Geodesic equivalence of metrics as a particular case of integrability of geodesic flows.

Vladimir S. Matveev[1], Petar J. Topalov[2] [3]

In Memoriam Mikhail Vladimirovich Saveliev

Let $g = (g_{ij})$ and $\bar{g} = (\bar{g}_{ij})$ be smooth metrics on the same manifold M^n.

Definition 1. *The metrics g and $\bar{g}$ are* geodesically equivalent*, if they have the same geodesics (considered as unparameterized curves).*

This is rather classical material. In 1869 Dini [6] formulated the problem of local classification of geodesically equivalent metrics, and solved it for dimension two. In 1896 Levi-Civita [7] obtained a local description of geodesically equivalent metrics on manifolds of arbitrary dimension.

Many interesting results in this area were obtained by P. Painlevé, H. Weyl, E. Cartan, P. A. Shirokov, S. Kobayashi, N. S. Sinyukov, A. Petrov, P. Venzi, J. Mikeš, A. Aminova, see [1, 9] for references.

The approach we would like to suggest helps to find an answer for the following questions:

- What closed manifolds admit geodesically equivalent metrics? [8]
- How many metrics are there geodesically equivalent to a given one? [9]

Firstly let us give two simple examples of geodesically equivalent metrics on closed manifolds.

There always exist a trivial example which is not interesting for our consideration: an arbitrary metric g is geodesically equivalent to the metric Cg, where C is a constant.

The first non-trivial example in this sense is the following. Take the torus $T^n = S^1 \times S^1 \times ... \times S^1$. Consider the coordinate system $x^1, ..., x^n$ on T^n, assuming x^k is the cyclic coordinate on the kth circle. Now consider the metrics $ds_1^2 = \sum_{i=1}^n (dx^i)^2$ and $ds_2^2 = \sum_{i,j=1}^n a_{ij}(dx^i)(dx^j)$, where (a_{ij}) is a positive definite symmetric matrix. The metrics ds_1^2 and ds_2^2 are evidently geodesically equivalent, and they are, generally speaking, not proportional.

The second non-trivial example is due to E. Beltrami [2, 3]. Consider the standard sphere $\sum_{i=1}^{n+1}(x^i)^2 = 1$, where $x^1, ..., x^{n+1}$ now are standard coordinates in the Euclidean space R^{n+1}. Take a linear transformation $L : R^{n+1} \to$

[1] vmatveev@mpim-bonn.mpg.de
[2] topalov@math.bas.bg
[3] Partially supported by MESC', grant number MM-810/98.

R^{n+1} of the space R^{n+1} and consider the corresponding projective transformation l of the sphere. Let x be a point of the sphere. Consider the ray $[0,x)$, where 0 is the zero point of R^{n+1}. Evidently the image $L([0,x))$ is also a ray with the origin in zero. Let the intersection of the ray $L([0,x))$ and the sphere be y. Then, by definition, put $l(x)=y$. In other words, the mapping l is given by the formula

$$l(x)=\frac{Lx}{\|Lx\|}.$$

It is easy to see, that the mapping l preserves the geodesics of the sphere. Actually, the geodesics of the sphere are intersections of the planes, which go through the origin, with the sphere. The linear mapping L takes planes to planes, therefore the projective mapping l takes geodesics to geodesics. Then the standard metric $g_{standard}$ of the sphere and the metric $l^*g_{standard}$ are geodesically equivalent.

We would like to point out the following common property of these two examples: the geodesic flows of the metrics are completely integrable, in the sense that there exist n integrals in involution. The main result of the paper is that this property is common to all geodesically equivalent metrics in general position.

Denote by G the linear operator $g^{-1}\bar g = (g^{i\alpha}\bar g_{\alpha j})$. In other words, the operator G "takes" metric $\bar g$ to g: for arbitrary vectors $\nu,\xi\in T_{x_0}M^n$ the dot product $\bar g(\xi,\nu)$ of vectors ξ,ν in the metric $\bar g$ is equal to the dot product $g(G\xi,\nu)$ of vectors $G\xi$, ν in the metric g.

