Representation Theory and Quantum Inverse Scattering Method: Open Toda Chain and Hyperbolic Sutherland Model

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Abstract

Using the representation theory of $\mathfrak{gl}(N,\mathbb{R})$ we explain the expression through integrals for $GL(N,\mathbb{R})$ Toda chain wave function obtained recently by Quantum Inverse Scattering Method. The main tool is the generalization of the Gelfand-Tsetlin method to the case of the infinite-dimensional representations of $\mathfrak{gl}(N,\mathbb{R})$. The interpretation of this generalized construction in terms of the coadjoint orbits is given and the connection with Yangian $Y(\mathfrak{gl}(N))$ is discussed. We also provide the representation by integrals for the hyperbolic Sutherland model eigenfunctions in Gelfand-Tsetlin representation. Using the example of the open Toda chain we discuss the connection between the Quantum Inverse Scattering Method and Representation Theory.

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1 Introduction

In the late seventies the two basic approaches to study the quantum integrable systems have been formulated.

M.Olshanetsky and A.Perelomov discovered the direct relation between the representation theory of non-compact semi-simple Lie groups and Sutherland-Calogero-Moser (SCM) systems [1]-[3]. Methods of Representation Theory provide the complete solution of the quantum SCM system. Soon B.Kostant and D.Kazhdan realized that the more general class of integrable systems, including open Toda chains could be solved by this method [4], [5]. The wave function of the SCM systems and Toda chains appears to be zonal spherical functions [6], [7] and Whittaker functions [8]-[11] respectively. There is an obvious generalization of the approach to a more general class of integrable systems corresponding formally to the affine Lie algebras. But up to now there was no progress in derivation of the explicit representation through integrals for the wave functions along this line due to analytic difficulties in the corresponding part of Representation theory.

At the same time another approach to the quantum integrability emerged. In the work of L.Faddeev, E.Sklyanin, P.Kulish and L.Takhtajan [12]-[15] [4] the Quantum Inverse Scattering Method (QISM) was formulated and successively applied to numerous quantum integrable systems [17]. The recent development of this approach is connected with the Separation of Variables (SoV) method [18], [19]. However, the understanding of the connection of QISM

[4] See the summary of the early period of the development in [16]

with the Representation Theory remains unclear up to now.

The starting point of this paper was the new method of deriving the wave function for $GL(N,\mathbb{R})$ Toda chain given in [20]. It had no obvious relation with the Representation Theory approach, but instead the QISM was used to construct the wave function. Understanding of this construction in terms of the Representation Theory approach and comparing the two basic approaches to quantum integrability for this simple integrable system may give additional insight into the nature of QISM. In this paper we give interpretation of the representation by integrals of the wave function in the open Toda chain [20] in terms of the Representation Theory approach to quantum integrability. The basic tool is a generalization of the Gelfand-Tsetlin method [21], [22] to the case of infinite-dimensional representations of $\mathfrak{gl}(N,\mathbb{R})$. This gives a construction of the representation of $\mathcal{U}(\mathfrak{gl}(N,\mathbb{R}))$ in terms of difference operators [5] which one can use for finding explicit expressions for Whittaker vectors. Thus, we reproduce the representation through integrals for Whittaker functions obtained previously in [20] by QISM.

Comparing the methods of [20] with those proposed in this paper reveals a deep connection between SoV [18] and Gelfand-Tsetlin's type representations. In [24], [18] the separated variables for the classical Toda chain were found. They could be successfully quantized to give an explicit solution of the corresponding quantum theory. In this paper we use the generalization of Gelfand-Tsetlin representation for $\mathfrak{gl}(N,\mathbb{R})$ which gives rise to a much larger set of the separated variables ($\frac{1}{2}N(N-1)$ instead of $(N-1)$). The idea of introducing the enlarged set of separated variables could be traced back to the work of I.M.Gelfand and A.A.Kirillov on the field of fractions of the universal enveloping algebras [25]. On the other hand, the representation we have constructed appears to be closely connected with the representation theory of the Yangian $Y(\mathfrak{gl}(N))$. The enlargement of the set of variables for solving the quantum problem is a crucial part of the Representation Theory approach to quantum integrability. One may hope that understanding of the correct notion of the separated variables in Representation Theory will turn out to be fruitful for both the theory of Quantum Integrable Systems and Representation Theory.

The plan of the paper is as follows. In section 2 we give a direct generalization of the Gelfand-Tsetlin method to the case of infinite-dimensional representations of $\mathcal{U}(\mathfrak{gl}(N))$. The crucial point is to use continuous unconstrained variables instead of integer valued Gelfand-Tsetlin parameters. We define Whittaker modules W and W' in this realization of the representation and show that there exists the hermitian pairing between W and W'.

In the section 3 we give interpretation of the representation through Kirillov-Kostant orbit method [26]. We give an explicit parameterization of the finite cover of the open part of the coadjoint orbit of $GL(N,\mathbb{R})$ with the stabilizer $H_N = GL(1,\mathbb{R})^{\otimes N}$. We show that the parameterization provides Darboux coordinates with respect to the canonical Kirillov-Kostant symplectic structure on the orbit. The above construction is a version of those by A.Alekseev, L.Faddeev and S.Shatashvilli [27], [28] adopted for non-compact groups. Our parameterization leads to the classical counterpart for generators of $\mathfrak{gl}(N)$ given in section 2. A subtle point of the orbit method interpretation of our representation is that we use only

[5] Accidentally the version of this construction for the affine algebra $\widehat{\mathfrak{sl}(2)}_k$ was discovered previously in [23]. The generalization to the case $\widehat{\mathfrak{sl}(N)}_k$ will be given elsewhere.

(the finite cover of) the open part of the orbit. This leads to a representation of $\mathfrak{gl}(N,\mathbb{R})$ which can not be integrated to a representation of the whole group. However it is obvious that the action of the Cartan subalgebra integrates to the action of a Cartan torus and this is enough for constructing solutions of the open Toda chain. We end this section giving contour integral formulas for the generators of $\mathfrak{gl}(N,\mathbb{R})$.

In section 4 we re-derive the Gelfand-Tsetlin parameterization using the representation theory of Yangian [29] (closely connected ideas for the classical Gelfand-Tsetlin construction were developed in [30]-[32]). Moreover, it reveals a deep connection of Gelfand-Tsetlin representation with QISM based on representation theory of Yangian.

In section 5 we apply the representation constructed in section 2 to find the explicit solution of the quantum open Toda chain and quantum hyperbolic Sutherland model. For Toda chain we reproduce the results of [20]. The corresponding representation of the wave functions of quantum hyperbolic Sutherland model appears to be new.

In the last section we compare QISM-based approach and those developed in this paper to the integrability of the open Toda chain.

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2 Whittaker modules in Gelfand-Tsetlin representation

2.1 Generalization of Gelfand-Tsetlin construction to the case of infinite-dimensional representations

We start recalling the original Gelfand-Tsetlin construction of the canonical basis in the space of an irreducible finite-dimensional representation of $GL(N,\mathbb{C})$ [21], [22]. It is based on the two well known facts:

Any irreducible finite-dimensional representation is uniquely determined by a n-dimensional vector $\boldsymbol{m}_N=(m_{N1},\dots,m_{NN})$ with integer entries such that: $m_{N1}\geq\dots\geq m_{NN}$. The vector $\boldsymbol{m}_N$ is called the *label* or the highest weight of the representation.

Consider $GL(N-1,\mathbb{C})$ as an upper left corner subgroup of $GL(N,\mathbb{C})$. Then in the decomposition of a finite-dimensional irreducible representation of $GL(N,\mathbb{C})$ each irreducible representations of the subgroup $GL(N-1,\mathbb{C})$ enters with multiplicity not greater than one. If we denote by $\boldsymbol{m}_{N-1}$ the labels of representations involved into the decomposition, then their entries satisfy the additional conditions: $m_{Nj} \geq m_{N-1,j} \geq m_{N,j+1}$ for all $j = 1, \ldots, N-1$.

If we continue the decomposition step by step, we obtain a decomposition of an irreducible finite-dimensional representation of $GL(N, \mathbb{C})$ into the direct sum of the one dimensional subspaces. Fixing an element in each one-dimensional summand one obtains the Gelfand-Tsetlin basis. The vectors of this basis are labeled by the triangular arrays, consisting of all $\boldsymbol{m}_n$ ($n = 1, \ldots, N$) with entries satisfying the above mentioned constraints.

Now we wish to generalize the Gelfand-Tsetlin construction to the case of infinite-dimensional representations of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(N))$. We start with the following generalization of the Gelfand-Tsetlin (GT) representation of $\mathfrak{gl}(N)$. Namely, let $(\boldsymbol{\gamma}_1 \ldots \boldsymbol{\gamma}_N)$ be a triangular array consisting of $\frac{1}{2}N(N+1)$ variables $\boldsymbol{\gamma}_n = (\gamma_{n1}, \ldots, \gamma_{nn})$; $n = 1, \ldots, N$.

Proposition 2.1 *Let M be the space of meromorphic functions in $\frac{1}{2}N(N-1)$ variables $\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_{N-1}$. Then the operators* [6]

$$E_{nn} = \frac{1}{i\hbar}\Big(\sum_{j=1}^{n} \gamma_{jn} - \sum_{j=1}^{n-1} \gamma_{j,n-1}\Big); \qquad (n = 1, \ldots, N), \tag{2.1a}$$

$$E_{n,n+1} = -\frac{1}{i\hbar} \sum_{j=1}^{n} \frac{\prod_{r=1}^{n+1} (\gamma_{nj} - \gamma_{n+1,r} - \frac{i\hbar}{2})}{\prod_{s \neq j} (\gamma_{nj} - \gamma_{ns})} e^{-i\hbar\partial_{\gamma_{nj}}}; \quad (n = 1, \ldots, N-1), \tag{2.1b}$$

$$E_{n+1,n} = \frac{1}{i\hbar} \sum_{j=1}^{n} \frac{\prod_{r=1}^{n-1} (\gamma_{nj} - \gamma_{n-1,r} + \frac{i\hbar}{2})}{\prod_{s \neq j} (\gamma_{nj} - \gamma_{ns})} e^{i\hbar\partial_{\gamma_{nj}}}; \quad (n = 1, \ldots, N-1) \tag{2.1c}$$

realize a representation of $\mathfrak{gl}(N)$ in M.

