Characteristic classes of flat bundles

by

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Introduction.

On an analytic manifold X, a bundle E is said to be flat if it is associated to a representation of the fundamental group, or, equivalently, if there is an holomorphic integrable connection ∇ on E. In this article we construct classes $c_p(E,\nabla) \in H^{2p}(X,\mathbb{Z}(p) \longrightarrow \mathbb{C})$, whose images in the Deligne cohomology $H^{2p}(X,\mathbb{Z}(p) \longrightarrow \mathcal{O}_X \longrightarrow \dots \longrightarrow \Omega^{p-1}_X)$ are the Chern classes $c_p^{\mathcal{D}}(E)$ in the Deligne cohomology. In particular their images in $H^{2p}(X,\mathbb{Z}(p))$ are the topological Chern classes $c_p^{top}(E)$ (and their images $c_p^{DR}(E)$ in $H^p(X,\Omega^p_X \longrightarrow \dots \longrightarrow \Omega^{\dim X}_X)$ vanish). Those classes $c_p(E,\nabla)$ are functorial and additive.

The group $H^2(X,\mathbb{Z}(1) \longrightarrow \mathcal{O}_X) = H^1(X,\mathcal{O}_X^*)$ is identified with the group of isomorphism classes of rank one bundles. P. Deligne ([1], (1.3)) remarked that the group $H^2(X,\mathbb{Z}(1) \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X)$ is identified with the group of isomorphism classes of rank one bundles with holomorphic connections (E,∇). Therefore one sees that ∇ is integrable if and only if the class (E,∇) lies in $H^2(X,\mathbb{Z}(1) \longrightarrow \Omega^{\cdot}_X) = H^1(X,\mathbb{C}^*)$. Our construction relies on this observation.

Suppose that E has a filtration by subbundles E_k such that $L_k = E_k/E_{k-1}$ is a rank one bundle and such that ∇ induces an integrable connection ∇_k on L_k. We call this a flat filtration. If we define a product

$$(\mathbb{Z}(p) \longrightarrow \mathbb{C}) \times (\mathbb{Z}(q) \longrightarrow \mathbb{C}) \longrightarrow (\mathbb{Z}(p+q) \longrightarrow \mathbb{C})$$

which is compatible with the standard cup product and Deligne product, we will define classes $c_p(E,\nabla)$ as symmetric sum of the p-products of (L_k,∇_k) which map to $c_p^{top}(E)$ and $c_p^{\mathcal{D}}(E)$.

However such a filtration does not exist in general, and of course if one considers a particular splitting morphism $f : P \longrightarrow X$ of E, the corresponding canonical filtration E_k is not flat. So one has to define a substitute for the flatness on P.

Assume first that rank $E = 2$, and consider the canonical filtration of f^*E on its projective bundle P by $\mathcal{O}(1)$ and $\Omega^1_{P/X}(1)$. The integrable connection ∇ defines a morphism $\tau : \Omega^{\bullet}_P \longrightarrow \Omega^{\bullet}_{\tau}$ from the De Rham complex of P to a complex whose image on X is $Rf_*\Omega^{\bullet}_{\tau} = \Omega^{\bullet}_X$, the De Rham complex of X. Further ∇ defines integrable τ-connections ∇_{τ} and ∇'_{τ} on $\mathcal{O}(1)$ and $\Omega^1_{P/X}(1)$, and classes $(\mathcal{O}(1),\nabla_{\tau})$ and $(\Omega^1_{P/X}(1),\nabla'_{\tau})$ in $H^2(P,\mathbb{Z}(1) \longrightarrow \Omega^{\bullet}_{\tau})$. We define a product by multiplying the class of $(\Omega^1_{P/X}(1),\nabla'_{\tau})$ by $c_1^{top}(\mathcal{O}(1))$ to get a class $c_2(f^*E,f^*\nabla) \in H^4(P,\mathbb{Z}(2) \longrightarrow \Omega^{\bullet}_{\tau})$, whose image in $H^4(P,\mathbb{Z}(2))$ is $c_2^{top}(f^*E)$. This implies in particular that $c_2(f^*E,f^*\nabla) = f^*c_2(E,\nabla)$ for a well defined class $c_2(E,\nabla) \in H^4(X,\mathbb{Z}(2) \longrightarrow \mathbb{C})$. It is not hard to compute the compatibility with $c_2^{\mathcal{D}}(E)$.

If one has now a flat filtration $L_1 \subset E$ and $L_2 = E/L_1$, one wishes the above construction gives the same class as before. As the τ-cohomology $H^{\bullet}(P,\mathbb{Z}(\cdot) \longrightarrow \Omega^{\bullet}_{\tau})$ is not a free module over $H^{\bullet}(X,\mathbb{Z}(\cdot) \longrightarrow \mathbb{C})$, one can not apply Hirzebruch-

Grothendieck's formalism to prove this additivity property. We show that the restriction of ∇_τ to the section of P over X corresponding to L_2 is precisely ∇_2. This proves essentially the additivity wanted.

For a higher rank bundle, one has to repeat this construction (rank E-1) times. To do this we have to start with general integrable τ-connections. The necessary study of formal operations (like pull-back...) makes the article a bit technical. But basically the general construction follows the same line as in the rank 2 case. One obtains the existence of similar classes for general integrable τ-connections with the usual properties.

J. Cheeger and J. Simons ([3]) constructed in a differential geometric framework classes $\hat{c}_p(E) \in H^{2p-1}(X, \mathbb{R}/\mathbb{Z})$ when X is a C^∞ manifold and E is a flat bundle. Following S. Bloch ([2]) their images in the Deligne cohomology are the classes $c_p^{\mathcal{D}}(E)$ in the unitary case. M. Karoubi ([7]) constructed with K-theory and cyclic homology classes $\check{c}_p(E) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p))$ when X is a simplicial set and E is a flat bundle. One may ask what is the relationship between $c_p(E,\nabla)$, $\hat{c}_p(E)$ and $\check{c}_p(E)$. However we don't consider this question here.

If D is a divisor with normal crossings on X, one may perform the same construction for bundles E with an integrable logarithmic connection ∇ along D. This leads to classes $c_p(E,\nabla,D) \in H^{2p}(X, \mathbb{Z}(p) \longrightarrow Rj_*\mathbb{C})$, where $j : X - D \longrightarrow X$ is the open embedding, whose images in $H^{2p}(X, \mathbb{Z}(p) \longrightarrow \mathcal{O}_X \longrightarrow \dots \longrightarrow \Omega_X^{p-1}\langle D\rangle)$ are the images of $c_p^{\mathcal{D}}(E)$. Those classes $c_p(E,\nabla,D)$ are functorial and additive.

One knows that if X has a Hodge structure and E is of rank one with vanishing Atiyah class, all the homomorphic connections ∇ on E are integrable. This can be easily seen in the language introduced before, namely one has $H^2(X, \mathbb{Z}(1) \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1) = H^2(X, \mathbb{Z}(1) \longrightarrow \Omega_X^{\cdot})$. The corresponding thing for higher rank bundles is: if one has a Hodge structure, then $H^2(P, \mathbb{Z}(1) \longrightarrow \mathcal{O}_P \xrightarrow{\tau d} f^*\Omega_X^1) = H^2(P, \mathbb{Z}(1) \xrightarrow{\tau d} \Omega^{\cdot}_{\tau})$, provided $\Omega^{\cdot}_{\tau}$ is a complex, i.e $(\tau d)^2 = 0$. The latter is therefore equivalent to the integrability of ∇ .

I like to thank B. Angéniol with whom I computed the important point (0.7) some time ago (see (0.8)), C. Soulé who told me a lot about Chern classes in the Deligne cohomology (and sent me [8]) , J.L. Verdier and E. Viehweg for very stimulating discussions. Finally I thank O. Gabber for pointing out to me an error in an earlier version of this work.

§ 0. Preliminaries.

(0.1) X: analytic manifold over $\mathbb{C}$ of complex dimension n

D: divisor with normal crossings on X

$j : X-D \longrightarrow X$: open embedding

$A(p) = (2i\pi)^p \cdot A$ for a $\mathbb{Z}$-module A

$\Omega_X^{\bullet}$: holomorphic De Rham complex with Kähler differential d

$\Omega_X^{\bullet}\langle D\rangle$: holomorphic De Rham complex with logarithmic singularities; it is quasi-isomorphic to $Rj_*\mathbb{C}$ ([4])

E: vector bundle of rank r on X

$\mathcal{E}nd\, E = \mathcal{O}_X \oplus \mathcal{E}nd^0 E$ via

$$\varphi = \frac{1}{r} \text{ trace } \varphi \cdot \text{id} \oplus \varphi^0$$

with trace $\varphi^0 = 0$: endomorphisms of E.

(0.2) An holomorphic connection

$\nabla: E \longrightarrow \Omega_X^1 \otimes E$ is a $\mathbb{C}$-linear morphism verifying the Leibnitz-rule

$\nabla(\lambda \cdot x) = \lambda \cdot \nabla(x) + d\lambda \cdot x$, for $\lambda \in \mathcal{O}_X$ and $x \in E$. One defines

$\nabla: \Omega_X^p \otimes E \longrightarrow \Omega_X^{p+1} \otimes E$ by

$\nabla(\omega \otimes x) = (-1)^p d\omega \otimes x + \omega \wedge \nabla x$, for $\omega \in \Omega_X^p$ and $x \in E$

One says that ∇ is integrable if $(\Omega_X^{\bullet} \otimes E, \nabla)$ is a complex, or equivalently, if the curvature $\nabla^2 \in \mathrm{Hom}_{\mathcal{O}_X}(E, \Omega_X^2 \otimes E)$ vanishes. The bundle E is said to be flat if some integrable connection exists. Flat bundles are in one-to-one correspondence with local constant systems by the Riemann-Hilbert correspondence

$$\{(E,\nabla)\} \longrightarrow \{L = \mathrm{Ker}\nabla\}, \quad \{L\} \longrightarrow \{L \otimes_{\mathbb{C}} \mathcal{O}_X, \ 1 \otimes d\}$$

(0.3) On a trivializing open cover U_i of E on X define ∇_i by declaring some basis to be flat. Then $\nabla_i - \nabla_j \in \Gamma(U_i \cap U_j, \Omega^1_X \otimes \mathcal{E}nd E)$ is a cocycle whose class in $H^1(X, \Omega^1_X \otimes \mathcal{E}nd E)$ is the <u>Atiyah class</u> $\mathrm{at}\,E$ of E. Its vanishing is the obstruction for E to have an holomorphic connection. One has $c_p^{DR}(E) = (-1)^p \operatorname{trace} \overset{p}{\Lambda} \mathrm{at}E \in H^p(X, \Omega^p_X)$, where $c_p^{DR}(E)$ is the De Rham Chern class. One has

$$\mathrm{at}\,E = -\frac{1}{r} c_1^{DR}(E) \cdot \text{identity} \oplus \mathrm{at}^0 E \quad .$$

If ξ_{ij} is a cocycle representing the class of E in $H^1(X, G\ell_r(\mathcal{O}_X))$, then $-\xi_{ij}^{-1} . d\xi_{ij}$ represents $\mathrm{at}E$.