Consider the characteristic polynomial

$$det(G-\mu E)=c_0\mu^n+c_1\mu^{n-1}+...+c_n.$$

The coefficients $c_1,..,c_n$ are smooth functions on the manifold M^n, and $c_0\equiv(-1)^n$. Consider the functions $I_k: TM^n\to R$, $k=0,...,n-1$, given by the general formula $I_k(\xi)=\left(\frac{det(g)}{det(\bar g)}\right)^{\frac{k+2}{n+1}}g(S_k\xi,\xi)$, where S_k is the linear operator given by $S_k\stackrel{\text{def}}{=}\sum_{i=0}^k c_iG^{k-i+1}$, and $g(\nu,\xi)$ denotes the dot product in the metrics g of the tangent vectors ν,ξ.

Theorem 1 ([14]). *If metrics g and $\bar g$ on M^n are geodesically equivalent then the functions I_k are integrals of the geodesic flow of the metric g and pairwise commute (if we identify with the help of g the tangent and the cotangent bundles of M^n and consider the canonical symplectic form on the cotangent bundle).*

Remark 1. *The integral $I_0=\left(\frac{det(g)}{det(\bar g)}\right)^{\frac{2}{n+1}}\bar g(\xi,\xi)$ was obtained by P. Painlevé, see [7]. The integral I_{n-1} is the energy integral (multiplied by minus two). The integrals $I_1,I_2,...,I_{n-2}$ seem to be new.*

Remark 2. *For the two-dimensional case, the theorem follows from results of P. Painlevé, see [7].*

Remark 3 ([14]). *The metrics $g, \bar{g}$ are* strictly non-proportional *if the characteristic polynomial $det(G - \mu E)$ has no multiple roots. It can be easily shown, that if the metrics $g, \bar{g}$ are geodesically equivalent and strictly non-proportional at some point then in the tangent bundle of a sufficiently small neighborhood of the point the integrals I_k are functionally independent almost everywhere.*

This theorem allows us to apply the theory of integrable geodesic flows to the theory of geodesically equivalent metrics and vice versa.

Corollary 1. *Suppose M^n is connected. Let metrics g, $\bar{g}$ on M^n be geodesically equivalent and strictly non-proportionally at least at one point of M^n. Then the metrics are strictly non-proportional almost everywhere on M^n.*

Theorem 2. *Let g, $\bar{g}$ be metrics on M^n. Consider the functions $I_0, I_1, ..., I_{n-1}$. Let they be functionally independent almost everywhere and commute (if we identify with the help of g the tangent and the cotangent bundles of M^n and consider the canonical symplectic form on the cotangent bundle). Then the metrics g, $\bar{g}$ are geodesically equivalent.*

Remark 4. *For the two-dimensional case, Theorem 2 was formulated in the book of E. T. Whittaker [16].*

Theorem 3 (Taimanov, [12]). *If a real-analytic closed manifold M^n with a real-analytic metric satisfies at least one of the conditions:*

a) $\pi_1(M^n)$ is not almost commutative

b) $dim H_1(M^n; \mathbf{Q}) > dim M^n$,

then the geodesic flow on M^n is not analytically integrable.

Corollary 2 ([14]). *Let M^n be a closed real-analytic manifold supplied with two real-analytic metrics $g, \bar{g}$ such that the metrics $g, \bar{g}$ are geodesically equivalent and strictly non-proportional at least at one point. Then the fundamental group $\pi_1(M^n)$ of the manifold M^n contains a commutative subgroup of finite index and the dimension of the homology group $H_1(M^n; \mathbf{Q})$ is no greater than n.*

Corollary 3 ([14]). *Let metrics $g, \bar{g}$ on a closed surface of negative Euler characteristic be geodesically equivalent. Then $g = C\bar{g}$, where C is a constant.*

Corollary 4 ([14]). *Let metrics $g, \bar{g}$ on the torus T^2 be geodesically equivalent. If they are proportional at a point $x \in T^2$, then $g = C\bar{g}$, where C is a positive constant.*

Corollary 5 ([14]). *Let metrics g, $\bar{g}$ on the sphere S^2 be geodesically equivalent. Then there are only three possibilities.*

1. *The metrics are proportional at exactly two points*

2. *The metrics are proportional at exactly four points.*
3. *The metrics are completely proportional, i.e. $g = C\bar{g}$, where C is a positive constant.*

In the first case the metrics admit a Killing vector field.