Proof. It is sufficient to show that the generators (2.1) satisfy the standard commutation relations

$$\begin{aligned} [E_{nn}, E_{k,k+1}] &= (\delta_{nk} - \delta_{n,k+1}) E_{k,k+1}, \\ [E_{nn}, E_{k+1,k}] &= -(\delta_{nk} - \delta_{n,k+1}) E_{k+1,k}, \\ [E_{n,n+1}, E_{m+1,m}] &= (E_{nn} - E_{n+1,n+1}) \delta_{nm} \end{aligned} \tag{2.2}$$

and obey the Serre identities

$$\begin{aligned} [E_{n,n+1}, [E_{n,n+1}, E_{n+1,n+2}]] &= 0, \\ [E_{n+1,n}, [E_{n+1,n}, E_{n+2,n+1}]] &= 0. \end{aligned} \tag{2.3}$$

[6] Actually, we construct the family of representations depending on the auxiliary parameter $\hbar$. One could rescale the variables $\gamma_{nj} \to \hbar\gamma_{nj}$ to get rid of this parameter. However, we will retain $\hbar$ in the formulae to make more explicit the connection with standard notations in the theory of quantum integrable systems.

This follows from the direct verification of above relations. Defining the composite root generators recursively as $E_{jk} = [E_{jm}, E_{mk}]$ for $j < m < k$ and $j > m > k$, one obtains the complete set of commutation relations

$$[E_{jk}, E_{lm}] = E_{jm}\delta_{lk} - E_{lk}\delta_{jm}. \tag{2.4}$$

■

The representation described above is parameterized by label $\boldsymbol{\gamma}_N = (\gamma_{N1}, \dots, \gamma_{NN})$ and extends to a representation of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(N))$. Obviously, there is a natural representation of any subalgebra $\mathcal{U}(\mathfrak{gl}(n)) \subset \mathcal{U}(\mathfrak{gl}(N))$, $(n = 1, \dots, N-1)$ on the same space M with the label $\boldsymbol{\gamma}_n = (\gamma_{n1}, \dots, \gamma_{nn})$. Let $\mathcal{Z}(\mathfrak{gl}(n))$ be the center of $\mathcal{U}(\mathfrak{gl}(n))$.

Definition 2.1 *We say that an* $\mathcal{U}(\mathfrak{gl}(n))$*-module* V *admits an* infinitesimal character ξ *if there is a homomorphism* $\xi : \mathcal{Z}(\mathfrak{gl}(n)) \to \mathbb{C}$ *such that* $zv = \xi(z)v$ *for all* $z \in \mathcal{Z}(\mathfrak{gl}(n)), v \in V$.

It is possible to show that $\mathcal{U}(\mathfrak{gl}(N))$-module M defined above admits an infinitesimal character. Actually, more general statement holds:

Lemma 2.1 *Each central element of* $\mathcal{U}(\mathfrak{gl}(n))$ *acts on* M *by the multiplication on symmetric polynomials of variables* γ_{nj}.

Proof. By construction, an operator $u \in \mathcal{U}(\mathfrak{gl}(n))$ acting on M is invariant under permutation of variables entering in each row $\boldsymbol{\gamma}_m$, $(m = 1, \dots, n)$. It is easy to prove that any symmetric operator commuting with the Cartan subalgebra is just a function in $\boldsymbol{\gamma}_m$. If this function commutes with all root generators it should be $i\hbar$ periodic with respect to any variable γ_{mj}, $(m = 1, \dots, n-1)$. The polynomial generators of the center of $\mathcal{U}(\mathfrak{gl}(n))$ may act only as a rational functions in this representation and thus we conclude that they do not depend on the variables γ_{mj}, $(m = 1, \dots, n-1)$. Taking into account that the generators of $\mathcal{U}(\mathfrak{gl}(n))$ depend polynomially on $\boldsymbol{\gamma}_n$ we conclude that the generators of the center $\mathcal{Z}(\mathfrak{gl}(n))$ act on M as symmetric polynomials in $\boldsymbol{\gamma}_n$. ■

Therefore, the elements $z \in \mathcal{Z}(\mathfrak{gl}(n))$ act on $\mathcal{U}(\mathfrak{gl}(n))$-module M as an operators of multiplication by scalar and the $\mathcal{U}(\mathfrak{gl}(n))$-module M $(n = 1, \dots, N)$ admits an infinitesimal character.

In the next subsection we calculate the explicit form of central elements acting on M using the notion of Whittaker vectors.

2.2 Whittaker modules

Let us now give a construction of Whittaker module using the representation of $\mathcal{U}(\mathfrak{gl}(N))$ described above. We recall some facts from [5] which we use in our construction.

Let n_+ and n_- be the subalgebras of $\mathfrak{gl}(N)$ generated by positive and negative root generators respectively. The characters $\chi_+ : n_+ \to \mathbb{C}$, $\chi_- : n_- \to \mathbb{C}$ are uniquely determined by their values on the simple root generators; they are called *non − degenerate* if the (complex) numbers $\chi_+(E_{n,n+1})$ and $\chi_-(E_{n+1,n})$ are non-zero for all $n = 1, \dots, N-1$.

Let V be any $\mathcal{U} = \mathcal{U}(\mathfrak{gl}(N))$-module. Denote the action of $u \in \mathcal{U}$ on $v \in V$ by uv. A vector $w \in V$ is called the Whittaker vector with respect to character χ_+ if

$$E_{n,n+1}w = \chi_+(E_{n,n+1})w\,, \qquad (n = 1, \dots, N-1), \tag{2.5}$$

and an element $w' \in V'$ is called the Whittaker vector with respect to character χ_- if

$$E_{n+1,n}w' = \chi_-(E_{n+1,n})w'\,, \qquad (n = 1, \dots, N-1). \tag{2.6}$$

A Whittaker vector is called the *cyclic* for V if $\mathcal{U}w = V$ and an $\mathcal{U}$-module is called a *Whittaker module* if it contains a cyclic Whittaker vector. The $\mathcal{U}$-modules V and V' are called *dual* if there exists a non-degenerate pairing of V' and V such that $\langle Xv', v\rangle = -\langle v', Xv\rangle$ for all $v \in V$, $v' \in V'$ and $X \in \mathfrak{gl}(N)$.

We proceed with explicit formulae for Whittaker vectors corresponding to representation given by (2.1).

Proposition 2.2 *Whittaker vectors satisfying the equations*

$$E_{n+1,n}w'_{_N} = -i\hbar^{-1}w'_{_N}, \tag{2.7}$$

$$E_{n,n+1}w_{_N} = -i\hbar^{-1}w_{_N} \tag{2.8}$$

for all $n = 1, \dots, N-1$, are given by the following formulae:

$$w'_{_N} = 1,$$
$$w_{_N} = e^{-\frac{\pi}{\hbar}\sum\limits_{n=1}^{N-1}(n-1)\sum\limits_{j=1}^{n}\gamma_{nj}} \prod_{n=1}^{N-1} g_n(\boldsymbol{\gamma}_n, \boldsymbol{\gamma}_{n+1}), \tag{2.9}$$

where

$$g_n(\boldsymbol{\gamma}_n, \boldsymbol{\gamma}_{n+1}) = \prod_{k=1}^{n}\prod_{m=1}^{n+1} \hbar^{\frac{\gamma_{nk}-\gamma_{n+1,m}}{i\hbar}+\frac{1}{2}}\,\Gamma\Big(\frac{\gamma_{nk}-\gamma_{n+1,m}}{i\hbar}+\frac{1}{2}\Big). \tag{2.10}$$

Proof.

$$E_{n,n+1}w_{_N} = -i\hbar^{-1}w_{_N}\sum_{j=1}^{n}\frac{\prod\limits_{r=1}^{n-1}(\gamma_{nj}-\gamma_{n-1,r}-\frac{i\hbar}{2})}{\prod\limits_{s\neq j}(\gamma_{nj}-\gamma_{ns})}\,. \tag{2.11}$$

Using the identity

$$\sum_{j=1}^{n} \frac{x_j^m}{\prod_{s\neq j}(x_j - x_s)} = \sum_{\substack{k_r\geq 0\\ k_1+\ldots+k_n=m-n+1}} x_1^{k_1}\ldots x_n^{k_n} \tag{2.12}$$

one arrives to (2.8). The equation (2.7) is proved similarly. ∎

The set of Whittaker vectors (2.9), (2.10) is closed under multiplication by an arbitrary $i\hbar$-periodic function of variables γ_{nj}. Hence, there are infinitely many invariant subspaces in M corresponding to infinitely many Whittaker vectors. To construct irreducible submodules, let us introduce the Whittaker modules W and W' cyclically generated by Whittaker vectors w_N and w'_N respectively.

Theorem 2.1 *The Whittaker module $W = \mathcal{U}w_N$ is spanned by the elements of the form*

$$w_{\boldsymbol{m}_1,\ldots,\boldsymbol{m}_{N-1}} = \prod_{n=1}^{N-1}\prod_{k=1}^{n} \sigma_k^{m_{nk}}(\boldsymbol{\gamma}_n) w_N, \tag{2.13}$$

where $\sigma_k(\boldsymbol{\gamma}_n)$ is the elementary symmetric function of the variables $\gamma_{n1},\ldots,\gamma_{nn}$ of order k:

$$\sigma_k(\boldsymbol{\gamma}_n) = \sum_{j_1<\ldots<j_k} \gamma_{nj_1}\cdots\gamma_{nj_k}. \tag{2.14}$$

Similarly, the Whittaker module $W' = \mathcal{U}w'_N$ is spanned by the polynomials

$$w'_{\boldsymbol{m}_1,\ldots,\boldsymbol{m}_{N-1}} = \prod_{n=1}^{N-1}\prod_{k=1}^{n} \sigma_k^{m_{nk}}(\boldsymbol{\gamma}_n). \tag{2.15}$$

The Whittaker modules W and W' are irreducible.

