STRICT POLYNOMIAL FUNCTORS AND COHERENT FUNCTORS

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Abstract. We build an explicit link between coherent functors in the sense of Auslander [1] and strict polynomial functors in the sense of Friedlander and Suslin [6].

1. Introduction

Since the foundational work of Schur in his thesis, the representation theory of general linear groups has been closely related to the representation theory of symmetric groups. Especially fruitful has been the study of tensor products $T^n(V) := V^{\otimes n}$ of a vector space V, endowed with the commuting actions of the general linear group $GL(V)$ (diagonally on each factor by linear substitution) and of the symmetric group $\mathfrak{S}_n$ (by permutation of the factors). For many purposes, the mysterious group ring of the general linear group can thus be replaced by the more manageable Schur algebra of $\mathfrak{S}_n$-equivariant linear maps of $V^{\otimes n}$.

The use of functors in representation theory, maybe first promoted by Auslander, is practical and efficient for formalizing the relations between symmetric groups and general linear groups. The classical work of Green [7] on representations of the Schur algebras pushes these ideas quite far and in great generality. In the example of interest to us, Green associates to every additive functor $\mathbf{f}$, defined on representations of the symmetric group, the representation over the Schur algebra given by $\mathbf{f}(V^{\otimes n})$. The main problem of constructing reverse correspondances is solved naturally by Green. It is one of the purpose of this paper to shed new light on these correspondances.

A few years later, Friedlander and Suslin [6] introduced strict polynomial functors, which are equivalent to representations of the Schur algebra when the dimension of V is at least n. This new formalization is aimed at cohomology computations, and has numerous applications, including a proof of finite generation of the cohomology of finite group schemes in the same paper [6]. The pleasing properties of the category $\mathscr{P}$ of strict polynomial functors lead to impressive cohomological computations in the difficult case of finite fields. These computations follow from the fundamental computation of $\mathrm{Ext}_{\mathscr{P}}(\mathsf{Id}^{(i)}, \mathsf{Id}^{(i)})$ given in [6] (the decoration (i) indicates Frobenius twists, that is base change along the Frobenius isomorphism). In a recent paper [2], Chałupnik proves elegant formulae computing functor cohomology, in the case of finite fields, for various fundamental functors of the form

$$V \mapsto \mathbf{f}(V^{\otimes n}).$$

He succeeds in comparing $\mathbf{f}(\mathrm{Ext}_{\mathscr{P}}(G, T^{n(i)}))$ and $\mathrm{Ext}_{\mathscr{P}}(G, \mathbf{f}(T^{n(i)}))$ for many important families of functors G and $\mathbf{f}$. [When the functor $\mathbf{f}$ is given by an idempotent in the group ring of the symmetric group, the two terms are easily seen to be isomorphic. However, this is very rarely the case in positive characteristic smaller than n.] To this end, Chałupnik considers certain adequate choices of functors $\mathbf{f}$, defined on representations of the symmetric group $\mathfrak{S}_n$, such that $\mathbf{f}(V^{\otimes n}) = F(V)$ for a given

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strict polynomial functor F, thus informally rediscovering the correspondances set up by Green. These methods motivated and inspired the present work.

Green's correspondances are best expressed in terms of adjoint functors and recollements of categories. The category $\mathscr{P}$ thus appears as a quotient category of the category of all additive functors defined on representations of the symmetric groups. Unfortunately, the latter is very large and stays quite mysterious. Representable functors, such as the functor $\mathrm{H}^0(\mathfrak{S}_n, -)$ taking invariants, are examples of functors obtained through the reverse correspondances of Green, but they are many more. We show however that one can restrict to considering *coherent functors*, that is functors which are presented by representable functors. The resulting category, if still very rich, is much better behaved. For instance, the global dimension of the category of coherent functors is two. This comes in sharp contrast with the rich functor cohomology obtained through homological algebra in the category of strict polynomial functors.

We revisit in this setting some of the properties of functors which make the category $\mathscr{P}$ much more tractable than coarse representations: tensor product, composition (or plethysm), linearization etc. and we try and find corresponding constructions for coherent functors. We also apply our insight to functor cohomology, and obtain Chałupnik's constructions in a natural way.

In Section 2, we give a new presentation of strict polynomial functors adapted to our purpose. Section 3 develops the general properties of coherent functors. Although we do not claim much originality, it contains a few results which we could not find in the literature. Section 4 contains our main results. It compares coherent functors and strict polynomial functors. Since the comparison is best stated in terms of recollements of abelian categories, we recall in an appendix A what is needed from this theory. Section 5 applies this new setting to functor cohomology and obtains natural versions of Chałupnik's results. Section 6 lifts the tensor product of polynomial functors to the level of coherent functors, and the final section 7 does the same for the composition of functors.

Notations. We fix a field $\mathbb{K}$ of positive characteristic p. All vector spaces are considered over $\mathbb{K}$, and Hom and $\otimes$ are taken over $\mathbb{K}$, unless otherwise decorated. Let $\mathscr{V}$ be the category of finite-dimensional vector spaces. For a finite group G, we let ${}_G\mathscr{V}$ denote the category of finite dimensional G-modules.

2. The category of strict polynomial functors

Strict polynomial functors were introduced by Friedlander and Suslin [6, §3]. To explain what they are in a given degree, fix a positive integer n. We start with the n-th divided power of a vector space V, defined by:

$$\Gamma^n(V) := \mathrm{H}^0(\mathfrak{S}_n, V^{\otimes n}) = (V^{\otimes n})^{\mathfrak{S}_n}$$

where the symmetric group on n-letters $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the factors. For any x in some vector space X, we let $\gamma(x)$ be the element $x^{\otimes n}$ in $\Gamma^n(X)$. This defines a natural set map $\gamma_X : X \to \Gamma^n(X)$. Reordering the factors

$$A^{\otimes n} \otimes B^{\otimes n} \to (A \otimes B)^{\otimes n}$$

induces a $\mathbb{K}$-linear natural transformation

$$\Gamma^n(A) \otimes \Gamma^n(B) \to \Gamma^n(A \otimes B)$$

sending $\gamma_A(a) \otimes \gamma_B(b)$ to $\gamma_{A\otimes B}(a \otimes b)$. Together with the composition law in $\mathscr{V}$, these maps define a composition map:

$$\Gamma^n(\mathrm{Hom}(V,W)) \otimes \Gamma^n(\mathrm{Hom}(U,V)) \to \Gamma^n(\mathrm{Hom}(V,W) \otimes \mathrm{Hom}(U,V)) \to \Gamma^n(\mathrm{Hom}(U,W)).$$

This defines a category $\Gamma^n\mathscr{V}$, with the same objects as $\mathscr{V}$, and with morphisms

$$\mathrm{Hom}_{\Gamma^n\mathscr{V}}(V,W) := \Gamma^n(\mathrm{Hom}(V,W)).$$

The following Lemma describes the category $\Gamma^n\mathscr{V}$ as a full subcategory of ${}_{\mathfrak{S}_n}\mathscr{V}$.

Lemma 2.1. *For a positive integer n, the functor*

$$\begin{aligned} \mathbf{i} : \Gamma^n\mathscr{V} &\to {}_{\mathfrak{S}_n}\mathscr{V} \\ V &\mapsto V^{\otimes n} \end{aligned}$$

is a full embedding.

Proof. This follows from the natural isomorphism:

$$\mathrm{Hom}_{\mathfrak{S}_n}(V^{\otimes n}, W^{\otimes n}) = (\mathrm{Hom}(V^{\otimes n}, W^{\otimes n}))^{\mathfrak{S}_n} \cong (\mathrm{Hom}(V,W)^{\otimes n})^{\mathfrak{S}_n} = \mathrm{Hom}_{\Gamma^n\mathscr{V}}(V,W).$$

□

According to [6], a *homogeneous strict polynomial functor of degree n* defined on $\mathscr{V}$ is a $\mathbb{K}$-linear functor $\Gamma^n\mathscr{V} \to \mathscr{V}$. We let $\mathscr{P}_n$ be the category of homogeneous strict polynomial functors of degree n. It is known that the category $\mathscr{P}_n$ is equivalent to the category of finite dimensional modules over the Schur algebra $S(n,n)$ [6, §3].

The collection of maps $\gamma_X : X \to \Gamma^n(X)$ yields a (nonlinear) functor $\gamma : \mathscr{V} \to \Gamma^n\mathscr{V}$. Precomposition with γ associates to any strict polynomial functor defined on $\mathscr{V}$ an usual functor on $\mathscr{V}$; it is called the underlying functor of the strict polynomial functor. It is usual to denote by the same letter a strict polynomial functor and its underlying functor. For example, the composite

$$\Gamma^n\mathscr{V} \xrightarrow{\mathbf{i}} {}_{\mathfrak{S}_n}\mathscr{V} \xrightarrow{H^0(\mathfrak{S}_n,-)} \mathscr{V}$$

is denoted by Γ^n, since its underlying functor is the n-th divided power functor, and Sym^n denotes the composite

$$\Gamma^n\mathscr{V} \xrightarrow{\mathbf{i}} {}_{\mathfrak{S}_n}\mathscr{V} \xrightarrow{H_0(\mathfrak{S}_n,-)} \mathscr{V}\ ,$$

because its underlying functor is the n-th symmetric power. Similarly the composite

$$\Gamma^n\mathscr{V} \xrightarrow{\mathbf{i}} {}_{\mathfrak{S}_n}\mathscr{V} \xrightarrow{\text{forget}} \mathscr{V}$$

is denoted by T^n, because the underlying functor is the n-th tensor power. We now recall from [6, §3] the basic properties of the category $\mathscr{P}_n$.

There is a well-defined tensor product of strict polynomial functors which corresponds to the usual tensor product of underlying functors, and it yields a bifunctor

$$-\otimes - : \mathscr{P}_n \times \mathscr{P}_m \to \mathscr{P}_{n+m}.$$

For example: $T^n = T^1 \otimes \cdots \otimes T^1$ (n-factors).

There is also a duality in $\mathscr{P}_n$. For an object F in $\mathscr{P}_n$, we let $\mathbb{D}F$ be the homogeneous strict polynomial functor given by

$$(\mathbb{D}F)(V) = (F(V^\vee))^\vee$$

where $W^\vee$ denotes the dual vector space of W. Since the values of any homogeneous strict polynomial functor are finite dimensional, $\mathbb{D}$ is an involution and defines an equivalence of categories $\mathbb{D} : \mathscr{P}_n^{op} \to \mathscr{P}_n$. The functor $\mathbb{D}F$ is called the *dual* of F.

The category $\mathscr{P}_n$ has enough projective and injective objects. A set of generators is indexed by partitions of n, that is decreasing sequences of positive integers adding up to n. For a partition $\lambda = (n_1 \geq n_2 \geq \cdots \geq n_k)$, put

$$\Gamma^\lambda := \Gamma^{n_1} \otimes \cdots \otimes \Gamma^{n_k}.$$

The functors Γ^λ, when λ runs through all partitions of the integer n, are small projective generators. Indeed, $\mathrm{Hom}_{\mathscr{P}_n}(\Gamma^\lambda, F)$ is the evaluation on the base field of the cross-effect of the functor F of homogeneous multidegree λ. Dually, the functors

$$Sym^\lambda := Sym^{n_1} \otimes \cdots \otimes Sym^{n_k}$$

form a set of injective cogenerators. In particular, the functor T^n is projective and injective in $\mathscr{P}_n$. Moreover, the action of $\mathfrak{S}_n$ by permuting factors yields an exact functor

$$\begin{aligned} c^* : \mathscr{P}_n &\to {}_{\mathfrak{S}_n}\mathscr{V} \\ F &\mapsto \mathrm{Hom}_{\mathscr{P}_n}(T^n, F). \end{aligned}$$

The representation $c^*(F)$ is often called the linearization of the functor F; we use the letter c for cross-effect. The functor c^* has both a left and a right adjoint functor given respectively by

$$(c_!(M))(V) = (M \otimes V^{\otimes n})_{\mathfrak{S}_n} \ , \quad c_*(M) = (M \otimes V^{\otimes n})^{\mathfrak{S}_n}.$$

Let $\mathscr{P}_n^0$ be the full subcategory of $\mathscr{P}_n$ whose objects are the strict polynomial functors F such that $c^*(F) = 0$. This condition means that the underlying functor has degree less than n in the additive sense of Eilenberg and MacLane. Let $d_* : \mathscr{P}_n^0 \to \mathscr{P}_n$ be the inclusion and let d^* and $d^!$ be the left and right adjoint of d_*. By Proposition A.2, this defines a recollement situation:

$$\begin{tikzcd} {\mathscr{P}_n}^0 \arrow[r, "d_*"] & \mathscr{P}_n \arrow[l, bend right, "d^*"'] \arrow[l, bend left, "d^!"] \arrow[r, "c^*"] & {}_{\mathfrak{S}_n}\mathscr{V} \arrow[l, bend right, "c_!"'] \arrow[l, bend left, "c_*"] \end{tikzcd} .$$

3. Coherent functors

A good reference for this section is a recent survey of Harsthorne [8]. All the results in this section are well-known to experts but some of them (e. g. Proposition 3.8 and Proposition 3.10) are not easy to find in the literature.

