &= \pi_* \left(\left(\prod_{i=1}^{d} \xi_i \right) \prod_{i \geq 1} c(\mathscr{L}_i^\vee)^{\alpha_i - \alpha_{i+1}} \, c(\mathscr{L}_i^\vee \otimes \mathscr{Q}_{i+1}) \cap [\mathbb{V}(\underline{\alpha})] \right) \\ &= \pi_* \left(\prod_{i \geq 1} c(\mathscr{L}_i^\vee)^{\alpha_i - \alpha_{i+1}} \, c(\mathscr{L}_i^\vee \otimes \mathscr{Q}_{i+1}) \cap [\mathbb{V}(\underline{\alpha}^-)] \right) \end{aligned}$$

Here $\mathscr{L}_i$, $\mathscr{Q}_i$ are the structure bundles on $\mathbb{V}(\underline{\alpha})$. Denoting by $\mathscr{L}_i^-$, $\mathscr{Q}_i^-$ the structure bundles on $\mathbb{V}(\underline{\alpha}^-)$ we have, as observed in §2.6,

$$\mathscr{L}_i|_{\mathbb{V}(\underline{\alpha}^-)} = \mathscr{L}_i^{-\vee} \quad , \quad \mathscr{Q}_i|_{\mathbb{V}(\underline{\alpha}^-)} = \mathscr{Q}_i^- \oplus \mathscr{O} \quad .$$

Therefore,

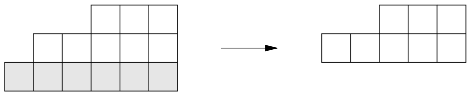
$$(\mathscr{L}_i^\vee \otimes \mathscr{Q}_{i+1})|_{\mathbb{V}(\underline{\alpha}^-)} = (\mathscr{L}_i^{-\vee} \otimes \mathscr{Q}_{i+1}^-) \oplus \mathscr{L}_i^{-\vee} \quad ,$$

giving

$$\begin{aligned} c_d(\mathscr{S}^\vee) \cap c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ) &= \pi_*^- \left(\prod_{i\ge 1} c(\mathscr{L}_i^{-\vee})^{\alpha_i-\alpha_{i+1}} \, c(\mathscr{L}_i^{-\vee} \otimes \mathscr{Q}_{i+1}^-) \, c(\mathscr{L}_i^{-\vee}) \cap [\mathbb{V}(\underline{\alpha}^-)] \right) \\ &= \pi_*^- \left(\left(\prod_{i\ge 1} c(\mathscr{L}_i^{-\vee}) \right) \prod_{i\ge 1} c(\mathscr{L}_i^{-\vee})^{\alpha_i-\alpha_{i+1}} \, c(\mathscr{L}_i^{-\vee} \otimes \mathscr{Q}_{i+1}^-) \cap [\mathbb{V}(\underline{\alpha}^-)] \right) \\ &= c(\mathscr{S}^\vee) \cap c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha}^-)^\circ) \end{aligned}$$

again by the projection formula and Corollary 3.3. □

By duality, Proposition 4.1 implies the following alternative formulation. For a non-empty diagram $\underline{\alpha}$, denote by $\underline{\alpha}'$ the diagram obtained by removing the *bottom row* from $\underline{\alpha}$:



That is, $\underline{\alpha}'$ is the truncation often used elsewhere in this article, for inductive arguments.

Proposition 4.2. *Let $\underline{\alpha}$ be a nonempty diagram; embed $\mathbb{S}(\underline{\alpha})$ in the Grassmannian $G_d(V)$, where $\dim V = \alpha_1 + d$, and let $\mathscr{Q}$ be the universal quotient bundle on $G_d(V)$. Then*

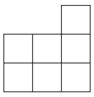
$$c_{\alpha_1}(\mathscr{Q}) \cap c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})) = c(\mathscr{Q}) \cap c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha}')) \quad ,$$

and the same relation holds for CSM classes of cells.

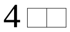
Proof. Apply Proposition 4.1 to the transpose of $\underline{\alpha}$; this gives the statement, since the isomorphism $G_d(V) \cong G_{\alpha_1}(V)$ interchanges the universal quotient bundle with the (dual) universal subbundle. □

These formulas imply that *some* of the coefficients in the CSM class of a Schubert cell (or variety) are determined by the CSM class of Schubert cells for smaller diagrams. For example: provided the class $c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha}^-)^\circ)$ is known, Proposition 4.1 determines the coefficients of the classes which survive the product by $c_d(\mathscr{S}^\vee)$, where d is the number of rows of the diagram $\underline{\alpha}$. These are the classes $[\mathbb{S}(\underline{\beta})]$ for subdiagrams $\underline{\beta} \le \underline{\alpha}$ also consisting of d rows.