(0.4) One defines $\mathrm{at}_D E$ to be the image of $\mathrm{at}E$ in $H^1(X, \Omega^1_X\langle D\rangle \otimes \mathcal{E}nd E)$. Its vanishing is the obstruction for E to have an holomorphic connection with logarithmic poles along D (same definition as in (0.2) where one replaces Ω^1_X by $\Omega^1_X\langle D\rangle$). Integrable logarithmic connections where studied by P. Deligne [4].

(0.5) Define $P = P(E) = \mathrm{Proj}_X(\underset{n\geq 0}{\oplus} S^{\cdot}(E))$ the <u>projective bundle</u> of E, where $S^{\cdot}(E)$ are the symmetric powers of E, $f : P \longrightarrow X$, $\mathcal{O}(1)$ as the relatively ample sheaf uniquely determined by the exact sequence

$$(1) \quad 0 \longrightarrow \Omega^1_{P/Z}(1) \longrightarrow f^*E \xrightarrow{q} \mathcal{O}(1) \longrightarrow 0$$

where $\Omega^1_{P/X}$ are the relative holomorphic one forms.

One has the other fundamental sequence

$$(0) \quad 0 \longrightarrow f^*\Omega^1_X \xrightarrow{i} \Omega^1_P \xrightarrow{p} \Omega^1_{P/X} \longrightarrow 0$$

Denote by $T^1_{P/X}$ the relative tangent sheaf.

(0.6) The sequence (0.5.0) is an extension class in

$$H^1(P, f^*\Omega^1_X \otimes T^1_{P/X}) = H^1(X, \Omega^1_X \otimes Rf_*T^1_{P/X})$$
$$= H^1(X, \Omega^1_X \otimes \mathcal{E}nd^0 E) .$$

Lemma. This class is $at^0 E$, up to the sign.

Proof. It is enough to see that on any trivializing open set U for E on X , some connection ∇ defines a section of p , and that $\Omega^1_U \otimes \mathcal{E}nd^0 E$ acts on the connections on U as $f^*\Omega^1_U \otimes T^1_{P/X}$ does on the sections of p on $f^{-1}U$.

∇ being given, define

$$\sigma \otimes 1_{\mathcal{O}(1)} = (1_{\Omega^1_P} \otimes q) f^*\nabla$$

where $f^*\nabla$ is defined to be $f^{-1}\nabla$ on $f^{-1}E$, d on $\mathcal{O}_P$ via the Leibnitz rule. Then $\sigma \otimes 1_{\mathcal{O}(1)}$ is $\mathcal{O}_P$-linear.

(0.6.1) Claim. $-\sigma$ is a section of p .

Proof. Let e^k be a basis of E on U . Define $t^k = q(e^k)$.

A basis of $\Omega^1_{P/X}(1)$ on $f^{-1}U \cap (t^0 \neq 0)$ is given by $x^k = e^k - \frac{t^k}{t^0} e^0$. One has

$$\sigma \otimes 1(x^k) = (1\otimes q)\ (f^*\nabla\, e^k - \frac{t^k}{t^0} f^*\nabla e^0) - d\left(\frac{t^k}{t^0}\right) \cdot t^0$$

$$p\,\sigma\left(\frac{x^k}{t^0}\right) = -\ pd\left(\frac{t^k}{t^0}\right) = -\ \frac{1}{t^0}(e^k - \frac{t^k}{t^0} e^0).$$

If $\nabla' = \nabla+\alpha$, with $\alpha = \alpha^{k\ell} \in \Gamma(U,\Omega^1_X \otimes \mathcal{E}nd\, E)$, one has

$$(\sigma'-\sigma) \otimes 1(x^k) = (1\otimes q)\ \left(\alpha e^k - \frac{t^k}{t^0}\, \alpha e^0\right)$$
$$= \sum_\ell \alpha^{k\ell} t^\ell - \frac{t^k}{t^0} \sum_\ell \alpha^{0\ell} t^\ell$$

One sees that $\frac{1}{r}$ trace $\alpha \cdot$id acts trivially, and that $\Omega^1_U \otimes \mathcal{E}nd^0 E$ acts as $f^*\Omega^1_U \otimes T^1_{P/X}$ does.

(0.7) Assume E to have an holomorphic connection ∇. This defines σ as in (0.6) and $\tau = 1 + \sigma p$ is a section of i. With the notations of (0.6) one has

$$\tau d\left(\frac{t^k}{t^0}\right) = \frac{1}{t^0}(1\otimes q)\ \ (f^*\nabla e^k - \frac{t^k}{t^0} f^*\nabla e^0)\ .$$

Define a τ-<u>connection</u> ∇_τ on a sheaf F on P to be a $\mathbb{C}$-linear morphism $\nabla_\tau : F \longrightarrow f^*\Omega^1_X \otimes F$ verifying the τ- <u>Leibnitz rule</u> $\nabla_\tau(\lambda.x) = \lambda.\nabla_\tau(x) + \tau d\lambda.x$, for $\lambda \in \mathcal{O}_P$ and $x \in F$.

<u>Lemma</u>. $\tau f^*\nabla$ <u>is a</u> τ-<u>connection on</u> f^*E <u>such that</u> $\tau f^*\nabla|_{\Omega^1_{P/X}(1)}$ <u>is a</u> τ-<u>connection</u> ∇'_τ on $\Omega^1_{P/X}(1)$ <u>and</u>

$(1\otimes q)\tau f^*\nabla$ <u>is a well defined</u> τ-<u>connection</u> ∇_τ <u>on</u> $\mathcal{O}(1)$.

<u>Proof</u>. As $-\sigma$ is a section of p,
$(1\otimes q)\ (\tau f^*\nabla)|_{\Omega^1_{P/X}(1)} = 0$. Therefore $\Omega^1_{P/X}(1)$ is stable under $\tau f^*\nabla$, and the quotient $(1\otimes q)\tau f^*\nabla$ is defined.

(0.8) <u>Remark</u>. In an effort to understand conditions for a bundle to be flat, we computed some time ago (0.6) and (0.7) with B. Angéniol. The point (0.6) is well known whereas the point (0.7) will play an important role in this article.

§ 1. Some conditions for a bundle to be flat.

(1.1) Let E be a rank one bundle. Its isomorphism class is a class in $H^1(X,\mathcal{O}^*) \xleftarrow[\exp]{\sim} H^2(X,\mathbb{Z}(1) \longrightarrow \mathcal{O})$, say of cocycle $\xi_{ij} \in \Gamma(U_i \cap U_j,\mathcal{O}^*)$ in a Čech cover U_i.

As $H^2(X,\ 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \Omega_X^2 \longrightarrow \dots \longrightarrow \Omega_X^n) = 0$, the morphism

$$\begin{array}{l} H^2(X,\mathbb{Z}(1) \longrightarrow \Omega_X^{\bullet}) \xrightarrow{\sim} H^1(X,\ \mathbb{C}^*) \\ \downarrow \\ H^2(X,\mathbb{Z}(1) \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1) \end{array}$$

is injective. One considers also the morphism

$$\begin{array}{l} H^2(X,\mathbb{Z}(1) \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1) \\ \downarrow \\ H^2(X,\mathbb{Z}(1) \longrightarrow \mathcal{O}_X) \xrightarrow{\sim} H^1(X,\mathcal{O}^*) \ . \end{array}$$

Lemma. i) The isomorphism classes of rank one bundles E with holomorphic connections ∇ build a group identified with $H^2(X,\mathbb{Z}(1) \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1)$. Denote by (E,∇) a class in $H^2(X,\mathbb{Z}(1) \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1)$. Its image (E) in $H^2(X,\mathbb{Z}(1) \longrightarrow \mathcal{O}_X)$ is the isomorphism class of E .

ii) ∇ is integrable if and only if $(E,\nabla) \in H^2(X,\mathbb{Z}(1) \longrightarrow \Omega_X^{\bullet})$.

Proof. i) This is Deligne's point of view.
In some Čech cover ∇ is given by one forms $\omega_i \in \Gamma(U_i,\Omega_X^1)$ verifying $\xi_{ij}^{-1}\cdot d\xi_{ij} = \omega_i - \omega_j$. (ξ_{ij},ω_i) is the class wanted. It is isomorphic to the class of $(\mathcal{O},d)$ if and only if they

are functions $f_i \in \Gamma(U_i, \mathcal{O})$ verifying $\xi_{ij} = f_i \cdot f_j^{-1}$ and $\omega_i = f_i^{-1} \cdot df_i$.

ii) The curvature $d\omega_i \in H^0(X, \Omega_X^2 \longrightarrow \ldots \longrightarrow \Omega_X^n)$ vanishes if and only if

$$(E, \nabla) \in \operatorname{Ker}(H^2(X, \mathbb{Z}(1) \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1) \xrightarrow[d]{} H^0(X, \Omega_X^2 \longrightarrow \ldots \longrightarrow \Omega_X^n))$$

$$= H^2(X, \mathbb{Z}(1) \longrightarrow \Omega_X^{\cdot}) \ .$$

(1.2) In this language it is easy to see the well known

<u>Claim</u>. <u>If</u> X <u>has an Hodge structure</u>, <u>then</u> $H^1(X, \mathbb{C}^*) = H^2(X, \mathbb{Z}(1) \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1)$. <u>Therefore if</u> E <u>is a rank one bundle with vanishing Atiyah class</u>, <u>all the holomorphic connections on</u> E <u>are integrable</u>.

<u>Proof</u>. The second statement is a trivial consequence of the first one.