In this section we fix a finite group G. Let ${}_G\mathscr{V}$ be the category of all finite dimensional G-modules, and let $\mathscr{A}(G)$ be the category of all covariant $\mathbb{K}$-linear functors from ${}_G\mathscr{V}$ to the category $\mathscr{V}$ of finite dimensional vector spaces. The following functors in $\mathscr{A}(G)$ are of special interest. For any M in ${}_G\mathscr{V}$, let $\mathbf{t}_M$ be the functor

$$\mathbf{t}_M = (-) \otimes_G M,$$

and let $\mathbf{h}_M$ be the functor represented by M:

$$\mathbf{h}_M = \mathrm{Hom}_G(M, -).$$

We shall use that $\mathbf{h}_M(\mathbb{K}[G])$ is isomorphic to the $\mathbb{K}$-dual $M^\vee$. This isomorphism is precisely defined as follows. Let τ be the element of $\mathbb{K}[G]^\vee$ given by

$$\tau(g) = 0, \quad \text{if} \quad g \neq 1 \quad \text{and} \quad \tau(1) = 1.$$

By the Yoneda lemma, the function τ yields a natural morphism

$$\begin{aligned} \tau_X : \mathrm{Hom}_G(X, \mathbb{K}[G]) &\to X^\vee \\ f &\mapsto \tau \circ f. \end{aligned}$$

The homomorphism τ_X is an isomorphism when $X = \mathbb{K}[G]$. Since $\mathbb{K}[G]$ is a self-injective algebra, $\mathrm{Hom}_G(-, \mathbb{K}[G])$ is an exact functor, and τ is a natural transformation between exact functors. It results that τ_X is an isomorphism for all X in ${}_G\mathscr{V}$. We shall use this fact without further reference.

The category $\mathscr{A}(G)$ is an abelian category. We state below its elementary properties.

Lemma 3.1. *For all object* $\mathbf{f}$ *of the category* $\mathscr{A}(G)$*, let* $\mathbb{D}(\mathbf{f})$ *be the object of* $\mathscr{A}(G)$ *defined by*

$$(\mathbb{D}\mathbf{f})(M) = (\mathbf{f}(M^\vee))^\vee.$$

The resulting functor $\mathbb{D}$ *is a duality in* $\mathscr{A}(G)$*.*

Lemma 3.2. *For any* M *in* ${}_G\mathscr{V}$*, the Yoneda lemma yields a natural isomorphism*

$$\mathrm{Hom}_{\mathscr{A}(G)}(\mathbf{h}_M, \mathbf{f}) \cong \mathbf{f}(M).$$

Thus the functor $\mathbf{h}_M$ *is a projective object in the category* $\mathscr{A}(G)$*. Moreover, for all* M, N *in* ${}_G\mathscr{V}$*, there is a natural isomorphism:*

$$\mathrm{Hom}_{\mathscr{A}(G)}(\mathbf{h}_M, \mathbf{h}_N) \cong \mathrm{Hom}_G(N, M).$$

Lemma 3.3. *For all* M *in* ${}_G\mathscr{V}$*, there is a natural isomorphism:* $\mathbb{D}(\mathbf{h}_M) \cong \mathbf{t}_M$*. Hence the functor* $\mathbf{t}_M$ *is an injective object in the category* $\mathscr{A}(G)$*. Moreover, for all* $\mathbf{f}$ *in* $\mathscr{A}(G)$*, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathscr{A}(G)}(\mathbf{f}, \mathbf{t}_M) \cong \mathbb{D}\mathbf{f}(M).$$

In particular, there is a natural isomorphism:

$$\mathrm{Hom}_{\mathscr{C}(G)}(\mathbf{t}_N, \mathbf{t}_M) = \mathrm{Hom}_G(N, M).$$

Proof. We have a canonical element θ_M in

$$(M^\vee \otimes_G M)^\vee \cong \mathrm{Hom}_G(\mathrm{Hom}_G(M, \mathbb{K}[G]) \otimes_G M, \mathbb{K}[G])$$

which is given by: $\theta_M(\xi \otimes m) = \xi(m)$. By the Yoneda lemma, it yields a natural transformation $\theta : \mathbf{h}_M \to \mathbb{D}(\mathbf{t}_M)$. For X in $\mathscr{V}$,

$$\begin{aligned} \theta_X : \mathrm{Hom}_G(M, X) \to& (X^\vee \otimes_G M)^\vee \\ \alpha \mapsto& \{\xi \otimes m \mapsto \xi(\alpha(m))\}. \end{aligned}$$

Since $\theta_{\mathbb{K}[G]}$ is an isomorphism and both $\mathbf{h}_M$ and $\mathbb{D}(\mathbf{t}_M)$ are left exact functors, it follows that θ is an isomorphism. The rest follows because $\mathbb{D}$ is a duality. □

An object $\mathbf{f}$ in $\mathscr{A}(G)$ is *finitely generated* if it is a quotient of $\mathbf{h}_M$ for some M in ${}_G\mathscr{V}$. Among finitely generated functors, coherent functors are defined by further requiring finiteness of the relations.

Definition 3.4. [8] An object $\mathbf{f}$ in $\mathscr{A}(G)$ is called *coherent* if it fits in an exact sequence

$$\mathbf{h}_N \to \mathbf{h}_M \to \mathbf{f} \to 0 \tag{1}$$

for some M, N in ${}_G\mathscr{V}$. We let $\mathscr{C}(G)$ be the category of all coherent functors and natural transformations between them.

Remark 3.5. The category $\mathscr{C}(G)$ has the following alternative description. Objects are all arrows of ${}_G\mathscr{V}$, while a morphism from $\alpha : M \to N$ to $\alpha' : M' \to N'$ is an equivalence classe of commutative diagrams

$$\begin{tikzcd} M' \arrow[r, "\alpha'"] \arrow[d, "\beta"'] & N' \arrow[d, "\beta'"] \\ M \arrow[r, "\alpha"'] & N \end{tikzcd} ,$$

such a commutative diagram being equivalent to the zero morphism if there exists $\delta : N' \to M$ such that $\delta\alpha' = \beta$. However we will use only the previous description of $\mathscr{C}(G)$.

It is a classical fact due to Auslander [1] that *the kernel and the cokernel of any morphism of coherent functors are still coherent functors.* As a consequence, the category $\mathscr{C}(G)$ is an abelian category.

We now give examples of coherent functors.

Proposition 3.6. (i) *For any M in ${}_G\mathscr{V}$, the functor $\mathbf{h}_M$ is coherent;*
(ii) *For any M in ${}_G\mathscr{V}$, the functor $\mathbf{t}_M$ is coherent;*
(iii) *If $\mathbf{f}$ is a coherent functor, then $\mathbb{D}\mathbf{f}$ is also a coherent functor;*
(iv) *For any integer $i \geq 0$, the homology and cohomology functors $H^i(G,-)$, $H_i(G,-)$ are coherent on ${}_G\mathscr{V}$;*
(v) *For any integer $i \geq 0$, the Tate homology and cohomology functors $\hat{H}^i(G,-)$ and $\hat{H}_i(G,-)$ are coherent on ${}_G\mathscr{V}$.*

Proof. Examples (i) to (iv) are also in [8, §2].
(i) Take $N = 0$ in the definition of coherent functors.
(ii) If M is free and finite dimensional, then $\mathbf{h}_M \cong \mathbf{t}_M$. For a general M, choose a presentation $K \to N \to M \to 0$ in the category ${}_F\mathscr{V}$, with free and finite dimensional K and N. Then $\mathbf{t}_M$ is a cokernel of $\mathbf{t}_K \to \mathbf{t}_M$ and hence it is coherent.
(iii) Assume $\mathbf{f}$ is a coherent functor. By definition, it is a cokernel of a morphism $\mathbf{h}_N \to \mathbf{h}_M$. Then $\mathbb{D}\mathbf{f}$ is the kernel of the dual morphism $\mathbf{t}_M \to \mathbf{t}_N$, hence it is also a coherent functor.
(iv) Since
$$H^0(G,-) = \mathbf{h}_{\mathbb{K}} \text{ and } H_0(G,-) = \mathbf{t}_{\mathbb{K}},$$
they are coherent. For a general i, choose a projective resolution P_* of $\mathbb{K}$ with finite dimensional P_i, $i \geq 0$. Then $H^i(G,-)$ is the i-th homology of the cochain complex $\mathbf{h}_{P_*}$ of coherent functors, therefore it is also a coherent functor.
(v) The functors $\hat{H}^0(G,-)$ and $\hat{H}^{-1}(G,-)$ are respectively the cokernel and the kernel of the norm homomorphism $H_0(G,-) \to H^0(G,-)$, so they are also coherent. We then proceed as for (iv).

□

Proposition 3.7. *The category $\mathscr{C}(G)$ has enough projective and injective objects. The category $\mathscr{C}(G)$ is semi-simple if, and only if, the order of G is invertible in K. In this case any coherent functor is of the form $\mathbf{h}_M$ and therefore the category $\mathscr{C}(G)$ is equivalent to ${}_G\mathscr{V}$. Otherwise the global dimension of $\mathscr{C}(G)$ is exactly two.*

Proof. Since $\mathbf{h}_M$ (resp. $\mathbf{t}_M$) is for any M in ${}_G\mathscr{V}$ a projective (resp. injective) object in $\mathscr{A}(G)$, it is also projective (resp. injective) in $\mathscr{C}(G)$.

By definition, any coherent functor $\mathbf{f}$ is a cokernel of a morphism $\mathbf{h}_N \to \mathbf{h}_M$. By the Yoneda lemma, this morphism is of the form $\mathbf{h}_\alpha$ for a uniquely defined $\alpha : M \to N$. It follows that there is an exact sequence

$$0 \to \mathbf{h}_{\mathrm{Coker}\,\alpha} \to \mathbf{h}_N \to \mathbf{h}_M \to \mathbf{f} \to 0 \tag{2}$$

This proves that $\mathscr{C}$ has enough projective objects and that $\mathsf{gl.dim.}\,\mathscr{C}(G) \leq 2$. Since $\mathbb{D}$ is a duality, we see that $\mathscr{C}$ has enough injective objects as well.

It remains to show that, if $\mathsf{gl.dim.}\,\mathscr{C}(G) \leq 1$, then the order of G is invertible in $\mathbb{K}$. To this end let us consider the augmentation ideal $I(G)$ of the group algebra $\mathbb{K}[G]$. There is an exact sequence in $\mathscr{C}$:
$$0 \to \mathbf{h}_{\mathbb{K}} \to \mathbf{h}_{\mathbb{K}[G]} \to \mathbf{h}_{I[G]} \to H^1(G,-) \to 0.$$
When $\mathsf{gl.dim}\,(\mathscr{C}(G)) \leq 1$, it follows that $\mathbf{h}_{\mathbb{K}}$ is a direct summand of $\mathbf{h}_{\mathbb{K}[G]}$. Thus $\mathbb{K}$, as a G-module, is a direct summand of $\mathbb{K}[G]$, hence it is a projective G-module. The fact that this holds if and only if the order of G is invertible in $\mathbb{K}$ is well-known. □

Proposition 3.8. *Let $\mathbf{f}$ be a coherent functor.*
(i) *The following are equivalent:*
(a) $\mathbf{f}$ *is projective;*
(b) *The functor* $\mathbf{f}$ *is left exact;*

(c) $\mathbf{f}$ *is of the form* $\mathbf{h}_M$ *for some* M.

(ii) *The following are equivalent:*

(a) $\mathbf{f}$ *is injective;*

(b) *The functor* $\mathbf{f}$ *is right exact;*

(c) $\mathbf{f}$ *is of the form* $\mathbf{t}_M$ *for some* M.

(iii) $pd(\mathbf{f}) \leq 1$ *if, and only if,* $\mathbf{f}$ *respect monomorphisms.*

(iv) $id(\mathbf{f}) \leq 1$ *if, and only if,* $\mathbf{f}$ *respect epimorphisms.*

Proof. We prove only (ii) and (iii). The rest follows by duality. Statement (i) is also proved in [8, Proposition 3.12 & 4.9].