Proof. Let us prove the statement for the module W. Using the identity $e^{i\hbar\partial_{\gamma_{nj}}}\sigma_k(\boldsymbol{\gamma}_n) = \sigma_k(\boldsymbol{\gamma}_n) + i\hbar\sum_{r=1}^{k-1}(-1)^r\sigma_{k-r-1}(\boldsymbol{\gamma}_n)\gamma_{nj}^r$ and the explicit formula for the action of $E_{n+1,n}$ on the Whittaker vector:

$$E_{n+1,n}w_N = i\hbar^{-1}w_N\sum_{j=1}^{n}\frac{\prod_{r=0}^{n+1}(\gamma_{nj}-\gamma_{n+1,r}+\frac{i\hbar}{2})}{\prod_{s\neq j}(\gamma_{nj}-\gamma_{ns})}, \tag{2.16}$$

one shows that the Whittaker module is spanned by the elements of the form (2.13).

To prove the irreducibility of Whittaker modules we need the following fact ([5], theorem 3.6.2):

Let M be any $\mathcal{U}$-module which admits an infinitesimal character. Assume $w \in M$ is a Whittaker vector. Then the submodule $\mathcal{U}w \subset M$ is irreducible.

Hence, the modules W and W' are irreducible according to Lemma 2.1 and the above mentioned theorem. ∎

Let us note that for any subalgebra $\mathcal{U}(\mathfrak{gl}(n)) \subset \mathcal{U}(\mathfrak{gl}(N))$, $2 \leq n < N$ the vector space with the basis $\prod_{s=1}^{n-1}\prod_{k=1}^{n} \sigma_k^{m_{sk}}(\boldsymbol{\gamma}_s) w_N$ is a $\mathcal{U}(\mathfrak{gl}(n))$ Whittaker module.

Now one can calculate the explicit form of the action of the central elements of $\mathcal{U}(\mathfrak{gl}(n))$ on the space M. It is well known that the generating function $\mathcal{A}_n(\lambda)$ of the central elements of $\mathcal{U}(\mathfrak{gl}(n))$ (the Casimir operators) can be represented as follows [33]:

$$\begin{aligned} &\mathcal{A}_n(\lambda) \\ &= \sum_{p\in P_n} \operatorname{sign} p \Big[(\lambda - i\hbar\rho_1^{(n)})\delta_{p(1),1} - i\hbar E_{p(1),1}\Big] \dots \Big[(\lambda - i\hbar\rho_n^{(n)})\delta_{p(n),n} - i\hbar E_{p(n),n}\Big], \end{aligned} \tag{2.17}$$

where $\rho_k^{(n)} = \frac{1}{2}(n-2k+1)$, $k = 1, \dots, n$ and summation is given over elements of the permutation group P_n.

Proposition 2.3 *The operators (2.17) has the following representation on M:*

$$\mathcal{A}_n(\lambda) = \prod_{j=1}^{n}(\lambda - \gamma_{nj}), \qquad (n = 1, \dots, N). \tag{2.18}$$

Proof. Due to Lemma 2.1 it is sufficient to calculate the action of the operators $\mathcal{A}_n(\lambda)$ on the Whittaker vector w_N (or w'_N) only. Due to (2.8) one has

$$\begin{aligned} \mathcal{A}_n(\lambda) w_N &= \mathcal{A}_{n-1}(\lambda - \tfrac{i\hbar}{2})[\lambda - i\hbar\rho_n^{(n)} - i\hbar E_{nn}] w_N \\ &- i\hbar \sum_{k=2}^{n} \mathcal{A}_{n-k}(\lambda - \tfrac{ik\hbar}{2}) E_{n,n-k+1} w_N . \end{aligned} \tag{2.19}$$

Further, using the relation (2.16), it is easy to calculate the action of the composite generators:

$$E_{n,n-k+1} w_N = i\hbar^{-1} w_N \sum_{j_1=1}^{n-1} \dots \sum_{j_k=1}^{n-k+1} \prod_{p=1}^{k-1} \frac{\prod\limits_{r_p \neq j_{p-1}} (\gamma_{n-p.j_p} - \gamma_{n-p+1,r_p} + \frac{i\hbar}{2})}{\prod\limits_{s_p \neq j_p} (\gamma_{n-p,j_p} - \gamma_{n-p,s_p})} . \tag{2.20}$$

Substituting (2.20) to (2.19) one arrives to (2.18) after some combinatorics. The similar calculation is obviously valid for the Whittaker module W'. ∎

One thing that remains is to construct the hermitian pairing between W and W' and proving that the Whittaker modules W and W' are dual with respect to this pairing.

Definition 2.2 *Let* $\phi \in W'$ *and* $\psi \in W$. *Define the hermitian pairing* $\langle\, , \rangle :\ W' \otimes W \to \mathbb{C}$ *by*

$$\langle \phi, \psi \rangle = \int\limits_{\mathbb{R}^{\frac{N(N-1)}{2}}} \mu_0(\boldsymbol{\gamma}) \overline{\phi}(\boldsymbol{\gamma})\, \psi(\boldsymbol{\gamma}) \prod_{\substack{n=1 \\ j \leq n}}^{N-1} d\gamma_{nj}, \tag{2.21}$$

where

$$\mu_0(\boldsymbol{\gamma}) = \prod_{n=2}^{N-1} \prod_{s<p} (\gamma_{ns} - \gamma_{np})(e^{\frac{2\pi\gamma_{np}}{\hbar}} - e^{\frac{2\pi\gamma_{ns}}{\hbar}}). \tag{2.22}$$

Then the integral (2.21) converges absolutely. Actually, this is a simple corollary of more general statement. Namely, denote the integrand in the r.h.s. of (2.21) by the $I_{N}(\boldsymbol{\gamma})$.

Lemma 2.2 *For the set of* $\operatorname{Im}\gamma_{nj}$ *such that* $\min_j\{\operatorname{Im}\gamma_{nj}\} > \max_k\{\operatorname{Im}\gamma_{n+1,k}\} - \frac{1}{2}$ *the following inequality holds:*

$$|I_{N}(\boldsymbol{\gamma})| \leq |P(\boldsymbol{\gamma})| \, \exp\Big\{ - \frac{1}{(2N-3)!!} \sum_{n=1}^{N-1} \sum_{j=1}^{n} |\operatorname{Re}\gamma_{nj}| \Big\}. \tag{2.23}$$

where $P(\boldsymbol{\gamma})$ *is a polynomial in* $\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_{N-1}$.

Proof. For any $x > 0$ the inequality $|\Gamma(x+iy)| \leq \Gamma(x) p_x(y) \cosh^{-\frac{1}{2}}(y)$ is valid for some polynomial $p_x(y)$ with degree linearly depending on x . Hence, the integrand satisfies:

$$|I_{N}(\boldsymbol{\gamma})| \leq |P(\boldsymbol{\gamma})| \, e^{\frac{\pi}{\hbar} S_N(\boldsymbol{\gamma})}, \tag{2.24}$$

where

$$\begin{aligned} S_{N}(\boldsymbol{\gamma}) = \sum_{n=2}^{N-1} \sum_{s<p} |\operatorname{Re}\gamma_{ns} - \operatorname{Re}\gamma_{sp}| \\ - \frac{1}{2} \sum_{n=1}^{N-2} \sum_{j,k} |\operatorname{Re}\gamma_{nj} - \operatorname{Re}\gamma_{n+1,k}| - \frac{N}{2} \sum_{j=1}^{N-1} |\operatorname{Re}\gamma_{N-1,j}|. \end{aligned} \tag{2.25}$$

Using inequality

$$\sum_{j,k=1}^{m} |a_j - b_k| - \sum_{j<k}^{m} |a_j - a_k| - \sum_{j<k}^{m} |b_j - b_k| \geq 0 \tag{2.26}$$

which holds for any set of real parameters $\{a_j, b_j\}$; $j = 1, \ldots, m$ (see (11.118) in [34]), one proves the recursive relation $S_{N} \leq \frac{1}{2N-3}(S_{N-1} - \sum_j |\operatorname{Re}\gamma_{N-1,j}|)$ thus arriving to (2.23). ∎

Proposition 2.4 *Let* $\boldsymbol{\gamma}_{N} \in \mathbb{R}^N$ *. Whittaker modules* W *and* W' *are dual with respect to the pairing defined by (2.21), which means that for any* $\phi \in W'$ *and* $\psi \in W$ *, the generators* $X \in \mathfrak{gl}(N, \mathbb{R})$ *possess the property*

$$\langle \phi, X\psi \rangle \, = \, - \, \langle X\phi, \psi \rangle. \tag{2.27}$$

Proof. For example, consider $\langle \phi, E_{n,n+1}\psi \rangle$, where $E_{n,n+1}$ is defined by (2.1b). It is easy to see that expression $\overline{E_{n,n+1}\phi}\,\psi$ does not have the poles in the upper hyper-plane and it is possible to deform the integration contour. Consider the shifts $\gamma_{nj} \to \gamma_{nj} + i\hbar$. Note that there is a difference equation

$$e^{i\hbar\partial_{\gamma_{nj}}}\mu_0(\boldsymbol{\gamma}) = \mu_0(\boldsymbol{\gamma}) \prod_{s\neq j} \frac{\gamma_{nj} - \gamma_{ns} + i\hbar}{\gamma_{nj} - \gamma_{ns}}\,. \tag{2.28}$$

Then using the deformation of the contour and the estimates (2.23), one obtains (2.27). ■

3 Orbit method for $GL(N, \mathbb{R})$ and Gelfand-Tsetlin representation

In this section we provide the orbit method interpretation for the explicit algebraic construction of the infinite-dimensional representation of $\mathcal{U}(\mathfrak{gl}(N))$ given in the previous section. Similar results for the case of the compact group were obtained previously in [27],[28].