One has the commutative square

$$\begin{array}{ccccccccc}
\mathbb{Z}(1) & \longrightarrow & \mathcal{O} & \longrightarrow & \Omega^1 & & & & \\
\downarrow & & & & \downarrow d_1 & & & & \\
\mathbb{Z}(1) & \longrightarrow & \mathcal{O} & & \Omega^2 & \longrightarrow & \ldots & \longrightarrow & \Omega^n \\
 & & \downarrow d_0 & & \downarrow & & & & \\
 & & \Omega^1 & \longrightarrow & \Omega^2 & \longrightarrow & \ldots & \longrightarrow & \Omega^n
\end{array}$$

This gives a commutative diagram

$$\begin{array}{ccc}
H^2(\mathbb{Z}(1) \longrightarrow \mathcal{O} \longrightarrow \Omega^1) & \xrightarrow{H(d_1)} & H^0(\Omega^2 \longrightarrow \ldots \longrightarrow \Omega^n) \\
\downarrow & & \downarrow \\
H^2(\mathbb{Z}(1) \longrightarrow \mathcal{O}) & \xrightarrow{H(d_0)} & H^1(\Omega^1 \longrightarrow \Omega^2 \longrightarrow \ldots \longrightarrow \Omega^n)
\end{array}$$

The first statement is equivalent to $H(d_1) = 0$. The image of $H(d_0)$ is contained in $H^1(\Omega^1)$ and therefore meets in 0 the injective image of $H^0(\Omega^2 \longrightarrow \ldots \longrightarrow \Omega^n)$. This implies $H(d_1) = 0$.

(1.3) Let E be a bundle of rank r with an holomorphic connection ∇ . Introduce $\tau, (\mathcal{O}(1), \nabla_\tau)$ and $(\Omega^1_{P/X}(1), \nabla'_\tau)$ as in (0.7). Define the τ-<u>flat sections</u> to be those which are annihilited by a τ-connection. If $(\tau d)^2 = 0$, denote by $\Omega^\cdot_\tau = \mathcal{O}_P \xrightarrow[\tau d]{} f^*\Omega^1_X \longrightarrow \ldots \xrightarrow[\tau d]{} f^*\Omega^n_X$ the τ-<u>Rham complex</u>.

<u>Lemma</u>.

i) <u>One has</u> $Rf_*(f^*\Omega^k_X \xrightarrow{\tau d} f^*\Omega^{k+1}_X) = \Omega^k_X \xrightarrow{d} \Omega^{k+1}_X$

ii) <u>One has</u> $Rf_* \nabla_\tau = \nabla$

iii) $(\tau d)^2 = 0$ <u>if and only if</u> ∇^2_τ <u>is</u> $\mathcal{O}_P$-<u>linear</u>. <u>In this case</u>, $\tau : \Omega^1_P \longrightarrow f^*\Omega^1_X$ <u>extends to a morphism of complexes</u> $\tau : \Omega^\cdot_P \longrightarrow \Omega^\cdot_\tau$. <u>This defines a morphism</u> $Rf_*\mathbb{C}_P \longrightarrow \mathbb{C}_X$ <u>in the derived category</u>. <u>One has</u> $Rf_*\Omega^\cdot_\tau = \Omega^\cdot_X$

iv) <u>One has</u> $\nabla^2 = 0$ <u>if and only if</u> $\nabla^2_\tau = 0$. <u>In this case one has</u> $\nabla'^2_\tau = 0$. <u>Moreover</u> $\mathcal{O}(1)$ <u>and</u> $\Omega^1_{P/X}(1)$ <u>are generated by</u> τ-<u>flat sections</u>.

<u>Proof</u>.

i) As $Rf_* f^*\Omega^k_X = \Omega^k_X$, one just has to see that $f_*\tau d = d$. This is a local condition on X . On an open set U on X , one has

$\Gamma(f^{-1}U, f^*\Omega^k_X) = \Gamma(f^{-1}U, f^{-1}\Omega^k_X)$ on which $\tau d = d$.

ii) As in i) , one just has to see $f_*\nabla_\tau = \nabla$. As f^*E is the sheaf generated by relative global sections of $\mathcal{O}(1)$, and as $\nabla_\tau = (1\otimes q)\tau f^*\nabla$, this is equivalent to see $f_*(\tau f^*\nabla) = \nabla$. This is the same as in i) .

iii) One has $\nabla^2_\tau(\lambda\cdot x) = \lambda\cdot\nabla^2_\tau(x) + (\tau d)^2(\lambda)\cdot x$, for $\lambda \in \mathcal{O}_P$ and $x \in \mathcal{O}(1)$.

Ω^k_P is additively generated by elements $y = \lambda\cdot d\omega$, for $\omega \in \Omega^{k-1}_P$, $\lambda \in \mathcal{O}_P$. Then $\tau dy = \tau d\lambda\wedge\tau d\omega$, whereas $\tau d(\lambda\cdot\tau d\omega) = \tau d\lambda\wedge\tau d\omega + \lambda(\tau d)^2\omega$. If $(\tau d)^2 = 0$, then $\tau dy = \tau d(\lambda\cdot\tau d\omega)$. In other words, one has a morphism of complexes $\tau : \Omega^\cdot_P \longrightarrow \Omega^\cdot_\tau$.

iv) If $\nabla^2 = 0$ then $E = L \otimes_{\mathbb{C}} \mathcal{O}_X$ where L is a local constant system, and $\nabla = 1\otimes d$. Then $\tau f^*\nabla = 1\otimes\tau d$. If e^k is a basis of L on U , one has (with the notations of (0.7)) $\tau d\left(\frac{t^k}{t^0}\right) = 0$. Therefore $(\tau d)^2 = 0$. This implies $(\tau f^*\nabla)^2 = 0$, as well as $\nabla^2_\tau = \nabla'^2_\tau = 0$.

Conversely if $\nabla^2_\tau = 0$, then $f_*\nabla^2_\tau = \nabla^2 = 0$. One may generate $\mathcal{O}(1)$ by t^k and $\Omega^1_{P/X}(1)$ by x^k , which are τ-flat sections.

(1.4) <u>Remark</u>. To see that $\nabla^2 = 0$ implies $(\tau d)^2 = 0$ (which means that $\Omega^\cdot_\tau$ is a complex), one does not need in iv) the description of E by its flat sections. If e^k is any basis of E on U, one has in the notations of (0.7)

$$\tau d\left(\frac{t^k}{t^0}\right) = \frac{1}{t^0}\left(\sum_s \omega^{ks} t^s - \frac{t^k}{t^0}\sum_s \omega^{0s} t^s\right)$$

for $\omega^{k\ell}$ the connection matrix of ∇ on U.

Therefore one has

$$(\tau d)^2\left(\frac{t^k}{t^0}\right) = -\sum_s d\omega^{ks}\frac{t^s}{t^0} + \sum_s \omega^{ks}\left[\sum_{s'}\omega^{ss'}\frac{t^{s'}}{t^0} - \frac{t^s}{t^0}\sum_{s'}\omega^{0s'}t^{s'}\right]$$

$$+ \frac{t^k}{t^0}\sum_s d\omega^{0s}\frac{t^s}{t^0} - \left[\sum_s \omega^{ks}\frac{t^s}{t^0} - \frac{t^k}{t^0}\sum_s \omega^{0s}\frac{t^s}{t^0}\right]\left(\sum_s \omega^{0s}\frac{t^s}{t^0}\right)$$

$$- \frac{t^k}{t^0}\sum \omega^{0s}\left[\sum_{s'}\omega^{ss'}\frac{t^{s'}}{t^0} - \sum_{s'}\omega^{0s'}\frac{t^{s'}}{t^0}\right].$$

Applying the integrability condition

$d\omega^{k\ell} + \sum_s \omega^{ks}\omega^{s\ell} = 0$, one finds $(\tau d)^2 = 0$.

(1.5) If $(\tau d)^2 = 0$, one has as in (1.1) an injection

$$\begin{array}{l} H^2(P, \mathbb{Z}(1) \longrightarrow \Omega^{\bullet}_{\tau}) \\ \quad\big\downarrow \\ H^2(P, \mathbb{Z}(1) \longrightarrow \mathcal{O}_P \xrightarrow{\tau d} f^*\Omega^1_X) \end{array}.$$

$(\mathcal{O}, \tau d)$ is the trivial (integrable) τ-connection. One considers the morphism

$$\begin{array}{l} H^2(P, \mathbb{Z}(1) \longrightarrow \mathcal{O}_P \xrightarrow{\tau d} f^*\Omega^1_X) \\ \quad\big\downarrow \\ H^2(P, \mathbb{Z}(1) \longrightarrow \mathcal{O}_P) \end{array}.$$

For (F,∇_τ) and (F',∇'_τ) two rank one bundles with (integrable) τ-connections, define the (integrable) τ-connection on $F \otimes F'$: $\nabla_\tau \otimes \nabla_{\tau'}(e\otimes e') = \nabla_\tau e \otimes e' + e \otimes \nabla'_\tau e'$. If $\varphi : F' \longrightarrow F$ is a $\mathcal{O}_P$-morphism, define on F' the (integrable) τ-connection: $\varphi^*\nabla_\tau(e') = \nabla_\tau(\varphi(e))$. Then φ is an isomorphism from (F',∇'_τ) to (F,∇_τ) if it is an isomorphism from F' to F verifying $\varphi^*\nabla_\tau = \nabla'_\tau$.

Lemma. i) The isomorphism classes of rank one bundles F with τ-connections ∇_τ build a group identified with $H^2(P,\mathbb{Z}(1) \longrightarrow \mathcal{O}_P \xrightarrow[\tau d]{} f^*\Omega^1_X)$. Denote by $(\mathcal{O}(1),\nabla_\tau)$ the class defined in (0.7) . Its image in $H^2(P,\mathbb{Z}(1) \longrightarrow \mathcal{O}_P)$ is the isomorphism class of $\mathcal{O}(1)$.

ii) Assume that $(\tau d)^2 = 0$. Then ∇ is integrable if and only if $(\mathcal{O}(1),\nabla_\tau) \in H^2(P,\mathbb{Z}(1) \longrightarrow \Omega^\bullet_\tau)$.

Proof. i) We mimic (1.1) . If $u_{\alpha\beta}$ is a cocycle representing F on some Čech cover, then ∇_τ is given by $\omega_\alpha \in \Gamma(U_\alpha,f^*\Omega^1_X)$ such that $u_{\alpha\beta}^{-1}\cdot\tau du_{\alpha\beta} = \omega_\alpha - \omega_\beta$. Then $(u_{\alpha\beta},\omega_\alpha)$ is the class wanted.

