Existence of Irreducible Representations for Homology Knot Complements with Nonconstant Equivariant Signature

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1 Introduction

The purpose of this paper is to show the existence of irreducible $SU(2)$ representations of homology knot complement fundamental groups near abelian representations where the equivariant knot signature changes. In [FK], Charles Frohman and Eric Klassen showed the existence of irreducible representations near abelian corresponding to square roots of simple roots of the Alexander polynomial, and they raised the question whether an analogous result holds for multiple roots. Our result shows existence of irreducibles for any knot whose equivariant signature is nonzero off of the square roots of roots of the Alexander polynomial.

The equivariant signature of a knot complement is defined as follows. Let Y be a homology knot complement, that is, a compact 3-manifold with torus boundary with the property that $H_*(Y;\mathbf{Z}) = H_*(S^1;\mathbf{Z})$. We choose a simple closed curve in ∂Y which represents a primitive element of $H_1(\partial Y;\mathbf{Z})$ in the kernel of the map $i_* : H_1(\partial Y;\mathbf{Z}) \to H_1(Y;\mathbf{Z})$. We will call this curve the longitude for Y and denote it by λ. We also choose a meridian μ, a simple closed curve in ∂Y which generates $H_1(Y;\mathbf{Z})$.

Let F be a Seifert surface, i.e., a surface in Y whose boundary is λ. Choose an orientation of the normal bundle of F in Y. If $\{x_i\}_{1\leq i\leq g}$ is a basis for $H_1(F;\mathbf{Z})$, let x_i^+ denote the pushoff of x_i in the positive direction. Finally, let V be the linking matrix whose entries are $V_{ij} = \ell k(x_i, x_j^+)$.

The symmetrized Alexander matrix for Y is the matrix $A(t) = t^{\frac{1}{2}}V - t^{-\frac{1}{2}}V^T$. We define $B(t) = (1-t)V + (1-t^{-1})V^T$. Note that $B(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})A(t)$, so the complex values of $t \neq \pm 1$ for which $B(t)$ is singular are exactly the roots of the Alexander polynomial $\Delta(t) = \det A(t)$.

If t is a unit complex number, then $B(t)$ is a skew hermitian matrix, and hence has only real eigenvalues. The equivariant knot signature of Y is the number of positive eigenvalues

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minus the number of negative eigenvalues for $B(t^2)$, counted with multiplicity. (See [KKR] or [H2] for details.) By the above comment, this signature is a map from $U(1)$ to $\mathbf{Z}$ which is continuous in $t \in U(1)$ except possibly at square roots of roots of the Alexander polynomial. Furthermore, $\mathrm{Sign}B(1) = 0$.

Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the standard orthonormal basis for $su(2)$ corresponding to the identification of $SU(2)$ with the space of unit quaternions. We will consider $U(1) = \{\exp(\mathbf{i}\theta)\} \subset SU(2)$, and we make the identifications $\mathrm{span}(\mathbf{i}) = \mathbf{R}$ and $\mathrm{span}(\mathbf{j}, \mathbf{k}) = \mathbf{C}$.

We now state the main result in this paper.

Theorem 1 *If the function $SignB(t) : U(1) \to \mathbf{Z}$ does not vanish on the complement of the set of unit roots of $\Delta(t)$, then there are irreducible representations $\rho : \pi_1(Y) \to SU(2)$. Furthermore, for any unit root $e^{i2\alpha}$ of $\Delta(t)$ where the right and left hand limits $\lim_{\beta \to \alpha^\pm} SignB(e^{i\beta})$ do not agree, there is a continuous family of irreducible representations limiting to the abelian one which takes μ to* $\exp(\mathbf{i}\alpha)$.

In the course of proving this we will also prove the following facts.

Corollary 2 *Suppose for some $0 < \theta < \pi$ the matrix $B(e^{i2\theta})$ has nontrivial kernel, and suppose that as $t \in U(1)$ moves through the value $t_0 = e^{i\theta}$, all eigenvalues of $B(t^2)$ crossing zero do so transversely, and all do so in the same direction. Then all the irreducible representations near the abelian one taking μ to $exp(\mathbf{i}\theta)$ send λ to $exp(\mathbf{i}\sigma)$ for some small $\sigma \neq 0$, where the sign of σ corresponds to the direction the eigenvalues go through* 0.

Corollary 3 *If κ is a knot and there exists any value $0 < \theta < \pi$ satisfying the above hypotheses, then for n sufficiently large, the homology spheres obtained by $\frac{1}{n}$ and $-\frac{1}{n}$ surgery on κ have nontrivial $SU(2)$ representations.*

Corollary 4 *Suppose κ is a knot in S^3 with only simple roots of the Alexander polynomial and suppose that the irreducible part of the character variety consists only of arcs (non-closed components) whose images in the character variety of the boundary torus all wrap monotonically around it. All torus knots, for example, satisfy this hypothesis. If X is the homology 3-sphere obtained by $\frac{1}{n}$-surgery on κ, then the absolute value of Casson's invariant for X is equal to twice the number of irreducible points in the character variety for X. That is to say, all these points count with the same sign when added to give the Casson invariant.*

The proof of the main theorem has the following outline. We begin by identifying the set of representations of $\pi_1(Y)$ into $SU(2)$ modulo conjugation with the moduli space of flat $SU(2)$ connections on $Y \times SU(2)$ modulo gauge equivalence by taking a representation ρ to the flat orbit which has ρ as its holonomy representation. We next show that there are arbitrarily small perturbations of the flatness equation for which there are irreducible orbits in the perturbed flat moduli space. Finally we make a limiting argument to show this property holds for the unperturbed flat moduli space.

The paper is organized as follows. Section 2 contains basic results about perturbing the flatness equation and the perturbed flat moduli space for 3-manifolds with torus boundary. Subsection 3.1 contains a statement of the basic existence theorem for irreducibles under certain assumptions of nondegeneracy. Subsection 3.2 provides a proof of this result. A proof of Corollary ?? is also given in this subsection. Section 4 then gives proofs of our main result along with Corollaries 3 and 4.

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2 The Structure of the Flat Moduli Space

We begin by describing $SU(2)$ gauge theory on 3-manifolds with torus boundary, recalling results from [H1].