Let G be a Lie group, $\mathfrak{g}$ corresponding Lie algebra and $\mathfrak{g}^*$ its dual. Coadjoint orbits $\mathcal{O}$ of G are supplied with the canonical symplectic structure, given by Kirillov-Kostant two-form [26]. The space of linear functions on $\mathfrak{g}^*$ is closed under Kirillov-Kostant Poisson bracket and coincides with $\mathfrak{g}$ as a Lie algebra.

It reproduces the relations in the algebra $\mathfrak{g}$. Consider coadjoint orbits of the group $GL(N, \mathbb{R})$ with stabilizer $H_{N} = GL(1, \mathbb{R})^{\otimes N}$. Below we construct Darboux coordinates on the finite cover $\widetilde{\mathcal{O}}^{(0)}$ of the open part $\mathcal{O}^{(0)}$ of such an orbit $\mathcal{O}$ of $G = GL(N, \mathbb{R})$. We also give expressions for the restriction onto $\mathcal{O}^{(0)}$ of the linear functions on $\mathfrak{g}^*$ in these coordinates. The corresponding Hamiltonian vector fields act on $\mathcal{O}^{(0)}$. The explicit realization of $\mathcal{U}(\mathfrak{gl}(N))$ by difference operators given in the previous section may be considered as a "quantization" of this Poisson algebra. Let us stress that this construction gives the representation of the $\mathcal{U}(\mathfrak{gl}(N))$ that can not be integrated to the representation of the whole group. From the picture that will be discussed in this section it is quite natural because the part of the orbit $\mathcal{O}^{(0)}$ is not stable with respect to the action of the group on the whole orbit $\mathcal{O}$.

Let $G = GL(N, \mathbb{R})$, $\mathfrak{g} = \mathfrak{gl}(N, \mathbb{R}))$ and we identify the algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ with the help of the Killing form. Consider the coadjoint orbit $\mathcal{O}$ of the diagonal element $u_{N} \in \mathfrak{g}^*$

with pairwise different diagonal entries: $\boldsymbol{\gamma}_N = (\gamma_{N1}, \gamma_{N2}, \cdots \gamma_{NN})$:

$$u_N = \begin{pmatrix} \gamma_{N1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \gamma_{NN} \end{pmatrix}. \tag{3.1}$$

The stabilizer of u_N is $H_N = GL(1, \mathbb{R})^{\otimes N}$ and the general element of the orbit may be parameterized as $u = g^{-1} u_N g$, where $g \in GL(N, \mathbb{R})$ is defined up to the left action of H_N. There is a canonical symplectic two-form on $\mathcal{O}$ which in this parameterization is given by:

$$\Omega = Tr\, u_N (dg\, g^{-1})^2 . \tag{3.2}$$

Consider the subspace in $GL(N, \mathbb{R})$ which consists of the elements of the form:

$$g = g_N g_{N-1} \cdots g_2 . \tag{3.3}$$

The matrices g_n differ from the unit matrix only in the upper-left $n \times n$ square and recursively defined as follows. Denote the upper-left $(n \times n)$ sub-matrix of g_n by f_n. There should exist such $\gamma_{n-1,j}$, $(j = 1, \dots, n-1)$ that the following condition holds:

$$f_n^{-1} \begin{pmatrix} \gamma_{n1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \gamma_{n,n-1} & 0 \\ 0 & \cdots & 0 & \gamma_{nn} \end{pmatrix} f_n = \begin{pmatrix} \gamma_{n-1,1} & 0 & \cdots & * \\ \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & \gamma_{n-1,n-1} & * \\ * & \cdots & * & * \end{pmatrix}. \tag{3.4}$$

This means that the diagonal matrix conjugated by f_n may have non-zero non-diagonal entries only in the last column and the last row. The matrix f_n is defined uniquely by $\gamma_{nj}, \gamma_{n-1,k}$ up to the left action by the diagonal matrix and the right action by the diagonal matrix with the last diagonal element being equal to one. Let us choose the following representatives for f_n in the coset of G by the left action of the diagonal matrices:

$$\begin{aligned} (f_n)_{jk} &= \frac{Q_{n-1,k}}{\gamma_{nj} - \gamma_{n-1,k}} \frac{\prod_{r=1}^{n-1} (\gamma_{nj} - \gamma_{n-1,r})}{\prod_{s \neq j} (\gamma_{nj} - \gamma_{ns})} \qquad (k = 1, \dots, n-1), \\ (f_n)_{jn} &= \frac{\prod_{r=1}^{n-1} (\gamma_{nj} - \gamma_{n-1,r})}{\prod_{s \neq j} (\gamma_{nj} - \gamma_{ns})}, \end{aligned} \tag{3.5}$$

where $Q_{n-1,k}$, $k = 1...(n-1)$ are additional coordinates on the orbit. This gives the map from $\mathbb{R}^{N(N-1)}/\Delta$ to the open part $\mathcal{O}^{(0)}$ of the orbit. Here Δ is a union of three subspaces of $\mathbb{R}^{N(N-1)}$: $\Delta = D_1 \cup D_2 \cup D_3$. The subspace D_1 is a subspace where at least two $\boldsymbol{\gamma}$-coordinates of the same level coincide: $\gamma_{nj} = \gamma_{nk}$. The subspace D_2 is defined as a subspace where at least two of $\boldsymbol{\gamma}$-coordinates from the consecutive levels coincide $\gamma_{nj} = \gamma_{n-1,k}$. Finally, the subspace D_3 is a subspace where at least one of Q_{nj} is zero. Now let us construct the inverse map. Consider the element of the orbit $u = g^{-1}u_N g$ shifted by the unit matrix multiplied by a formal variable λ:

$$T(\lambda) = \mathbf{1}\lambda - u\,. \tag{3.6}$$

The n-th principal minor $a_n(\lambda)$ of $T(\lambda)$ does not depend on g_m, $m < n+1$. For the general point of the orbit the roots of the minors may be complex. Consider the open part $\mathcal{O}^{(0)}$ of the orbit $\mathcal{O}$ such that all upper-left sub-matrices are diagonalizable and their determinants have distinct real roots. We also impose the condition that there are no identical roots of the minors of the consecutive ranks. Then define $\boldsymbol{\gamma}$-variables as roots of the minors:

$$a_n(\lambda) = \prod_{j=1}^{n}(\lambda - \gamma_{nj})\,. \tag{3.7}$$

Note that by this condition the roots are defined only up to the action of symmetric group S_n. As a result we get the coordinates on the finite cover of the part of the orbit.

Now let us give the explicit expressions for the $\mathbf{Q}$-coordinates. Let $b_n(\lambda)$ be the determinant of $n \times n$ corner sub-matrix of the matrix which is obtained from $T(\lambda)$ by the interchanging the n-th and the $(n+1)$-th columns, i.e. $b_n(\lambda) = \det\limits_{n\times n} ||T(\lambda)S_{n,n+1}||$ where $N \times N$ matrix $S_{n,n+1}$ is a unit matrix with the diagonal 2×2 sub-matrix in the n, $n+1$ columns and n, $n+1$ rows being replaced by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Similarly, let us define $c_n(\lambda)$ as the determinant of the $n \times n$ corner sub-matrix obtained from $T(\lambda)$ by interchanging the n-th and the $(n+1)$-th rows, i.e. $c_n(\lambda) = \det\limits_{n\times n} ||S_{n,n+1}T(\lambda)||$. Then the other part of the variables may be introduced as follows:

$$Q_{nj} = b_n^{-1}(\gamma_{nj})a_{n+1}(\gamma_{nj}) = -c_n(\gamma_{nj})a_{n-1}^{-1}(\gamma_{nj})\,, \tag{3.8}$$

where the last equality can be proved by straightforward check. It is not difficult to verify that this definition is compatible with the explicit parameterization in terms of (γ_{nj}, Q_{nj}) given above. Due to the condition (3.4) the n-th left-upper corner sub-matrix of the matrix $u_n = g_{n+1}^{-1}\cdots g_N^{-1}u_N g_N \cdots g_{n+1}$ is the diagonal matrix, $a_n(\lambda)$ is given by (3.7). The determinant $b_n(\lambda)$ is invariant under conjugation by the matrix which differs from the unit matrix only in the upper-left $(n-1)\times(n-1)$ corner. Thus, $b_n(\lambda)$ may be represented as:

$$b_n(\lambda) = \det(g_n^{-1}) \det_{n\times n} ||(\lambda - u_n)g_n S_{n,n+1}||\,. \tag{3.9}$$

Straightforward calculation gives

$$b_n(\lambda) = \sum_{j=1}^{n} Q_{nj}^{-1}\prod_{r=1}^{n+1}(\gamma_{nj} - \gamma_{n+1,r})\prod_{s\neq j}\frac{\lambda - \gamma_{ns}}{\gamma_{nj} - \gamma_{ns}}\,. \tag{3.10}$$

Similarly, for $c_n(\lambda)$ one has

$$c_n(\lambda) = -\sum_{j=1}^{n} Q_{nj}\prod_{r=1}^{n-1}(\gamma_{nj} - \gamma_{n-1,r})\prod_{s\neq j}\frac{\lambda - \gamma_{ns}}{\gamma_{nj} - \gamma_{ns}}\,. \tag{3.11}$$

This gives the inverse transformation from the finite cover of the open part of the orbit $\widetilde{\mathcal{O}}^{(0)}$ to $\mathbb{R}^{\frac{N(N-1)}{2}}/\Delta$.