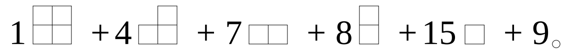
Example 4.3. Let $\underline{\alpha} = (3 \ge 3 \ge 1)$:



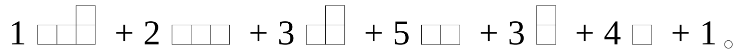
Then the coefficients in $c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ)$ of the six classes corresponding to subdiagrams of $(3 \ge 3 \ge 1)$ with exactly three rows are determined by Proposition 4.1 from $c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha}^-)^\circ) = c_{\mathrm{SM}}(\mathbb{S}(2 \ge 2)^\circ)$, that is

1  +3  + 4  + 4  + 4  +1 $\circ$

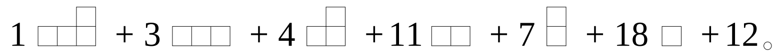
(denoting each class by the corresponding diagram). Multiplying by the Chern class of the dual of the universal subbundle $\mathscr{S}^\vee$ amounts to simple applications of Pieri's formula, and yields



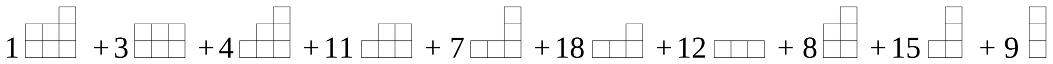
According to Proposition 4.1, the classes of diagrams obtained by adding one vertical 3-column to each of these diagrams appear with the same coefficients in $c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ)$. Similarly, applying Proposition 4.2 allows us to recover the coefficients of classes corresponding to subdiagrams with the same number of columns as $\underline{\alpha}$, provided one knows $c_{\mathrm{SM}}(\mathbb{S}(3 \geq 1)^\circ)$, that is



Multiplying by $c(\mathscr{Q})$ gives



giving the coefficients of classes for diagrams obtained from these by adding one 3-row. The conclusion is that $c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ)$ must equal



plus classes for diagrams contained in the 2×2 rectangle. That is, recursion suffices to compute most coefficients of $c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ)$.

4.3. **Positivity for two rows: first proof.** The relevance of these considerations to the positivity question is the following: assuming that positivity has been established for all diagrams strictly smaller than $\underline{\alpha}$, then Propositions 4.1 and 4.2 imply that the coefficients of $[\mathbb{S}(\underline{\beta})]$ in $c_{\mathrm{SM}}(\mathbb{S}(\underline{\alpha})^\circ)$ are nonnegative, for all $\underline{\beta} \leq \underline{\alpha}$ with the same number of rows or the same number of columns. Indeed, multiplication by $c(\mathscr{S}^\vee)$ or $c(\mathscr{Q})$ preserve effectivity. Therefore:

Lemma 4.4. *In order to prove Conjecture 1, it suffices to show that $\gamma_{\underline{\alpha},\underline{\beta}} \geq 0$ for all $\underline{\alpha}$ and all $\underline{\beta} < \underline{\alpha}$ with strictly fewer rows and columns than $\underline{\alpha}$.*

The positivity of CSM classes for two-row diagrams follows immediately from Lemma 4.4 and Corollary 3.11. Indeed, positivity for one-row diagrams holds (as pointed out in §4.1); by Lemma 4.4, verifying the conjecture for two-row diagrams $\underline{\alpha}$ amounts to verifying that $\gamma_{\underline{\alpha},\underline{\beta}} \geq 0$ for all one-row diagrams $\underline{\beta}$, and Corollary 3.11 shows that this is indeed the case: for $\underline{\alpha} = (\alpha_1 \geq \alpha_2)$ and $\underline{\beta} = (r)$,

$$\sum_{r \geq 0} \gamma_{\underline{\alpha},(r)}\, u^r = (1+u)^{\alpha_1 - \alpha_2}(1+2u)^{\alpha_2}$$

has nonnegative coefficients.