If $\mathbf{f}$ is an injective object, then it is a direct summand of $\mathbf{t}_M$ for some M. The corresponding projector of $\mathbf{t}_M$ has the form $\mathbf{t}_\alpha$, where α is a projector of M. Thus $\mathbf{f} \cong \mathbf{t}_{\mathsf{Im}(\alpha)}$. In particular any injective object is a right exact functor.

Conversely, assume that $\mathbf{f}$ is right exact. The G-module $M = \mathbf{f}(\mathbb{K}[G])$ is finitely generated and therefore one can consider the functor $\mathbf{t}_M$. There is a well-defined transformation $\alpha : \mathbf{t}_M \to \mathbf{f}$ given by: $\alpha_X(m \otimes x) = \mathbf{f}(\hat{x})(m)$, where, for x in X, $\hat{x} : \mathbb{K}[G] \to X$ is the G-homomorphism defined by $\hat{x}(1) = x$. By construction this map is an isomorphism when $X = \mathbb{K}[G]$. Because $\mathbf{f}$ is right exact and X is a finitely generated G-module, it follows that α_X is an isomorphism for all X in ${}_G\mathscr{V}$, so that $\mathbf{f}$ is injective in $\mathscr{C}(G)$.

Suppose that $\mathbf{f}$ respects monomorphisms. Consider an exact sequence of functors

$$0 \to \mathbf{f}' \to \mathbf{h}_M \to \mathbf{f} \to 0.$$

We want to show that $\mathbf{f}'$ is projective. By (i), we need to prove that $\mathbf{f}'$ is left exact. For any short exact sequence in ${}_G\mathscr{V}$

$$0 \to A \to B \to C \to 0,$$

there is a commutative diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & \mathbf{f}'(A) \arrow[r] \arrow[d] & \mathbf{f}'(B) \arrow[r] \arrow[d] & \mathbf{f}'(C) \arrow[d] \\
0 \arrow[r] & \mathbf{h}_M(A) \arrow[r] \arrow[d] & \mathbf{h}_M(B) \arrow[r] \arrow[d] & \mathbf{h}_M(C) \arrow[d] \\
 & \mathbf{f}(A) \arrow[r, "\alpha"] \arrow[d] & \mathbf{f}(B) \arrow[r] \arrow[d] & \mathbf{f}(C) \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

In this diagram the middle row is also exact and a diagram chase shows that α is a monomorphism. It follows that $\mathbf{f}'$ is left exact and hence projective. This shows that $pd(\mathbf{f}) \leq 1$.

Conversely, suppose that the projective dimension of $\mathbf{f}$ is ≤ 1. There is a short exact sequence of functors

$$0 \to \mathbf{h}_N \to \mathbf{h}_M \to \mathbf{f} \to 0.$$

If

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in ${}_G\mathscr{V}$, there is a commutative diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathbf{h}_N(A) \arrow[r] \arrow[d] & \mathbf{h}_N(B) \arrow[r] \arrow[d] & \mathbf{h}_N(C) \arrow[d] \\
0 \arrow[r] & \mathbf{h}_M(A) \arrow[r] \arrow[d] & \mathbf{h}_M(B) \arrow[r] \arrow[d] & \mathbf{h}_M(C) \arrow[d] \\
 & \mathbf{f}(A) \arrow[r, "\alpha"] \arrow[d] & \mathbf{f}(B) \arrow[r] \arrow[d] & \mathbf{f}(C) \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

The first two rows in this diagram are also exact, and α is a monomorphism. This shows that $\mathbf{f}$ respects monomorphisms. □

Following Auslander [1, 7], we now consider a recollement situation involving the categories $\mathscr{C}(G)$ and ${}_G\mathscr{V}$.

One considers the functors

$$t^* : \mathscr{C}(G) \to {}_G\mathscr{V}, \quad t_* : {}_G\mathscr{V} \to \mathscr{C}(G) \quad \text{and} \quad t_! : {}_G\mathscr{V} \to \mathscr{C}(G)$$

given respectively by

$$t^*(\mathbf{f}) = \mathbf{f}(\mathbb{K}[G]), \qquad t_*(M) = \mathbf{h}_{M^\vee} \quad \text{and} \quad t_!(M) := \mathbf{t}_M.$$

Lemma 3.9. (i) *The functor $t_!$ is left adjoint to t^* and, for all M in ${}_G\mathscr{V}$, the G-module $t^*t_!(M)$ is naturally isomorphic to M.*

(ii) *The functor t_* is right adjoint to t^* and, for all M in ${}_G\mathscr{V}$, the G-module $t^*t_!(M)$ is naturally isomorphic to M.*

Proof. (i) For any coherent functor $\mathbf{f}$, the functor $\mathbb{D}\mathbf{f}$ is also coherent, so one can assume that $\mathbb{D}\mathbf{f} = \operatorname{Coker}(\mathbf{h}_\alpha : \mathbf{h}_F \to \mathbf{h}_E)$, for some linear map $\alpha : E \to F$. By duality, $\mathbf{f} = \operatorname{Ker}(\mathbf{t}_\alpha)$ and we get an exact sequence

$$0 \to \mathbf{f} \to \mathbf{t}_E \to \mathbf{t}_F. \tag{3}$$

It shows that: $t^*(\mathbf{f}) = \mathbf{f}(\mathbb{K}[G]) = \operatorname{Ker}(\alpha)$. Moreover, there are natural isomorphisms:

$$\begin{aligned}
\operatorname{Hom}_{\mathscr{C}(G)}(\mathbf{t}_M, \mathbf{f}) &= \operatorname{Hom}_{\mathscr{C}(G)}(\mathbf{t}_M, \operatorname{Ker}(\mathbf{t}_\alpha)) \\
&= \operatorname{Ker}(\operatorname{Hom}_{\mathscr{C}(G)}(\mathbf{t}_M, \mathbf{t}_\alpha)) \\
&= \operatorname{Ker}(\operatorname{Hom}_G(M, \alpha)) \\
&= \operatorname{Hom}_G(M, \operatorname{Ker}\alpha) \\
&= \operatorname{Hom}_G(M, t^*(\mathbf{f})).
\end{aligned}$$

¿From this follows the first statement of (1). The second one follows from the natural isomorphism: $t^*t_!(M) = \mathbf{t}_M(\mathbb{K}[G]) \cong M$.

(ii) We use the duality: $\operatorname{Hom}_G(M, \mathbb{K}[G]) \cong M^\vee$. Take $\mathbf{f}$ as in the exact sequence (1): $\mathbf{f} = \operatorname{Coker}\mathbf{h}_\alpha$ for some linear map α. Because $\mathbb{K}[G]$ is a selfinjective algebra, it follows:

$$t^*(\mathbf{f}) = \mathbf{f}(\mathbb{K}[G]) = \operatorname{Coker}(\operatorname{Hom}_G(\alpha, \mathbb{K}[G])) \cong \operatorname{Hom}_G(\operatorname{Ker}(\alpha), \mathbb{K}[G]) \cong (\operatorname{Ker}\,\alpha)^\vee.$$

Moreover, there are natural isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}}(\mathbf{f},\mathbf{h}_{X^\vee}) &= \mathrm{Ker}(\mathbf{h}_{X^\vee}(\alpha)) \\
&= \mathrm{Ker}(\mathrm{Hom}_G(X^\vee,\alpha)) \\
&= \mathrm{Hom}_G(X^\vee,\mathrm{Ker}\,\alpha) \\
&\cong \mathrm{Hom}_G((\mathrm{Ker}\ \alpha)^\vee,X) \\
&\cong \mathrm{Hom}_G(t^*(\mathbf{f}),X)
\end{aligned}$$

Finally, for M in ${}_G\mathscr{V}$: $t^*t_*(M)=t^*(\mathbf{h}_{M^\vee})=\mathrm{Hom}_G(M^\vee,\mathbb{K}[G])\cong M$.

□

We let $\mathscr{C}^0(G)$ be the full subcategory of $\mathscr{C}(G)$ whose objects $\mathbf{f}$ are such that: $t^*(\mathbf{f})=0$. Thus the category $\mathscr{C}^0(G)$ consists exactly of coherent functors which vanish on projective objects. Since t^* is exact, the subcategory $\mathscr{C}^0(G)$ is abelian. Indeed, it is a Serre subcategory of $\mathscr{C}(G)$. We let

$$r_*:\mathscr{C}^0(G)\to\mathscr{C}(G)$$

be the inclusion. It is an exact functor. It is a consequence of Proposition A.2 that the functors $r_*,t^*,t_*,t_!$ are part of a recollement situation

$$\begin{tikzcd}
\mathscr{C}^0(G) \arrow[r, "r_*"] & \mathscr{C}(G) \arrow[l, bend right, "r^*"'] \arrow[l, bend left, "r^!"] \arrow[r, "t^*"] & {}_G\mathscr{V} \arrow[l, bend right, "t_!"'] \arrow[l, bend left, "t_*"]
\end{tikzcd}.$$

where r^* and $r^!$, left and right adjoint to r_*, are defined by the following exact sequences:

$$0\to r_*r^!(\mathbf{f})\to\mathbf{f}\to t_*t^*(\mathbf{f}),\quad t_!t^*(M)\to M\to r_*r^*(\mathbf{f})\to 0.$$

The following Proposition gives another description of r^* and $r^!$.

Proposition 3.10. (i) *For a functor $\mathbf{f}:{}_G\mathscr{V}\to\mathscr{V}$, let $\tau:L_0\mathbf{f}\to\mathbf{f}$ be the natural transformation from the 0-th left derived functor. There is an isomorphism*

$$r^*(\mathbf{f})\cong\mathrm{Coker}(\tau).$$

(ii) *For M in ${}_G\mathscr{V}$, let ΣM be a finitely generated G-module which fits in a short exact sequence*

$$0\to M\to P\to\Sigma M\to 0 \tag{4}$$

where P is a projective G-module. There is an isomorphism

$$r^!(\mathbf{t}_M)\cong\mathrm{Tor}_1^G(-,\Sigma M).$$

(iii) *For a functor $\mathbf{f}:{}_G\mathscr{V}\to\mathscr{V}$, write $\mathbf{f}=\mathrm{Ker}(\mathbf{t}_\alpha)$ for some linear map $\alpha:E\to F$, as in the exact sequence (3). There is an isomorphism*

$$r^!(\mathbf{f})\cong\mathrm{Ker}(\mathrm{Tor}_1^G(-,\Sigma\alpha)).$$

Proof. (i) Let us consider a natural transformation

$$\xi:\mathbf{f}\to\mathbf{g}$$

where $\mathbf{f}$ is a coherent functor and $\mathbf{g}$ is in $\mathscr{C}^0(G)$. We have to prove that ξ factors trough $\mathrm{Coker}(\tau)$. In other words we have to show that the composite $\xi\circ\tau:L_0\mathbf{f}\to\mathbf{g}$ is zero. To this end, for an object M in ${}_G\mathscr{V}$ choose an exact sequence $0 \to N \to P \to M \to 0$ with projective P. The following commutative diagram with exact top row implies the result:

$$\begin{tikzcd}
L_0\mathbf{f}(P) \arrow[r] \arrow[d, "\cong"'] & L_0\mathbf{f}(M) \arrow[r] \arrow[d] & 0\,. \\
\mathbf{f}(P) \arrow[r] \arrow[d] & \mathbf{f}(M) \arrow[d] & \\
0 = \mathbf{g}(P) \arrow[r] & \mathbf{g}(M) &
\end{tikzcd}$$

(ii) The long exact sequence for Tor-groups on $0 \to M \to P \to \Sigma M \to 0$ yields an exact sequence

$$0 \to \mathrm{Tor}_1^G(-, \Sigma M) \to \mathbf{t}_M \to \mathbf{t}_P \to \mathbf{t}_{\Sigma M} \to 0.$$

Let $\psi : \mathbf{t}_M \to \mathbf{h}_{M^\vee}$ be the natural transformation defined by:

$$\begin{array}{rcl} \psi_X : X \otimes_G M & \to & \mathrm{Hom}_G(M^\vee, X) \\ x \otimes m & \mapsto & (\xi \mapsto \xi(m)x). \end{array}$$

It is an isomorphism when M is a projective object in ${}_G\mathscr{V}$. There is a commutative diagram with exact rows

$$\begin{tikzcd}
& 0 \arrow[r] & \mathbf{h}_{M^\vee} \arrow[r] & \mathbf{h}_{P^\vee} \arrow[r] & \mathbf{h}_{\Sigma M^\vee} \arrow[d] & \\
0 \arrow[r] & \mathrm{Tor}_1^G(-, \Sigma_M) \arrow[r] & \mathbf{t}_M \arrow[r] \arrow[u] & \mathbf{t}_P \arrow[r] \arrow[u, "\cong"'] & \mathbf{t}_{\Sigma M} \arrow[r] & 0.
\end{tikzcd}$$

It follows that there is an exact sequence

$$0 \to \mathrm{Tor}_1^G(-, \Sigma M) \to \mathbf{t}_M \to \mathbf{h}_{M^\vee}.$$

The result follows from the comparison with the exact sequence

$$0 \to r_* r^!(\mathbf{t}_M) \to \mathbf{t}_M \to t_* t^*(\mathbf{t}_M)) = t_*(M) \cong \mathbf{h}_{M^\vee}.$$

(iii) Apply $r^!$ to the exact sequence (3). Because the functor $r^!$ is left exact:

$$r^!(\mathbf{f}) = r^!(\mathrm{Ker}(\mathbf{t}_\alpha) = \mathrm{Ker}(r^!(\mathbf{t}_\alpha)),$$

and the result follows from (ii).