Simple calculation of the Kirillov-Kostant simplectic form in the coordinates (γ_{nj}, Q_{nj}) gives the following

Proposition 3.1 *1. The coordinates (γ_{nj}, Q_{nj}) parameterize the finite cover $\widetilde{\mathcal{O}}^{(0)}$ of the open part of the orbit $\mathcal{O}^{(0)}$. The product of the symmetric groups $S = \prod_{n=2}^{N} S_n$ acts freely on the fibers of the projection: $\pi : \widetilde{\mathcal{O}}^{(0)} \to \mathcal{O}^{(0)}$ and the factor is $\mathcal{O}^{(0)}$.*

2. The lift of 2-form Ω on the $\widetilde{\mathcal{O}}^{(0)}$ has the canonical form

$$\pi^*\Omega = \sum_{n=1}^{N-1}\sum_{j=1}^{n} d\gamma_{nj} \wedge Q_{nj}^{-1} dQ_{nj}. \tag{3.12}$$

Coordinates (γ_{nj}, Q_{nj}) give Darboux coordinates on the finite cover of the open part $\mathcal{O}^{(0)}$ of the coadjoint orbit of $GL(N, \mathbb{R})$ with the stabilizer conjugated to $H_N = GL(1, \mathbb{R})^{\otimes N}$.

Now let us give explicit expressions for the elements on the sub-diagonal, diagonal and over-diagonal of the matrix $u = g^{-1} u_N g$ in the GT-coordinates. This gives the classical counterparts of the generators (2.1).

Proposition 3.2 *The linear functions u_{nn}, $u_{n,n+1}$, $u_{n+1,n}$ on $\mathfrak{gl}(N, \mathbb{R})^*$, being restricted to $\mathcal{O}^{(0)}$ and lifted by the projection map $\pi : \widetilde{\mathcal{O}}^{(0)} \to \mathcal{O}^{(0)}$ to $\widetilde{\mathcal{O}}^{(0)}$, have the following form in the GT-coordinates [7] :*

$$\begin{aligned} u_{nn} &= \sum_{j=1}^{n} \gamma_{nj} - \sum_{j=1}^{n-1} \gamma_{n-1,j}, \\ u_{n,n+1} &= -\sum_{j=1}^{n} \frac{\prod_{r=1}^{n+1} (\gamma_{nj} - \gamma_{n+1,r})}{\prod_{s\neq j} (\gamma_{nj} - \gamma_{ns})} Q_{nj}^{-1}, \\ u_{n+1,n} &= \sum_{j=1}^{n} \frac{\prod_{r=1}^{n-1} (\gamma_{nj} - \gamma_{n-1,r})}{\prod_{s\neq j} (\gamma_{nj} - \gamma_{ns})} Q_{nj}. \end{aligned} \tag{3.13}$$

Proof. Let us start with the diagonal elements. Coadjoint action of the diagonal matrix $\Phi = \mathrm{diag}(e^{\phi_1}, \cdots e^{\phi_N})$ on u may be represented as the right action on the element g (3.3). It is easy to see that this is equivalent to the shift of the variables:

$$\begin{aligned} Q_{nj} &\to Q_{nj} e^{\phi_n - \phi_{n+1}}, \\ Q_{Nj} &\to Q_{Nj} e^{\phi_N}. \end{aligned} \tag{3.14}$$

[7] There is another representation with “square roots” which is more close to the one given in [21]. It may be obtained by a different choice of the parameterization of the solution of (3.4).

Thus, the corresponding Hamiltonian functions u_{nn} are:

$$u_{nn} = \sum_{j=1}^{n} \gamma_{nj} - \sum_{j=1}^{n-1} \gamma_{n-1,j} \,. \tag{3.15}$$

This result could also be derived from the explicit representation of the matrix u. Moreover, from the structure of the matrices u_n, f_n one could easily obtain the expressions for the other generators. For example,

$$u_{n+1,n} = \sum_{j=1}^{n} (u_n)_{n+1,j} (f_n)_{jn} = \sum_{j=1}^{n} \frac{\prod_{r=1}^{n-1} (\gamma_{nj} - \gamma_{n-1,r})}{\prod_{s \neq j} (\gamma_{nj} - \gamma_{ns})} Q_{nj} \,. \tag{3.16}$$

Quite similarly,

$$u_{n,n+1} = - \sum_{j=1}^{n} \frac{\prod_{r=1}^{n+1} (\gamma_{nj} - \gamma_{n+1,r})}{\prod_{s \neq j} (\gamma_{nj} - \gamma_{ns})} Q_{nj}^{-1} \,. \tag{3.17}$$

■

There is a representation for $u_{n,n}$, $u_{n,n+1}$, and $u_{n+1,n}$ by the contour integrals in terms of the polynomials $a_n(\lambda), b_n(\lambda)$, and $c_n(\lambda)$:

$$\begin{aligned} u_{n,n}(\lambda) &= - \oint_{\infty} \frac{a_n(\lambda)}{a_{n-1}(\lambda)} \frac{d\lambda}{\lambda} \,, \\ u_{n,n+1}(\lambda) &= - \oint_{\infty} a_n^{-1}(\lambda) b_n(\lambda) d\lambda \,, \\ u_{n+1,n}(\lambda) &= - \oint_{\infty} c_n(\lambda) a_n^{-1}(\lambda) d\lambda \,. \end{aligned} \tag{3.18}$$

In the next section we discuss the connection of this representation with the Drinfeld's "new realization" of the Yangian [29].

4 Yangian $Y(\mathfrak{gl}(N))$ and Gelfand-Tsetlin representation

In the previous section we outline the construction of the Gelfand-Tsetlin parameterization of the coadjoint orbit of $GL(N, \mathbb{R})$. The polynomials $a_n(\lambda), b_n(\lambda)$, and $c_n(\lambda)$ that appear in the invariant formulation of this parameterization bear obvious similarity to the basic ingredients of Drinfeld's "new realization" of Yangian. In this section we systematically use the Yangian relations to derive Gelfand-Tsetlin representation starting with the representation theory of Yangian $Y(\mathfrak{gl}(N))$. [8] It appears that this approach provides us with a new interpretation of the QISM to the solution of the open Toda chain [20].

We recall some well known facts about the Yangian $Y(\mathfrak{gl}(N))$ [29]. The Yangian is an associative Hopf algebra generated by the elements $T_{ij}^{(r)}$ where $i,j=1,\dots,N$ and $r=0,\dots,\infty$ subject the following relations. Consider an $N\times N$ matrix $||T_{ij}(\lambda)||_{ij=1}^{N}$ with the operator valued entries

$$T_{ij}(\lambda)=\delta_{ij}\lambda+\sum_{r=0}^{\infty}T_{ij}^{(r)}\lambda^{-r}. \tag{4.1}$$

Let

$$R_{N}(\lambda)\;=\;I\otimes I+i\hbar P/\lambda\;;\qquad\qquad P_{ik,jl}=\delta_{il}\delta_{kj} \tag{4.2}$$

be an $N^2\times N^2$ numerical matrix (the Yang R-matrix). Then the relations between the generators $T_{ij}^{(r)}$ can be written in standard form:

$$R_{N}(\lambda-\mu)(T(\lambda)\otimes I)(I\otimes T(\mu))=(T(\mu)\otimes I)(T(\lambda)\otimes I)R_{N}(\lambda-\mu). \tag{4.3}$$

Let $||X_{ij}(\lambda)||_{i,j=1}^{n}$ be a $n\times n$ sub-matrix of the matrix $||T_{ij}(\lambda)||_{i,j=1}^{N}$. It is obvious from the explicit form of $R_N(\lambda)$ that this sub-matrix satisfies RTT relations with the corresponding $n^2\times n^2$ matrix $R_n(\lambda)$. The center of the Yangian is generated by the coefficients of the following polynomial (quantum determinant of $T_{ij}(\lambda)$):

$$\begin{aligned}&\det{}_qT(\lambda)\\ &=\sum_{p\in P_n}\operatorname{sign}p\;T_{p(1),1}(\lambda-i\hbar\rho_1^{(n)})\dots T_{p(k),k}(\lambda-i\hbar\rho_k^{(n)})\dots T_{p(n),n}(\lambda-i\hbar\rho_n^{(n)})\,.\end{aligned} \tag{4.4}$$

The following way to describe the Yangian $Y(\mathfrak{gl}(N)))$ was introduced in [29]. Let $\mathcal{A}_n(\lambda)$ ($n=1,\dots,N$) be the quantum determinants of sub-matrices, consisting of the first n rows and columns, while the operators $\mathcal{B}_n(\lambda),\mathcal{C}_n(\lambda)$ be the quantum determinants of sub-matrices with the elements $T_{ij}(\lambda)$, where $i=1,\dots,n$; $j=1,\dots,n-1,n+1$ and $i=1,\dots,n-1,n+1$; $j=1,\dots,n$ respectively. Finally, let $\mathcal{D}_n(\lambda)$ be the quantum determinant of sub-matrix with the elements $T_{ij}(\lambda)$ with $i=1,\dots,n-1,n+1;j=1,\dots,n-1,n+1$. The expansion coefficients of $\mathcal{A}_n(\lambda),\mathcal{B}_n(\lambda),\mathcal{C}_n(\lambda),\mathcal{D}_n(\lambda)$ ($n=1,\dots,N-1$) with respect to λ together with those of $\mathcal{A}_N(\lambda)$ generate the algebra $Y(\mathfrak{gl}(N))$. The whole set of relations between $\mathcal{A}_n(\lambda)$, $\mathcal{B}_n(\lambda)$, $\mathcal{C}_n(\lambda)$ and $\mathcal{D}_n(\lambda)$ could be found in [32]. The part which is relevant for what follows is given by the relations

$$\begin{aligned}&[\mathcal{A}_n(\lambda),\mathcal{A}_m(\mu)]=0\;;\qquad (n,m=1,\dots,N),\\ &[\mathcal{B}_n(\lambda),\mathcal{B}_n(\mu)]=0;\quad [\mathcal{C}_n(\lambda),\mathcal{C}_m(\mu)]=0\;;\qquad (m\neq n\pm1),\\ &(\lambda-\mu+i\hbar)\mathcal{A}_n(\lambda)\mathcal{B}_n(\mu)\;=\;(\lambda-\mu)\mathcal{B}_n(\mu)\mathcal{A}_n(\lambda)+i\hbar\mathcal{A}_n(\mu)\mathcal{B}_n(\lambda),\\ &(\lambda-\mu+i\hbar)\mathcal{A}_n(\mu)\mathcal{C}_n(\lambda)\;=\;(\lambda-\mu)\mathcal{C}_n(\lambda)\mathcal{A}_n(\mu)+i\hbar\mathcal{A}_n(\lambda)\mathcal{C}_n(\mu).\end{aligned} \tag{4.5}$$

[8] For the connection of finite-dimensional Gelfand-Tsetlin representation with Yangian see [30], [32].