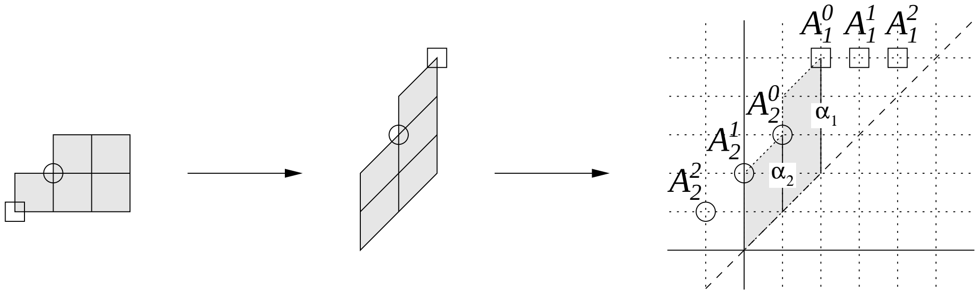
4.4. **Nonintersecting lattice paths, and second proof of positivity for two rows.** Positivity for cells corresponding to two-row diagrams is also immediate from the following concrete interpretation of the coefficients $\gamma_{\underline{\alpha},\underline{\beta}}$ in this case: for $\underline{\alpha} = (\alpha_1 \geq \alpha_2)$, $\underline{\beta} = (\beta_1 \geq \beta_2)$, with $\beta \leq \alpha$, $\gamma_{\underline{\alpha},\underline{\beta}}$ *equals the number of certain non-intersecting lattice paths joining pairs of points in the plane.*

Here is the precise statement. A *lattice path* joining two points A, B of the integer lattice in the plane is a sequence of lattice points starting from A and ending in B, such that each point is one step to the right or down from the preceding one. Given $\alpha_1 \geq \alpha_2$, consider the two sets of $\alpha_2 + 1$ points in the plane

$$A_1 = \{A_1^\ell\}_{0\leq \ell\leq \alpha_2} \quad , \quad A_2 = \{A_2^\ell\}_{0\leq \ell\leq \alpha_2} \quad ,$$

where

$$A_1^\ell = (\ell+2, \alpha_1+2) \quad , \quad A_2^\ell = (1-\ell, \alpha_2+1-\ell) \quad .$$



Given $\beta_1 \geq \beta_2$, also consider the pair of points

$$B_1 = (\beta_1+2, \beta_1+2) \quad , \quad B_2 = (\beta_2+1, \beta_2+1)$$

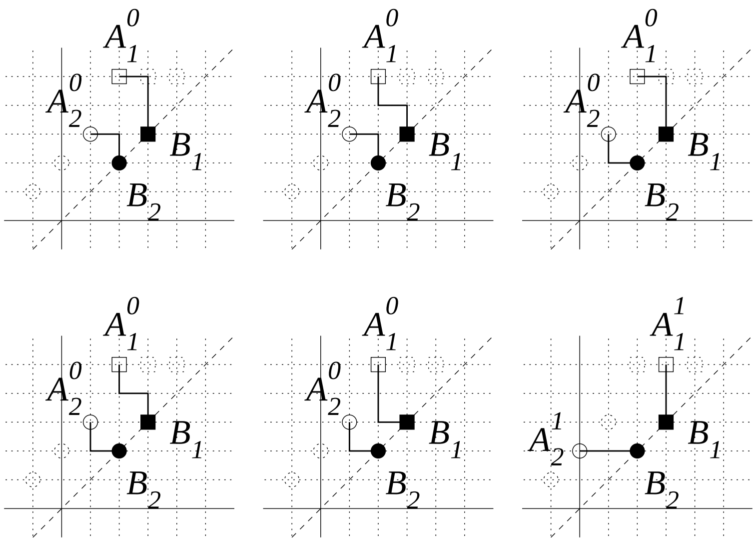
on the main diagonal. Let $\Pi(\ell)$, for $0 \leqslant \ell \leqslant \alpha_2$, be the set of pairs (π_1, π_2) such that π_r is a path from A_r^ℓ to B_r and π_1 doesn't intersect π_2.

Theorem 4.5. *With notation as above, the coefficient $\gamma_{\underline{\alpha},\underline{\beta}}$ is*

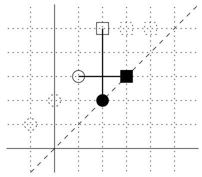
$$\sum_{l=0}^{\alpha_2} |\Pi(\ell)|$$

where $|\Pi(\ell)|$ denotes the cardinality of the set $\Pi(\ell)$.

Example 4.6. The contribution of $[\mathbb{S}(1 \geq 1)]$ to $c_{\rm SM}(\mathbb{S}(3 \geq 2)^\circ)$ is $\gamma_{3\geq 2,1\geq 1} = 6$; the 6 pairs of paths are



There are 5 nonintersecting paths from the first pair of points in A_1, A_2 to B_1, B_2; 1 path from the second pair, and no paths from the third pair. One pair of lattice paths $A_1^0 \to B_1$, $A_2^0 \to B_2$:



is not counted since the paths meet.