$\square$

4. The relation between $\mathscr{P}_n$ and $\mathscr{A}_n$

For simplicity, we write $\mathscr{A}_n$ and $\mathscr{C}_n$ instead of $\mathscr{A}(\mathfrak{S}_n)$ and $\mathscr{C}(\mathfrak{S}_n)$. For an object $\mathbf{f}$ in $\mathscr{A}_n$, the composite

$$\Gamma^n\mathscr{V} \xrightarrow{\mathbf{i}} {}_{\mathfrak{S}_n}\mathscr{V} \xrightarrow{\mathbf{f}} \mathscr{V}$$

defines a strict polynomial functor, which is denoted by $j^*(\mathbf{f})$. This construction defines a functor

$$j^* : \mathscr{A}_n \to \mathscr{P}_n$$
$$j^*(\mathbf{f})(V) = \mathbf{f}(V^{\otimes d}).$$

The same functor was constructed, in terms of Schur algebras, by Green [7, §5, pp 275–276]. The functor j^* was also considered by Chałupnik [2] in his work on functor cohomology.

Lemma 4.1. *The functor j^* respects duality: $\mathbb{D} \circ j^* \cong j^* \circ \mathbb{D}$.*

Proof. $\mathbb{D}j^*(\mathbf{f})(V) = (j^*(\mathbf{f})(V^\vee))^\vee = (\mathbf{f}(V^{\vee \otimes n}))^\vee \cong (\mathbb{D}\mathbf{f})(V^{\otimes n}) = j^*\mathbb{D}(\mathbf{f})(V)$. $\square$

Proposition 4.2. *The functor $j^* : \mathscr{A}_n \to \mathscr{P}_n$ has a right adjoint functor defined by:*

$$j_*(F)(M) = \mathrm{Hom}_{\mathscr{P}_n}(j^*(\mathbf{h}_M), F) = \mathrm{Hom}_{\mathscr{P}_n}(\mathrm{Hom}_{\mathfrak{S}_n}(M, (-)^{\otimes n}), F)$$

where M is representation of $\mathfrak{S}_n$. It has also a left adjoint functor defined by:

$$j_!(F) = \mathbb{D}(j_*(\mathbb{D}F)).$$

In other words:

$$(j_!F)(M)^\vee = \mathrm{Hom}_{\mathscr{P}_n}(F, (-)^{\otimes n} \otimes_{\mathfrak{S}_n} M^\vee).$$

Proof. Since hom's in the category $\mathscr{P}_n$ are finite dimensional vector spaces, we see that $j_*(F)$ belongs to $\mathscr{A}_n$. The fact that it is right adjoint of j^* follows from the Yoneda lemma. The dual formula is formal:

$$\begin{aligned}\mathrm{Hom}_{\mathscr{A}_n}(\mathbb{D}(j_*(\mathbb{D}F)), \mathbf{f}) &\cong \mathrm{Hom}_{\mathscr{A}_n}(\mathbb{D}\mathbf{f}, j_*(\mathbb{D}F)) \cong \mathrm{Hom}_{\mathscr{P}_n}(j^*(\mathbb{D}\mathbf{f}), \mathbb{D}F)\\ &\cong \mathrm{Hom}_{\mathscr{P}_n}(\mathbb{D}j^*(\mathbf{f}), \mathbb{D}F) \cong \mathrm{Hom}_{\mathscr{P}_n}(F, j^*(\mathbf{f})).\end{aligned}$$

To check the last formula, observe that:

$$j_*(\mathbb{D}F)(M^\vee) = \mathrm{Hom}_{\mathscr{P}_n}(j^*(\mathbf{h}_{M^\vee}), \mathbb{D}F) \cong \mathrm{Hom}_{\mathscr{P}_n}(F, j^*(\mathbb{D}\mathbf{h}_{M^\vee}))$$

and

$$j^*(\mathbb{D}\mathbf{h}_{M^\vee}) \cong j^*\mathbf{t}_{M^\vee} = (-)^{\otimes n} \otimes_{\mathfrak{S}_n} M^\vee.$$

□

Remark 4.3. In particular j_* and $j_!$ are a functorial choice of, respectively, an injective and projective symmetrization of [2, Section 3].

Remark 4.4. The existence of adjoints of a precomposition functor is quite a general phenomenon, see Example A.4.

We now study these adjoint functors. For a partition λ of a positive integer n, we let $\mathfrak{S}_\lambda$ be the corresponding Young subgroup of $\mathfrak{S}_n$.

Lemma 4.5. *Let λ be a partition of n. For all finite dimensional $\mathfrak{S}_n$-module M, there are natural isomorphisms:*

$$\begin{aligned}j_*(Sym^\lambda)(M) &\cong H_0(\mathfrak{S}_\lambda, M),\\ j_!(\Gamma^\lambda)(M) &\cong H^0(\mathfrak{S}_\lambda, M)\end{aligned}$$

Proof.

$$\begin{aligned}j_*(Sym^\lambda)(M) &= \mathrm{Hom}_{\mathscr{P}_n}(\mathrm{Hom}_{\mathfrak{S}_n}(M, (-)^{\otimes n}), Sym^\lambda)\\ &\cong \mathrm{Hom}_{\mathscr{P}_n}(\Gamma^\lambda, \mathbb{D}(\mathrm{Hom}_{\mathfrak{S}_n}(M, (-)^{\otimes n})))\\ &\cong \mathrm{Hom}_{\mathscr{P}_n}(\Gamma^\lambda, (-)^{\otimes n} \otimes_{\mathfrak{S}_n} M).\end{aligned}$$

The first isomorphism follows from [6, Corollary 2.12]. The second follows by duality. □

Proposition 4.6. *The values of the functors j_* and $j_!$ are coherent functors.*

Proof. By duality, it is enough to consider the functor j_*. By Proposition 4.5:

$$j_*(Sym^\lambda) = H_0(\mathfrak{S}_\lambda, -) = \mathbf{t}_{\mathbb{K}[\mathfrak{S}_n/\mathfrak{S}_\lambda]},$$

so the statement is true for injective cogenerators of $\mathscr{P}_n$. Since j_* is left exact the result follows by a resolution argument. □

Proposition 4.7. *The unit $\mathsf{Id}_{\mathscr{P}_n} \to j^*j_!$ and the counit $j^*j_* \to \mathsf{Id}_{\mathscr{P}_n}$ are isomorphisms.*

Proof. We prove only the second isomorphism, the first one follows by duality. It is clear that $j^*(H_0(\mathfrak{S}_\lambda, -)) \cong Sym^\lambda$. Thus Lemma 4.5 shows that the statement is true for injective cogenerators of $\mathscr{P}_n$. Since j^* is exact and j_* is left exact, the result follows by taking resolutions. □

Remark 4.8. Since the functor j_* is a full embedding, this gives a new proof of [2, Lemma 3.4].

Proposition 4.9. *Let $\mathscr{C}_n^{\curlyvee}$ be the full subcategory of $\mathscr{C}_n$ whose objects are the coherent functors $\mathbf{f}$ such that, for all partitions λ of n:*

$$\mathbf{f}(\mathbb{K}[\mathfrak{S}_n/\mathfrak{S}_\lambda]) = 0.$$

The functors j^* and its adjoints $j_*, j_!$ are part of a recollement of abelian categories:

$$\begin{tikzcd} \mathscr{C}_n^{\curlyvee} \arrow[r, "i_*"] & \mathscr{C}_n \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathscr{P}_n \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Proof. According to Proposition A.2, the functor j^* and its adjoints give rise to a recollement situation. To determine the kernel category, it is enough to notice that every permutation representation $\mathbb{K}[\mathfrak{S}_n/\mathfrak{S}_\lambda]$ occurs as a direct factor in the tensor product $V^{\otimes n}$ as soon as the dimension of V is n. $\square$

Example 4.10. $\mathscr{C}_n^{\curlyvee} = 0$ for $p = 2$ and $n = 2$ or 3.

Proposition 4.11. *The counit $j_!j^*(\mathbf{f}) \to \mathbf{f}$ is an isomorphism when $\mathbf{f} = \mathbf{t}_M$. Dually, the unit $\mathbf{f} \to j_*j^*(\mathbf{f})$ is an isomorphism when $\mathbf{f} = \mathbf{h}_M$.*

Proof. We prove only the first assertion. Since both $j_!j^*(\mathbf{t}_M)$ and $\mathbf{t}_M$ are right exact functors of M, it is enough to consider the case when $M = \mathbb{K}[\mathfrak{S}_n]$. In this case $\mathbf{t}_M$ is the forgetful functor $\mathbf{u}$. Therefore: $j^*(\mathbf{t}_M) = \otimes^n = \Gamma^{11\cdots 1}$ and

$$j_!j^*(\mathbf{t}_M) = j_!(\Gamma^{11\cdots 1}) = H^0(\mathfrak{S}_{11\cdots 1}, -) = \mathbf{u} = \mathbf{t}_M.$$

$\square$

Proposition 4.12. *The norm transformation (see Appendix A) for the previous recollement situation is an isomorphism on projective and injective objects.*

Proof. By Lemma 4.5 we have $j_!(\Gamma^\lambda) = \mathbf{h}_M$, for $M = \mathbb{K}[\mathfrak{S}_n/\mathfrak{S}_\lambda]$. By Proposition 4.11 we have also $j_*(\Gamma^\lambda) = j_*j^*(\mathbf{h}_M) = \mathbf{h}_M$, thus the norm is an isomorphism on projective objects. By duality it is also an isomorphism on injective objects. $\square$

The following examples gather some other known values of the adjoint functors $j_*, j_!$.

Example 4.13. The relation of Proposition 4.11:

$$j_*(j^*\mathbf{t}_M) \cong \mathbf{t}_M \cong j_!(j^*\mathbf{t}_M)$$

applies in particular when M is the signature, or when M is induced from the signature of a Young subgroup $\mathfrak{S}_\mu$. This shows that, when p is odd, the norm is an isomorphism on a tensor product of exterior powers Λ^μ.

Example 4.14. Let $\mathsf{Id}^{(1)}$ be the Frobenius twist in $\mathscr{P}_p$, that is base change along the Frobenius [6]. It is related to the norm N by an exact sequence:

$$0 \longrightarrow \mathsf{Id}^{(1)} \longrightarrow Sym^p \xrightarrow{N} \Gamma^p \longrightarrow \mathsf{Id}^{(1)} \longrightarrow 0$$

It follows that there are exact sequences:

$$0 \to j_*(\mathsf{Id}^{(1)}) \to H_0(\mathfrak{S}_p, -) \to H^0(\mathfrak{S}_p, -)$$

$$H_0(\mathfrak{S}_p, -) \to H^0(\mathfrak{S}_p, -) \to j_!(\mathsf{Id}^{(1)}) \to 0.$$

Thus $j_*(\mathsf{Id}^{(1)}) = \hat{H}^{-1}(\mathfrak{S}_p, -)$ and $j_!(\mathsf{Id}^{(1)}) = \hat{H}^0(\mathfrak{S}_p, -)$.