This class is isomorphic to $(\mathcal{O},\tau d)$ if and only if they are $f_\alpha \in \Gamma(U_\alpha,\mathcal{O}^*_P)$ verifying $u_{\alpha\beta} = f_\alpha\cdot f_\beta^{-1}$, and $\omega_\alpha = f_\alpha^{-1}\cdot\tau df_\alpha$.

ii) By (1.3)iv) , $\nabla^2 = 0$ if and only if $\nabla^2_\tau = 0$.
This is equivalent to $0 = \tau d\omega_\alpha \in H^0(P,f^*\Omega^2_X \longrightarrow \ldots \longrightarrow f^*\Omega^n_X)$ or

$$(\mathcal{O}(1),\nabla_\tau) \in \operatorname{Ker}\left(H^2(P,\mathbb{Z}(1) \longrightarrow \mathcal{O}_P \longrightarrow f^*\Omega^1_X) \downarrow H^2(P,f^*\Omega^2_X \longrightarrow \ldots \longrightarrow f^*\Omega^n_X)\right)$$

$$= H^2(P,\mathbb{Z}(1) \longrightarrow \Omega^\bullet_\tau) .$$

(1.6) Claim. If X has an Hodge structure, and E is a bundle on X with an holomorphic connection ∇ such that $(\tau d)^2 = 0$, then one has $H^2(P, \mathbb{Z}(1) \longrightarrow \Omega^{\cdot}_{\tau}) = H^2(P, \mathbb{Z}(1) \longrightarrow \mathcal{O}_P \xrightarrow{\tau d} f^*\Omega^1_X)$. In particular $\nabla^2 = 0$ if and only if $(\tau d)^2 = 0$.

Proof. The second statement is a trivial consequence of the first one.

From the commutative diagram

$$
\begin{array}{ccccccccc}
\mathbb{Z}(1) & \longrightarrow & \mathcal{O}_P & \longrightarrow & f^*\Omega^1_X & & & & \\
\downarrow & & & & \downarrow d_1 & & & & \\
\mathbb{Z}(1) & \longrightarrow & \mathcal{O}_P & & & & & & \\
& & \downarrow d_0 & & f^*\Omega^2_X & \longrightarrow & \cdots & \longrightarrow & f^*\Omega^n_X \\
& & & & \downarrow & & & & \\
& & \Omega^1_P & \longrightarrow & \Omega^2_P & \longrightarrow & \cdots & \longrightarrow & \Omega^{n+r-1} \\
& & \downarrow & & \downarrow & & & & \\
& & f^*\Omega^1_X & \longrightarrow & f^*\Omega^2_X & \longrightarrow & \cdots & \longrightarrow & f^*\Omega^n_X
\end{array}
$$

one has the commutative diagram

$$
\begin{array}{ccccc}
H^2(\mathbb{Z}(1) \longrightarrow \mathcal{O}_P \longrightarrow f^*\Omega^1_X) & \xrightarrow{\qquad H(d_1) \qquad} & & & H^0(f^*\Omega^2_X \longrightarrow \cdots \longrightarrow f^*\Omega^n_X) \\
\downarrow & & & & \downarrow \\
H^2(\mathbb{Z}(1) \longrightarrow \mathcal{O}_P) & \xrightarrow[H(d_0)]{} & H^1(\Omega^1_P \longrightarrow \cdots \longrightarrow \Omega^{n+r-1}_P) & \xrightarrow[H(\tau)]{} & H^1(f^*\Omega^1_X \longrightarrow \cdots \longrightarrow f^*\Omega^n_X)
\end{array}
$$

The first statement is equivalent to $H(d_1) = 0$. The image of $H(d_0)$ is contained in $H^1(\Omega^1_P)$, therefore the image of $H(\tau)H(d_0)$ is contained in $H^1(f^*\Omega^1_X)$.

It meets in 0 the injective image of $H^0(f^*\Omega_X^2 \longrightarrow \ldots \longrightarrow f^*\Omega_X^n)$. Therefore one has $H(d_1) = 0$.

Remark. Compare (1.3)iv) and (1.6) .
In general one has $\nabla^2 = 0$ if and only if $\nabla_\tau^2 = 0$. With an Hodge structure, one has $\nabla^2 = 0$ if and only if $(\tau d)^2 = 0$. This is slightly weaker. This corresponds to (1.2) .

§ 2. Characteristic classes of a bundle E with an integrable connection .

(2.1) Let Y be a smooth analytic variety. Let $(A^k, k\geq 0)$ be a complex such that there is a morphism of complexes $\tau : \Omega_Y^\cdot \longrightarrow A^\cdot$ where $A^0 = \mathcal{O}_Y$, $\overset{k}{\wedge} A^1 = A^k$ is a quotient bundle of Ω_Y^k . Define $B^1 = \operatorname{Ker} \tau : \Omega_Y^1 \longrightarrow A^1$. As the differential of $A^\cdot$ is the factorization on $A^\cdot$ of τd , write simply τd for it.

A bundle F is said to have a τ-connection if there is a $\mathbb{C}$-linear morphism $\nabla : F \longrightarrow A^1 \otimes F$ verifying the τ-Leibnitz rule $\nabla_\tau(\lambda\cdot x) = \lambda\cdot\nabla_\tau(x) + \tau d(\lambda)\otimes x$.
∇_τ is said to be integrable if $\nabla_\tau^2 = 0$.
F is said to be generated by τ-flat sections if locally sections x generate F with $\nabla_\tau x = 0$. In this case one may find a cocycle $u_{\alpha\beta}$ representing F with $u_{\alpha\beta}^{-1}\cdot\tau du_{\alpha\beta} = 0.$
$(\mathcal{O},\tau d)$ is the trivial (integrable) τ-connection. As in (1.5) the isomorphism class of (F,∇_τ) is in $H^2(Y,\mathbb{Z}(1) \longrightarrow \mathcal{O}_Y \longrightarrow A^1)$, and $\nabla_\tau{}^2 = 0$ if and only if (F,∇_τ) is in $H^2(Y,\mathbb{Z}(1) \longrightarrow A^\cdot)$.

(2.2) One has the standard operations for bundles with τ-connections.
Let F and F' be bundles with (integrable) τ-connections ∇_τ and ∇'_τ . One defines (integrable) τ-connections on

$\overset{\ell}{\Lambda}F$ by $(\overset{\ell}{\Lambda}\nabla)(f_1\wedge\ldots\wedge f_\ell) = \sum f_1\wedge\ldots\wedge f_{i-1}\wedge\nabla f_i\wedge f_{i+1}\wedge\ldots\wedge f_\ell$

$F\otimes F'$ by $\nabla_\tau\otimes\nabla_{\tau'}(f\otimes f') = \nabla_\tau(f)\otimes f' + f\otimes\nabla'_\tau(f')$

$Hom_{\mathcal{O}_Y}(F,F')$ by $(\nabla\varphi)(f) = \nabla'_\tau\,\varphi(f) - \varphi(\nabla_\tau f)$.

Denote by $\nabla^\vee_\tau$ the connection on $Hom_{\mathcal{O}_Y}(F,\mathcal{O}_Y) = F^\vee$.

If (F,∇_τ) and (F',∇'_τ) are of rank one, of cocycles $(u_{\alpha\beta},\omega_\alpha)$ and $(u'_{\alpha\beta},\omega'_\alpha)$, then $(F\otimes F',\nabla_\tau\otimes\nabla'_\tau)$ is of cocycle $(u_{\alpha\beta}\cdot u'_{\alpha\beta},\omega_\alpha+\omega'_\alpha)$. Therefore $(F\otimes F',\nabla_\tau\otimes\nabla'_\tau) = (F,\nabla_\tau)+(F',\nabla'_\tau)$ in $H^2(Y,\mathbb{Z}(1)\longrightarrow\mathcal{O}_Y\longrightarrow A^1)$ (resp. in $H^2(Y,\mathbb{Z}(1)\longrightarrow A^\cdot)$) Similarly $(F^\vee,\nabla^\vee_\tau) = -(F,\nabla_\tau)$ in $H^2(Y,\mathbb{Z}(1)\longrightarrow\mathcal{O}_Y\longrightarrow A^1)$ (resp. $H^2(Y,\mathbb{Z}(1)\longrightarrow A^\cdot)$) .

A <u>filtration</u> $F_{k-1}\subset F_k$ of a higher rank bundle F by sub-bundles F_k such that $\nabla_\tau F_k\subset A^1\otimes F_k$ is said to be τ-compatible (τ-<u>flat</u> if $\nabla^2_\tau = 0$) . This defines (integrable) τ-connections $\nabla_{\tau,k}$ on F_k/F_{k-1} .

An <u>exact sequence</u> $0\longrightarrow F'\longrightarrow F\longrightarrow F''\longrightarrow 0$ is said to be τ-compatible (τ-flat) if the filtration $F'\subset F$ is .

(2.3) Let $g: Z\longrightarrow Y$ be a morphism between two manifolds, an F and τ be as in (2.1). Define the exact sequence

$$g^*B^1\longrightarrow\Omega^1_Z\xrightarrow{r}\Omega^1_{Z,\tau}\longrightarrow 0 .$$

One has the exact sequence

$$g^*A^1\longrightarrow\Omega^1_{Z,\tau}\xrightarrow{p'}\Omega^1_{Z/Y}\longrightarrow 0$$

Define $\overset{k}{\Lambda}\Omega^1_{Z,\tau} = \Omega^k_{Z,\tau}$.

Claim. r extends to a morphism of complexes $r : \Omega^{\cdot}_Z \longrightarrow \Omega^{\cdot}_{Z,\tau}$

Proof. The kernel of $\Omega^k_Z \longrightarrow \Omega^k_{Z,\tau}$ is generated by $g^*B^1 \wedge \Omega^{k-1}_Z$. One has to see that $d(g^*B^1 \wedge \Omega^{k-1}_Z) \subset g^*B^1 \wedge \Omega^k_Z$. One has $dB^1 \subset B^1 \wedge \Omega^1_Y$. Write $g^*B^1 = \mathcal{O}_Z \otimes_{g^{-1}\mathcal{O}_Y} g^{-1}B^1$.

One has

$$
\begin{aligned}
d(g^*B^1 \wedge \Omega^{k-1}_Z) &\subset \Omega^1_Z \wedge g^*B^1 \wedge \Omega^{k-1}_Z \\
&+ \mathcal{O}_Z \otimes_{g^{-1}\mathcal{O}_Y} g^{-1}(B^1 \wedge \Omega^1_Y) \wedge \Omega^{k-1}_Z \\
&+ g^*B^1 \wedge \Omega^k_Z \\
&\subset g^*B^1 \wedge \Omega^k_Z
\end{aligned}
$$

Denote by rd the differential on $\Omega^{\cdot}_{Z,\tau}$. One has $(rd)^2 = 0$.

