The geometric realisation of C-groups

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0. In [K2], one class of groups, so called C-groups, was introduced (see the definition of a C-group in n.2). This class contains naturally the class of knot and link groups (with the Wirtinger corepresentation) and the class of the fundamental groups of the complements of algebraic curves in $\mathbb{C}^2$ (with the corepresentation from [K1]).

Denote by $\mathcal{C}$ the class of C-groups, and let $\mathcal{L}$ be the class of knot and link groups, $\mathcal{A}$ the class of the fundamental groups of the complement of algebraic curves in $\mathbb{C}^2$. It was shown in [K2] that

$$\mathcal{A} \not\subseteq \mathcal{L}, \qquad \mathcal{L} \not\subseteq \mathcal{A}, \qquad \mathcal{A} \cup \mathcal{L} \subsetneqq \mathcal{C}.$$

It follows from [L], that there exists an irreducible C-group G such that G can not be a group of a 2-knot, i.e. G can not be the fundamental group of the complement of a sphere S^2 imbedded in $\mathbb{R}^4$.

The purpose of this note is to prove the following

Theorem. *For each C-group G, there exists a smooth orientable compact Riemannian surface $S \subset \mathbb{R}^4$ such that*

$$\pi_1(\mathbb{R}^4 \setminus S) \simeq G.$$

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1.1. Let $S \subset \mathbb{R}^4$ be a smooth orientable compact Riemannian surface. Choose a point $o \notin S$. Let $K \subset \mathbb{R}^4$ be the cone over S with the vertex o. The cone K is a singular real hypersurface, $\dim K = 3$. Denote by $SingK$ the set of singular points of K, and let $K(2)$ be its double locus, i.e. $K(2)$ is a subset of K such that at each point $x \in K(2)$, K is locally a union of two nonsingular hypersurfaces meeting transversally at x.

Let us choose $o \in \mathbb{R}^4$ in general position with respect to S. In this case $SingK \setminus K(2)$ is a set of a finite number of straight lines $\{L_1, ..., L_k\}$. Moreover we can assume that the following conditions are satisfied:

(i) If L_i touches S at a point x, then $L_i \cap S = \{x\}$.

(ii) If L_i meets S at more than two distinct points, then $L_i \cap S = \{x_1, x_2, x_3\}$ and the tangent spaces $T_{x_1}S$, $T_{x_2}S$, $T_{x_3}S$ are in general position.

1.2. We fix a point $o \in \mathbf{R}^4$, which is in general position with respect to S, and choose a coordinate system in $\mathbf{R}^4$ such that o is the origin of this system. Denote by l_x the ray begining at o and passing throuth x.

We shall say that a point $x \in S$ is *invisible*, if there exists $t \in \mathbf{R}, 0 < t < 1$, such that the point tx also belongs to S, where $tx = (tx_1, ..., tx_4)$ for $x = (x_1, ..., x_4)$.

Let IS be the closure of the set of invisible points. The set

$$SS = \{x \in S \mid tx \in S \text{ for some } t > 1\}$$

is called a *screen*.

The surface S divides K into two parts. Let

$$ES = \{x \in \mathbb{R}^4 \mid tx \in S \text{ for some } t,\ 0 < t < 1\}$$

be the part of $K \setminus S$, which does not contain the origin o. ES is called *a shade* of S (or *the external part* of K).

Let $S \setminus IS = S_1 \cup ... \cup S_n$ be the decomposition into the connected components. Denote by ES_i the shade of S_i and let EIS be the shade of the set of invisible points. The open hypersurface $K_i = ES_i \setminus EIS$ will be called *a wall.*

1.3. Fix an orientation on S and on $\mathbb{R}^4$. The orientation on S induces an orientation on each wall K_i, because $ES_i \simeq S_i \times \{t \in \mathbf{R} \mid 0 < t < 1\}$. Thus the orientations on K_i and on $\mathbf{R}^4$ allow us to consider each K_i as a two-sided hypersurface each side of which is coloured: one of the sides is painted into "positive" colour and the other side into "negative" colour.