Let $\mathcal{A}$ denote the space of connections on $Y \times SU(2)$. Given a fixed trivialization of this principle bundle, we may identify $\mathcal{A}$ with the space of $su(2)$ valued 1-forms on Y, $\Omega^1(Y; su(2))$. We complete this space using the L^2_2 Sobolev norm. Let $\mathcal{G} = \mathrm{Aut}(Y \times SU(2))$ be the gauge group, with the L^2_3 completion. To each connection A is associated its curvature 2-form, $F(A) = dA + A \wedge A$, and A is said to be flat if $F(A) = 0$.

The flat moduli space is the quotient $\mathcal{M} = F^{-1}(0)/\mathcal{G}$. There is a standard method of perturbing the flatness equation in order to obtain a moduli space which is nondegenerate (nondegenerate will be given a precise definition below), given by Taubes [T] and Floer [F]. We sketch it below; see [H1] for more details.

Let $\{\gamma_i : S^1 \times D^2 \to Y\}_{1 \leq i \leq n}$ be a collection of embeddings of the solid torus into Y whose images are disjoint. Let η be a radially symmetric bump function on D^2 with support in the interior of D^2, multiplied by the standard volume form. Let $\{\overline{h}_i : \mathbf{R} \to \mathbf{R}\}_{1 \leq i \leq n}$ be a collection of smooth functions. Let $\mathrm{tr\ hol}_{\gamma_i}(x, A)$ be the trace of the holonomy of the connection A around the curve $\gamma_i(S^1 \times \{x\})$. We define a function $h : \mathcal{A} \to \mathbf{R}$ by the formula

$$h(A) = \sum_{i=1}^{n} \int_{D^2} \overline{h}_i(\mathrm{tr\ hol}_{\gamma_i}(x, A))\eta(x).$$

We call functions h constructed in this way admissible perturbation functions.

Now fix a Riemannian metric on Y, and let $*$ denote the Hodge star operator on $su(2)$ valued forms. Given an admissible perturbation function h, we call a connection perturbed flat if it satisfies the equation

$$\zeta_h(A) \stackrel{def}{=} *\frac{1}{2\pi}F(A) + \nabla h(A) = 0,$$

where ∇ denotes the L^2 gradient of h. We can now define the perturbed flat moduli space by

$$\mathcal{M}_h = \zeta_h^{-1}(0)/\mathcal{G}.$$

The structure of the perturbed flat moduli space for a 3-manifold with boundary was described in [H1]. We summarize the results below. Let $*d_{A,h} \stackrel{def}{=} *\frac{1}{2\pi}d_A + \mathrm{Hess}h(A)$. Let $\mathcal{H}^1_{A,h}(Y;su(2))$ and $\mathcal{H}^1_{A,h}(Y,\partial Y;su(2))$ be (the harmonic spaces representing) the first and second cohomology groups of the following elliptic complex (where the grading goes 0,1,2,3):

$$0 \to \Omega^0(Y;su(2)) \xrightarrow{d_A} \Omega^1(Y;su(2)) \xrightarrow{*d_{A,h}} \Omega^1(Y;su(2)) \xrightarrow{d_A^*} \Omega^0(Y;su(2)) \to 0.$$

Let $\mathcal{M}_h^*$, $\mathcal{M}_h^{U(1)}$, and $\mathcal{M}_h^{SU(2)}$ denote the portions of $\mathcal{M}_h$ consisting of irreducible, abelian (noncentral), and central orbits, respectively. We will say that $\mathcal{M}_h$ is nondegenerate if it satisfies the following 5 properties (and otherwise degenerate):

(a) The only orbits in $\mathcal{M}_h$ which are central when restricted to ∂Y are central on Y.

(b) At every $[A] \in \mathcal{M}_h^{SU(2)}$, $\mathcal{H}^1_{A,h}(Y,\partial Y;su(2)) = 0$.

(c) At all but finitely many orbits $[A] \in \mathcal{M}_h^{U(1)}$, $\dim\mathcal{H}^1_{A,h}(Y,\partial Y;su(2)) = 0$, and the remaining abelian orbits $\dim\mathcal{H}^1_{A,h}(Y,\partial Y;su(2)) = 2$.

(d) At each abelian orbit where there is nontrivial relative first cohomology the family of symmetric matrices H_t (defined below) has transverse spectral flow.

(e) For each $[A] \in \mathcal{M}_h^*$, dim $\mathcal{H}^1_{A,h}(Y,\partial Y;su(2)) = 1$.

To define the matrix H_t in condition (d) we first choose a family of connections $[A_t] \in \mathcal{M}_h^{U(1)}$ with $\mathcal{H}^1_{A_0,h}(Y,\partial Y;su(2))$ nonzero. For the moment let V denote the orthogonal complement of $T_{[A_0]}\mathcal{M}_h^{U(1)}$ in $\mathcal{H}^1_{A_0,h}(Y;su(2))$. Then we define a 1-parameter family of bilinear forms on V by the formula

$$H_t(\alpha,\beta) = \langle *d_{A_t,h}\alpha, \beta\rangle.$$

H_t is symmetric and $U(1)$ invariant.

When $h = 0$, a jump in $\mathcal{H}^1_A(Y;su(2))$ occurs (for abelian connections) exactly at connections where, up to gauge, $\mathrm{hol}_\mu A = \exp(\mathbf{i}\theta)$ and $\mathrm{hol}_\lambda A = 1$ where $B_\kappa(e^{\mathbf{i}}2\theta)$ is degenerate, and Propositions 9 and 11 of [H2] imply that the spectral flows of H_t and $B_\kappa(e^{\mathbf{i}}2\theta)$ through such points are equal.

Theorem 5 *(Theorem 15, [H1])*
If $\mathcal{M}_h$ is nondegenerate, then $\mathcal{M}_h$ is compact. $\mathcal{M}_h^{SU(2)}$ consists of 2 points. $\mathcal{M}_h^{U(1)}$ is a smooth 1-dimensional manifold, compact except for one open end limiting to each central orbit. $\mathcal{M}_h^*$ is a 1-dimensional manifold, compact except for open ends which limit to distinct points on $\mathcal{M}_h^{U(1)}$, i.e. points where dim $\mathcal{H}^1_{A,h}(Y,\partial Y;su(2)) = 2$. Each such abelian orbit where the relative cohomology jumps is the limit of exactly one such irreducible end.

Remark: The last claim is the foundation of our existence result. There is a gap in the proof of this in [H1], so after describing some more background, we state this claim as Theorem 11 and provide a complete proof of this in Section 3.