One defines the r-connection

$$g^*\nabla_\tau : g^*F \longrightarrow \Omega^1_{Z,\tau} \otimes_{\mathcal{O}_Z} g^*F$$

by writing $g^*F = \mathcal{O}_Z \otimes_{g^{-1}\mathcal{O}_Z} g^{-1}F$

and $g^*\nabla_\tau(\lambda \otimes \varphi) = rd\lambda \otimes_{\mathcal{O}_Z} \varphi + \lambda \otimes_{g^{-1}\mathcal{O}_Y} g^{-1}\nabla_\tau \varphi$

for $\varphi \in g^{-1}F$ and $\lambda \in \mathcal{O}_Z$.

The corresponding B'^1 is the image of g^*B^1 in Ω^1_Z .
As $(rd)^2 = 0$, $g^*\nabla_\tau$ <u>is integrable if</u> ∇_τ <u>is</u>, and g^*F is generated by r-flat sections if F is generated by τ-flat sections.

(2.4) Set $Z = P(F)$ the projective bundle of F . One has the other exact sequence

$$0 \longrightarrow \Omega^1_{Z/Y}(1) \longrightarrow g^*F \xrightarrow{q} \mathcal{O}(1) \longrightarrow 0$$

Define as in (0.6) $\sigma : \Omega^1_{Z/Y} \longrightarrow \Omega^1_{Z,\tau}$
by $\sigma \otimes 1 = (1 \otimes q)\ g^*\nabla_\tau$.
By the same computation as in (0.6.1) one has

$-\sigma$ <u>is a section of</u> p' .

In this case, g^*A^1 is embedded in $\Omega^1_{Z,\tau}$.
One obtains a section

$$\tau' = (1 + p'\sigma) : \Omega^1_{Z,\tau} \longrightarrow g^*A^1$$

which may be written with the notations (0.7) as
$\tau' rd\left(\frac{t^k}{t^0}\right) = \frac{1}{t^0}(1 \otimes q)\ (g^*\nabla_\tau e^k - \frac{t^k}{t^0}\ g^*\nabla_\tau e^0)$.

(2.5) Assume now that ∇_τ is integrable.
By (1.4) one has $(\tau' rd)^2 = 0$. This defines (1.3)iii) a <u>morphism of complexes</u>

$$\tau' r : \Omega_Z^{\cdot} \longrightarrow g^*A^{\cdot}$$

where the differential on $g^*A^{\cdot}$ is defined by $\tau' rd$. As in (1.3) one has $Rf_*f^*A^{\cdot} = A^{\cdot}$. The morphism $\tau' r$ defines a morphism in the derived category $\tau' r : Rg_*\mathbb{C}_Z \longrightarrow A^{\cdot}$.

(2.6) Further one may define: $\tau' r$-<u>connections</u> $\nabla_{\tau' r}$ and $\nabla'_{\tau' r}$ on $\mathcal{O}(1)$ and $\Omega^1_{Z/Y}(1)$ by

$$\nabla'_{\tau' r} = \tau' g^* \nabla_\tau \Big|_{\Omega^1_{Z/Y}(1)}$$

$$\nabla_{\tau' r} = (1 \otimes q)\tau' g^* \nabla_\tau \quad .$$

They are <u>integrable</u> if ∇_τ is .

(2.7) <u>Through the rest of</u> § 2 , <u>one considers on a manifold</u> X <u>a morphism of complexes</u> $\tau_0 : \Omega_\tau^{\cdot} \longrightarrow A^{\cdot}$ <u>as in</u> (2.1) <u>and</u> <u>a bundle</u> E <u>with an integrable</u> τ_0-<u>connection</u> ∇ .

On the projective bundle P(E) one has defined $r\tau_0$ and integrable $r\tau_0$-connections on $\mathcal{O}(1)$ and $\Omega^1_{P(E)/X}(1)$. One may repeat this construction (rank E-1) times.
One has the following data on <u>the flag bundle</u> of E which we call $f : P \longrightarrow X$, with f <u>the splitting morphism</u>.

i) There is a morphism $\tau : \Omega^1_P \longrightarrow f^*A^1$ with $(\tau d)^2 = 0$.
The complex $A_\tau^{\cdot} = \mathcal{O}_P \xrightarrow[\tau d]{} f^*A^1 \longrightarrow \dots \xrightarrow[\tau d]{} f^*A^n$ verifies $Rf_*A_\tau^{\cdot} = A^{\cdot}$. If τ_0 = identity (which means E flat) ,

write $\Omega^{\bullet}_{\tau}$ for $A^{\bullet}_{\tau}$. One has $Rf_*\Omega^{\bullet}_{\tau} = \Omega^{\bullet}_{X}$.

τ extends to a morphism of complexes

$\tau : \Omega^{\bullet}_{P} \longrightarrow \Omega^{\bullet}_{\tau}$.

ii) The integrable τ_0-connection ∇ defines an integrable τ-connection $(f^*\nabla)_\tau$ on f^*E . The canonical filtration $0 = E_0 \subset \ldots \subset E_r = f^*E$ of f^*E is τ-flat (see 2.2) . This defines an integrable τ-connection $\nabla_{\tau,k}$ on the splitting rank one bundle $L_k = E_k/E_{k-1}$, and therefore a class $(L_k, \nabla_{\tau,k}) \in H^2(P, \mathbb{Z}(1) \longrightarrow A^{\bullet}_{\tau})$ whose image in $H^2(P, \mathbb{Z}(1) \longrightarrow \mathcal{O}_P)$ is the isomorphism class of L_k (and whose image in $H^2(P, \mathbb{Z}(1))$ is $c_1^{top}(L_k)$, the topological Chern class ((2.1)) . This class is represented on some Čech cover by $(u^k_{\alpha\beta}, \omega_\alpha{}^k) \in \Gamma(U_{\alpha\beta}, \mathcal{O}^*) \times \Gamma(U_\alpha, A^1_\tau)$ such that $\delta u = 0$, $u^{-1}\cdot \tau du = \delta\omega$, $\tau d\omega = 0$ . .

(2.8) The Deligne complexes (see [1]) on a manifold Z are

$$\mathbb{Z}(p)_{\mathcal{D}} = \mathbb{Z}(p) \longrightarrow \mathcal{O}_Z \longrightarrow \ldots \longrightarrow \Omega^{p-1}_Z$$
$$= \mathrm{cone}(\mathbb{Z}(p) \oplus F^p \xrightarrow[\alpha+i]{} \Omega^{\bullet}_Z)[-1]$$

where $\alpha : \mathbb{Z}(p) \longrightarrow \mathbb{C}$ is the natural embedding and $i : F^p \longrightarrow \Omega^{\bullet}$ is the Hodge-Deligne F-filtration. There is a product

$$\mathbb{Z}(p)_{\mathcal{D}} \times \mathbb{Z}(q)_{\mathcal{D}} \longrightarrow \mathbb{Z}(p+q)_{\mathcal{D}}$$

which is uniquely defined by

$$x.x' = \alpha(x).x' \quad \text{if} \quad \deg x = 0$$
$$\phantom{x.x' = {}} xdx' \quad \text{if} \quad \deg x > 0 \quad \text{and} \quad \deg x' = q$$
$$\phantom{x.x' = {}} 0 \quad \text{otherwise,}$$

for x homogeneous in $\mathbb{Z}(p)_{\mathcal{D}}$ and x' homogeneous in $\mathbb{Z}(q)_{\mathcal{D}}$.

In the cone language this corresponds to

$(n\oplus f\oplus\omega).(n'\oplus f'\oplus\omega') = (n.n'+f.f', \alpha(n).\omega'+\omega\wedge i(f))$, for

$(n\oplus f) \in \mathbb{Z}(p)\oplus F^p$, $(n'\oplus f') \in \mathbb{Z}(q)\oplus F^q, \omega$ and $\omega' \in \Omega^{\cdot}$.

This defines a product in the cohomology

$$H^{p'}(\mathbb{Z}(p)_{\mathcal{D}}) \times H^{q'}(\mathbb{Z}q)_{\mathcal{D}}) \longrightarrow H^{p'+q'}(\mathbb{Z}(p+q)_{\mathcal{D}})$$

and therefore classes

$$c_p^{\mathcal{D}}(f^*E) \in H^{2p}(P, \mathbb{Z}(p)_{\mathcal{D}}) \quad \text{on the flag bundle } P \text{ of } E .$$

Define $\mathbb{Z}(p)_{\mathcal{D},\tau_0} = \mathbb{Z}(p) \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \dots \longrightarrow A^{p-1}$.

One has the morphism $\tau_0 : \mathbb{Z}(p)_{\mathcal{D}} \longrightarrow \mathbb{Z}(p)_{\mathcal{D},\tau_0}$.

(2.9) On a manifold Y with a morphism $\tau : \Omega_Y^{\cdot} \longrightarrow A^{\cdot}$ as in (2.1), define $\mathbb{Z}(p)_{\tau} = \mathbb{Z}(p) \xrightarrow{\tau} A^{\cdot}$ and a product

$$\mathbb{Z}(p)_{\tau} \times \mathbb{Z}(q)_{\tau} \longrightarrow \mathbb{Z}(p+q)_{\tau}$$

by $(x,x') = \tau(x).x'$ if $\deg x = 0$

$\phantom{by (x,x') = {}}$ 0 otherwise

for x and x' homogeneous in $\mathbb{Z}(p)_{\tau}$ and $\mathbb{Z}(q)_{\tau}$.

This defines a τ-product in the τ-cohomology

$$H^{p'}(\mathbb{Z}(p)_\tau) \times H^{q'}(\mathbb{Z}(q)_\tau) \longrightarrow H^{p'+q'}(\mathbb{Z}(p+q)_\tau)$$

as one easily sees in the Čech representation.

Write

$y = (y^{p'}+y^{p'1}+\dots) \in C^{p'}(\mathbb{Z}(p)) + C^{p'-1}(\mathcal{O}) + \dots$

such that $\delta y^{p'} = 0$, $\tau y^{p'} = \delta y^{p'-1}$, $\tau d y^{p'-1} = -\delta y^{p'-2}$ etc. ...

similarly for $z = (z^{q'}, z^{q'-1}, \dots)$.