1.4. Consider a point $z \in ES \cap K(2)$. In a small neighborhood U_z of the point z, we have that $K \cap U_z = K' \cup K''$, where K' and K'' are nonsingular hypersurfaces intersecting transversally along nonsingular surface $K' \cap K''$.

The ray l_z intersects S in two points a and b, where $a \in SS$ and $b \in IS$. In a neighborhood of the points a and b, the surface S splits into two disjoint connected components S' and S'' ($a \in S'$ and $b \in S''$) so that K' is the shade of S' and K'' is the shade of S''.

In the neighborhood U_z the intersection $K' \cap K''$ divides each K' and K'' into two parts K'_1, K'_2, K''_1, K''_2. The parts K''_1 and K''_2 belong to some walls, say K_q and K_r (it is possible that $K_q = K_r$). In U_z, the set K' is divided by $K' \cap K''$ into two parts K'_1 and K'_2. But it is easy to see that these parts belong to the same wall. Denote by K_p this wall. The walls K_p, K_q, K_r will be called *adjacent walls* at the point z.

The hypersurfaces K' and K'' divide U_z into four parts E_1, E_2, E_3, E_4. Let E_1 be the part whose internal boundary is coloured into positive colour.

We shall say that the triple K_p, K_q, K_r is *well-ordered* if the boundary of E_1 consists of K_p and K_q (and not K_p and K_r). Of course, it is possible that the walls K_p, K_q, K_r are the adjacent walls at some other point z_1 and for this point, the triple K_p, K_r, K_q is well-ordered.

1.5. We associate a group Γ_S to the surface S. The generators of Γ_S are the walls K_i and the complete set of relations are

$$K_q K_p = K_p K_r \tag{1}$$

for each $z \in ES \cap K(2)$, where K_p, K_q, K_r are well-ordered adjacent walls at z.

Theorem 1. *Let $S \subset \mathbf{R}^4$ be a smooth orientable compact Riemannian surface. Then*

$$\pi_1(\mathbf{R}^4 \setminus S, o) \simeq \Gamma_S.$$

Proof. The same as the proof of Theorem 3.1 in [K1].

Remark 1. *It is easy to see that Γ_S is a finitely generated group, because S is compact.*

2. In this section we shall recall the definition and some simple properties of C-groups.

2.1. Let $I_q = \{1, 2, ..., q\}$ be a segment of $\mathbb{N}$, $M \subset I_q^3 = I_q \times I_q \times I_q$ a subset and $|M| = \#M$ the cardinality of M.

Definition. *A group G together with a corepresentation*

$$G =< x_1, ..., x_q \mid \{R_\alpha(x)\}_{\alpha \in M} > \tag{2}$$

is called a C-group of type M, where for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ the relation

$$R_\alpha(x) = x_{\alpha_1} x_{\alpha_2} x_{\alpha_1}^{-1} x_{\alpha_3}^{-1}.$$

is a conjugation (the letter "C" in "C-group" is the first letter of the word "conjugation").

A homomorphism $f : G_1 \to G_2$ is a homomorphism of C-groups if for each generator x_i of the C-group G_1, $f(x_i)$ is conjugated to some generator of the C-group G_2.

Remark 2. *If we add one more generator, say y, and one more relation $x_i y x_i^{-1} x_j^{-1}$ to the corepresantation (2), then we obtain the group which is isomorphic to G as a C-group.*

2.2. To any C-corepresentation of type M we can associate an oriented graph Γ_M with vertices $v_1, ..., v_q$, and with edges $e_\alpha, \alpha \in M$. The edge e_α connects the vertex v_{α_2} with v_{α_3} , where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

It is easy to prove the following

Lemma 1. *(cf. [K2]) Let G be a C-group of type M , and $G' = [G, G]$. Then $G/G' = \mathbf{Z}^n$, where n is the number of connected components of the graph Γ_M.*

A C-group G of type M is called an *irreducible* C-group if its graph Γ_M is connected.