Example 4.15. Assume $p = 2$. Let S be a set with n elements. For each $0 \leq k \leq n$ we let B_k be the vector space spanned on the set of all subsets of S with exactly k-elements. Define $d : B_k \to B_{k+1}$ by

$$d(X) = \sum_{X \subset Y \in B_{k+1}} Y$$

Then $d^2 = 0$ and B_* is a cochain complex of $\mathfrak{S}_n$-modules. One checks that $H_*(B_*) = 0$. For an integer $m \geq 1$ and $n = 2^{m+1}$, the explicit injective resolution of $Sym^{2^m(1)}$ of [3, §8] allows to compute:

$$R^* j_*(Sym^{2^m(1)}) \cong H^*(\mathbf{t}_{B_*}). \tag{5}$$

In particular, $j_*(Sym^{2^m(1)})$ is the kernel of the obvious map $H_0(\mathfrak{S}_n, -) \to H_0(\mathfrak{S}_{n-1,1}, -)$. Another consequence of (5) is the fact that $R^k j_*(Sym^{2^m(1)}) = 0$ when $k \geq m$.

We now show the compatibility of the different recollement situations.

Proposition 4.16. *There are commutative diagrams of categories and functors:*

$$\begin{tikzcd} \mathscr{C}_n \arrow[d, phantom] & \mathscr{P}_n \arrow[l, "j_*"'] \\ {}_{\mathfrak{S}_n}\mathscr{V} \arrow[u, "t_*"] \arrow[r, equal] & {}_{\mathfrak{S}_n}\mathscr{V} \arrow[u, "c_*"'] \end{tikzcd}
\qquad
\begin{tikzcd} \mathscr{C}_n^{\mathsf{Y}} \arrow[r] \arrow[d, equal] & \mathscr{C}_n^0 \arrow[r] \arrow[d, "r_*"'] & \mathscr{P}_n^0 \arrow[d, "d_*"] \\ \mathscr{C}_n^{\mathsf{Y}} \arrow[r, "i_*"] & \mathscr{C}_n \arrow[r, "j^*"] \arrow[d, "t^*"'] & \mathscr{P}_n \arrow[d, "c^*"'] \\ & {}_{\mathfrak{S}_n}\mathscr{V} \arrow[r, equal] & {}_{\mathfrak{S}_n}\mathscr{V} \end{tikzcd}
\qquad
\begin{tikzcd} \mathscr{C}_n & \mathscr{P}_n \arrow[l, "j_!"'] \\ {}_{\mathfrak{S}_n}\mathscr{V} \arrow[u, "t_!"] \arrow[r, equal] & {}_{\mathfrak{S}_n}\mathscr{V}. \arrow[u, "c_!"'] \end{tikzcd}$$

Proof. To show that $c^* \circ j^* = t^*$, note that the three functors involved are exact. It is therefore enough to check that they coincide on $\mathbf{h}_M$ for each M in ${}_{\mathfrak{S}_n}\mathscr{V}$. This means that we need to show that:

$$\mathrm{Hom}_{\mathscr{P}_n}(T^n, \mathrm{Hom}_{\mathfrak{S}_n}(M, (-)^{\otimes n})) \cong M^\vee$$

Since T^n is projective, the left hand side of the expected isomorphism is left exact as a functor of M. So it suffices to consider the case when M is injective, and it reduces to the case when $M = \mathbb{K}[\mathfrak{S}_n]$. In this case it is a well-known isomorphism:

$$\mathrm{Hom}_{\mathscr{P}_n}(T^n, T^n) \cong \mathbb{K}[\mathfrak{S}_n].$$

To show that $j_! c! = t!$, note that both sides are right exact. It is therefore enough to check that they coincide on $\mathbb{K}[\mathfrak{S}_n]$. In this case, $j_! c_!(\mathbb{K}[\mathfrak{S}_n]) = j_!(T^n)$ has already been seen (see the proof of Proposition 4.11) to be $\mathbf{u} = t_{\mathbb{K}[\mathfrak{S}_n]}$.

The rest is quite similar. □

We end the section with the following immediate consequence of Proposition A.5 and Proposition 3.7.

Corollary 4.17. *Let $F \in \mathscr{P}_n$ and $\mathbf{g} \in \mathscr{C}_n$. For every $k \geq 2$, there exist a functorially defined subgroup $E^k(F, \mathbf{g})$ of $\mathrm{Ext}^k_{\mathscr{P}}(F, j^*(\mathbf{g}))$ and exact sequences:*

$$0 \to \mathrm{Ext}^1_{\mathscr{C}}(j_! F, \mathbf{g}) \to \mathrm{Ext}^1_{\mathscr{P}}(F, j^*\mathbf{g}) \to \mathrm{Hom}_{\mathscr{C}}(L_1 j_!(F), \mathbf{g}) \to$$
$$\to \mathrm{Ext}^2_{\mathscr{C}}(j_! F, \mathbf{g}) \to E^2(F, \mathbf{g}) \to \mathrm{Ext}^1_{\mathscr{C}}(L_1 j_!(F), \mathbf{g}) \to 0,$$
$$0 \to E^k(F, \mathbf{g}) \to \mathrm{Ext}^k_{\mathscr{P}}(F, j^*\mathbf{g}) \to \mathrm{Hom}_{\mathscr{C}}(L_k j_!(F), \mathbf{g}) \to$$
$$\to \mathrm{Ext}^k_{\mathscr{C}}(L_{k-1} j_! F, \mathbf{g}) \to E^{k+1}(F, \mathbf{g}) \to \mathrm{Ext}^1_{\mathscr{C}}(L_k j_!(F), \mathbf{g}) \to 0, \quad k \geq 2.$$

Dually, for every $k \geq 2$, there exist a functorially defined subgroup $E^k(\mathbf{g}, F)$ of $\operatorname{Ext}^k_{\mathscr{P}}(j^*(\mathbf{g}), F)$ and exact sequences:

$$0 \to \operatorname{Ext}^1_{\mathscr{C}}(\mathbf{g}, j_*F) \to \operatorname{Ext}^1_{\mathscr{P}}(j^*\mathbf{g}, F) \to \operatorname{Hom}_{\mathscr{C}}(\mathbf{g}, R^1j_*(F)) \to$$
$$\to \operatorname{Ext}^2_{\mathscr{C}}(\mathbf{g}, j_*F) \to E^2(\mathbf{g}, F) \to \operatorname{Ext}^1_{\mathscr{C}}(\mathbf{g}, R^1j_*(F)) \to 0,$$
$$0 \to E^k(\mathbf{g}, F) \to \operatorname{Ext}^k_{\mathscr{P}}(j^*\mathbf{g}, F) \to \operatorname{Hom}_{\mathscr{C}}(\mathbf{g}, R^kj_*(F)) \to$$
$$\to \operatorname{Ext}^k_{\mathscr{C}}(\mathbf{g}, R^{k-1}j_*F) \to E^{k+1}(\mathbf{g}, F) \to \operatorname{Ext}^1_{\mathscr{C}}(\mathbf{g}, R^kj_*(F)) \to 0, \quad k \geq 2.$$

5. Application to functor cohomology

M. Chałupnik [2, Theorem 4.3] has generalized the Ext-computations in the category $\mathscr{P}_n$ obtained in [5]. We show in this section how one can obtain his results through natural isomorphisms.

Let i be a non negative integer; for F in $\mathscr{P}_n$, we denote its Frobenius twist in $\mathscr{P}_{np^i}$ by $F^{(i)}$. Any functor $\mathbf{f}$ in $\mathscr{A}_n$ is naturally extended to graded objects as in [2]. We want to compare $\mathbf{f}(\operatorname{Ext}_{\mathscr{P}_{np^i}}(G, T^{n(i)}))$ and $\operatorname{Ext}_{\mathscr{P}_{np^i}}(G, j^*\mathbf{f}^{(i)})$.

Lemma 5.1. *For all functors $\mathbf{f}$ in $\mathscr{A}_n$, there is a natural transformation*

$$\mathbf{f}(\operatorname{Hom}_{\mathscr{P}_{np^i}}(G, T^{n(i)})) \to \operatorname{Hom}_{\mathscr{P}_{np^i}}(G, j^*\mathbf{f}^{(i)}),$$

which is a isomorphism when $\mathbf{f}$ is of the form $\mathbf{h}_M$, and a monomorphism for all coherent functors $\mathbf{f}$.

Proof. We start with the structure map:

$$\operatorname{Hom}_{\mathfrak{S}_n}(M, N) \to \operatorname{Hom}(\mathbf{f}(M), \mathbf{f}(N)),$$

rewritten as:

$$\mathbf{f}(M) \to \operatorname{Hom}(\operatorname{Hom}_{\mathfrak{S}_n}(M, N), \mathbf{f}(N)).$$

In case $M = \operatorname{Hom}(E, N)$ for E in $\mathscr{V}$ and N in ${}_{\mathfrak{S}_n}\mathscr{V}$, we compose with the evaluation map $E \to \operatorname{Hom}_{\mathfrak{S}_n}(\operatorname{Hom}(E, N), N)$ and get a natural map:

$$\mathbf{f}(\operatorname{Hom}(E, N)) \to \operatorname{Hom}(\operatorname{Hom}_{\mathfrak{S}_n}(\operatorname{Hom}(E, N), N), f(N)) \to \operatorname{Hom}(E, \mathbf{f}(N)).$$

This map is an isomorphism when $\mathbf{f}$ is of the form $\mathbf{h}_M$ or $\mathbf{t}_M$.

Using freely the notion of end of a bifunctor,we obtain a natural map:

$$\mathbf{f}\int_{V \in \Gamma^{np^i}\mathscr{V}} \operatorname{Hom}(GV, V^{\otimes n(i)}) \to \int_{V \in \Gamma^{np^i}\mathscr{V}} \mathbf{f}\operatorname{Hom}(GV, V^{\otimes n(i)}) \to \int_V \operatorname{Hom}(GV, \mathbf{f}(V^{\otimes n(i)}))$$

which, in view of the description of $\operatorname{Hom}_{\mathscr{P}_{np^i}}$ as end of the bifunctor Hom, is a natural transformation:

$$\mathbf{f}(\operatorname{Hom}_{\mathscr{P}_{np^i}}(G, T^{n(i)})) \to \operatorname{Hom}_{\mathscr{P}_{np^i}}(G, j^*\mathbf{f}^{(i)}).$$

It is an isomorphism when $\mathbf{f}$ is of the form $\mathbf{h}_M$. For, in this case, both sides of:

$$\operatorname{Hom}_{\mathfrak{S}_n}(M, \operatorname{Hom}_{\mathscr{P}_{np^i}}(G, T^{n(i)})) \to \operatorname{Hom}_{\mathscr{P}_{np^i}}(G, \operatorname{Hom}_{\mathfrak{S}_n}(M, T^{n(i)}))$$

are left exact functors of M which coincide when $M = \mathbb{K}[\mathfrak{S}_n]$.

For a general coherent $\mathbf{f}$, write $\mathbf{f} = \operatorname{Coker}\mathbf{h}_\alpha$ for $\alpha : M \to N$. There is a commutative diagram:

$$\begin{tikzcd}
\mathbf{h}_N \operatorname{Hom}_{\mathscr{P}_{np^i}}(G, T^{n(i)})) \arrow[r, "\mathbf{h}_\alpha"] \arrow[d, "\cong"] & \mathbf{h}_M(\operatorname{Hom}_{\mathscr{P}_{np^i}}(G, T^{n(i)})) \arrow[r] \arrow[d, "\cong"] & \mathbf{f}(\operatorname{Hom}_{\mathscr{P}_{np^i}}(G, T^{n(i)})) \arrow[r] \arrow[d] & 0 \\
\operatorname{Hom}_{\mathscr{P}_{np^i}}(G, j^*\mathbf{h}^{(i)}_N) \arrow[r, "\mathbf{h}_{\alpha *}"] & \operatorname{Hom}_{\mathscr{P}_{np^i}}(G, j^*\mathbf{h}^{(i)}_M) \arrow[r] & \operatorname{Hom}_{\mathscr{P}_{np^i}}(G, j^*\mathbf{f}^{(i)}) &
\end{tikzcd},$$

with an exact first row, the composition of the second row being zero. Thus the third vertical map is a monomorphism. □

In general, this is not an isomorphism: consider for example the case when $p = n = 2$, G is the Frobenius twist, and $\mathbf{f}$ is $\mathrm{H}_0(\mathfrak{S}_2, -)$.

We now follow closely the proof of [2, Theorem 4.3]. For $\mathbf{f}$ in $\mathscr{A}_n$, consider an injective resolution $S^\bullet$ of $j^*\mathbf{f}$ in $\mathscr{P}_n$. Note that $j_*S^\bullet$ is a complex, and that, because j_* is left exact, $H_0(j_*S^\bullet) = \mathbf{f}$. Remark [2, Proposition 3.4.2] that, when evaluated at a permutation module M, the complex $j_*S^\bullet(M)$ is a resolution of $f(M)$.