Then

$$yz = (y^{p'}.z^{q'},\ y^{p'}.z^{q'-1},\ y^{p'}.z^{q'-2},\ \dots)$$
$$\in C^{p'+q'}(\mathbb{Z}(p+q)) + C^{p'+q'-1}(\mathcal{O}) \dots .$$

This fullfilles trivially the cocycle condition.

For $p'=q'=2$, $p=q=1$, $yz-zy = (0,\delta(y^1.z^1)),\tau d(y^1.z^1) - \delta(y^1.z^2-z^1.y^2))$

is a co-boundary.

Therefore the τ-product $H^2(\mathbb{Z}(1)_\tau) \times H^2(\mathbb{Z}(1)_\tau)$ is commutative.

The τ-product factorizes over the product

$$\mathbb{Z}(p) \times \mathbb{Z}(q)_\tau \longrightarrow \mathbb{Z}(p+q)_\tau \qquad \text{defined}$$

by $(x,x') \longrightarrow \tau(x).x'$.

Therefore the τ-product in the τ-cohomology factorizes over the product

$$H^{p'}(\mathbb{Z}(p)) \times H^{q'}(\mathbb{Z}(q)_\tau) \longrightarrow H^{p'+q'}(\mathbb{Z}(p+q)_\tau)$$

which is defined by $\tau(x)\cdot x'$.

Finally the product on $\mathbb{Z}(p)_\tau$ maps to the cup-product $\mathbb{Z}(p)$. Therefore the τ-product in the τ-cohomology maps to the cup-product in cohomology: the following diagram

$$\begin{array}{ccc} H^{p'}(\mathbb{Z}(p)_\tau) \times H^{q'}(\mathbb{Z}(q)_\tau) & \longrightarrow & H^{p'+q'}(\mathbb{Z}(p+q)_\tau) \\ \downarrow & & \downarrow \\ H^{p'}(\mathbb{Z}(p)) \times H^{q'}(\mathbb{Z}(q)) & \longrightarrow & H^{p'+q'}(\mathbb{Z}(p+q)) \end{array}$$

is commutative.

(2.10) Define the characteristic classes of $(f^*E, f^*\nabla)$ by

$c_p(f^*E, f^*\nabla)$ = p-th symmetric function of
$(L_k, \nabla_{\tau,k}) \quad \in \quad H^{2p}(P, \mathbb{Z}(p)_\tau)$

for E as in (2.7) .

(2.11) Denote by a_p the morphism

$$A_\tau^{\cdot} \longrightarrow (\mathcal{O}_P \xrightarrow[\tau d]{} \ldots \longrightarrow f^*A^{p-1}) \ ,$$

by τ the morphism

$$(\mathcal{O}_P \longrightarrow \ldots \longrightarrow \Omega_P^{p-1}) \longrightarrow (\mathcal{O}_P \xrightarrow[\tau d]{} \ldots \longrightarrow f^*A^{p-1}) \ ,$$

by $\mathbb{Z}(p)_{\mathcal{D},\tau}$ the complex

$$\mathbb{Z}(p) \longrightarrow \mathcal{O}_P \xrightarrow[\tau d]{} \ldots . \longrightarrow f^*A^{p-1}$$

and similarly for τ_0 .

Proposition. One has

$$\tau c_p^{\mathcal{D}}(f^*E) = a_p c_p(f^*E, f^*\nabla) \quad \underline{\text{in}} \quad H^{2p}(P, \mathbb{Z}(p)_{\mathcal{D},\tau}) \quad .$$

Proof. Compute it in the Čech representation. One may represent $(L_k, \nabla_{\tau,k})$ by

$$(n^k, \nu^k, \omega^k) \quad \in \quad C^2(\mathbb{Z}(1)) + C^1(\mathcal{O}) + C^0(A_\tau^1)$$

with $\tau n^k = \delta\nu^k$, $\exp\nu^k = u^k$ = cocycle of L_k , $\tau d\nu = \delta\omega$, $\tau d\omega = 0$.

Then $c_p(f^*E, f^*\nabla)$ may be represented by the symmetric sum of

$$(n^{k_1} \ldots n^{k_p}, \tau(n^{k_1}) \ldots \tau(n^{k_{p-1}}) \cdot \nu^{k_p}, \tau(n^{k_1}) \ldots \tau(n^{k_{p-1}}) \omega^{k_p}, 0 \ldots)$$

in $C^{2p}(\mathbb{Z}(p)) + C^{2p-1}(\mathcal{O}) + C^{2p-2}(A_\tau^1) + \ldots$.

Now $c_p^{\mathcal{D}}(f^*E)$ may be represented by the symmetric sum of

$$(n^{k_1} \ldots n^{k_p}, \alpha(n^{k_1}) \ldots \alpha(n^{k_{p-1}}) \cdot \nu^{k_p}, \alpha(n^{k_1}) \ldots \alpha(n^{k_{p-2}}) .$$

$$\nu^{k_{p-1}} d\nu^{k_p}, \ldots, \nu^{k_1} d\nu^{k_2} \wedge \ldots \wedge d\nu^{k_p}) \quad \text{in}$$

$$C^{2p}(\mathbb{Z}(p)) + C^{2p-1}(\mathcal{O}) + C^{2p-2}(\Omega_P^1) + \ldots .$$

Then $c_p(f^*E, f^*\nabla) - \tau c_p^{\mathcal{D}}(f^*E)$ may be represented by the symmetric sum of

$$(0, 0, \delta(\tau(n^{k_1}) \ldots \tau(n^{k_{p-2}}) \nu^{k_{p-1}} \omega^{k_p}) ,$$

$$\tau(n^{k_1}) \ldots \tau(n^{k_{p-2}}) \omega^{k_{p-1}} \omega^{k_p} - \delta(\tau(n^{k_1}) \ldots \tau(n^{k_{p-3}}) \nu^{k_{p-2}} \omega^{k_{p-1}} \omega^{k_p}), \ldots .$$

This is precisely a coboundary.

Remark that if L_k is generated by τ-flat sections, then the computation is trivial; one has $\omega = 0$ and $\tau d\nu = 0$.

(2.12) If $g : M \longrightarrow X$ is the projective bundle of E , then one has ([1], 1.7.2)

$$H^q(M,\mathbf{Z}(p)_{\mathcal{D}}) = \bigoplus_{\substack{0 \le j \le r-1 \\ 0 \le q-2j \\ 0 \le p-j}} g^{-1}H^{q-2j}(X,\mathbf{Z}(p-j)_{\mathcal{D}})\cdot\mathcal{O}(1)^j$$

The Deligne cohomology of M is a free module over the Deligne cohomology of X , with bases $\mathcal{O}(1)^j$, $0 \le j \le r-1$. By taking the coefficients of the expansion of $\mathcal{O}(1)^r$, one defines the Chern classes $c_p^{\mathcal{D}}(E) \in H^{2p}(X,\mathbf{Z}(p)_{\mathcal{D}})$. With the formalism of Hirzebruch-Grothendieck ([5]) , one proves they are functorial and additive, and thereby verify $f^{-1}c_p^{\mathcal{D}}(E) = c_p^{\mathcal{D}}(f^*E)$, where $c_p^{\mathcal{D}}(f^*E)$ was defined in (2.8) (see [1], 1.7.2 and 1.7.3) . The image of $c_p^{\mathcal{D}}(f^*E)$ in $H^{2p}(P,\mathbf{Z}(p))$ is the topological Chern class $c_p^{top}(f^*E) = f^{-1}c_p^{top}(E)$, where $c_p^{top}(E)$ is the image of $c_p^{\mathcal{D}}(E)$ in $H^{2p}(X,\mathbf{Z}(p))$.

(2.13) The formula (2.12) is no longer true for the τ-cohomology: $H^\cdot(M,\mathbf{Z}(\cdot)_\tau)$ is not a free module over $H^\cdot(X,\mathbf{Z}(\cdot)_{\tau_0})$. Therefore one can not use Hirzebruch-Grothendieck's formalism to prove that our classes $c_p(f^*E,f^*\nabla)$ verify the standard properties of Chern classes.

The rest of this chapter is essentially devoted to the definition of classes $c_p(E,\nabla)$ on X (2.15), to the proof of the functoriality (2.16) and the additivity (2.17), and to some simple comments (2.20), (2.21) and (2.22).

(2.14) Lemma. With the notations of (2.7) and (2.9), one has the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^{-1}H^q(X,\mathbb{Z}(p)_{\tau_0}) & \longrightarrow & H^q(P,\mathbb{Z}(p)_\tau) & \longrightarrow & H^q(P,\mathbb{Z}(p))/f^{-1}H^q(X,\mathbb{Z}(p)) & \longrightarrow & 0 \\ & & \downarrow a_p & & \downarrow a_p & & \| & & \\ 0 & \longrightarrow & f^{-1}H^q(X,\mathbb{Z}(p)_{\mathcal{D},\tau_0}) & \longrightarrow & H^q(P,\mathbb{Z}(p)_{\mathcal{D},\tau}) & \longrightarrow & H^q(P,\mathbb{Z}(p))/f^{-1}H^q(X,\mathbb{Z}(p)) & \longrightarrow & 0 \end{array}.$$

Proof. Just write

$$\begin{array}{lcl} \mathbb{Z}(p)_\tau & = & \mathrm{cone}(\mathbb{Z}(p) \longrightarrow A^\cdot)[-1] \\ \downarrow a_p & & \\ \mathbb{Z}(p)_{\mathcal{D},\tau} & = & \mathrm{cone}(\mathbb{Z}(p) \longrightarrow (\mathcal{O}_P \xrightarrow{\tau d} \dots \longrightarrow (A_\tau^{p-1}))[-1] \end{array}$$

and remember that

$$Rf_*(A_\tau^k \xrightarrow{\tau d} A_\tau^{k+1}) = A^k \xrightarrow{\tau_0 d} A^{k+1}$$

(2.15) Theorem. Let E be a bundle of rank r on a manifold X with an integrable τ_0-connection ∇. They are classes $c_p(E,\nabla) \in H^{2p}(X,\mathbb{Z}(p)_{\tau_0})$ whose images in $H^{2p}(X,\mathbb{Z}(p)_{\mathcal{D},\tau_0})$ are the images by τ_0 of the Chern classes $c_p^{\mathcal{D}}(E) \in H^{2p}(X,\mathbb{Z}(p)_{\mathcal{D}})$ in the Deligne cohomology, and whose

<u>images in</u> $H^{2p}(X,\mathbb{Z}(p))$ <u>are the topological Chern classes</u> $c_p^{top}(E)$.