Let $\Gamma_M = \Gamma_1 \cup ... \cup \Gamma_n$ be the decomposition into connected components. For each Γ_j, put $I(j) = \{i \in I_q | v_i \notin \Gamma_j\}$. The group

$$G_j =< x_1, ..., x_q \mid \{R_\alpha\}_{\alpha \in M} \cup \{x_i\}_{i \in I(j)} >$$

is called *an irreducible component* of a C-group G of type M, and we shall say that the C-group G is composed of n irreducible components G_j.

Remark 3. *It is easy to see from Theorem 1 that for each smooth orientable compact Riemannian surface $S \subset \mathbb{R}^4$, $\pi_1(\mathbb{R}^4 \setminus S)$ is a C-group composed of n irreducible components, where n is the number of connected components of S.*

2.3. Denote by $i : I_q^3 \to I_q^3$ the involution defined by

$$i : (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_1, \alpha_3, \alpha_2),$$

and let $M^* = i(M)$ be the image of $M \subset I_q^3$.

The C-group G^* of type M^* is called *conjugate* to a C-group G of type M.

Lemma 2. *Let G and G^* be conjugated C-groups. Then G and G^* are isomorphic groups.*

Proof. Indeed, if in the group

$$G = < x_1, ..., x_q \mid \{R_\alpha(x)\}_{\alpha \in M} >$$

we shall take the generators $y_1 = x_1^{-1}, ..., y_q = x_q^{-1}$ instead of the generators $x_1, ..., x_q$, then the group G will have the following corepresentation

$$G = < y_1, ..., y_q \mid \{R_\alpha(y)\}_{\alpha \in M^*} > .$$

2.4. Let G be a C-group. Denote by $\mathcal{M}_G$ the collection of the sets $M_i \subset I_{q_i}^3$ such that the C-groups G_{M_i} of type M_i are isomorphic to G as C-groups. The number

$$r(G) = \min_{M \in \mathcal{M}_G} rk\, \pi_1(\Gamma_M)$$

is called *the rank* of the C-group G, where $rk\, \pi_1(\Gamma_M)$ is the rank of a free group $\pi_1(\Gamma_M)$.

3.1. We shall say that $S \subset \mathbf{R}^4$ is *a tamely imbedding of the simplest kind*, if for S, there exists a projection $p : \mathbf{R}^4 \to \mathbb{R}^3$ from a point $o \in \mathbf{R}^4$ such that the image $p(S)$ satisfies the following condition:

(s) Locally at each point $z \in p(S)$ either $p(S)$ is smooth, or $p(S)$ is a union of two smooth surfaces meeting transversally.

The main theorem follows from

Theorem 2. *For each C-group G, there exists a tamely imbedding of the simplest kind $S \subset \mathbb{R}^4$ of a smooth orientable compact Riemannian surface S such that*

$$\pi_1(\mathbf{R}^4 \setminus S) \simeq G.$$

Proof. Fix a C-group

$$G = < x_1, ..., x_q \mid \{R_\alpha(x)\}_{\alpha \in M} > .$$

At the beginning, we shall construct the image $p(S)$ of the projection $p : \mathbb{R}^4 \to \mathbb{R}^3$ from the point o, where S is a desired surface. The surface $p(S)$ will be glued from standart pieces step by step. Now we shall describe these standart pieces.

For this, let $n(i)$ be the number of all edges of Γ_M either starting or ending at the vertex e_i, $1 \le i \le q$. Let $r(i)$ be the number of the relations R_α with $\alpha = (i, \cdot, \cdot)$. Put $n_i = n(i) + r(i)$.

The standart piece $A_i(n_i) = S_i^2 \setminus \bigcup_{1 \le j \le n_i} \Delta_{i,j}$, $1 \le i \le q$, is a sphere $S_i^2 \subset \mathbb{R}^3$ from which n_i non-intersecting disks $\Delta i, j \subset S_i^2$ are cut out. We assume that $S_i^2 \cap S_j^2 = \varnothing$ for $i \ne j$. Let $z_{i,j}$ be the center of $\Delta i, j$.