Drinfeld's "new realization" emerges if we consider the coefficients of the expansions with respect to λ of some simple combinations of the generators $\mathcal{A}_n(\lambda), \mathcal{B}_n(\lambda), \mathcal{C}_n(\lambda)$ [29].

There is a natural homomorphism $\pi_{_N} : Y(\mathfrak{gl}(N)) \to \mathcal{U}(\mathfrak{gl}(N))$ defined by

$$T_{jk}(\lambda) = \lambda\delta_{jk} - i\hbar E_{jk}\,, \qquad (j,k = 1,\ldots,N). \tag{4.6}$$

Let us describe the images of the generators $\mathcal{A}_n(\lambda), \mathcal{B}_n(\lambda)$, and $\mathcal{C}_n(\lambda)$ explicitly. The image of $\mathcal{A}_{_N}(\lambda)$ which is a polynomial of order N gives a generating function for the generators of the center $\mathcal{Z} \subset \mathcal{U}(\mathfrak{gl}(N))$. Consider the chain of Yangians

$$Y(\mathfrak{gl}(1)) \subset \ldots \subset Y(\mathfrak{gl}(N)) \tag{4.7}$$

defined by the standard embeddings. Let $\mathcal{A}$ be the maximal commutative sub-algebra of $Y(\mathfrak{gl}(N)))$. The images of the generators of $\mathcal{A}$ under the homomorphism (4.6) have the form (2.17). The other generators $\mathcal{B}_n(\lambda)$, $\mathcal{C}_n(\lambda)$ can be written as

Lemma 4.1

$$\mathcal{B}_n(\lambda) = [\mathcal{A}_n(\lambda), E_{n,n+1}], \tag{4.8}$$

$$\mathcal{C}_n(\lambda) = [E_{n+1,n}, \mathcal{A}_n(\lambda)], \tag{4.9}$$

where $n = 1,\ldots,N-1$.

The homomorphism $\pi_{_N} : Y(\mathfrak{gl}(N)) \to \mathcal{U}(\mathfrak{gl}(N))$ is really an epimorphism. The following integral formula expresses the generators of $\mathcal{U}(\mathfrak{gl}(N))$ in terms of $\mathcal{A}_n(\lambda), \mathcal{B}_n(\lambda), \mathcal{C}_n(\lambda)$:

$$\begin{aligned} E_{n,n+1} &= -\frac{1}{i\hbar}\oint_{\infty} \mathcal{A}_n^{-1}(\lambda)\mathcal{B}_n(\lambda)d\lambda\,, \\ E_{n+1,n} &= -\frac{1}{i\hbar}\oint_{\infty} \mathcal{C}_n(\lambda)\mathcal{A}_n^{-1}(\lambda)d\lambda\,, \\ E_{nn} &= -\frac{1}{i\hbar}\oint_{\infty} \frac{\mathcal{A}_n(\lambda)}{\mathcal{A}_{n-1}(\lambda - \frac{i\hbar}{2})}\frac{d\lambda}{\lambda} - \frac{1}{2}(n-1)\,, \end{aligned} \tag{4.10}$$

where the contours of integrations are taken around ∞. These expressions may be considered as a "quantization" of the classical relations (3.18). Thus it is not surprising that Drinfeld's construction of the "new realization" of the Yangian leads naturally to the the generalized Gelfand-Tsetlin representation discussed in sections 2,3. Consider the model for representation of the $\mathcal{U}(\mathfrak{gl}(N))$ based on the commutative subalgebra $\mathcal{A}$. In this realization all polynomial generators $\mathcal{A}_n(\lambda)$ act by multiplication by the polynomials depending on λ and some auxiliary variables. We restrict the representation by the condition that all roots of these polynomials are real (this is compatible with commutation relations (4.5)). Thus, in this realization we have

$$\mathcal{A}_n(\lambda) = \prod_{j=1}^{n}(\lambda - \gamma_{nj}). \tag{4.11}$$

As auxiliary variables one could choose γ_{nj}. Using the commutation relations between the generators $\mathcal{A}_n(\lambda), \mathcal{B}_n(\lambda)$, and $\mathcal{C}_n(\lambda)$ one could derive the expression for these generators in γ_{nj} - parameterization.

Proposition 4.1 *The operators*

$$\mathcal{A}_n(\lambda) = \prod_{j=1}^{n}(\lambda - \gamma_{nj}),$$
$$\mathcal{B}_n(\lambda) = \sum_{j=1}^{n}\prod_{s\neq j}\frac{\lambda-\gamma_{ns}}{\gamma_{nj}-\gamma_{ns}}\prod_{r=1}^{n+1}(\gamma_{nj}-\gamma_{n+1,r}-\frac{i\hbar}{2})\, e^{-i\hbar\partial_{\gamma_{nj}}}, \tag{4.12}$$
$$\mathcal{C}_n(\lambda) = -\sum_{j=1}^{n}\prod_{s\neq j}\frac{\lambda-\gamma_{ns}}{\gamma_{nj}-\gamma_{ns}}\prod_{r=1}^{n-1}(\gamma_{nj}-\gamma_{n-1,r}+\frac{i\hbar}{2})\, e^{i\hbar\partial_{\gamma_{nj}}}$$

satisfy the full set of the Yangian relations [32]. In particular, (4.5) hold.

The formulae (4.12) have also the "classical" counterpart (3.7), (3.10), (3.11) derived in section 2. The easy check shows that the combination of Proposition 4.1 and the integral representation (4.10) leads to the representation (2.1) of $\mathfrak{gl}(N)$ derived in section 2.

5 Application to the quantum integrable systems

5.1 The open Toda chain

The open Toda chain corresponding to $\mathfrak{gl}(N,\mathbb{R}))$ is one of the simplest examples of the integrable quantum mechanical system with N degrees of freedom (see [35] and references therein). It has N mutually commuting Hamiltonians, the first two of which are given by

$$h_1 = \sum_{j=1}^{N} p_j\,,$$
$$h_2 = \sum_{j<k} p_j p_k - \sum_{j=1}^{N-1} e^{x_j - x_{j+1}}\,, \tag{5.1}$$

where $[x_n, p_m] = i\hbar\delta_{nm}$.

Recall the general idea behind the Representation Theory approach to the quantum integrability in the case of the open Toda chain. Let V and V' be any dual irreducible Whittaker $\mathcal{U}(\mathfrak{gl}(N,\mathbb{R}))$ - modules and $w \in V$ and $w' \in V'$ be the corresponding cyclic Whittaker vectors with respective characters $\chi_+(E_{n,n+1}) = -i\hbar^{-1}$ and $\chi_-(E_{n+1,n}) = -i\hbar^{-1}$. We assume that the action of the Cartan subalgebra is integrated to the action of the Cartan torus so the following function is well defined

$$\psi_{\gamma_{N1},\dots,\gamma_{NN}} = e^{-\boldsymbol{x}\cdot\boldsymbol{\rho}^{(N)}}\langle w'_N, e^{-\sum_{k=1}^{N} x_k E_{kk}} w_N\rangle\,. \tag{5.2}$$

In each irreducible representation the action of the elements of the center $\mathcal{Z}$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(N,\mathbb{R}))$ is proportional to the action of the unit operator. Consider the function

$$\widetilde{\psi}_{\gamma_{N1},\dots,\gamma_{NN}} = e^{-\boldsymbol{x}\cdot\boldsymbol{\rho}^{(N)}}\langle w'_N, e^{-\sum\limits_{k=1}^{N} x_k E_{kk}} z\, w_N\rangle\,, \tag{5.3}$$

where z belongs to the center $\mathcal{Z}$. Using the properties of the Whittaker vectors one could show that there is a differential operator $\mathcal{D}_z$ in the variables x_k such that $\widetilde{\psi}_{\gamma_{N1},\dots,\gamma_{NN}} = \mathcal{D}_z\psi_{\gamma_{N1},\dots,\gamma_{NN}}$ Thus, taking the first two Casimir operators, one gets the first two Hamiltonians (5.1). On the other hand, z is an element of the center $\mathcal{Z}$ and $\widetilde{\psi}_{\gamma_{N1},\dots,\gamma_{NN}}$ is proportional to $\psi_{\gamma_{N1},\dots,\gamma_{NN}}$ with some numerical coefficient. Hence, we get the common eigenfunction for the set of the differential operators corresponding to the elements of the center $\mathcal{Z}$.