Dynamical Background:
We suggest a construction that, given an orbital diffeomorphism between two Hamiltonian systems, produces integrals of them.

Let v and $\bar{v}$ be Hamiltonian systems on symplectic manifolds (N, ω) and $(\bar{N}, \bar{\omega})$ with Hamiltonians H and $\bar{H}$, respectively.

Consider the isoenergy surfaces

$$Q \stackrel{\text{def}}{=} \{x \in N : H(x) = h\}, \quad \bar{Q} \stackrel{\text{def}}{=} \left\{x \in \bar{N} : \bar{H}(x) = \bar{h}\right\},$$

where h and $\bar{h}$ are regular values of the functions H, $\bar{H}$, respectively.

Definition 2. *A diffeomorphism $\phi : Q \longrightarrow \bar{Q}$ is said to be* orbital*, if it takes the orbits of the system v to the orbits of the system $\bar{v}$.*

Given orbital diffeomorphism, we can invariantly construct integrals. Denote by σ, $\bar{\sigma}$ the restrictions of the forms $\omega, \bar{\omega}$ to Q, $\bar{Q}$ respectively. Consider the form $\phi^*\bar{\sigma}$ on Q.

Lemma 1 (Topalov, [13]). *The flow v preserves the form $\phi^*\bar{\sigma}$.*

Proof of Lemma 1. The Lie derivative L_v of the form $\phi^*\bar{\sigma}$ along the vector field v satisfies

$$L_v \phi^* \bar{\sigma} = \mathrm{d}\left[\imath_v \phi^* \bar{\sigma}\right] + \imath_v \mathrm{d}\left[\phi^* \bar{\sigma}\right].$$

On the right side both terms vanish. Since the form $\bar{\omega}$ is closed, the form $\bar{\sigma}$ is also closed and $\mathrm{d}\left[\phi^*\bar{\sigma}\right] = \phi^*(\mathrm{d}\bar{\sigma}) = 0$. Since the diffeomorphism takes the orbits to orbits, it takes the kernel of the form σ to the kernel of the form $\bar{\sigma}$, so that $\imath_v \phi^* \bar{\sigma}$ is equal to zero. The lemma is proved.

It is obvious that the kernels of the forms σ and $\phi^*\bar{\sigma}$ coincide (in the space $\mathcal{T}_x Q$ at each point $x \in Q$) with the linear span of the vector v. Therefore these forms induce two non-degenerate tensor fields on the quotient bundle $\mathcal{T}Q/\langle v\rangle$. We shall denote the corresponding forms on $\mathcal{T}Q/\langle v\rangle$ also by the letters $\sigma, \bar{\sigma}$.

Lemma 2. *The characteristic polynomial of the operator $(\sigma)^{-1}(\phi^*\bar{\sigma})$ on $\mathcal{T}Q/\langle v\rangle$ is preserved by the flow v.*

Proof of Lemma 2. Since the flow v preserves the Hamiltonian H and the form ω, the flow v preserves the form σ. Since the flow v preserves both forms, it preserves the characteristic polynomial of the operator $(\sigma)^{-1}(\phi^*\bar{\sigma})$. The lemma is proved.

Since both forms are skew-symmetric, each root of the characteristic polynomial of the operator $(\sigma)^{-1}(\Phi^*\bar\sigma)$ has an even multiplicity. Then the characteristic polynomial is the square of a polynomial $\delta^{n-1}(t)$ of degree $n-1$. Hence the polynomial $\delta^{n-1}(t)$ is also preserved by the flow v. Therefore the coefficients of the polynomial $\delta^{n-1}(t)$ are integrals of the system v.

Geodesic flows of geodesically equivalent metrics can be considered as orbitally equivalent systems. The manifold $N = \bar N = TM^n$, the forms ω, $\bar\omega$ are given by

$$\omega \stackrel{\text{def}}{=} \mathrm{d}[g_{ij}\xi^j dx^i], \quad \bar\omega \stackrel{\text{def}}{=} \mathrm{d}[\bar g_{ij}\xi^j dx^i]$$

and the orbital diffeomorphism ϕ is given by

$$\phi(x,\xi) = \left(x, \xi\frac{||\xi||_g}{||\xi||_{\bar g}}\right).$$

Direct calculations give us the formulae for the integrals I_k from Theorem 1.