The flat moduli space for the torus is equal to $\mathcal{M}_{T^2} = T^2/\mathbf{Z}_2$, known as the pillowcase. It is topologically a 2-sphere, but has 4 "corners" corresponding to the central orbits, i.e., the fixed points under the involution. There is a restriction map $r : \mathcal{M}_h \to \mathcal{M}_{T^2}$.

Theorem 6 *(Theorem 15, [H1]) Suppose $\mathcal{M}_h$ is nondegenerate. Then r is an immersion on each stratum. Then there is a double cover $\tilde{\mathcal{M}}_h$ of $\mathcal{M}_h$ and a map to the double cover $\tilde{\mathcal{M}}_{T^2} = T^2$ of $\mathcal{M}_{T^2}$, where both are branched over the central orbits. Furthermore, given an orientation on $H_1(Y;\mathbf{R})$ there is a canonical orientation on the 1-dimensional strata of $\tilde{\mathcal{M}}_h$. The $\mathbf{Z}_2$ action is orientation reversing on $\tilde{\mathcal{M}}_h$ (but orientation preserving on $\tilde{\mathcal{M}}_{T^2}$).*

Remark: By Proposition 17 of [H2], the absolute value of the Casson invariant of the $\frac{1}{n}$-surgery on κ may be computed in terms of an oriented intersection number of $\tilde{\mathcal{M}}^*_h$ with a particular oriented curve in $\tilde{\mathcal{M}}_{T^2}$. This fact is used in the proof of Corollary 4.

Given a flat abelian connection A, let Sym_A denote the set of symmetric bilinear forms on $\mathcal{H}^1_A(Y, \partial Y; su(2))$ which are Stab$A$ invariant. For any loop ℓ in Y, let $\mathrm{Hess}_A\mathrm{tr}\ \mathrm{hol}_\ell$ be the element of Sym_A given by the Hessian of $\mathrm{tr}\ \mathrm{hol}_\ell(A)$ restricted to V.

Proposition 7 *(Lemma 38, Lemma 60, and Theorem 15, [H1]) There is a finite collection of embedded loops $\{\ell_i\}_{1\leq i\leq n}$ with the following properties:*

1. *For all flat connections A the map $D\ tr\ hol_{\ell_I} : \mathbf{R}^n \to Hom(\mathcal{H}^1_A(Y; su(2)), \mathbf{R})$ given by*
$$D\ tr\ hol_{\ell_I}(A)(b_1, \dots, b_n)(\alpha) = \sum_{i=1}^{n} b_i\ D(tr\ hol_{\ell_i}(A))(\alpha)$$
is surjective.

2. *For all abelian flat connections A the map $D\ tr\ hol_{\ell_I} \oplus Hess_A tr\ hol_{\ell_I} : \mathbf{R}^n \to Hom(\mathcal{H}^1_A(Y; su(2)), \mathbf{R}) \oplus Sym_A$ is surjective, where the second component takes $(b_1, \dots, b_n)$ to $\sum_{i=1}^{n} b_i Hess_A tr\ hol_{\ell_i}(A)$.*

Choose a collection of loops $\{\ell_i\}$ as in the previous proposition, and let $\{\gamma_i\}$ be a corresponding collection of maps of solid tori onto disjoint tubular neighborhoods of the loops. Let $\overline{\mathcal{E}} = C^2(\mathbf{R}, \mathbf{R})$. Let $\mathcal{E} = \overline{\mathcal{E}}^n$, and let $\mathcal{E}_1 \subset \mathcal{E}$ be the subset consisting of n-tuples $(\overline{h}_1, \dots, \overline{h}_n)$ which give rise admissible perturbations h for which the perturbed flat moduli space is degenerate.

Theorem 8 *(Theorem 15, [H1]) There is a neighborhood U of $(0, \dots, 0) \in \overline{\mathcal{H}}^n$ such that $U_1 = \mathcal{E}_1 \cap U$ has codimension 1.*

For any path $h_t : [0, \epsilon] \to U$, define
$$\mathcal{M}_{\{h_t\}} = \{([A], t) \in \mathcal{A}/\mathcal{G} \times [0, \epsilon] |\ \zeta_{h_t}(A) = 0\}.$$

Proposition 9 *(Proposition 49, [H1]) $\mathcal{M}_{\{h_t\}}$ is compact.*

3 Existence of Irreducible Orbits in the Nondegenerate Case

3.1 Statement of the theorem and some comments

Let h be a perturbation with $\mathcal{M}_h$ nondegenerate. Assume that the abelian stratum of $\mathcal{M}_h$ is as in Theorem 5. Let $[A_t] \in \mathcal{M}_h^{U(1)}$ be a path with $[A_0]$ a point with $\mathcal{H}^1_{A_0,h}(Y, \partial Y; su(2))$ nonzero. We define a path H_t in Sym_{A_0} as follows.

$$H_t(\alpha, \beta) \stackrel{def}{=} \langle *d_{A_t,h}\alpha, \beta\rangle.$$

Proposition 10 *(Proposition 6 of [H2]) For each t the matrix H_t is equal to a real valued function $\lambda(t)$ multiplied by the identity matrix.*

Our nondegeneracy requirement (d) insures that the $\lambda(t)$ has a transverse zero at $t = 0$. The existence theorem for irreducibles in the nondegenerate situation is the following.

Theorem 11 *Suppose that $\mathcal{M}_h$ satisfies nondegeneracy conditions (c) and (d). Then there is a neighborhood $U \subset \mathcal{B}(Y)$ of $[A_0]$ such that $U \cap \mathcal{M}^*_h(Y)$ is a smooth arc with one open end limiting to $[A_0]$.*

A proof of this theorem by a somewhat indirect route is the content of the next subsection. We conclude this subsection by mentioning the subtlety standing in the way of a more direct proof.