The standart pieces C_α, $\alpha \in M$ and $C_{\alpha'} \cap C_{\alpha''} = \varnothing$ for $\alpha' \ne \alpha''$, are the unions

$$C_\alpha = C_{\alpha,1} \cup C_{\alpha,2} \cup \Delta_\alpha$$

for each $\alpha \in M$, such that there exists a neighborhood U_α of C_α which is diffeomorphic (orientation being preserved) to

$$I^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x_i| \le 1 \,\mathrm{for}\, i = 1,\, 2,\, 3\ \}$$

and such that C_α is diffeomorphic (via the same diffeomorphism) to $C_1 \cup C_2 \cup \Delta$, where

$$\begin{aligned} C_1 &= \{(x_1,\, x_2,\, x_3) \in I^3 \mid x_1^2 + x_2^2 = R^2,\, R \ll 1\ \},\\ C_2 &= \{(x_1,\, x_2,\, x_3) \in I^3 \mid x_2^2 + x_3^2 = r^2,\, r \ll R\ \},\\ \Delta &= \{(x_1,\, x_2,\, x_3) \in I^3 \mid x_1^2 + x_2^2 \le R^2,\, x_3 = -1\ \}. \end{aligned}$$

Put $y_1 = (0,0,1)$, $y_2 = (-1,0,0)$, $y_3 = (1,0,0)$, and let $y_{i,\alpha}$, $i = 1,\, 2,\, 3$, be the corresponding points in U_α.

Let us connect for $\alpha = (\alpha_1,\, \alpha_2,\, \alpha_3)$, the point $y_{1,\alpha}$ with a free (that is not used at previous steps) point z_{α_1, j_1} by a smooth path $\gamma_{1,\alpha}$ which does not intersect the standart pieces. One connects the points $y_{2,\alpha}$ and $y_{3,\alpha}$ with free points z_{α_2, j_2} and z_{α_3, j_3} by smooth paths $\gamma_{2,\alpha}$ and $\gamma_{3,\alpha}$, respectively. Let $B_{i,\alpha}$ be the boundaries of tubular neighborhoods of the paths $\gamma_{i,\alpha}$. Each $B_{i,\alpha}$ is diffeomorphic to $S^1 \times [0,1]$. We choose the tubular neighborhoods of the paths $\gamma_{i,\alpha}$ so that one of the connected components of the boundary of $B_{i,\alpha}$ would coinside with one of the components of the boundary of $A_{\alpha_i}(n_i)$ and the other one would coinside with one of the components of the boundary of C_α. We glue each $B_{i,\alpha}$ with $A_{\alpha_i}(n_i)$ and C_α along these boundaries (see Fig.1). After these glueing, we obtain $p(S)$ which is immersed into $\mathbb{R}^3$.

Now we construct the surface S. For this let $\mathbb{R}^3$, considered above, be defined in $\mathbb{R}^4$ by the equation $x_4 = 0$ and let $o = (o_1, ..., o_4)$ be coordinates of the center of the projection p, where $o_4 \gg 0$. We shall say that the point o lies higher than the hypersurface $\mathbb{R}^3$.

For each standart piece C_α, the intersection $C_{\alpha,1} \cap C_{\alpha,2}$ is the disjoint union of two loops $\nu_{\alpha,+}$ and $\nu_{\alpha,-}$, where $\nu_{\alpha,+}$ corresponds to the circle