Let us given an explicit realization of the representation of $\mathcal{U}(\mathfrak{gl}(N,\mathbb{R}))$ in terms of some difference/differential operators which integrates to the representation of the Cartan group. This leads to the integral formula for the wave function of the open Toda chain. To obtain such a formula for GT-representation, we substitute the expressions (2.9), (2.10), and (2.1a) into (5.2) thus getting

$$\psi_{\gamma_{N1},\dots,\gamma_{NN}} = e^{-\boldsymbol{x}\cdot\boldsymbol{\rho}^{(N)}}$$

$$\times \int\limits_{\mathbb{R}^{\frac{N(N-1)}{2}}} \prod_{n=1}^{N-1} \frac{\prod\limits_{k=1}^{n}\prod\limits_{m=1}^{n+1} \hbar^{\frac{\gamma_{nk}-\gamma_{n+1,m}}{i\hbar}+\frac{1}{2}}\,\Gamma\big(\frac{\gamma_{nk}-\gamma_{n+1,m}}{i\hbar}+\frac{1}{2}\big)}{\prod\limits_{s<p}\big|\Gamma\big(\frac{\gamma_{ns}-\gamma_{np}}{i\hbar}\big)\big|^2}\, e^{\frac{i}{\hbar}\sum\limits_{n,j=1}^{N}(\gamma_{nj}-\gamma_{n-1,j})x_n} \prod_{\substack{n=1\\ j\le n}}^{N-1} d\gamma_{nj}\,. \tag{5.4}$$

In the study of the analytic properties over $\boldsymbol{\gamma}_N$ of this solution, the following reformulation is very usefull:

Proposition 5.1 *An analytical continuation of the eigenfunction (5.4) as a function of* $\boldsymbol{\gamma}_N$ *can be expressed in the form*

$$\psi_{\boldsymbol{\gamma}_N}(x_1,\dots,x_N)$$

$$= \int\limits_{\mathcal{S}} \prod_{n=1}^{N-1} \frac{\prod\limits_{k=1}^{n}\prod\limits_{m=1}^{n+1} \hbar^{\frac{\gamma_{nk}-\gamma_{n+1,m}}{i\hbar}}\,\Gamma\big(\frac{\gamma_{nk}-\gamma_{n+1,m}}{i\hbar}\big)}{\prod\limits_{s\neq p}\Gamma\big(\frac{\gamma_{ns}-\gamma_{np}}{i\hbar}\big)}\, e^{\frac{i}{\hbar}\sum\limits_{n,j=1}^{N}(\gamma_{nj}-\gamma_{n-1,j})x_n} \prod_{\substack{n=1\\ j\le n}}^{N-1} d\gamma_{nj}\,, \tag{5.5}$$

where the domain of integration $\mathcal{S}$ *is defined by conditions* $\min_j\{\mathrm{Im}\,\gamma_{kj}\} > \max_m\{\mathrm{Im}\,\gamma_{k+1,m}\}$ *for all* $k=1,\dots,N-1$. *Recall that we assume that* $\gamma_{nj}=0$ *for* $j>n$. *The integral (5.5) converges absolutely.*

Proof. Let us change the variables of integration in (5.4):

$$\gamma_{nj} \to \gamma_{nj} - \frac{i\hbar}{n}\sum_{s=1}^{n}\rho_s^{(N)}\,; \qquad (n=1,\dots,N-1)\,. \tag{5.6}$$

We recall that $\rho_s^{(N)} = \frac{1}{2}(N-2s+1)$. After elementary calculations, the integral in (5.4) acquires the form (5.5). It is worth mentioning that after the change of variables (5.6) we shift the domain of integration $\mathbb{R}^{\frac{N(N-1)}{2}}$ to the complex plain in such a way that the domain of integration over the variables $\gamma_{n-1,j}$ lies above the domain of integration over the variables γ_{nj}. Thus, we arrive at the analytic continuation of the wave function described in the theorem. ∎

The explicit form of the eigenfunction and absolute convergence of the integral leads to recursive relation between the wave functions corresponding to $\mathfrak{gl}(n)$ and $\mathfrak{gl}(n-1)$; this relation is a direct consequence of the Gelfand-Tsetlin inductive procedure:

Corollary 5.1 *Let us fix the solution $\psi_{\gamma_{11}}(x_1) = e^{\frac{i}{\hbar}x_1\gamma_{11}}$ of the $GL(1,\mathbb{R})$ Toda chain. For any $n=2,\ldots,N$ there is the recursive relation between the wave functions $\psi_{\boldsymbol{\gamma}_n}$ and $\psi_{\boldsymbol{\gamma}_{n-1}}$*

$$\begin{aligned}
&\psi_{\boldsymbol{\gamma}_n}(x_1,\ldots,x_n) \\
&= \int\limits_{\mathcal{S}_n} \mu^{(n-1)}(\boldsymbol{\gamma}_{n-1}) f_n(\boldsymbol{\gamma}_n,\boldsymbol{\gamma}_{n-1})\, e^{-\frac{i}{\hbar}x_n(\sum\limits_{j=1}^{n}\gamma_{nj}-\sum\limits_{k=1}^{n-1}\gamma_{n-1,k})} \psi_{\boldsymbol{\gamma}_{n-1}}(x_1,\ldots,x_{n-1}) \prod_{j=1}^{n-1} d\gamma_{n-1,j},
\end{aligned} \tag{5.7}$$

where

$$\mu^{(n-1)}(\boldsymbol{\gamma}_{n-1}) = \prod_{s\neq p}\left[\Gamma\Big(\frac{\gamma_{n-1,s}-\gamma_{n-1,p}}{i\hbar}\Big)\right]^{-1}$$

and

$$f_n(\boldsymbol{\gamma}_n,\boldsymbol{\gamma}_{n-1}) = \prod_{k=1}^{n-1}\prod_{m=1}^{n} \hbar^{\frac{\gamma_{n-1,k}-\gamma_{nm}}{i\hbar}}\,\Gamma\Big(\frac{\gamma_{n-1,k}-\gamma_{nm}}{i\hbar}\Big). \tag{5.8}$$

The domain of integration $\mathcal{S}_n$ is defined by conditions $\min_j\{\mathrm{Im}\,\gamma_{n-1,j}\} > \max_m\{\mathrm{Im}\gamma_{nm}\}$.

The above results are in complete agreement with the recursive relations between the wave functions and the explicit solution in terms of Mellin-Barnes representation obtained previously by the QISM-based approach in [20].

5.2 The hyperbolic Sutherland model

In this section we give a short derivation of the representation through integrals for a wave function of the hyperbolic Sutherland model.

The quantum hyperbolic Sutherland model corresponding to $\mathfrak{gl}(N)$ is the integrable quantum mechanical system with N degrees of freedom (see [2] and references therein). It has N mutually commuting Hamiltonians, the first two of which are given by

$$\begin{aligned}
h_1 &= \sum_{n=1}^{N} p_n\,, \\
h_2 &= \sum_{m<n}\Big\{p_n p_m + \frac{\hbar^2/4}{\sinh^2(x_m-x_n)}\Big\}.
\end{aligned} \tag{5.9}$$

The eigenfunction of the Hamiltonians of the quantum hyperbolic Sutherland model is defined by the equations:

$$\begin{gathered}\sum_{n=1}^{N} p_n \, \Psi = \sigma_1(\boldsymbol{\gamma}_{N})\Psi\,, \\ \sum_{m<n} \Big\{ p_n p_m + \frac{\hbar^2/4}{\sinh^2(x_m - x_n)} \Big\} \Psi = \sigma_2(\boldsymbol{\gamma}_{N})\Psi\,. \end{gathered} \tag{5.10}$$

Similarly to the open Toda chain, the hyperbolic Sutherland model could be explicitly solved using the Representation Theory approach. Let V be a representation of $\mathcal{U}(\mathfrak{gl}(N,\mathbb{R}))$ and v_N is a vector in V such that

$$(E_{n,n+1} - E_{n+1,n})v_N = 0\,, \qquad (n = 1, \dots , N-1). \tag{5.11}$$

Consider the matrix element $\Phi(x_1, \dots , x_N)$ of the form

$$\Phi = \langle v_N, e^{-\sum\limits_{k=1}^{N} x_k E_{kk}} v_N \rangle\,. \tag{5.12}$$

One could show that (5.12) gives (up to a simple factor) the eigenfunction of the Hamiltonian of the hyperbolic Sutherland model. To derive this let us consider the more general matrix elements:

$$\Phi_r = \langle v_N, e^{-\sum\limits_{k=1}^{N} x_k E_{kk}} a_{Nr} v_N \rangle\,, \tag{5.13}$$

where a_{Nr} are the coefficients of the expansion of $\mathcal{A}_N(\lambda)$ (see (2.17)): $\mathcal{A}_N(\lambda) = \lambda^N - i\hbar\lambda^{N-1}a_{N1} + \lambda^{N-2}a_{N2} + \dots$. The first two have the form:

$$\begin{gathered} a_{N1} = -i\hbar \sum_{n=1}^{N} E_{nn}\,, \\ a_{N2} = -\hbar^2 \Big\{ \sum_{m<n} E_{mm}E_{nn} - \sum_{n=1}^{N} \rho_n^{(N)} E_{nn} - \sum_{m<n} E_{nm}E_{mn} + \sigma_2(\boldsymbol{\rho}^{(N)}) \Big\}. \end{gathered} \tag{5.14}$$

This should be compared with the discussion of the open Toda chain (5.3). The same line of reasoning shows that the matrix element (5.12) satisfies the differential equations

$$\begin{gathered} i\hbar \sum_{n=1}^{N} \partial_{x_n} \Phi = \sigma_1(\boldsymbol{\gamma}_N)\Phi\,, \\ -\hbar^2 \sum_{m<n} \Big\{ \partial_{x_m}\partial_{x_n} - \tfrac{1}{2}\coth(x_m - x_n)(\partial_{x_m} - \partial_{x_n}) \Big\} \Phi = \Big(\sigma_2(\boldsymbol{\gamma}_N) + \hbar^2 \sigma_2(\boldsymbol{\rho}^{(N)}) \Big) \Phi\,. \end{gathered} \tag{5.15}$$

Finally, the wave function Ψ defined by

$$\Psi := \Phi \prod_{j<k} \sinh^{1/2}(x_j - x_k) \tag{5.16}$$

satisfies the equations (5.10).

Now we give explicit integral formula for the eigenfunction of the hyperbolic Sutherland model using the GT representation discussed in section 2. First, let us find the vector v_N satisfying (5.11).