The perturbed flat moduli space near $[A_0]$ is homeomorphic to the zero set of the Kuranishi map $\Phi : \mathcal{H}^1_{A_0,h}(Y; su(2)) \cong \mathbf{R} \oplus \mathbf{C} \to \mathcal{H}^1_{A_0,h}(Y, \partial Y; su(2)) \cong \mathbf{C}$. Φ is defined as follows (see [H1], Section 6.4, for details). The implicit function theorem gives a map $\psi : \mathcal{H}^1_{A_0,h}(Y; su(2)) \to *d_{A_0,h}\Omega^1_\nu(Y; su(2))$ (here ν denotes the Neumann boundary conditions) with the property that $\Pi_{\ker d^*_{A_0}} \zeta_h(A + \alpha + \psi(\alpha)) \in \mathcal{H}^1_{A_0,h}(Y, \partial Y; su(2))$. The map Φ is defined by $\Phi(\alpha) = \Pi_{\ker d^*_{A_0}} \circ \zeta_h(A + \alpha + \psi(\alpha))$. These maps are $\mathrm{Stab}A_0 \cong U(1)$ equivariant. The linearization of Φ at $(t, 0)$ in the $\mathbf{C}$ direction, composed with inclusion of relative cohomology into absolute, is equal to H_t. One would like to argue that Φ is a 1-parameter family of gradient vector fields on $\mathbf{C}$ (here we identify the relative and absolute $\mathbf{C}$ valued cohomology through the inclusion of relative into absolute) and hence must look, up to change of coordinates, like the family of gradient vector fields coming from the function $f(t, z) = t|z|^2$.

The situation is complicated by the fact that we don't know that $\Phi(t, z)$ is really be a gradient vector field on $\mathbf{C}$. Recall from [H1] that $\frac{1}{2\pi} * F(A) + \nabla h(A)$ is not the L^2 gradient of a function on $\mathcal{A}$, but rather the gradient of a section of a $U(1)$ bundle defined with respect to a connection on that bundle. This connection (restricted to the graph of the function $\psi : H^1_{A_0,h}(Y; su(2)) \to *d_{A_0,h}\Omega^1_\tau(Y; su(2)))$ may not be flat, and hence the gradient with respect to this connection may not in fact be a conservative vector field.

The problem is to rule out a family of vector fields on $\mathbf{C}$ like

$$(t,x,y) \mapsto (tx - y(x^2+y^2)^n, ty - x(x^2+y^2)^n),$$

which has the same linearization along $\mathbf{R}\oplus\{0\}$, and is $U(1)$ equivariant, but has no zeros off of $\mathbf{R}\oplus\{0\}$. The existence of such functions was pointed out to me by Eric Klassen.

Although we certainly expect there to be a direct proof, we avoid this difficulty by using a somewhat different argument. We consider the closed manifold obtained by 0-surgery on the knot whose complement is Y. We then demonstrate in this setting, where $\frac{1}{2\pi} * F(A) + \nabla h(A)$ is truly the L^2 gradient of $cs(A)+h(A) : \mathcal{A} \to \mathbf{R}$, a family of irreducible connections whose restrictions to the knot complement Y are flat and limit to the orbit $[A_0]$ as required. This approach has the added benefit that we learn something about the image of the nearby irreducibles in the pillowcase.

3.2 The picture on Y_0

In this subsection we consider connections on a closed manifold Y_0. We will begin with a completely general description of the perturbed flat moduli space near a flat connection (with no assumptions of nondegeneracy), and then add three assumptions, one at a time, in the course of the subsection. In particular, we wait as long as possible to impose the assumption that the second part of nondegeneracy condition (c) is satisfied, in order to prove Corollary 19. This corollary is an ingredient in the proofs of Corollaries 3 and 4.

As before when we considered manifolds with boundary, let $\zeta_h(A)$ denote $\frac{1}{2\pi}F(A) + \nabla h(A)$, and let $X_A = \{A_0 + a \mid d^*_{A_0} a = 0\}$. For any closed subspace $W \subset \Omega^1(Y; su(2))$ we let Π_W denote the orthogonal projection onto W. The next lemma describes the Kuranishi picture for the perturbed flat moduli space near $[A_0]$. For a proof see [H1], Section 6.4, or [MMR], Section 12.1.

Lemma 12 *Let A_0 be a smooth perturbed flat connection. There exist:*

(a) *a* $\mathrm{Stab}(A_0)$ *equivariant neighborhood V_{A_0} of 0 in $\mathcal{H}^1_{A_0,h}(Y_0; su(2))$,*

(b) *a $\mathcal{G}$ equivariant neighborhood U_{A_0} of A_0 in $\mathcal{A}$,*

(c) *a* $\mathrm{Stab}(A_0)$ *equivariant real analytic embedding*

$$\phi : V_{A_0} \to X_A$$

whose differential at 0 is just the inclusion of $\mathcal{H}^1_{A_0,h}(Y_0; su(2))$ into $\ker d^*_{A_0} \cap \Omega^1(Y_0; su(2))$,

(d) *and a* $\mathrm{Stab}(A_0)$ *equivariant map*

$$\Phi : V_{A_0} \to \mathcal{H}^1_{A_0,h}(Y_0; su(2))$$

such that ϕ maps $\Phi^{-1}(0)$ homeomorphically onto the zero set of $\zeta_h|_{X_A \cap U_A}$.

An important point for us will be that the map ϕ is defined by $\phi(\alpha) = A_0 + \alpha + \psi(\alpha)$ where $\psi(\alpha) \in *d_{A_0}\Omega^1(Y_0; su(2))$ solves

$$\Pi_{*d_{A_0}\Omega^1(Y_0;su(2))}(*d_{A_0}\psi(\alpha) + *\frac{1}{2}[\alpha + \psi(\alpha) \wedge \alpha + \psi(\alpha)]) = 0.$$

In other words, the graph of ψ has the property that for any $\alpha \in V_{A_0}$,

$$\zeta_h(A_0 + \alpha + \psi(\alpha)) \perp *d_{A_0,h}\Omega^1(Y_0; su(2)).$$

Proposition 13 *For any $\alpha \in V_{A_0}$, $\zeta_h(A_0+\alpha+\psi(\alpha)) = 0$ if and only iff $\nabla(cs+h)|_{\phi(V_{A_0})}(A_0+\alpha + \psi(\alpha)) = 0$. Here ∇ denotes the L^2 gradient, of course.*

Proof: By a standard argument, the first condition is equivalent to $\Pi_{\mathcal{H}^1_{A_0,h}(Y_0;su(2))}(\nabla(cs + h)(A_0 + \alpha + \psi(\alpha))) = 0$. The lemma now follows from the fact that the tangent space to the graph of ψ projects onto the tangent space of its domain. □

Assumption 1: Assume that Y_0 is the zero framed surgery on the knot complement Y. Let $[A_0] \in \mathcal{M}_h^{U(1)}$ and assume that $\mathcal{H}^1_{A_0,h}(Y; \mathbf{R}) \cong \mathbf{R}$, which guarantees that $\mathcal{M}_h^{U(1)}(Y)$ meets nondegeneracy condition (c) near $[A_0]$. Finally, assume that $\mathcal{H}^1_{A_0,h}(Y; \mathbf{C})$ is nonempty, and the graphs of the eigenvalues of the family of bilinear forms H_t defined earlier are transverse to zero at $t = 0$ and all cross it in the same direction.