For G in $\mathscr{P}_{np^i}$, applying $\mathrm{Hom}_{\mathscr{P}_{np^i}}(G, -)$ yields a spectral sequence:

$$\mathrm{E}_1^{s,*} = \mathrm{Ext}^s_{\mathscr{P}_{np^i}}(G, S^{*(i)})$$

converging to $\mathrm{Ext}_{\mathscr{P}_{np^i}}(G, F^{(i)})$. It is related to $\mathbf{f}(\mathrm{Ext}_{\mathscr{P}_{np^i}}(G, T^{n(i)}))$ through two maps:

$$\mathbf{f}(\mathrm{Ext}_{\mathscr{P}_{np^i}}(G, T^{n(i)})) = \mathrm{H}_0(j_*S^\bullet\,\mathrm{Ext}_{\mathscr{P}_{np^i}}(G, T^{n(i)})) \to \mathrm{H}(j_*S^\bullet\,\mathrm{Ext}_{\mathscr{P}_{np^i}}(G, T^{n(i)})),$$

and, using the map in Lemma 5.1:

$$\mathrm{H}(j_*S^\bullet\,\mathrm{Ext}_{\mathscr{P}_{np^i}}(G, T^{n(i)})) \to \mathrm{H}(\mathrm{Ext}_{\mathscr{P}_{np^i}}(G, S^{\bullet(i)})) = \mathrm{E}_2.$$

In the case when $\mathrm{Ext}_{\mathscr{P}_{np^i}}(G, T^{n(i)})$ is a permutation module, the first map is an isomorphism and the spectral sequence collapses at E_2. This is the case for instance when G is the twist $\Gamma^{\mu(i)}$ of a projective. For $G = \Gamma^{\mu(i)}$, the second map is an isomorphism as well [2, Proposition 4.1]. We thus recover the isomorphisms of [2] in a natural manner. For instance, for $G = \Gamma^{n(i)}$ and $\mathbf{f} = j_*F$, we obtain:

Proposition 5.2. *For all F in $\mathscr{P}_n$, there is a natural isomorphism:*

$$\mathrm{Ext}_{\mathscr{P}_{np^i}}(\Gamma^{n(i)}, F^{(i)}) \cong F(\mathrm{Ext}_{\mathscr{P}_{p^i}}(\mathsf{Id}^{(i)}, \mathsf{Id}^{(i)})).$$

6. Tensor products of coherent functors

The aim of this section is to lift the bifunctor $\otimes : \mathscr{P}_n \times \mathscr{P}_m \to \mathscr{P}_{n+m}$ given by $(F \otimes G)(V) = F(V) \otimes G(V)$ at the level of coherent functors. Not surprisingly, it involves the induction functor

$$\mathsf{Ind}^{\mathfrak{S}_{m+n}}_{\mathfrak{S}_m \times \mathfrak{S}_n} : {}_{\mathfrak{S}_m \times \mathfrak{S}_n}\mathscr{V} \to {}_{\mathfrak{S}_{n+m}}\mathscr{V}.$$

For M and N in ${}_{\mathfrak{S}_n}\mathscr{V}$, let $M \odot N$ denote the module $\mathsf{Ind}^{\mathfrak{S}_{m+n}}_{\mathfrak{S}_m \times \mathfrak{S}_n}(M \otimes N)$. The operation $\odot$ yields a symmetric monoidal structure on the category $\bigoplus_{n \geq 0} {}_{\mathfrak{S}_n}\mathscr{V}$. Because the objects $\mathbf{h}_M$ are projective generators in the category of coherent functors, one readily obtains:

Proposition 6.1. *There exists a unique biadditive functor*

$$\blacktriangledown : \mathscr{C}_m \times \mathscr{C}_n \to \mathscr{C}_{m+n}$$

such that:

(i) $\mathbf{h}_M \blacktriangledown \mathbf{h}_N = \mathbf{h}_{M \odot N}$*;*
(ii) *The bifunctor $\blacktriangledown$ is right exact with respect to both variables.*

Moreover $\blacktriangledown$ equips the category $\bigoplus_{d \geq 0} \mathscr{C}_d$ with a symmetric monoidal structure.

We show next that the bifunctor $\blacktriangledown$ is balanced, which means that $(-)\blacktriangledown\mathbf{g}$ is an exact functor when $\mathbf{g}$ is projective in $\mathscr{C}_n$. We start by defining another product, using restriction. Consider $\mathfrak{S}_n$ as the subgroup of $\mathfrak{S}_{m+n}$ which fixes the first m elements. For X in ${}_{\mathfrak{S}_{m+n}}\mathscr{V}$, denote by $\mathsf{Res}^{\mathfrak{S}_{n+m}}_{1_m \times \mathfrak{S}_n}(X)$ the corresponding restriction. Then, for any $\mathbf{g}$ in $\mathscr{C}_n$, the vector space $\mathbf{g}(\mathsf{Res}^{\mathfrak{S}_{m+n}}_{1_m \times \mathfrak{S}_n}(X))$ has a natural structure of representation of $\mathfrak{S}_m$, because $\mathfrak{S}_m \times 1_n$ commutes with $1_m \times \mathfrak{S}_n$. Thus, for $\mathbf{f}$ in $\mathscr{C}_m$, the evaluation $\mathbf{f}(\mathbf{g}(\mathsf{Res}^{\mathfrak{S}_{m+n}}_{1_m \times \mathfrak{S}_n} X))$ makes sense. Denote it $(\mathbf{f}\blacklozenge g)(X)$. It defines a biadditive functor

$$\blacklozenge : \mathscr{C}_m \times \mathscr{C}_n \to \mathscr{C}_{m+n}.$$

The bifunctor $\blacklozenge$ is clearly exact with respect to the first argument. The bifunctor $\blacklozenge$ is highly nonsymmetric.

Lemma 6.2. *For all* $\mathbf{f}$ *in* $\mathscr{C}_m$ *and all* N *in* ${}_{\mathfrak{S}_n}\mathscr{V}$*, there is a natural isomorphism*

$$\mathbf{f}\blacktriangledown\mathbf{h}_N \cong \mathbf{f}\blacklozenge\mathbf{h}_N.$$

Proof. Since both sides are right exact on $\mathbf{f}$, one can assume that $\mathbf{f} = \mathbf{h}_M$ for some M in ${}_{\mathfrak{S}_m}\mathscr{V}$. For X in ${}_{\mathfrak{S}_{m+n}}\mathscr{V}$, one has:

$$\begin{aligned}
\mathbf{h}_{M\odot N}(X) &= \mathrm{Hom}_{\mathfrak{S}_{m+n}}(\mathsf{Ind}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(M\otimes N), X)\\
&= \mathrm{Hom}_{\mathfrak{S}_m\times\mathfrak{S}_n}(M\otimes N, \mathsf{Res}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(X))\\
&= \mathrm{Hom}_{\mathfrak{S}_m}(M, \mathrm{Hom}_{\mathfrak{S}_n}(N, \mathsf{Res}_{1_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}} X)\\
&= \mathbf{h}_M\blacklozenge\mathbf{h}_N(X).
\end{aligned}$$

$\square$

Proposition 6.3. *The bifunctor* $\blacktriangledown$ *is balanced.*

Proof. Since $\blacktriangledown$ is symmetric, it suffices to prove that $(-)\blacktriangledown\mathbf{h}_N$ is an exact functor. By the lemma, this is the same as $(-)\blacklozenge\mathbf{h}_N$, which is exact. $\square$

Before stating our next result, let us recall that a *composition* is a finite sequence of positive integers. The integers of any composition are the parts of a unique partition. In particular, the concatenation of two partitions λ and μ is in general not a partition, but only a composition. We let $\lambda\cup\mu$ be the associated partition. There is an isomorphism:

$$\mathsf{Ind}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(\mathsf{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_m}\mathbb{K}\otimes\mathsf{Ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_n}\mathbb{K}) \cong \mathsf{Ind}_{\mathfrak{S}_{\lambda\cup\mu}}^{\mathfrak{S}_{m+m}}\mathbb{K}. \tag{6}$$

Proposition 6.4. (i) *For all* $\mathbf{f}$ *in* $\mathscr{C}_m$ *and* $\mathbf{g}$ *in* $\mathscr{C}_n$*, there is a natural isomorphism*

$$j^*(\mathbf{f})\otimes j^*(\mathbf{g}) \cong j^*(\mathbf{f}\blacktriangledown\mathbf{g});$$

(ii) *For all* F *in* $\mathscr{P}_m$ *and* G *in* $\mathscr{P}_n$*, there is a natural isomorphism*

$$j_!(F)\blacktriangledown j_!(G) \cong j_!(F\otimes G).$$

Proof. (i) Since j^* and $\blacktriangledown$ are right exact functors, it is enough to consider the case when $\mathbf{f} = \mathbf{h}_M$ and $\mathbf{g} = \mathbf{h}_N$. Then we have to prove that

$$\mathbf{h}_{M\odot N}(V^{\otimes n+m}) \cong \mathbf{h}_M(V^{\otimes m})\otimes\mathbf{h}_N(V^{\otimes n}).$$

This follows from the isomorphism:

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{S}_{m+n}}(\mathsf{Ind}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(M\otimes N), V^{\otimes m+n}) &= \mathrm{Hom}_{\mathfrak{S}_m\times\mathfrak{S}_n}(M\otimes N, V^{\otimes m}\otimes V^{\otimes n})\\
&= \mathrm{Hom}_{\mathfrak{S}_m}(M, V^{\otimes m})\otimes\mathrm{Hom}_{\mathfrak{S}_n}(N, V^{\otimes n}).
\end{aligned}$$

(ii) It is enough to assume that F and G are projective generators: $F = \Gamma^\lambda$ and $G = \Gamma^\mu$. Thanks to Lemma 4.5, we have

$$j_!(F) = H^0(\mathfrak{S}_\lambda, -) = \mathbf{h}_{\mathbb{K}[\mathfrak{S}_m/\mathfrak{S}_\lambda]}\ ,\quad j_!(G) = H^0(\mathfrak{S}_\mu, -) = \mathbf{h}_{\mathbb{K}[\mathfrak{S}_n/\mathfrak{S}_\mu]}$$

So the result follows from the isomorphism (6).

$\square$

Since the functors $\mathbf{t}_M$, $M \in {}_{\mathfrak{S}_m}\mathscr{V}$, form a family of injective cogenerators, we obtain the following result, which is dual to Proposition 6.1.

Proposition 6.5. *There exists a unique biadditive functor*

$$\blacktriangle : \mathscr{C}_m\times\mathscr{C}_n \to \mathscr{C}_{m+n}$$

such that:

(i) $\mathbf{t}_M\blacktriangle\mathbf{t}_N = \mathbf{t}_{M\odot N}$*;*

(ii) *the functor* $\blacktriangle$ *is left exact with respect to both variables;*
(iii) *the functor* $\blacktriangle$ *equips the category* $\bigoplus_{d\geq 0}\mathscr{C}_d$ *with a symmetric monoidal structure;*
(iv) *For all* $\mathbf{f}$ *in* $\mathscr{C}_m$ *and* $\mathbf{g}$ *in* $\mathscr{C}_n$*, there is a natural isomorphism*

$$j^*(\mathbf{f})\otimes j^*(\mathbf{g})\cong j^*(\mathbf{f}\blacktriangle\mathbf{g});$$

(v) *For all* F *in* $\mathscr{P}_m$ *and* G *in* $\mathscr{P}_n$*, there is a natural isomorphism*

$$j_*(F)\blacktriangle j_*(G)\cong j_*(F\otimes G).$$

The rest of this section is devoted to the relationship between the three bifunctors $\blacktriangledown, \blacktriangle, \blacklozenge$.