$$\nu_+ = \{x_1^2 + x_2^2 = R^2,\ \ x_2^2 + x_3^2 = r^2,\ \ x_1 > 0\}.$$

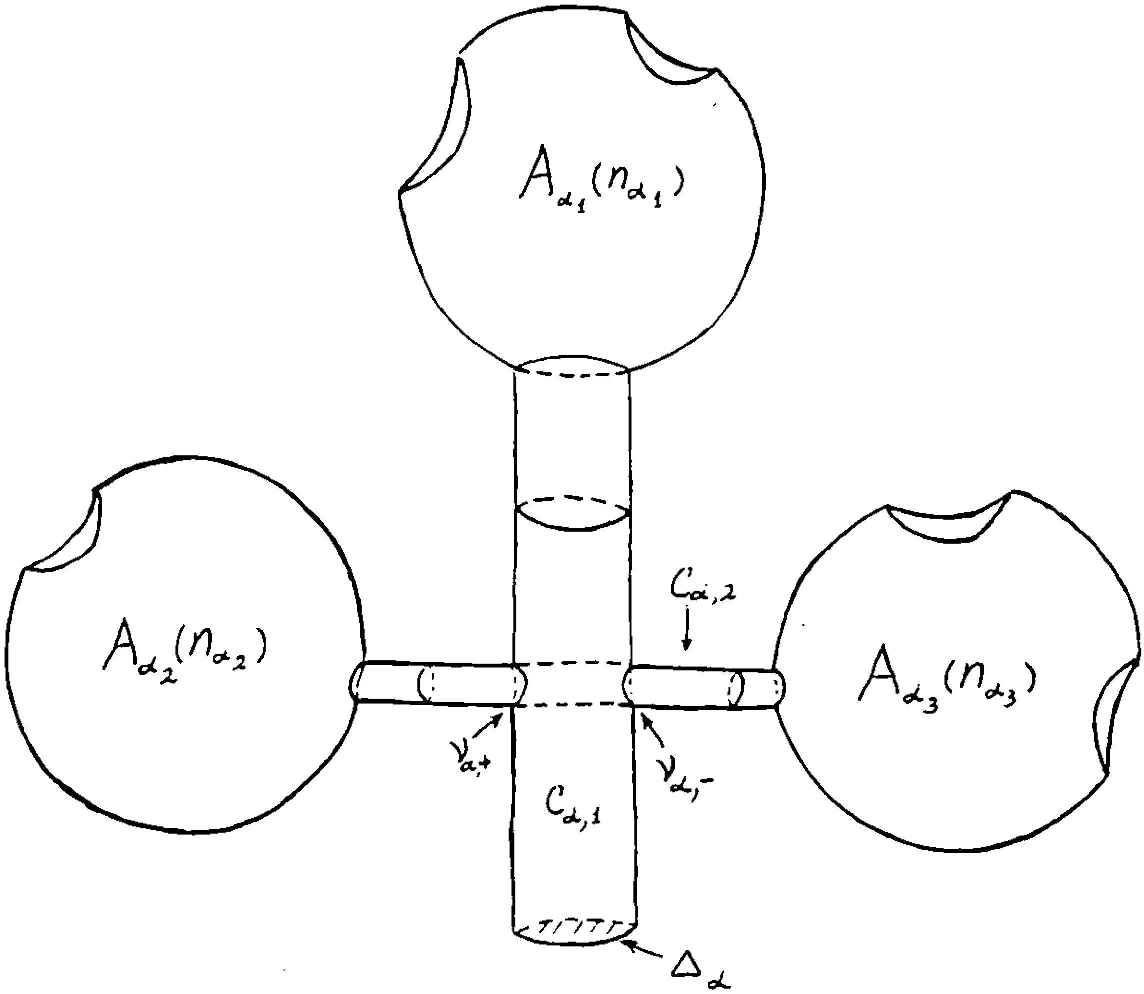

Fig.1

Let $U_{\alpha,+}$ and $U_{\alpha,-}$ be small open subsets of $\mathbf{R}^4$, $U_{\alpha,+} \cap U_{\alpha,-} = \emptyset$, and let $V_{\alpha,+} \subset U_{\alpha,+}$, $V_{\alpha,-} \subset U_{\alpha,-}$ be compactly imbedded neighborhoods of $\nu_{\alpha,+}$ and $\nu_{\alpha,-}$, respectively. We make a small shift up of the intersection $C_{\alpha,2} \cap V_{\alpha,+}$ and glue the shifted surface with $C_{\alpha,2} \setminus U_{\alpha,+}$ using smoothing functions. Similarly, we make a small shift down of the intersection $C_{\alpha,2} \cap V_{\alpha,-}$ and glue the shifted surface with $C_{\alpha,2} \setminus U_{\alpha,-}$.