Proof. Note that A_2 lies entirely due south-west of A_1, both are north of the main diagonal, and B_2 is south-west of B_1. This implies that every lattice path $A_1^\ell \to B_2$ meets every lattice path from $A_2^\ell \to B_1$. Under this hypothesis, the number of nonintersecting paths considered in the statement equals

$$\sum_{\ell=0}^{\alpha_2} \det \begin{bmatrix} P(A_1^\ell \to B_1) & P(A_1^\ell \to B_2) \\ P(A_2^\ell \to B_1) & P(A_2^\ell \to B_2) \end{bmatrix}$$

where $P(A \to B)$ denotes the number of lattice paths from A to B; this follows from the 'Lindström-Gessel-Viennot theorem' (Theorem 6 in [Kra05]).

Now recall that, for a, b nonnegative integers, the binomial coefficient $\binom{a}{b}$ equals the number of lattice paths joining the points $(0, a)$ and (b, b). Applying evident translations,

$$P(A_1^\ell \to B_1) = \binom{\alpha_1 - \ell}{\beta_1 - \ell} \quad , \quad P(A_1^\ell \to B_2) = \binom{\alpha_1 - \ell}{\beta_2 - 1 - \ell}$$
$$P(A_2^\ell \to B_1) = \binom{\alpha_2}{\beta_1 + 1 + \ell} \quad , \quad P(A_2^\ell \to B_2) = \binom{\alpha_2}{\beta_2 + \ell}$$

Therefore, the number of nonintersecting paths given in the statement is

$$\sum_{\ell=0}^{\alpha_2} \det \begin{bmatrix} \binom{\alpha_1-\ell}{\beta_1-\ell} & \binom{\alpha_1-\ell}{\beta_2-1-\ell} \\ \binom{\alpha_2}{\beta_1+1+\ell} & \binom{\alpha_2}{\beta_2+\ell} \end{bmatrix} \quad .$$

By Theorem 3.8 this equals $\gamma_{\underline{\alpha},\underline{\beta}}$, concluding the proof. □

References

[Alu] P. Aluffi. Limits of Chow groups, and a new construction of Chern-Schwartz-MacPherson classes. FSU05-14, arXiv:math.AG/0507029.

[Alu99] Paolo Aluffi. Differential forms with logarithmic poles and Chern-Schwartz-MacPherson classes of singular varieties. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(7):619–624, 1999.

[Alu06] Paolo Aluffi. Classes de Chern des variétés singulières, revisitées. *C. R. Math. Acad. Sci. Paris*, 342(6):405–410, 2006.

[Dem74] Michel Demazure. Désingularisation des variétés de Schubert généralisées. *Ann. Sci. École Norm. Sup. (4)*, 7:53–88, 1974. Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I.

[Ful84] William Fulton. *Intersection theory*. Springer-Verlag, Berlin, 1984.

[GP02] Mark Goresky and William Pardon. Chern classes of automorphic vector bundles. *Invent. Math.*, 147(3):561–612, 2002.

[Kra99] C. Krattenthaler. Advanced determinant calculus. *Sém. Lothar. Combin.*, 42:Art. B42q, 67 pp. (electronic), 1999. The Andrews Festschrift (Maratea, 1998).

[Kra05] C. Krattenthaler. Advanced determinant calculus: a complement. *Linear Algebra Appl.*, 411:68–166, 2005.

[Mac74] Robert D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.

[PP95] Adam Parusiński and Piotr Pragacz. Chern-Schwartz-MacPherson classes and the Euler characteristic of degeneracy loci and special divisors. *J. Amer. Math. Soc.*, 8(4):793–817, 1995.

[Sch65a] M.-H. Schwartz. Classes caractéristiques définies par une stratification d'une variété analytique complexe. I. *C. R. Acad. Sci. Paris*, 260:3262–3264, 1965.

[Sch65b] M.-H. Schwartz. Classes caractéristiques définies par une stratification d'une variété analytique complexe. II. *C. R. Acad. Sci. Paris*, 260:3535–3537, 1965.

[Vak] Ravi Vakil. A geometric Littlewood-Richardson rule, arXiv:math.AG/0302294. to appear in the Annals of Math.

Mathematics Department, Florida State University, Tallahassee FL 32306, U.S.A.
E-mail address: `aluffi@math.fsu.edu`

Mathematics Department, Florida State University, Tallahassee FL 32306, U.S.A.
E-mail address: `mihalcea@math.fsu.edu`