Lemma 6.6. *For all* M *in* ${}_{\mathfrak{S}_m}\mathscr{V}$ *and* N *in* ${}_{\mathfrak{S}_n}\mathscr{V}$*, there are natural isomorphisms*

$$\mathbf{t}_M\blacklozenge\mathbf{t}_N\cong\mathbf{t}_{M\odot N}\cong\mathbf{t}_M\blacktriangledown\mathbf{t}_N,$$

$$\mathbf{h}_M\blacklozenge\mathbf{h}_N\cong\mathbf{h}_{M\odot N}\cong\mathbf{h}_M\blacktriangle\mathbf{h}_N.$$

Proof.

$$\begin{aligned}(\mathbf{t}_M\blacklozenge\mathbf{t}_N)(X) &= M\otimes_{\mathfrak{S}_m}(N\otimes_{\mathfrak{S}_n}(\mathsf{Res}^{\mathfrak{S}_{m+n}}_{1_m\times\mathfrak{S}_n}(X))\\ &\cong (M\otimes N)_{\mathfrak{S}_m\times\mathfrak{S}_n}(\mathsf{Res}^{\mathfrak{S}_{m+n}}_{\mathfrak{S}_m\times\mathfrak{S}_n}(X))\\ &\cong ((M\otimes N)_{\mathfrak{S}_m\otimes\mathfrak{S}_n}\mathbb{K}[\mathfrak{S}_{m+n}])\otimes_{\mathfrak{S}_{m+n}}X\\ &\cong \mathbf{t}_{M\odot N}(X).\end{aligned}$$

Similarly:

$$\begin{aligned}(\mathbf{h}_M\blacklozenge\mathbf{h}_N)(X) &= \mathrm{Hom}_{\mathfrak{S}_m}(M,\mathrm{Hom}_{\mathfrak{S}_n}(N,\mathsf{Res}^{\mathfrak{S}_{m+n}}_{1_m\times\mathfrak{S}_n}(X)))\\ &\cong \mathrm{Hom}_{\mathfrak{S}_m\times\mathfrak{S}_n}(M\otimes N,\mathsf{Res}^{\mathfrak{S}_{m+n}}_{\mathfrak{S}_m\times\mathfrak{S}_n}(X))\\ &\cong \mathrm{Hom}_{\mathfrak{S}_{n+m}}(M\odot N,X)\\ &\cong \mathbf{h}_{M\odot N}(X).\end{aligned}$$

To show that $\mathbf{t}_{M\odot N}\cong\mathbf{t}_M\blacktriangledown\mathbf{t}_N$,, observe that it holds when M is free, because $\mathbf{t}_M\cong\mathbf{h}_{M^\vee}$. It follows for a general M from the right exactness of the tensor and $\blacktriangledown$-products. The proof that $\mathbf{h}_{M\odot N}\cong\mathbf{h}_M\blacktriangle\mathbf{h}_N$ is similar. ☐

Corollary 6.7. *The left exact bifunctor* $\blacktriangle$ *is balanced, meaning that it is exact provided one of the arguments is injective.*

Let us observe that the bifunctors $\blacktriangledown$ and $\blacklozenge$ (resp. $\blacktriangle$ and $\blacklozenge$) take the same values on projective (resp. on injective) objects and $\blacktriangledown$ (resp. $\blacktriangle$) is right (resp. left) exact with respect to both arguments. It follows that there are natural transformations: $\mathbf{f}\blacktriangledown\mathbf{g}\to\mathbf{f}\blacklozenge\mathbf{g}$ and $\mathbf{f}\blacklozenge\mathbf{g}\to\mathbf{f}\blacktriangle\mathbf{g}$.

Lemma 6.8. *The natural transformation* $\mathbf{f}\blacktriangledown\mathbf{g}\to\mathbf{f}\blacklozenge\mathbf{g}$ *is an isomorphism, provided* $\mathbf{g}$ *is projective, and the natural transformation* $\mathbf{f}\blacklozenge\mathbf{g}\to\mathbf{f}\blacktriangle\mathbf{g}$ *is an isomorphism provided* $\mathbf{g}$ *is injective. Moreover the composite transformation* $\mathbf{f}\blacktriangledown\mathbf{g}\to\mathbf{f}\blacktriangle\mathbf{g}$ *is an isomorphism when both arguments are simultaneously injective or projective.*

Lemma 6.9. *For all* $\mathbf{f}$ *in* $\mathscr{C}_m$ *and* $\mathbf{g}$ *in* $\mathscr{C}_n$*, there is a natural isomorphism*

$$j^*(\mathbf{f})\otimes j^*(\mathbf{g})\cong j^*(\mathbf{f}\blacklozenge\mathbf{g}).$$

Proof. We have:

$$j^*(\mathbf{f}\blacklozenge\mathbf{g})(V)=\mathbf{f}(\mathbf{g}(\mathsf{Res}^{\mathfrak{S}_{m+n}}_{1_m\times\mathfrak{S}_n}V^{\otimes m+n}))$$

For all $W\in\mathscr{V}$ and $N\in{}_{\mathfrak{S}_n}\mathscr{V}$, there is a natural isomorphism

$$\mathbf{g}(W\otimes N)\cong W\otimes\mathbf{g}(N),$$

where $\mathfrak{S}_n$ acts on the second factor of $W \otimes N$. Now: $V^{\otimes m+n} = V^{\otimes m} \otimes V^{\otimes n}$ as $\mathfrak{S}_n$-modules, with trivial action on the first factor, so:

$$\mathsf{g}(\mathsf{Res}^{\mathfrak{S}_{m+n}}_{1_m \times \mathfrak{S}_n} V^{\otimes m+n}) = V^{\otimes m} \otimes \mathbf{g}(V^{\otimes n}).$$

Similarly, the group $\mathfrak{S}_m$ acts trivially on $\mathbf{g}(V^{\otimes n})$, and we obtain:

$$j^*(\mathbf{f}\blacklozenge\mathbf{g})(V) \cong \mathbf{f}(V^{\otimes m}) \otimes \mathbf{g}(V^{\otimes n}).$$

$\square$

Corollary 6.10. *The functor j^* sends the morphisms $\mathbf{f}\blacktriangledown\mathbf{f}' \to \mathbf{f}\blacklozenge\mathbf{f}' \to \mathbf{f}\blacktriangle\mathbf{f}'$ to isomorphisms.*

We end this section with the derived functors of $\blacktriangledown$ and $\blacktriangle$. We let $\mathrm{Tor}^{\blacktriangledown}_*(\mathbf{f}, \mathbf{g})$ be the left derived functors of the bifunctor $\blacktriangledown$. Since $\blacktriangledown$ is balanced, taking projective resolutions $\mathbf{p}_\bullet \to \mathbf{f}$ and $\mathbf{q}_\bullet \to \mathbf{g}$, we get isomorphisms:

$$\mathrm{Tor}^{\blacktriangledown}_*(\mathbf{f}, \mathbf{g}) = H_*(\mathbf{p}_\bullet\blacktriangledown\mathbf{g}) \cong H_*(\mathbf{f}\blacktriangledown\mathbf{q}_\bullet) \cong H_*(\mathbf{p}_\bullet\blacktriangledown\mathbf{q}_\bullet).$$

Similarly, we let $\mathrm{Tor}^{\blacktriangle}_*(\mathbf{f}, \mathbf{g})$ be the right derived functors of the bifunctor $\blacktriangle$. Since $\blacktriangle$ is balanced, it can be computed via injective resolutions of $\mathbf{f}$ and $\mathbf{g}$. By Proposition 3.7: $\mathrm{Tor}^{\blacktriangledown}_k = 0 = \mathrm{Tor}^{\blacktriangle}_k$ when $k > 2$. Moreover, the following variant of the Künneth spectral sequence holds.

Lemma 6.11. *For F in $\mathscr{P}_m$ and G in $\mathscr{P}_n$, there are spectral sequences*

$$\mathrm{E}^2_{pq} = \bigoplus_{s+t=q} \mathrm{Tor}^{\blacktriangledown}_p(L_s j_!(F), L_q j_!(G)) \Rightarrow L_{p+q} j_!(F \otimes G)$$

and

$$\mathrm{E}_2^{pq} = \bigoplus_{s+t=q} \mathrm{Tor}^{\blacktriangle}_p(R^s j_*(F), R^q j_*(G)) \Rightarrow R^{p+q} j_*(F \otimes G)$$

Moreover $\mathrm{E}^2_{pq} = 0 = \mathrm{E}_2^{pq}$ provided $p > 2$.

Proof. Let $P_\bullet \to F$ and $Q_\bullet \to G$ be projective resolutions. Then $P_\bullet \otimes Q_\bullet \to F \to G$ is also a projective resolution. Thus $L_* j_!(F \otimes G) = H_*(j_!(P_\bullet \otimes Q_\bullet))$. We have $j_!(P_\bullet \otimes Q_\bullet) \cong j_!(P_\bullet)\blacktriangledown j_!(Q_\bullet)$. Since $\blacktriangledown$ is balanced, and both $j_!(P_\bullet)$ and $j_!(Q_\bullet)$ are degreewise projective complexes whose homology is respectively $L_* j_!(F)$ and $L_* j_!(T)$, the result follows by repeating the proof of the classical Künneth spectral sequence. $\square$

7. Composition and coherent functors

The composite of two strict polynomial functors is a strict polynomial functor [6]. The aim of this section is to lift the resulting bifunctor:

$$\circ : \mathscr{P}_n \times \mathscr{P}_m \to \mathscr{P}_{nm}$$
$$(F, G) \mapsto F \circ G$$

at the level of coherent functors.

Composition of functors is exact with respect to the first variable. Although the functor $G \mapsto F \circ G$ is not additive for $n > 1$, it still has some exactness properties with respect to the second variable.

Definition 7.1. Let $\mathbb{A}$ and $\mathbb{B}$ be abelian categories. For any short exact sequence

$$0 \longrightarrow A_1 \xrightarrow{\alpha} A \xrightarrow{\beta} A_2 \longrightarrow 0 \tag{7}$$

in $\mathbb{A}$, define $\delta_1, \delta_2 : A \oplus A_1 \to A$ by

$$\delta_1(a, a_1) = a + \alpha(a_1)\ , \quad \delta_2(a, a_1) = a,$$

and $\gamma = T(\delta_1) - T(\delta_2)$. A covariant functor $T : \mathbb{A} \to \mathbb{B}$ *preserves reflective coequalizers* if the following sequence is exact:

$$T(A \oplus A_1) \xrightarrow{\gamma} T(A) \xrightarrow{T(\beta)} T(A_2) \longrightarrow 0\ . \tag{8}$$

Observe that when T is an additive, then it preserves reflective coequalizers if, and only if, T is right exact. Let us observe also that if the exact sequence (7) splits then the sequence (8) is exact for any functor T. If $\mathbb{A}$ has enough projective objects, then any (possibly nonadditive) functor, from the category of projective objects in $\mathbb{A}$ to the category $\mathbb{B}$, has a unique (up to unique isomorphism) extension as a functor $\mathbb{A} \to \mathbb{B}$ which preserves reflective coequalizers.

We leave to the reader to define the dual notion, a functor preserving coreflective equalizers. An additive functor has this property if, and only if, it is left exact.

Lemma 7.2. *For any F in $\mathscr{P}_n$, the functor*

$$\begin{aligned} \mathscr{P}_m \to & \mathscr{P}_{nm} \\ G \mapsto & F \circ G \end{aligned}$$

preserves reflective coequalizers and coreflective equalizers.

Proof. Take any short exact sequence in $\mathscr{P}_m$:

$$0 \longrightarrow G_1 \xrightarrow{\alpha} G \xrightarrow{\beta} G_2 \longrightarrow 0\ .$$

After evaluating at $V \in \mathscr{V}$, the corresponding sequence

$$0 \to G_1(V) \to G(V) \to G_2(V) \to 0$$

splits. Therefore for any F, the sequence

$$F(G(V) \oplus G_1(V)) \to F(G(V)) \to F(G_2(V)) \to 0$$

is exact. This shows that $F \circ (-)$ respects reflective coequalizers. Similarly for coreflective equalizers. □

For a natural number m and a group G, let $\mathfrak{S}_m \wr G$ be the wreath product, which by definition is the semi-direct product $G^m \rtimes \mathfrak{S}_m$. For M in ${}_{\mathfrak{S}_m}\mathscr{V}$ and N in ${}_G\mathscr{V}$, it acts on $M \otimes N^{\otimes m}$. In particular, for $G = \mathfrak{S}_n$, let

$$M \bullet N := \mathsf{Ind}_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{mn}} (M \otimes N^{\otimes m})$$

It defines a functor:

$$\bullet : {}_{\mathfrak{S}_m}\mathscr{V} \times {}_{\mathfrak{S}_n}\mathscr{V} \to {}_{\mathfrak{S}_{mn}}\mathscr{V}$$

Lemma 7.3. *There is a unique (up to isomorphism) bifunctor*

$$\diamond : \mathscr{C}_m \times \mathscr{C}_n \to \mathscr{C}_{mn}$$

with the following properties

(i) *The functor $\diamond$ preserves reflective coequalizers with respect to each variables;*
(ii) $\mathbf{h}_M \diamond \mathbf{h}_N = \mathbf{h}_{M \bullet N}$.