<u>Proof</u>. The τ-product is compatible with the cup-product (2.9). Therefore the image of $c_p(f^*E,f^*\nabla)$ in $H^{2p}(P,\mathbb{Z}(p))$ is precisely $f^{-1}c_p^{top}(E)$. This shows via (2.14) that $c_p(f^*E,f^*\nabla) = f^{-1}c_p(E,\nabla)$ for a class $c_p(E,\nabla) \in H^{2p}(X,\mathbb{Z}(p)_{\tau_0})$ which is uniquely determined.
Its image c' in $H^{2p}(X,\mathbb{Z}(p)_{\mathcal{D},\tau_0})$ verifies

$$\begin{aligned} f^{-1}c' &= a_p c_p(f^*E,f^*\nabla) \\ &= \tau c_p^{\mathcal{D}}(f^*E) \qquad (2.11) \\ &= \tau f^{-1}c_p^{\mathcal{D}}(E) \qquad (2.12) \quad . \end{aligned}$$

One has the commutative diagram

$$\begin{array}{ccc} f^{-1}\, H^q(X,\mathbb{Z}(p)_{\mathcal{D}}) & \hookrightarrow & H^q(P,\mathbb{Z}(p)_{\mathcal{D}}) \\ \tau_0 \downarrow & & \downarrow \tau \\ f^{-1}\, H^q(X,\mathbb{Z}(p)_{\mathcal{D},\tau_0}) & \hookrightarrow & H^q(P,\mathbb{Z}(p)_{\mathcal{D},\tau}) \end{array}$$

Therefore $c' = \tau_0 c_p^{\mathcal{D}}(E)$.

This proves also that the image of $c_p(E,\nabla)$ is the topological class $c_p^{top}(E)$.

(2.16) In this point we want to prove the <u>functoriality</u> .
Let $g : Y \longrightarrow X$ be a morphism between two manifolds, and

E be a bundle with an integrable τ_0-connection on X (2.1). As in (2.3) τ_0 defines a morphism $\tau'_0 : \Omega^{\cdot}_Y \longrightarrow \Omega^{\cdot}_{Y,\tau'_0}$. Write for simplicity $A'^{\cdot} = \Omega^{\cdot}_{Y,\tau'_0}$ and set $B'^1 = \mathrm{im}\ g^*B^1$ in Ω^1_Y. Let $r : A'^{\cdot} \longrightarrow A''^{\cdot}$ be a morphism of complexes with $A''^0 = \mathcal{O}_Y$, A''^k is a quotient bundle of A'^k. Set $B'^1 \subset B''^1 = \mathcal{K}er(\Omega^1_Y \longrightarrow A''^1)$. Then $rg^*\nabla = r\nabla'$ is a well defined integrable τ''_0-connection on $g^*E = E'$ for $\tau''_0 : \Omega^{\cdot}_Y \longrightarrow A''^{\cdot}$.

Define $g^{-1}A^{\cdot}$ by $g^{-1}A^0 \longrightarrow g^{-1}A^1 \longrightarrow \dots$ as a complex of $\mathbb{C}$-modules. One has a natural map of $\mathbb{C}$-complexes $\rho : g^{-1}(\mathbb{Z}(p)_{\tau_0}) \longrightarrow \mathbb{Z}(p)_{\tau''_0}$. This defines

$$\rho g^{-1} : H^{2p}(X, \mathbb{Z}(p)_{\tau_0}) \longrightarrow H^{2p}(Y, \mathbb{Z}(p)_{\tau''_0}) .$$

<u>Proposition</u>. i) <u>One has</u> $\rho g^{-1} c_p(E,\nabla) = c_p(E',\nabla')$.

ii) <u>One has</u> $c_1(E,\nabla) = (\overset{r}{\Lambda} E, \overset{r}{\Lambda}\nabla)$ <u>as defined in</u> (2.15) <u>and</u> (2.2).

<u>Proof</u>. The second statement is a consequence of the first.

If i) is true, then one has $\rho' f^{-1}(\overset{r}{\Lambda} E, \overset{r}{\Lambda}\nabla) = (\overset{r}{\Lambda} f^*E, \overset{r}{\Lambda}(f^*\nabla)_\tau)$, for

$$\rho' : f^{-1}\,\mathbb{Z}(p)_{\tau_0} \longrightarrow \mathbb{Z}(p)_\tau .$$

One has

$$\begin{aligned}(\overset{r}{\Lambda} f^*E, \overset{r}{\Lambda}(f^*\nabla)_\tau) &= (\otimes_j L_j, \otimes_j \nabla_{\tau,j}) && \text{by construction}\\ &= \sum_j (L_j, \nabla_{\tau,j}) && (2.2)\\ &= c_1(f^*E, f^*\nabla) && (2.10)\\ &= f^{-1} c_1(E,\nabla) && (2.15)\end{aligned}$$

Therefore one has $(\overset{r}{\Lambda} E, \overset{r}{\Lambda}\nabla) = c_1(E,\nabla)$.

Let us prove the first statement. Consider the cartesian product

$$\begin{array}{ccc} P' & \xrightarrow{h} & P \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & X \end{array}$$

where P is the flag bundle of E and P' is the flag bundle of E'(2.7) . The canonical filtration E'_k (resp. splitting L'_k) of f'^*E' is the pull-back by h of the canonical filtration E_k (resp. splitting L_k) of f^*E .

On P and P' one has $\tau : \Omega^{\cdot}_P \longrightarrow A^{\cdot}_{\tau}$ and $\tau" : \Omega^{\cdot}_{P'} \longrightarrow A"^{\cdot}_{\tau"}$ as defined in (2.7) .

One wants to see that there is a natural map

$$\rho' : h^{-1}\mathbb{Z}(p)_{\tau} \longrightarrow \mathbb{Z}(p)_{\tau"}$$

such that the image by ρ' of

$(L_j,\nabla_{\tau,j}) \in H^2(P,\mathbb{Z}(1)_{\tau})$ is $(L'_j,\nabla_{\tau",j})$ in $H^2(P',\mathbb{Z}(1)_{\tau"})$.

Assume that $P = P(E)$ and $P' = P'(E')$ (this means that rank $E \leq 2$) .

One has the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & h^*f^*A^1 & \longrightarrow & h^*\Omega^1_P/f^*B^1 & \longrightarrow & h^*\Omega^1_{P/X} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow\wr & & \\ 0 & \longrightarrow & f'^*A"^1 & \longrightarrow & \Omega^1_{P'}/f'^*B"^1 & \longrightarrow & \Omega^1_{P'/Y} & \longrightarrow & 0 \end{array}$$

Recall that σ is defined by

$$\begin{array}{ccc}
\Omega^1_{P/X}(1) & \longrightarrow & f^*E \\
 & \searrow & \big\downarrow f^*\nabla \\
\sigma\otimes 1 & & \Omega^1_P/f^*B^1 \otimes f^*E \\
 & & \big\downarrow 1\otimes q \\
 & & \Omega^1_P/f^*B^1 \otimes \mathcal{O}(1)
\end{array}$$

This gives a commutative diagram

$$\begin{array}{ccccc}
h^*A^1_\tau = h^*f^*A^1 & \xleftarrow{h^*\tau} & h^*\Omega^1_P/f^*B^1 & & \\
\big\downarrow & & & \searrow & (0) \\
A'^1_{\tau'} = f'^*A''^1 & \xleftarrow{\tau'} & \Omega^1_P/f'^*B''^1 & \xleftarrow{\alpha} & \Omega^1_{P'}/f'^*B'^1
\end{array}$$

One has $\tau'\alpha h^*f^*\nabla = \tau' f'^*g^*\nabla$.

Define $C^1 = \mathit{Ker}\ \Omega^1_P \longrightarrow A^1_\tau$. Then $h^*(f^*\nabla)_\tau$ is a connection with values in $\Omega^1_{P'}/h^*C^1$. Define the morphisms r' and r''

$$\Omega^1_{P'}/f'^*B'^1 \xrightarrow{r'} \Omega^1_{P'}/h^*C^1 \xrightarrow{r''} A''^{\cdot}_{\tau''} \quad .$$

One has $r''r' = \tau'\alpha$. Therefore one has

(1) $$r''h^*(f^*\nabla)_\tau = \tau'\alpha h^*f^*\nabla \ .$$

Call ∇_τ and ∇'_τ the integrable τ-connections on $\mathcal{O}_P(1)$ and $\Omega^1_{P/X}(1)$, $\nabla_{\tau''}$ and $\nabla'_{\tau''}$ the integrable τ''-connections on $\mathcal{O}_{P'}(1)$ and $\Omega^1_{P'/Y}(1)$.

(1) implies
$$r''h^*\nabla_\tau = \nabla_{\tau''}$$
$$r''h^*\nabla'_\tau = \nabla'_{\tau''} \quad .$$

Now (0) implies that the map $h^{-1}A^k_\tau \longrightarrow A''^k_{\tau''}$ extends to well defined maps of complexes $h^{-1}A^\bullet_\tau \longrightarrow A''^\bullet_{\tau''}$ and $\rho' : h^{-1}\mathbb{Z}(p)_\tau \longrightarrow \mathbb{Z}(p)_{\tau''}$ such that

$$\rho'(\mathcal{O}_P(1),\nabla_\tau) = (\mathcal{O}_{P'}(1),\nabla_{\tau'}) \quad \text{and}$$
$$\rho'(\Omega^1_{P/X}(1),\nabla'_\tau) = (\Omega^1_{P'/Y}(1),\nabla'_{\tau''}) \ .$$

One repeats the construction inductively for $(\Omega^1_{P/X}(1),\nabla'_\tau)$ and $(\Omega^1_{P'/Y}(1),\nabla'_{\tau''})$.

(2.17) The next points (2.18) and (2.19) are devoted to the following additivity property.

Let $0 \longrightarrow (G,\nabla) \longrightarrow (E,\nabla) \xrightarrow{\pi} (E,\nabla) \longrightarrow 0$ be a τ_0-flat sequence (all 2.2)), with $r = \text{rank } E$ and $s = \text{rank } G$.

Proposition. One has $c_p(E,\nabla) = \sum_{k+\ell=p} c_k(G,\nabla) \cdot c_\ell(F,\nabla)$.