Lemma 5.1 *Let*

$$\phi_n(\boldsymbol{\gamma}_n,\boldsymbol{\gamma}_{n+1})=\prod_{k=1}^{n}\prod_{m=1}^{n+1}\hbar^{\frac{\gamma_{nk}-\gamma_{n+1,m}}{2i\hbar}+\frac{1}{4}}\,\Gamma\Big(\frac{\gamma_{nk}-\gamma_{n+1,m}}{2i\hbar}+\frac{1}{4}\Big). \tag{5.17}$$

Then the vector

$$v_N=e^{-\frac{\pi}{2\hbar}\sum\limits_{n=1}^{N-1}(n-1)\sum\limits_{j=1}^{n}\gamma_{nj}}\prod_{n=1}^{N-1}\phi_n(\boldsymbol{\gamma}_n,\boldsymbol{\gamma}_{n+1}) \tag{5.18}$$

satisfies the equations (5.11).

Proof. Straightforward check. ∎

Now we have the following

Proposition 5.2 *The solution to (5.10) admits the integral representation:*

$$\begin{aligned}\Psi_{\gamma_{N1},\dots,\gamma_{NN}}(x_1,\dots,x_N)\;&=\prod_{j<k}\sinh^{1/2}(x_j-x_k)\\ \times\int\limits_{\mathbb{R}^{\frac{N(N-1)}{2}}}\prod_{n=1}^{N-1}&\frac{\prod\limits_{k=1}^{n}\prod\limits_{m=1}^{n+1}\Big|\Gamma\big(\frac{\gamma_{nk}-\gamma_{n+1,m}}{2i\hbar}+\frac{1}{4}\big)\Big|^2}{\prod\limits_{s<p}\Big|\Gamma\big(\frac{\gamma_{ns}-\gamma_{np}}{i\hbar}\big)\Big|^2}\,e^{\frac{i}{\hbar}\sum\limits_{n,j=1}^{N}(\gamma_{nj}-\gamma_{n-1,j})x_n}\prod_{\substack{n=1\\ j\leq n}}^{N-1}d\gamma_{nj}.\end{aligned} \tag{5.19}$$

Proof. The only non obvious thing is a convergence of the integral. The integral converges absolutely due to the inequality (2.23). ∎

6 Connection with QISM-based approach

In section 5 we have outlined the derivation of the integral representation for the wave function of open Toda chain based on the GT representation of $\mathcal{U}(\mathfrak{gl}(N,\mathbb{R}))$. As was stressed in section 4, this representation has a direct relation to $Y(\mathfrak{gl}(N,\mathbb{R}))$. This provides us with another point of view on the way of obtaining the wave functions for the the open Toda chain. In particular, the algebra generated by the images of the polynomials $\mathcal{A}_n(\lambda)$, $\mathcal{B}_n(\lambda)$, $\mathcal{C}_n(\lambda)$ under homomorphism $\pi_N: Y(\mathfrak{gl}(N))\to\mathcal{U}(\mathfrak{gl}(N))$ could be successfully used to derive directly the recursive relations for the wave functions.

We begin with a short summary of QISM-based approach to $GL(N,\mathbb{R})$ Toda chain due to Sklyanin [18]. The model is described with the help of auxiliary 2×2 matrix Lax operator with spectral parameter:

$$L_n(\lambda) = \begin{pmatrix} \lambda - p_n & e^{-x_n} \\ -e^{x_n} & 0 \end{pmatrix}. \tag{6.1}$$

The operator (6.1) satisfies the usual quadratic relation

$$R_2(\lambda-\mu)(L_n(\lambda))\otimes I)(I\otimes L_n(\mu)) = (I\otimes L_n(\mu))(L_n(\lambda)\otimes I)R_2(\lambda-\mu) \tag{6.2}$$

with the rational 4×4 R-matrix (see (4.2)). The monodromy matrix

$$T_n(\lambda) = L_n(\lambda)\dots L_1(\lambda) \equiv \begin{pmatrix} A_n(\lambda) & B_n(\lambda) \\ C_n(\lambda) & D_n(\lambda) \end{pmatrix} \tag{6.3}$$

also satisfies the relation:

$$R_2(\lambda-\mu)(T_n(\lambda)\otimes I)(I\otimes T_n(\mu)) = (I\otimes T_n(\mu))(T_n(\lambda)\otimes I)R_2(\lambda-\mu). \tag{6.4}$$

In particular, the following commutation relations hold:

$$\begin{gathered}[A_n(\lambda), A_n(\mu)] = [B_n(\lambda), B_n(\mu)] = [C_n(\lambda), C_n(\mu)] = 0, \\ (\lambda-\mu+i\hbar)A_n(\mu)C_n(\lambda) = (\lambda-\mu)C_n(\lambda)A_n(\mu) + i\hbar A_n(\lambda)C_n(\mu)\,, \\ (\lambda-\mu+i\hbar)A_n(\lambda)B_n(\mu) = (\lambda-\mu)B_n(\mu)A_n(\lambda) + i\hbar A_n(\mu)B_n(\lambda). \end{gathered} \tag{6.5}$$

The polynomial $A_n(\lambda)$ is a generating function for the quantum Hamiltonians of $GL(n,\mathbb{R})$ Toda chain and the wave functions $\psi_{\boldsymbol{\gamma}_n}(x_1,\dots,x_n)$, $n=1,\dots,N$ are defined as solutions of the differential equations

$$A_n(\lambda)\psi_{\gamma_{n1},\dots,\gamma_{nn}} = \prod_{j=1}^{n}(\lambda-\gamma_{nj})\ \psi_{\gamma_{n1},\dots,\gamma_{nn}}\,. \tag{6.6}$$

The idea of the QISM approach is to solve iteratively these equations for $\psi_{\boldsymbol{\gamma}_n}$, $n=1,\dots,N$ using the additional operators $B_n(\lambda)$ and $C_n(\lambda)$. ¿From the results of [20] the following expressions for the operators $A_n(\lambda)$, $B_n(\lambda)$, and $C_n(\lambda)$ in $\boldsymbol{\gamma}$-representation can be obtained:

$$\begin{gathered} A_n(\lambda) = \prod_{j=1}^{n}(\lambda-\gamma_{nj})\,, \\ B_n(\lambda) = i^{n-1}\sum_{j=1}^{n}\prod_{s\neq j}\frac{\lambda-\gamma_{ns}}{\gamma_{nj}-\gamma_{ns}}\ e^{-i\hbar\partial_{\gamma_{nj}}}\,, \\ C_n(\lambda) = i^{-n-1}\sum_{j=1}^{n}\prod_{s\neq j}\frac{\lambda-\gamma_{ns}}{\gamma_{nj}-\gamma_{ns}}\ e^{i\hbar\partial_{\gamma_{nj}}}\,. \end{gathered} \tag{6.7}$$

The simple check shows that thus defined operators (6.7) satisfy the relations (6.5). Hence, one establishes the connection between the generators $\mathcal{A}_n(\lambda)$, $\mathcal{B}_n(\lambda)$, and $\mathcal{C}_n(\lambda)$ entering the description of the GT representation (4.12) and the corresponding QISM generators (6.7):

$$\begin{aligned} A_n(\lambda) &= \mathcal{A}_n(\lambda)\,, \\ B_n(\lambda) &= g_n^{-1}\circ \mathcal{B}_n(\lambda)\circ g_n\,, \\ C_n(\lambda) &= g_{n-1}^{-1}\circ \mathcal{C}_n(\lambda)\circ g_{n-1}\,, \end{aligned} \tag{6.8}$$

where g_n is defined by (2.10).

At the end of the section we illustrate the connection between the differential operator $A_n(\lambda)$ entering in (6.3) and generator $\mathcal{A}_n(\lambda)$ defined by (2.17). First, let us note that (6.1), (6.3) define the relations between the operators $A_n(\lambda)$ for different levels:

$$A_n(\lambda) = (\lambda - p_n)A_{n-1}(\lambda) - e^{x_{n-1}-x_n}A_{n-2}(\lambda)\,, \tag{6.9}$$

where $p_n = -i\hbar\partial_{x_n}$.

Lemma 6.1 *The operators* $A_n(\lambda)$ *and* $\mathcal{A}_n(\lambda)$ *are related as*

$$A_n(\lambda)\psi_{\gamma_{N1},\dots,\gamma_{NN}} = e^{-\boldsymbol{x}\cdot\boldsymbol{\rho}^{(N)}}\langle w'_N, e^{-\sum\limits_{k=1}^{N} x_kE_{kk}}\mathcal{A}_n(\lambda - \tfrac{i(N-n)\hbar}{2})w_N\rangle. \tag{6.10}$$

Proof. We prove the statement by showing that the operators $A_n(\lambda)$ defined by (6.10) satisfy the recursive relations (6.9). Consider the relation (2.19). The generators E_{nn} and $E_{n,n-k+1}$ commute with $\mathcal{A}_{n-1}(\lambda)$ and $\mathcal{A}_{n-k}(\lambda)$ respectively. Therefore, due to (2.27) and (2.7)

$$\begin{aligned} &\langle w'_N, e^{-\sum\limits_{k=1}^{N} x_kE_{kk}}\mathcal{A}_n(\lambda - \tfrac{i(N-n)\hbar}{2})w_N\rangle \\ = \ &(\lambda - i\hbar\rho_n^{(N)} + i\hbar\partial_{x_n})\langle w'_N, e^{-\sum\limits_{k=1}^{N} x_kE_{kk}}\mathcal{A}_{n-1}(\lambda - \tfrac{i(N-n+1)\hbar}{2})w_N\rangle \\ &- e^{x_{n-1}-x_n}\langle w'_N, e^{-\sum\limits_{k=1}^{N} x_kE_{kk}}\mathcal{A}_{n-2}(\lambda - \tfrac{i(N-n+2)\hbar}{2})w_N\rangle. \end{aligned} \tag{6.11}$$

∎

We leave a more systematic discussion of the interrelation between two sets of operators $\mathcal{A}_n(\lambda)\,,\mathcal{B}_n(\lambda)\,,\mathcal{C}_n(\lambda)$ and $A_n(\lambda)\,,B_n(\lambda)\,,C_n(\lambda)$ for the future publication.

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