After these shifts, we shall obtain a surface S. For this surface, the loop $\nu_{\alpha,+} \subset C_{\alpha,1}$ and the shifted loop $\nu_{\alpha,-}$, which belongs to the shifted down surface $C_{\alpha,2}$, are the subsets of the set of invisible points.

The loop $\nu_{\alpha,+}$ divides $C_{\alpha,1}$ into two parts. One of them is homeomorphic to a disk. Denote this disk by y_α.

We identify each standart piece $A_i(n_i)$ with the generator x_i of the C-group G. Then $S \setminus IS$ is the disjoint union of the connected components x_i, $i = 1, ..., q$, and y_α, $\alpha \in M$.

The relations in $\pi_1(\mathbb{R}^4 \setminus S, o)$ between x'es are either the relations $\{R_\alpha(x)\}_{\alpha \in M}$ or the relations $\{R_\alpha(x)\}_{\alpha \in M^*}$, and it depends on a choice of orientations on $\mathbb{R}^4$ and S. If it is necessary, we change the orientation on S such that these relations will be $\{R_\alpha(x)\}_{\alpha \in M}$.

The relations in $\pi_1(\mathbb{R}^4 \setminus S, o)$ between x'es and y_α's are the added relations as in Remark 2. Thus $\pi_1(\mathbb{R}^4 \setminus S, o) \simeq G$ and Theorem 2 is proven.

Remark 4. *It is easy to see that the genus $g(S)$ of the surface S constructed in the proof of Theorem 2 is equal to*

$$g(S) = rk\, \pi_1(\Gamma_M).$$

4. *Example.* The simplest (non-trivial) irreducible C-group has at least three generators and two relations. There exist unique C-groups G_1 and G_2 of types $M_1 \subset I_3^3$, and $M_2 \subset I_3^3$, respectively, $\#M_1 = \#M_2 = 2$. Their graphs Γ_{M_1} and Γ_{M_2} are presented on Fig.2.

Γ_{M_1} Γ_{M_2}

*Fig.*2

The group G_1 is the clover-leaf knot group.