Proposition 14 *There is an additional perturbation which does not change the topological structure of $\mathcal{M}_h(Y)$ near $[A_0]$, but changes its image in the pillowcase by a diffeomorphism of the pillowcase (minus the corners) in such a way that the new abelian arc lines up with the flat connections on the Dehn filling in Y_0. That is to say, we may assume after this perturbation that all the perturbed flat abelian connections on Y near $[A_0]$ extend as perturbed flat connections over Y_0.*

Proof: The proof is not hard using the description of perturbed flat connections in Lemma 61, [H1]. We outline it below. If the abelian stratum around $[A_0]$ maps into the pillowcase to a curve which is transverse to the circles $\{\mathrm{hol}_\mu = \text{constant}\}$, then by doing an additional perturbation using a perturbation curve in a tubular neighborhood of the boundary torus ∂Y with trivial framing we can make this piece of the abelian stratum lie on the $\{\mathrm{hol}_\lambda = \mathrm{id}\}$ arc in the pillow case. Specifically, let the core of the perturbation curve be a meridian and a framing curve is a parallel meridian in the same $T^2 \subset T^2 \times [0, 1]$, and choose the function of trace appropriately.

If the tangent direction to the abelian stratum at $[A_0]$ is vertical, then first do a perturbation using a trivially framed longitude to tip it slightly so that it satisfies the former hypothesis. Then perturb as above. The crucial fact is that perturbations using trivially framed longitudes and meridians change the picture of $r : \mathcal{M}_h \to \mathcal{M}_{T^2}$ by diffeomorphisms of $\mathcal{M}_{T^2} \setminus \{\text{centrals}\}$, so this doesn't affect the existence or nonexistence of an arc of irreducibles limiting to $[A_0]$, nor does it affect whether the image of these irreducibles coincides with the image of the abelians. □

Proposition 15 *The additional perturbation in the Proposition 14 does not alter the cohomology at A_0 nor does it affect the transversality condition on the eigenvalues of H_t.*

Proof: We may identify Y with the additional perturbation, $(Y, h + h')$, with the union $(Y, h) \cup_{T^2 \times \{0\}} (T^2 \times [0,1], h')$.

We sketch the proof, leaving the details as an exercise for the reader. A_0 extends uniquely (up to gauge) over $Y \cup (T^2 \times [0,1])$ to a perturbed flat connection. We will use the same notation for this extension.

$\mathcal{H}^1_{A_0,h'}(Y^2 \times [0,1]; su(2))$ is two dimensional, and the restriction map to the cohomology of either boundary component is a surjection. The way to see this (in the harder case, when h' consists of two perturbation curves) is to consider first $\mathcal{H}^1_{A_0,h'}(T^2 \times [0,1] \setminus \{\text{the two perturbation curves}\}; su(2))$, which equals the ordinary real cohomology of this space with $\mathbf{R}$ coeffients (4-dimensional). Then use the Mayer Vietoris argument to check that the subspace consisting of cohomology classes whose restrictions to the boundaries of the perturbation curves lie in the image of the (perturbed) cohomology on the solid tori has the required properties. (Note that the second claim does not contradict the fact that the image of $\mathcal{H}^1_{A_0,h'}(T^2 \times [0,1]; su(2))$ must map to a Lagrangian subspace of the direct sum $\mathcal{H}^1_{A_0,h'}(T^2 \times \{0\} \cup T^2 \times \{1\}; su(2))$ with its symplectic structure. Recall that this symplectic structure will be the difference of the pull backs of the two pillowcase symplectic structures because the orientations on the tori differ, and then this Lagrangian property is also easily verified.)

The Mayer Vietoris sequence applied to $Y \cup T^2 \times [0,1]$ now implies that $\mathcal{H}^1_{A_0,h+h'}(Y \cup T^2 \times [0,1]; su(2)) \cong \mathcal{H}^1_{A_0,h}(Y; su(2))$, and similarly for relative first cohomology. In addition, it implies that relative 1-dimensional classes on the union are represented by forms which are exact on $T^2 \times [0,1]$. The signs of the derivatives of the eigenvalues of H_t as they pass through 0 is detected by a cohomology pairing on $\mathcal{H}^1_{A_0,h+h'}(Y \cup T^2 \times [0,1])$ (see [H2], Section 5, for a similar argument), which then must agree for Y and $Y \cup T^2 \times [0,1]$. □

By Propositions 14 and 15, from the point of view of studying irreducibles near $[A_0]$, we can without any loss add the following condition to our first assumption.

Assumption 2: The abelian perturbed flat connections on Y in the arc through $[A_0]$ extend over the 0-surgery Y_0. Furthermore, $H_t(Y_0)$ has transverse spectral flow at $t = 0$, all in the same direction.

Remark: This is not a generic situation; reducible and irreducible orbits on Y_0 are isolated for generic perturbations. We are deliberately putting ourselves in this degenerate situation. Also, there is nothing special about the 0-surgery. We choose this particular Dehn filling simply because in the unperturbed case, there is no perturbation required for the abelians on Y to extend over this closed 3-manifold.

For the remainder of this subsection we will work on Y_0, and the connections, Chern-Simons function, etc., are on this closed 3-manifold unless otherwise specified. For perturbed abelian flat connections, for example A_0, we will use the same notation to denote the connection on Y_0 and its restriction to Y.

We can assume after gauge transformation that A_0 takes values in the a fixed 1-dimensional subspace $\mathbf{R} \subset su(2)$, and then the stabilizer $U(1)$ action on $su(2)$ is compatible with our decomposition $su(2) = \mathbf{R} \oplus \mathbf{C}$. The space of $su(2)$ valued forms, and in particular the perturbed flat de Rham cohomology, decompose accordingly. That is, $\mathcal{H}^1_{A_0,h}(Y_0; su(2))$ reduces under the action of $\mathrm{Stab}(A_0) \cong U(1)$ to $\mathcal{H}^1_{A_0,h}(Y_0; \mathbf{R}) \oplus \mathcal{H}^1_{A_0,h}(Y_0; \mathbf{C})$.