We define another bifunctor

$$\bar{\circ} : \mathscr{C}_m \times \mathscr{C}_n \to \mathscr{C}_{mn}.$$

Because $\blacktriangledown$ is a symmetric monoidal structure on $\oplus_{d \geq 0} \mathscr{C}_d$, the functor $\mathbf{g}^{\blacktriangledown m}$ has a natural action of $\mathfrak{S}_m$ for $\mathbf{g}$ in $\mathscr{C}_n$. We put (compare with [4]):

$$\mathbf{f} \bar{\circ} \mathbf{g} := \mathbf{f}(\mathbf{g}^{\blacktriangledown m}).$$

Lemma 7.4. *For all $\mathbf{f}$ in $\mathscr{C}_m$ and N in ${}_{\mathfrak{S}_n}\mathscr{V}$, there is a natural isomorphism:*

$$\mathbf{f} \diamond \mathbf{h}_N \cong \mathbf{f} \bar{\circ} \mathbf{h}_N.$$

In particular, $\mathbf{f} \diamond \mathbf{g}$ is an exact functor on $\mathbf{f}$, provided $\mathbf{g}$ is projective.

Proof. Since $\mathbf{f} \diamond \mathbf{g}$ is right exact on $\mathbf{f}$ and $\mathbf{f}\bar{\circ}\mathbf{g}$ is exact on $\mathbf{f}$, it is enough to consider the case $\mathbf{f} = \mathbf{h}_M$. Then:

$$\begin{aligned}
\mathbf{h}_{M\bullet N}(X) &= \mathrm{Hom}_{\mathfrak{S}_{mn}}(\mathsf{Ind}^{\mathfrak{S}_{mn}}_{\mathfrak{S}_n \wr \mathfrak{S}_m}(M \otimes N^{\otimes m}), X) \\
&= \mathrm{Hom}_{\mathfrak{S}_n \wr \mathfrak{S}_m}(M \otimes N^{\otimes m}, \mathsf{Res}^{\mathfrak{S}_{mn}}_{\mathfrak{S}_n \wr \mathfrak{S}_m}(X)) \\
&= \mathrm{Hom}_{\mathfrak{S}_m}(M, \mathrm{Hom}_{(\mathfrak{S}_n)^m}(N^{\otimes m}, \mathsf{Res}^{\mathfrak{S}_{mn}}_{(\mathfrak{S}_n)^m}(X))) \\
&= \mathbf{h}_M(\mathbf{h}_{N^{\odot m}})(X) \\
&= (\mathbf{h}_m \bar{\circ} \mathbf{h}_N)(X).
\end{aligned}$$

□

Proposition 7.5. (i) *For all* $\mathbf{f}$ *in* $\mathscr{C}_m$ *and* $\mathbf{g}$ *in* $\mathscr{C}_n$*, there is a natural isomorphism*

$$j^*(\mathbf{f}) \circ j^*(\mathbf{g}) \cong j^*(\mathbf{f} \diamond \mathbf{g}).$$

(ii) *For all* F *in* $\mathscr{P}_m$ *and* G *in* $\mathscr{P}_n$*, there is a natural isomorphism*

$$j_!(F) \diamond j_!(G) \cong j_!(F \circ G).$$

Proof. (i) Since j^* is right exact and $\diamond$ respects reflective coequalizers, it suffices to consider the case when $\mathbf{f} = \mathbf{h}_M$ and $\mathbf{g} = \mathbf{h}_N$. Then we have:

$$\begin{aligned}
j^*(\mathbf{f}) \circ j^*(\mathbf{g}) &\cong \mathrm{Hom}_{\mathfrak{S}_m}(M, (\mathrm{Hom}_{es_n}(N, V^{\otimes n}))^{\otimes m}) \\
&= \mathrm{Hom}_{\mathfrak{S}_m}(M, \mathrm{Hom}_{(\mathfrak{S}_n)^m}(N^{\otimes m}, \mathsf{Res}^{\mathfrak{S}_{mn}}_{(\mathfrak{S}_n)^m}(V^{\otimes mn})))
\end{aligned}$$

By the previous computation the last group is isomorphic to $\mathbf{h}_{M\bullet N}(V^{\otimes mn})$.

(ii) It is enough to assume that F and G are projective generators: $F = \Gamma^\mu$, $G = \Gamma^\nu$. We set $M = \mathbb{K}[\mathfrak{S}_m/\mathfrak{S}_\mu]$ and $N = \mathbb{K}[\mathfrak{S}_n/\mathfrak{S}_\nu]$. Then we have $j_!(F) = \mathbf{h}_M$ and $j_!(T) = \mathbf{h}_N$. Therefore $j_!(F) \diamond j_!(G) \cong \mathbf{h}_{M\bullet N}$. On the other hand

$$\begin{aligned}
F \circ G(V) = \Gamma^m u(\Gamma^\nu(V))\, &\mathrm{Hom}_{\mathfrak{S}_m}(M, (\mathrm{Hom}_{\mathfrak{S}_n}(N, V^{\otimes n}))^{\otimes m}) \\
&= \mathrm{Hom}_{\mathfrak{S}_m}(M, \mathrm{Hom}_{\mathfrak{S}_n \times \cdots \times \mathfrak{S}_n}(N^{\otimes m} \mathsf{Res}^{\mathbb{K}[\mathfrak{S}_{mn}]}_{\mathfrak{S}_n \times \cdots \times \mathfrak{S}_n}(V^{\otimes mn}))) \\
&= \mathrm{Hom}_{\mathfrak{S}_m \wr \mathfrak{S}_n}(M \otimes N^{\otimes m}, \mathsf{Res}^{\mathbb{K}[\mathfrak{S}_{mn}]}_{es_m \wr \mathfrak{S}_n}(V^{mn})) \\
&= \mathrm{Hom}_{\mathfrak{S}_{mn}}(N \bullet M, V^{\otimes mn})
\end{aligned}$$

It follows that: $j_!(F \circ G) = \mathbf{h}_{M\bullet N}$.

□

Since the functors $\mathbf{t}_M$, $M \in {}_{\mathfrak{S}_m}\mathscr{V}$ form the family of injective cogenerators, by duality we obtain the following result.

Proposition 7.6. *There is a unique bifunctor*

$$* : \mathscr{C}_m \times \mathscr{C}_n \to \mathscr{C}_{mn}$$

such that:

(i) $\mathbf{t}_M * \mathbf{t}_N = \mathbf{t}_{M\bullet N}$*;*
(ii) *the functor* $*$ *respect coreflective equalizers with respect to both variables;*
(iii) *for all* F *in* $\mathscr{P}_m$ *and* G *in* $\mathscr{P}_n$*, there is a natural isomorphism*

$$j_*(F) * j_*(G) \cong j_*(F \circ G),$$

(iv) *for all* $\mathbf{f}$ *in* $\mathscr{C}_m$ *and* $\mathbf{g}$ *in* $\mathscr{C}_n$*, there is a natural isomorphism*

$$j^*(\mathbf{f}) \circ j^*(\mathbf{g}) \cong j^*(\mathbf{f} * \mathbf{g}).$$

Appendix A. Recollement of abelian categories

To reveal the relationship between the different abelian categories, we use the language of recollements (see e.g. [3]). A recollement of abelian categories consists of a diagram of abelian categories and additive functors

$$\begin{tikzcd} \mathscr{A}' \arrow[r, "i_*"] & \mathscr{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathscr{A}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

satisfying the following conditions:

(i) the functor $j_!$ is left adjoint to j^* and the functor j^* is left adjoint of j_*;
(ii) the unit $Id_{\mathscr{A}''} \to j^*j_!$ and the counit $j^*j_* \to Id_{\mathscr{A}''}$ are isomorphisms;
(iii) the functor i^* is left adjoint of i_* and i_* is left adjoint of $i^!$;
(iv) the unit $Id_{\mathscr{A}'} \to i^!i_*$ and the counit $i^*i_* \to Id_{\mathscr{A}'}$ are isomorphisms;
(v) the functor $i_* : \mathscr{A}' \to \mathsf{Ker}(j^*)$ is an equivalence of categories.

Example A.1. The following example is the paradigm of a recollement situation. Let X be a space, C is a closed subset in X and $U = X \setminus C$ its open complement. Extension and restriction yield a recollement of sheaves categories:

$$\begin{tikzcd} \mathsf{Sh}(C) \arrow[r, "i_*"] & \mathsf{Sh}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathsf{Sh}(U) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} .$$

The list of properties (i)-(v) can be somewhat shortened.

Proposition A.2. *Let $j^* : \mathscr{A} \to \mathscr{A}''$ be an exact functor which satisfies* (i) *and* (ii)*: it admits both a left adjoint $j_!$ and a right adjoint j_*, and the unit $Id_{\mathscr{A}''} \to j^*j_!$ and counit $j^*j_* \to Id_{\mathscr{A}''}$ are isomorphisms. Let $\mathscr{A}'$ be the full subcategory of $\mathscr{A}$ with objects those A such that $j^*A = 0$. Then the full embedding $i_* : \mathscr{A}' \to \mathscr{A}$ has adjoint functors $(i^*, i^!)$ and the unit $Id_{\mathscr{A}'} \to i^!i_*$ and counit $i^*i_* \to Id_{\mathscr{A}'}$ are isomorphisms. In other words we have a recollement situation.*

Proof. Let A in $\mathscr{A}$ and let ϵ_A: $j_!j^*A \to A$ be the counit of the adjoint pair $(j_!, j^*)$. Because $Id_{\mathscr{A}''} \to j^*j_!$ is an isomorphism, we have $j^*(\mathrm{Coker}\,\epsilon_A) = 0$. It follows that $\mathrm{Coker}(\epsilon_A)$ lies in the subcategory $\mathscr{A}'$. So there is a well-defined functor $i^* : \mathscr{A} \to \mathscr{A}'$ such that $\mathrm{Coker}(\epsilon_A) = i_*i^*A$. The rest follows, using the short exact sequence of natural transformations:

$$j_!j^* \xrightarrow{\epsilon} Id_{\mathscr{A}} \to i_*i^* \to 0$$

and the dual study of the unit of adjonction η which sums up in the following exact sequence:

$$0 \to i_*i^! \to Id_{\mathscr{A}} \xrightarrow{\eta} j_*j^* .$$

□

Remark A.3. Actually, if $\mathscr{A}$ is a category of modules over a ring (or, more generally, if $\mathscr{A}$ is a Grothendieck category), then it is enough to assume that j^* is an exact functor which has a left adjoint functor $j_!$ such that the unit of adjonction: $Id_{\mathscr{A}''} \to j^*j_!$ is an isomorphism. The existence of j_* follows from [10, Proposition 2.2].

Example A.4. Recollements arise naturally when relating functor categories through precomposition. Indeed, starting with a functor $\mathbf{i} : \mathscr{A} \to \mathscr{B}$, precomposition is an exact functor:

$$j^* : \ \mathscr{V}^{\mathscr{B}} \to \mathscr{V}^{\mathscr{A}}$$
$$F \mapsto F \circ \mathbf{i}.$$

A classic result of D. Kan tells that it always admits adjoint functors, called the left and the right Kan extension. By [9, §X.3, Corollary 3], the unit and the counit of adjonction are isomorphisms when the functor **i** is a full embedding. A recollement situation then arises by Proposition A.2.

In the case when **i** is a full embedding of $\mathbb{K}$-linear categories, the functor j^* and its adjoints restrict to the subcategories of $\mathbb{K}$-linear functors. Proposition 4.9 describes the resulting recollement when the functor **i** is the full embedding of Lemma 2.1.

Another useful functor arises from a recollement: the functor $j_{!*} : \mathscr{A}'' \to \mathscr{A}$ is the image of the norm $N : j_! \to j_*$, the natural transformation which corresponds to 1_X, under the isomorphism

$$\mathrm{Hom}_{\mathscr{A}}(j_!X, j_*X) \cong \mathrm{Hom}_{\mathscr{A}''}(X, j^*j_*X) \cong \mathrm{Hom}_{\mathscr{A}''}(X, X).$$

The functor $j_{!*}$ preserves simple objects, and every simple object in $\mathscr{A}$ is either the image of a simple in $\mathscr{A}'$ by the functor i_*, or the image of a simple in $\mathscr{A}''$ by the functor $j_{!*}$.

We close with an immediate consequence of the Grothendieck spectral sequence for a composite functor.

Proposition A.5. *Assume in a recollement situation all abelian categories have enough projective objects. For X in $\mathscr{A}''$ and B in $\mathscr{A}$, there are spectral sequences:*

$$\mathrm{E}_2^{pq} = \mathrm{Ext}^p_{\mathscr{A}}(L_q j_!(X), B) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathscr{A}''}(X, j^*B)$$

and

$$\mathrm{E}_2^{pq} = \mathrm{Ext}^p_{\mathscr{A}}(B, R^q j_*(X)) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathscr{A}''}(j^*B, X).$$

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