Are there interesting examples of geodesically equivalent metrics on closed manifolds?

The following theorem (essentially due to N. Sinyukov [10]) gives us a construction that, given a pair of geodesically equivalent metrics, produces another pair of geodesically equivalent metrics. Starting from the metric of constant curvature on the sphere, we obtain the metric of the ellipsoid and the metric of the Poisson sphere.

Let $g, \bar g$ be Riemannian metrics on the manifold M^n. Consider the linear operator $B : TM^n \to TM^n$ given by $B^i_j = \left(\frac{det(\bar g)}{det(g)}\right)^{\frac{1}{n+1}} \bar g^{ip} g_{pj}$. By definition put metric gB equal $g_{ip}B^p_j$. By definition put metric $\bar g B$ equal $\bar g_{ip}B^p_j$. In other words, the dot product $gB(\xi,\nu)$ of arbitrary vectors $\xi,\nu \in T_{x_0}M^n$ is equal to $g(B\xi,\nu)$ and the dot product $\bar gB(\xi,\nu)$ is equal to $\bar g(B\xi,\nu)$. Evidently the linear operator B is self-adjoint with respect to metrics g and $\bar g$ and the metrics gB, $\bar gB$ are well-definite.

Theorem 4. *The metrics g, $\bar g$ are geodesically equivalent if and only if the metrics gB, $\bar gB$ are geodesically equivalent.*

Clearly the metrics g, $\bar g$ are strictly non-proportional (almost everywhere) if and only if the metrics gB, $\bar gB$ are strictly non-proportional (almost everywhere).

Thus if we have a pair of geodesically equivalent metrics g, $\bar g$ then we can construct the other pair of geodesically equivalent metrics. By applying the construction once more the result is another pair of geodesically equivalent metrics. It is natural to denote it by gB^2, $\bar gB^2$, since this pair is given by

$$gB^2(\xi,\nu) = g(B^2\xi,\nu), \quad \bar gB^2(\xi,\nu) = \bar g(B^2\xi,\nu).$$

We can go in other direction and consider the metrics gB^{-1}, $\bar gB^{-1}$ given by

$$gB^{-1}(\xi,\nu) = g(B^{-1}\xi,\nu), \quad \bar gB^{-1}(\xi,\nu) = \bar g(B^{-1}\xi,\nu).$$

They are geodesically equivalent also.

To start the process, we need an initial pair of geodesically equivalent metrics g, $\bar{g}$.

We take the pair $g_{standard}$, $l^* g_{standard}$ from the second example. Then the metric gB is the metric of an ellipsoid and the metric $\bar{g}B^2$ is the metric of a Poisson sphere. Varying the linear operator L, we can obtain metrics of all possible ellipsoids and Poisson spheres.

Theorem 5 (independently obtained by S. Tabachnikov [11]). *The restriction of the Euclidean metric*

$$(dx^1)^2 + ... + (dx^{n+1})^2$$

to the standard ellipsoid

$$E^n = \{(x^1, x^2, ..., x^{n+1}) \in R^{n+1} : \frac{(x^1)^2}{a_1} + ... + \frac{(x^{n+1})^2}{a_{n+1}} = 1\}$$

is geodesically equivalent to the restriction of the metric

$$\frac{1}{\left(\frac{x^1}{a_1}\right)^2 + ... + \left(\frac{x^{n+1}}{a_{n+1}}\right)^2} \left(\frac{(dx^1)^2}{a_1} + ... + \frac{(dx^{n+1})^2}{a_{n+1}}\right)$$

to the same ellipsoid.

Because of [5], we can consider the metric of the Poisson sphere as the restriction of the metric

$$\frac{1}{\frac{(x^1)^2}{a_1^2} + ... + \frac{(x^{n+1})^2}{a_{n+1}^2}} \left((dx^1)^2 + ... + (dx^{n+1})^2\right)$$

to the ellipsoid

$$E^n = \{(x^1, x^2, ..., x^{n+1}) \in R^{n+1} : \frac{(x^1)^2}{a_1} + ... + \frac{(x^{n+1})^2}{a_{n+1}} = 1\}.$$