Proposition 16 *Any perturbed flat abelian connection on any 3-manifold is gauge equivalent to a smooth connection.*

Proof: First assume that $h = 0$. Then we can assume, after gauge transformation that $A \in \Omega^1(Y; \mathbf{R})$. Since $dA = 0$, $A = \alpha + da$ for some harmonic form α (smooth by elliptic regularity) and a 0-form a. Let g be the gauge transformation $g = \exp(-a)$. Then $gA = g^{-1}Ag + g^{-1}dg = A - da = \alpha$..

If $h \neq 0$, then A is still gauge equivalent to a smooth connection off the perturbation solid tori. On the solid tori, A can be put into the canonical form described in Corollary 62, [H1], which is smooth. □

Assume A_0 is smooth and consider the Kuranishi picture near A_0. Let t be a real coordinate on $\mathcal{H}^1_{A_0,h}(Y_0; \mathbf{R})$. Then the abelian stratum near $[A_0]$ is parameterized by $A_t = A_0 + t + \psi(t, 0)$.

Let $H_t(Y_0)$ denote the symmetric bilinear form on $\mathcal{H}^1_{A_0,h}(Y_0; \mathbf{C})$ given by

$$H_t(Y_0)(\alpha, \beta) = \langle *d_{A_t,h}\alpha, \beta\rangle_{L^2(Y_0)}.$$

By a similar argument to the proof of Proposition same hess after realign, the spectral flow of this bilinear form concides with the one on Y.

Let $M = \phi(V_{A_0})$. Rather than work with $\zeta_h|_M$ directly, we consider the gradient vector field $\xi : \mathcal{H}^1_{A_0,h}(Y_0; su(2)) \to \mathcal{H}^1_{A_0,h}(Y_0; su(2))$ of the function $(cs+h)\circ\phi : \mathcal{H}^1_{A_0,h}(Y_0; su(2)) \to \mathbf{R}$ given by $\alpha \mapsto (cs + h)(A_0 + \alpha + \psi(\alpha))$.

Proposition 17 *The linearization of ξ at $(t, 0, \ldots, 0)$ in the $\mathbf{C}^n$ direction agrees with H_t to the order of $\|t\|^2_{L^2_2}$. In particular, these two symmetric bilinear forms on $\mathcal{H}^1_{A_0,h}(Y_0; \mathbf{C})$ have the same spectral flow and the transversality requirement on the eigenvalues of H_t implies the same for the Hessian of $(cs + h) \circ \phi$.*

Proof: For any $\alpha \in \mathcal{H}^1_{A_0,h}(Y_0; su(2))$, $\xi(\alpha) = \Pi_{TM}\Phi(\alpha)$. Recall that

$$\Phi(\alpha) = \Pi_{\ker d^*_{A_0}}(*d_{A_0,h}\psi(\alpha) + *\frac{1}{2}[\alpha + \psi(\alpha) \wedge \alpha + \psi(\alpha)]).$$

The linearization of Φ at $\alpha = (t, 0)$ is

$$\Phi_*(t,0)(\alpha) = \Pi_{\ker d^*_{A_0}}(*d_{A_0,h}\psi_*(t,0)(\alpha) + *[(t,0) + \psi(t,0) \wedge \alpha + \psi_*(t,0)(\alpha)]),$$

and $\xi_*(\alpha + \psi_*(t,0)(\alpha)) = \Pi_{TM}\Phi_*(t,0)(\alpha)$. Thus we can compute

$$\begin{aligned}\langle \xi_*(\alpha + \psi_*(t,0)(\alpha)), \beta + \psi_*(t,0)(\beta)\rangle &= \langle *[(t,0) + \psi(t,0) \wedge \alpha], \beta\rangle + \\ &\quad \langle *[(t,0) + \psi(t,0) \wedge \psi_*(t,0)(\alpha)], \beta\rangle \\ &\quad + \langle *[(t,0) + \psi(t,0) \wedge \alpha], \psi_*(t,0)(\beta)\rangle \\ &\quad + \langle *[(t,0) + \psi(t,0) \wedge \psi_*(t,0)(\alpha)], \psi_*(t,0)(\beta)\rangle.\end{aligned}$$

The first term is exactly $H_t(\alpha,\beta)$.

Since ψ is a real analytic map and $\psi(0,0) = 0$ and $\psi_*(0,0) = 0$, there is a constant C such that whenever $\|t\|_{L^2_2} \le 1$,

$$\|\psi(t,0)\|_{L^2_2} \le C\|t\|^2_{L^2_2}$$

and

$$\|\psi_*(t,0)(\alpha)\|_{L^2_2} \le C\|t\|_{L^2_2}\|\alpha\|_{L^2_2}.$$

In the following we will use the same letter C to denote any constant which only depends on A_0. By the Multiplication Theorem for Sobolev spaces,

$$\| * [t + \psi(t,0) \wedge \alpha]\|_{L^2} \le C(\|t\|_{L^2_1} + \|\psi(t,0)\|_{L^2_1})\|\alpha\|_{L^2_1} \le C\|t\|^2_{L^2_2}\|\alpha\|_{L^2_2}.$$

Similarly,

$$\| * [t + \psi(t,0) \wedge \psi_*(t,0)(\alpha)]\|_{L^2} \le C\|t\|^2_{L^2_2}\|\alpha\|_{L^2_2}.$$

Arguing in this manner gives the necessary bounds on the remaining terms. □

Proposition 18 *Under assumptions (1) and (2), there are no irreducibles on Y_0 near $[A_0]$.*

Proof: Let $\lambda_i(t)$, $i = 1,\ldots,n$, be the eigenvalues of $H_t(Y_0)$. In terms of the coordinates above

Let $(x_1(t), y_1(t), \ldots, x_n(t), y_n(t))$ be 1-parameter family of real coordinates on $\mathcal{H}^1_{A_0,h}(Y_0;\mathbf{C})$ corresponding to a basis of eigenvectors (depending on t, but we suppress this from the notation) $\{\vec{x}_1, \vec{y}_1, \ldots, \vec{x}_n, \vec{y}_n\}$ such that $J\vec{x}_i = \vec{y}_i$ and $\vec{x}_i$ and $\vec{y}_i$ are $\lambda_i(t)$ eigenvectors of $H_t(Y_0)$.