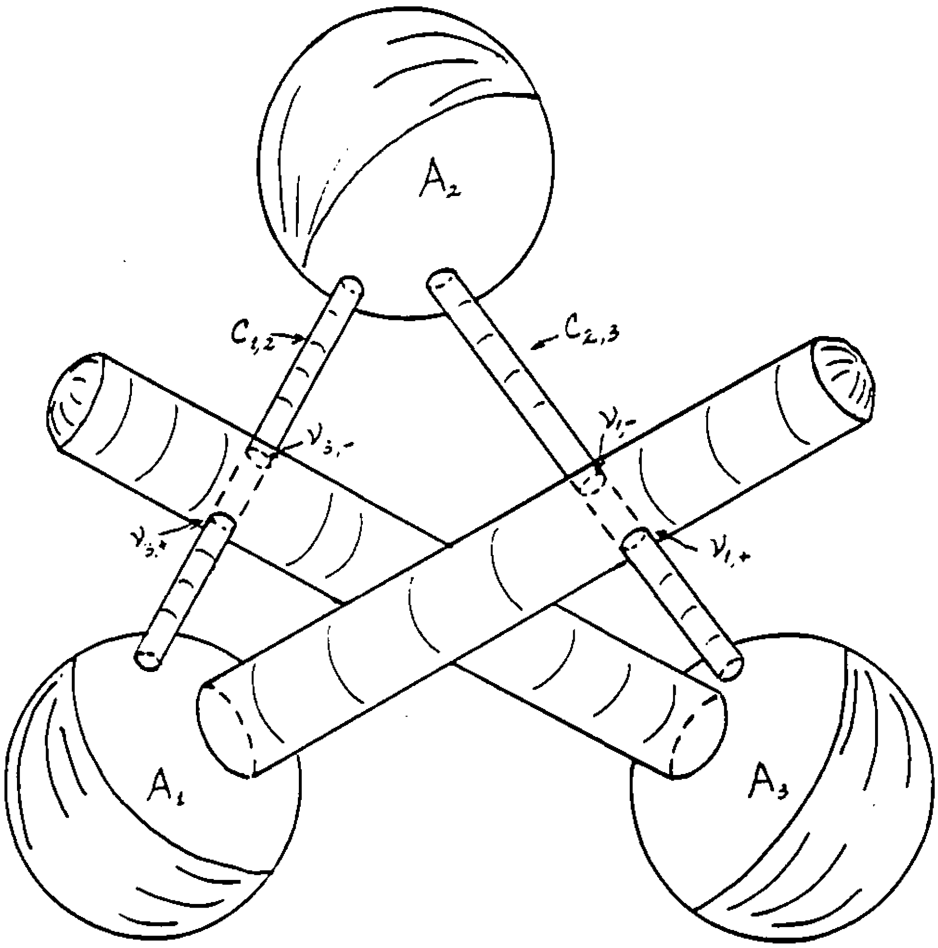

*Fig.*3

The groups G_1 and G_2 are not isomorphic C-groups, because the Alexander polynomial $\Delta_{G_1}(t)$ of G_1 is $\Delta_{G_1}(t) = t^2 - t + 1$ and the Alexander polynomial $\Delta_{G_2}(t)$ of G_2 is $\Delta_{G_2}(t) = t - 2$.

The group G_1 can be realized as the fundamental group of the complement of a surface S_1 in $\mathbf{R}^4$ whose image $p(S)$ is pictured on Fig.3.

To obtain S_1 from $p(S)$ we must shift up $C_{1,2}$ and $C_{2,3}$ in neighborhoods of $\nu_{3,+}$ and $\nu_{1,+}$, respectively, and then shift down $C_{1,2}$ and $C_{2,3}$ in neighborhoods of $\nu_{3,-}$ and $\nu_{1,-}$, respectively.

The group G_2 can be realized as the fundamental group of the complement of a surface S_2 in $\mathbf{R}^4$. To construct S_2, one can use the same image $p(S)$. Only it is necessary to make shifts in the another directions. One must shift up $C_{1,2}$ and $C_{2,3}$ in neighborhoods of $\nu_{3,+}$ and $\nu_{1,-}$ and shift down $C_{1,2}$ and $C_{2,3}$ in neighborhoods of $\nu_{3,-}$ and $\nu_{1,+}$, respectively.

5. Fix an irreducible C-group G. Let $\mathcal{S}_G$ be the set of smooth connected compact orientable Riemannien surfaces $S \subset \mathbf{R}^4$ such that G is isomorphic to $\pi_1(\mathbb{R}^4 \setminus S)$ as a C-group.

We shall call

$$g(G) = \min_{S \in \mathcal{S}_G} g(S)$$

the genus of the irreducible C-group G, where $g(S)$ is the genus of a Riemannian surface S.

Let $\mathcal{S}_{s,G}$ be the set of tamely imbeddings of the simplest kind of smooth connected orientable compact Riemannian surface $S \subset \mathbb{R}^4$ such that $\pi_1(\mathbb{R}^4 \setminus S)$ is isomorphic to G.

We shall call

$$g_s(G) = \min_{S \in \mathcal{S}_{s,G}} g(S)$$

the *s-genus* of a irreducible C-group G.

Theorem 3. *Let G be an irreducible C-group. Then*

$$r(G) = g_s(G).$$

Proof. It follows from Remark 4.

Corollary. *Let G be an irreducible C-group. Then*

$$g(G) \leq r(G).$$

Remark 4. *It follows from [L], that there exists an irreducible C-group such that its genus $g(G) > 0$.*

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