Theorem 6 (Topalov, [15]). *The restriction of the metric*

$$\frac{1}{\frac{(x^1)^2}{a_1^2} + ... + \frac{(x^{n+1})^2}{a_{n+1}^2}} \left((dx^1)^2 + ... + (dx^{n+1})^2\right)$$

to the ellipsoid

$$E^n = \{(x^1, x^2, ..., x^{n+1}) \in R^{n+1} : \frac{(x^1)^2}{a_1} + ... + \frac{(x^{n+1})^2}{a_{n+1}} = 1\}$$

is geodesically equivalent to the restriction of the metric

$$a_1(dx^1)^2 + ... + a_{n+1}(dx^{n+1})^2 - (x^1 dx^1 + x^2 dx^2 + ... + x^{n+1} dx^{n+1})^2)$$

to the same ellipsoid.

Quantum integrability of the Beltrami-Laplace operator for geodesically equivalent metrics:
Consider the Beltrami-Laplace operator $\Delta := g^{ij}\nabla_i\nabla_j$, where ∇ denotes the covariant derivative, corresponding to the metric g. Beltrami-Laplace operator acts on the space L_2 of measurable functions $\psi : M^n \to R$ such that the Lebesgue integral

$$\int_{M^n} \psi^2 \sqrt{det(g)} dx^1 dx^2 ... \, dx^n$$

is finite.

Consider the operators

$$\mathcal{I}_0, \ \mathcal{I}_1, ..., \mathcal{I}_{n-1} : L_2 \to L_2$$

given by the general formula $\mathcal{I}_k := g^{ij}\nabla_i \left(\frac{det(g)}{det(\bar g)}\right)^{\frac{k+2}{n+1}} (S_k)^p_j \nabla_p$.

Remark 5. *The operator $\mathcal{I}_{n-1}$ is exactly the operator $-\Delta$.*

Theorem 7. *If metrics g and $\bar g$ on M^n are geodesically equivalent then the operators $\mathcal{I}_k$ pairwise commute. If the manifold M^n is closed then the operators $\mathcal{I}_k$ are self-adjoint.*

Corollary 6. *Suppose M^n is connected. Metrics $g, \ \bar g$ on M^n are geodesically equivalent and strictly non-proportional at least at one point of M^n if and only if the operators $\mathcal{I}_0, ..., \mathcal{I}_{n-1}$ are linearly independent.*

So that if the manifold is closed and if the metrics $g, \ \bar g$ on it are geodesically equivalent and strictly non-proportional at least at one point then we have the complete quantum integrability of the Beltrami-Laplace operator of the metric g.

Evidently the symbols of the operators $\mathcal{I}_k$ are I_k. Then if the operators $\mathcal{I}_k$ commute then the functions I_k commute.

Corollary 7. *Suppose the manifold M^n is connected. Let $g, \ \bar g$ be metrics on it. Let the operators $\mathcal{I}_k$ commute and let they be linear independent. Then the metrics $g, \bar g$ are geodesically equivalent.*

The space of the metrics, geodesically equivalent to a given one:
If two metrics are geodesically equivalent then there exist at least one-parametric family of geodesically equivalent metrics and this family includes this two metrics [4]. It could be shown (at least in real-analytic case) that the set of metrics, geodesically equivalent to a given metric, is a manifold. What is the dimension of this manifold?

Even locally, there exist metrics that admit no (non-trivial) geodesically equivalent [9].

Even locally, the dimension of this space does not exceed $\frac{(n+1)(n+2)}{2}$ and is equal to $\frac{(n+1)(n+2)}{2}$ only for the metrics of constant curvature [9].

Let M^n be closed, and let $g, \bar{g}$ be geodesically equivalent and strictly non-proportional at almost everywhere dense set of points. Then the geodesic flow of the metric g is completely integrable, and almost all orbits lie at the corresponding Liouville tori. Suppose, that the geodesic flow is non-resonant. Then the Liouville foliation is uniquely definite, and any integral of the geodesic flow commutes with the integrals $I_0, ..., I_{n-1}$. Assume in addition, that there are sufficiently many caustics of the Liouville tori of the geodesic flow: almost each point of the surface is an intersection of n caustics. Then the dimension of the space of the metrics (modulo multiplication by a constant), geodesically equivalent to the metric g, is equal one.

The geodesic flows of the metric of the ellipsoid and of the Poisson sphere satisfies all these conditions.

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