This gives a set of coordinates $(t, x_1, y_1, \ldots, x_n, y_n)$ on $\mathcal{H}^1_{A_0,h}(Y_0; su(2))$. Let

$$r(t, x_1, y_1, \ldots, x_n, y_n) = (x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2)^{\frac{1}{2}}.$$

Then

$$cs \circ \phi(t, x_1, y_1, \ldots, x_n, y_n) = \sum_{i=1}^{n} \frac{\lambda_i(t)}{2}(x_i^2 + y_i^2) + O(r^3).$$

Thus the t component of the gradient ξ is $\sum_{i=1}^n \frac{\lambda_i'(t)}{2}(x_i^2+y_i^2) + O(r^3)$, which has no zeros for $r > 0$ very small. □

Corollary 19 *Under the assumption (1), $[A_0]$ (on Y) has a neighborhood U in $\mathcal{M}_h(Y)$ such that $r(U \cap \mathcal{M}_h^{U(1)}(Y))$ and $r(U \cap \mathcal{M}_h^*)$ are disjoint in $\mathcal{M}_{T^2}$.*

Remark: The assumption of transverse spectral flow is necessary. In fact, if Y is the complement of the connect sum of a right handed trefoil and a left handed trefoil, for example, then there are components of $\mathcal{M}^*(Y)$ which limit to orbits in $\mathcal{M}^{U(1)}(Y)$ and whose image in $\mathcal{M}_{T^2}$ coincides with that of (part of) $\mathcal{M}^{U(1)}(Y)$. In this example, the total spectral flow of H_t through these abelian limit points is zero. By taking a sum of two right handed and one left handed trefoils, however, we get an example of this behavior where the spectral flow is algebraically nonzero.

We now make our final assumption to prove Theorem 11.

Assumption 3: Suppose now, in addition, $\mathcal{H}^1_{A_0,h}(Y, \partial Y; su(2))$ has complex dimension 1. By Theorem 5 this is the case at each abelian orbit where this cohomology is nonempty for generic h.

Proof of Theorem 5: When the extra cohomology at A_0 is only of complex dimension 1, the $U(1)$ invariance of the functions becomes a much stronger condition on the function $(cs+h)\circ\phi$, namely that it depends only on t and r. Specifically, $(cs+h)\circ\phi(t, r\cos\theta, r\sin\theta) = (cs+h)\circ\phi(t, r, 0) \stackrel{def}{=} f(t, r)$.

To complete the proof, we perturb slightly once again, so that the abelian parts of $\mathcal{M}_h(Y)$ and $\mathcal{M}_{S^1\times D^2}$ no longer match up. For simplicity, we leave the existing perturbation on Y alone and add a new function of trace of holonomy around the Dehn filling core to $cs+h$. Basically, we want to gradually perturb $\mathcal{M}(S^1 \times D^2)$ across the pillowcase to detect what irreducibles there are in $\mathcal{M}_h(Y)$ whose images lie on either side of $r(\mathcal{M}_h^{U(1)}(Y))$.

Choose an admissible function $h' : \mathcal{A}(Y_0) \to \mathbf{R}$ defined using the core of the Dehn filling, in such a way that $\langle \nabla(h'\circ\phi)(t,0), \vec{t}\rangle = 1$ and $h'\circ\phi(0,0) = 0$. The crucial observation is that any connection on Y_0 which is $h+h'$ perturbed flat restricts to Y to give an h perturbed flat connection.

We will abuse the notation and let h' also denote the corresponding function of t and r. Consider the function $f_\epsilon(t,r) = (f+\epsilon h')(t,r)$. Then $f_\epsilon(t,0) = \epsilon t + f(0,0)$ and

$$f_\epsilon(t,r) = f(0,0) + f_\epsilon(t,0) + \lambda(t)r^2 + O(t^2r^2) + O(r^3).$$

The lower order terms depend not only on t and r but on ϵ.

A local model for the flat moduli space of Y near $[A_0]$ is the quotient under the $\mathbf{Z}_2$ symmetry $(r \mapsto -r)$ of the set

$$\{(t,r)|\ \frac{\partial f_\epsilon(t,r)}{\partial t} = \frac{\partial f_\epsilon(t,r)}{\partial r} = 0 \text{ for some } \epsilon\}.$$

This set is the union of $\{(t,0)\}$ and the image under projection onto the (t,r) coordinate plane of

$$N = \{(t,r,\epsilon) |\; r \neq 0, \frac{\partial f_\epsilon(t,r)}{\partial t} = \frac{1}{r}\frac{\partial f_\epsilon(t,r)}{\partial r} = 0\} \cup \{(0,0)\}.$$

Let $P = (P_1, P_2) : \mathbf{R}^3 \to \mathbf{R}^2$ where $P_1(t,r,\epsilon) = \frac{\partial f_\epsilon(t,r)}{\partial t}$ and

$$P_2(t,r,\epsilon) = \begin{cases} 0 & r = 0 \\ \frac{1}{r}\frac{\partial f_\epsilon(t,r)}{\partial r} & r \neq 0 \end{cases}.$$

The linearization of P at $(0,0,0)$ is

$$DP(0,0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & \lambda'(0) \end{bmatrix}.$$

The implicit function theorem now implies that there are smooth functions $\epsilon(r)$ and $t(r)$ such that for r small, $(t(r), r, \epsilon(r))$ parameterizes N near $(0,0,0)$. This shows that up to gauge equivalence there is a smooth 1-dimensional family of irreducible connections on Y limiting to $[A_0]$. $\square$

Corollary 20 *Given assumptions (1)-(3), if* $hol_\mu A_0 = \exp(\mathrm{i}\theta)$ *for* $0 < \theta < \pi$, *then the nearby irreducibles have* $hol_\lambda A = \exp(\mathrm{i}\sigma)$ *where* $\sigma \neq 0$ *is small and either positive or negative, according to whether* $\lambda'(0) > 0$ *or* $\lambda'(0) < 0$.

Remark: The sign of $\lambda'(0)$ also determines the orientation of the irreducible arc emanating from $[A_0]$ in $\tilde{\mathcal{M}}_h$. Therefore this orientation depends on whether the arc leaves the irreducibles pointing up the front or back of the pillowcase. Corollary 4 follows from this fact and Proposition 17 of [H2], which relate the Casson invariant to the oriented intersection number of $\tilde{\mathcal{M}}^*$ with a circle in $\tilde{\mathcal{M}}_{T^2}$.

4 General Existence Theorem

In this section we use Theorem 11 to prove the main theorem of this paper, which we restate below.

Theorem 21 *Let Y be the complement of a tubular neighborhood of a knot κ in a homology 3-sphere. If the function* $SignB_\kappa(t) : U(1) \to \mathbf{Z}$ *does not vanish on the complement of the set of unit roots of* $\Delta(t)$, *then there are irreducible representations* $\rho : \pi_1(Y) \to SU(2)$. *Furthermore, for any unit root* $e^{\mathrm{i}2\alpha}$ *of* $\Delta(t)$ *where the right and left hand limits* $\lim_{\beta \to \alpha^\pm} SignB(e^{\mathrm{i}\beta})$ *do not agree, there is a continuous family of irreducible representations limiting to the abelian one which takes* μ *to* $\exp(\mathrm{i}\alpha)$.

Proof: Find a collection of curves in Y which meets the criterion described in Proposition 7, and let U and U_1 be as in Theorem 8. Choose a path $h_s : [-\epsilon, \epsilon] \to U$ with $h_0 = 0$ which is transverse to U_1. We can take ϵ small enough that $\mathcal{M}_{h_s}$ is nondegenerate when $0 < s \leq \epsilon$.

There is a 2-parameter family of abelian connections $A_{s,t}$ near the one $A_{0,0}$ which has holonomy $\mu \mapsto \exp(\mathrm{i}\alpha)$, $\lambda \mapsto 1$ such that $[A_{s,t}] \in \mathcal{M}_{h_s}^{U(1)}$. Let

$$H_{s,t}(\alpha, \beta) = \langle *d_{A_{s,t},h_s}\alpha, \beta\rangle$$

be the corresponding 2-parameter family of bilinear forms on $\mathcal{H}^1_{A_{0,0}}(Y; \mathbf{C})$.

Let B be an arbitrarily small ball in $\mathcal{M}_{T^2}$. By shrinking B and ϵ if necessary, we can assume there is a $\delta > 0$ such that

1. $\det H_{s,t} = 0$ implies $r[A_{s,t}] \in B$ if $(s,t) \in [0,\epsilon] \times [-\delta, \delta]$.
2. $r[A_{s,\pm\delta}] \notin B$ if $s \in [0, s]$.
3. Each curve $\{r[A_{s,t}] | \, t \in [-\delta, \delta]\}$ for $s \in [0, \epsilon]$ intersects B.

Let $\overline{\mathcal{M}^*_{h_s}}$ denote the closure of the irreducible stratum of $\mathcal{M}_{h_s}$, i.e. the irreducible stratum compactified by adding the abelian limit points. For all $0 < s \leq \epsilon$, $r(\overline{\mathcal{M}^*_{h_s}})$ consists of an immersed compact 1 manifold with an odd number of endpoints in the interior of B. Therefore $r(\overline{\mathcal{M}^*_{h_s}}) \cap \partial B \neq \emptyset$ for all $0 < s \leq \epsilon$.

By Proposition 9, $r(\overline{\mathcal{M}^*_{\{h_s\}}}) \cap \partial B$ (where s ranges over $[0, \epsilon]$) is compact, and hence $r(\overline{\mathcal{M}^*_{h_0}}) \cap \partial B = r(\overline{\mathcal{M}^*}) \cap \partial B \neq \emptyset$. Since the same is true for arbitrarily small B, $[A_{0,0}]$ is in $\overline{\mathcal{M}^*}$. If there were no continuous path in $r(\overline{\mathcal{M}^*})$ connecting $[A_0]$ to ∂B, then we could separate $[A_0]$ and $r(\overline{\mathcal{M}^*}) \cap \partial B$ by a continuous loop $\gamma : S^1 \to (B \setminus r(\overline{\mathcal{M}^*})$. The above argument showing that $r(\overline{\mathcal{M}^*}) \cap \partial B \neq \emptyset$ may be applied to $r(\overline{\mathcal{M}^*}) \cap \gamma(S^1)$ to give a contradiction. □

Corollary 22 *Let κ be a knot in a homology sphere X. If for some $t \neq \pm 1$ in $U(1)$ the matrix $B_\kappa(t)$ is degenerate and all eigenvalues due so transversely and in the same direction, for $|n|$ large enough the homology sphere obtained by $\frac{1}{n}$ surgery on κ has irreducible $SU(2)$ representations.*

Proof: Suppose such a $t = e^{i2\theta}$ exists. Then, since $B_\kappa(\bar{t}) = B_\kappa(t)^T$, the eigenvalues of $B_\kappa(t)$ are invariant under $t \mapsto t^{-1}$, and so there is a corresponding point $t' = -e{-}i2\theta \in U(1)$ where there are the same number of eigenvalues crossing zero but in the opposite direction (as t' moves in the positive direction around $U(1)$). Theorem 21 then says that there are continuous families of irreducibles near the corresponding abelian representations $\mu \mapsto \exp(\mathrm{i}\theta)$ and $\mu \mapsto \exp(\mathrm{i}(\pi - \theta))$. By Corollary 19 these do not lie along the arc $\mathrm{hol}_\lambda = \mathrm{id}$ in the pillowcase. By the observation in the proof of Theorem 11 that there are no solutions when the sign of the spectral flow and the sign of ϵ agree, we see that the image of the irreducible families near the abelian points corresponding to $\exp(\mathrm{i}\theta)$ and $\exp(\mathrm{i}(\pi - \theta))$ are on opposite sides of the abelian arc. This may also be seen by noting that $r(\mathcal{M}(Y))$ is symmetric under the involution $(\mathrm{hol}_\lambda, \mathrm{hol}_\mu) \mapsto (\mathrm{hol}_\lambda, -\mathrm{hol}_\mu)$. Thus for $|n|$ large enough, this family of irreducibles of $\pi_1 Y$ will intersect the curve of slope $\pm\frac{1}{n}$ in the pillowcase, which corresponds to the set of representations of $\pi_1 T^2$ which extend over the corresponding Dehn surgeries. □

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