To prove it we need a standard geometrical compatibility of the flag bundles of F,G,E and further we need that this compatibility respects the complexes $\mathbb{Z}(p)_\tau$.

(2.18) We consider the flat exact sequence and

$$\begin{array}{ccccc} P(F) & & & & P(E) \\ & \searrow^{\varepsilon} & & {}^{\varepsilon'}\swarrow & \\ & & X & & \end{array}$$

The surjective morphism $\varepsilon^*E \longrightarrow \mathcal{O}_{P(F)}(1)$ defines an injection $j : P(F) \longrightarrow P(E)$ such that $j^*\mathcal{O}_{P(E)}(1) = \mathcal{O}_{P(F)}(1)$ ([6]) .
One obtains the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \varepsilon^*G & \simeq & \varepsilon^*G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^1_{P(E)/X}(1)\big|_{P(F)} & \longrightarrow & \varepsilon^*E & \xrightarrow{j^*q_E} & \mathcal{O}_{P(F)}(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \varepsilon^*\pi} & & \| & & \qquad (*) \\
0 & \longrightarrow & \Omega^1_{P(F)/X}(1) & \longrightarrow & \varepsilon^*F & \xrightarrow{q_F} & \mathcal{O}_{P(F)}(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Call σ_E and σ_F the sections defined in (2.4) . $\varepsilon'^*\nabla$ connection with values in $\Omega^1_{P(E)}/\varepsilon'^*B^1$, and we have $j^*\varepsilon'^*\nabla = \varepsilon^*\nabla$ by construction. Call $j^*\varepsilon'^*\nabla$ simply the <u>restriction of</u> $\varepsilon'^*\nabla$ to $P(F)$.

One has

$$\begin{aligned} \varepsilon^*\pi j^*\sigma_E \otimes 1 &= \varepsilon^*\pi j^*(1 \otimes q_E)\ (\varepsilon'^*\nabla) \\ &= (1 \otimes q_F)\,\varepsilon^*\nabla \\ &= \sigma_F \otimes 1 \end{aligned}$$

Therefore the diagram

$$\begin{array}{ccc} \Omega^1_{P(E)|P(F)} & \xrightarrow{\tau_{E|P(F)}} & \varepsilon^* A^1 \\ \downarrow & \nearrow_{\tau_F} & \\ \Omega^1_{P(F)} & & \end{array}$$

is commutative and extends to the commutative diagram

$$\begin{array}{ccc} \Omega^{\bullet}_{P(E)|P(F)} & \xrightarrow{\tau_{E|P(F)}} & A^{\bullet}_{\tau_F} \\ \downarrow & \nearrow_{\tau_F} & \\ \Omega^{\bullet}_{P(F)} & & \end{array}$$

Especially the restriction of the τ_E-connection of $\mathcal{O}_{P(E)}(1)$ to $P(F)$ is the τ_F-connection of $\mathcal{O}_{P(F)}(1)$, and the vertical left hand side sequence of (*) is an exact sequence of integrable τ_F-connections. This shows that our situation is inductive. We repeat the previous first step to reach the following state at the $(r-(s+1))$-st step.

One has the commutative diagram

$$\begin{array}{ccc} D(F) & \xrightarrow{i'} & Z' \\ f \downarrow & \swarrow_{h'} & \\ X & & \end{array}$$

where $D(F)$ is the flag bundle of F, i' is injective. On Z' one has the canonical "half-splitting" of

$E : E'_s \subset E'_{s+1} \subset \ldots \subset E'_r = h^*E$

such that $i'^*E'_s = f^*G$

$$i'^*E'_k/E_{k-1} = F_k/F_{k-1} \quad \text{for} \quad s+1 \leq k \leq r$$

where F_k is the canonical splitting of F :

$$0 = F_s \subset F_{s+1} \subset \ldots \subset F_r = f^*F \quad .$$

Call $\tau_F : \Omega^{\cdot}_{D(F)} \longrightarrow A^{\cdot}_{\tau_F}$ the morphism defined in (2.7) , with $A^k_{\tau(F)} = f^*A^k$, $Rf_*A^{\cdot}_{\tau(F)} = A^{\cdot}$, and $\tau_1 : \Omega^{\cdot}_{Z'} \longrightarrow A^{\cdot}_{\tau_1}$ the morphism in Z' defined in (2.5) and (2.7) . The filtration E'_k (F_k) is τ_1- (τ_F-) flat. The restriction of the integrable τ_1-connection $\nabla_{\tau_1,k}$ on E'_k/E'_{k-1} to $D(F)$ is the integrable τ_F-connection $\nabla_{\tau_F,k}$ on F_k/F_{k-1} .

One has the commutative diagram of complexes

$$\begin{array}{ccc} \Omega^{\cdot}_{Z'}|D(F) & \xrightarrow{\tau_1|D(F)} & A^{\cdot}_{\tau_F} \\ \downarrow & \nearrow_{\tau_F} & \\ \Omega^{\cdot}_{D(F)} & & \end{array}$$

This defines a morphism $\mathbb{Z}(p)_{\tau_1|D(F)} \longrightarrow \mathbb{Z}(p)_{\tau_F}$. The classes $(E'_k/E'_{k-1}, \nabla_{\tau_1,k})$ in $H^2(Z', \mathbb{Z}(1)_{\tau})$ are mapped to the classes $(F_k/F_{k-1}, \nabla_{\tau_F,k})$ in $H^2(D(F), \mathbb{Z}(1)_{\tau_F})$.

(2.19) Consider now the cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{i} & Z \\ \beta\downarrow & & \downarrow h'' \\ D(F) & \xrightarrow{i'} & Z' \end{array}$$

where Y is the flag bundle of f^*G, Z is the flag bundle of E'_s. Of course $Y = D(F) \times_X D(G)$ and Z is also the flag bundle of E. Write E_k the canonical filtration of $h^*E = h''^*h'^*E$. Then $0 = E_0 \subset E_1 \subset \ldots \subset E_s = h''^*E'_s$ is the canonical filtration of $h''^*E'_s$ and $E_k = h''^*E'_k$ for $s+1 \leq k \leq r$. Call $0 = G_0 \subset G_1 \subset \ldots \subset G_s = \gamma^*G = \beta^*f^*G = i^*E_s$ the canonical filtration of γ^*G, and set $F'_k = \beta^*F_k$. One has $i^*E_k = G_k$ for $0 \leq k \leq s$ and $i^*E_k/E_{k-1} = F'_k/F'_{k-1}$ for $s+1 \leq k \leq r$.

Call $\tau' : \Omega^{\cdot}_Y \longrightarrow A^{\cdot}_{\tau'}$ the morphism defined in (2.7) (with respect to f^*G and $\tau_F : \Omega^{\cdot}_{D(F)} \longrightarrow A^{\cdot}_{\tau_F}$ on $D(F)$).

One has $A^k_{\tau'} = \gamma^*A^k$ and $R\gamma_*A^{\cdot}_{\tau'} = A^{\cdot}$. Call $\tau : \Omega^{\cdot}_Z \longrightarrow A^{\cdot}_{\tau}$ the morphism defined in (2.7) (with respect to $h''^*E'_s$ and $\tau_1 : \Omega^{\cdot}_{Z'} \longrightarrow A^{\cdot}_{\tau_1}$, or if one prefers, with respect to h^*E and $\tau_0 : \Omega^{\cdot}_X \longrightarrow A^{\cdot}$).

We apply now the functoriality (2.16). There is a morphism $\rho' : \mathbb{Z}(p)_{\tau} \longrightarrow \mathbb{Z}(p)_{\tau'}$ which sends the class of $(E_k/E_{k-1}, \nabla_{\tau,k})$ in $H^2(Z, \mathbb{Z}(p)_{\tau})$ to the class of $(F'_k/F'_{k-1}, \nabla_{\tau',k})$ for $s+1 \leq k \leq r$ or to the class of $(G_k/G_{k-1}, \nabla_{\tau',k})$ for $0 \leq k \leq s$ in $H^2(Y, \mathbb{Z}(p)_{\tau'})$.

By the functoriality again, one knows that

$$\gamma^{-1}c_p(F,\nabla) = c_p(\gamma^*F,(\gamma^*\nabla)_{\tau'}) \quad \text{and}$$
$$\gamma^{-1}c_p(G,\nabla) = c_p(\gamma^*G,(\gamma^*\nabla)_{\tau'}) \quad .$$

Therefore one obtains

$$\rho' i^{-1}c_p(h^*E,(h^*\nabla)_\tau) = \sum_{k+\ell=p} \gamma^{-1}c_k(G,\nabla)\cdot\gamma^{-1}c_\ell(F,\nabla) \ .$$

The later is $\gamma^{-1} \sum_{k+\ell=p} c_k(G,\nabla)\cdot c_\ell(F,\nabla)$.

This finishes the proof.

(2.20) Corollary. Let $g : Y \longrightarrow X$ be as in (2.16) . Assume that $(g^*E, rg^*\nabla)$ has a τ''_0-flat filtration $(E_k, rg^*\nabla = \nabla'_k)$. For $c(E_k/E_{k-1},\nabla'_k) = \sum_i c_i(E_k/E_{k-i},\nabla'_k)$ one has $\rho g^{-1}c(E,\nabla) = \prod_k c(E_k/E_{k-1},\nabla'_k)$.

Proof. Apply the functoriality and the additivity.

(2.21) Corollary. Let X be a smooth projective variety. Let $0 \longrightarrow (G,\nabla) \longrightarrow (E,\nabla) \longrightarrow (F,\nabla) \longrightarrow 0$ be a flat exact sequence with rank $E = r$ and rank $G = s$. Then $c_p(E,\nabla)$ is torsion for $p \geq \sup(s,r-s)+1$.

Proof. One has ((2.17) and (2.9)), assuming $r-s < p$ and $s < p$:

$$c_p(E,\nabla) = \sum_{k+\ell=p} c_k^{top}(G)\cdot c_\ell(F,\nabla)$$

As $c_k^{top}(G)$ is torsion for $k \geq 1$ and as $\ell < p$, one obtains (2.21).

<u>Remark</u>. This implies (2.15) that the image $c_p^{\mathcal{D}}(E)$ is torsion also.

(2.22) <u>Multiplicativity</u>.

Let E and F be two bundles on X with integrable τ_0-connections ∇ and ∇'. Consider a morphism $f : P \longrightarrow X$ realizing a splitting L_i of E, M_j of F with integrable τ-connections ∇_i and ∇_j'. One has the splitting of $f^*(E \otimes F)$ by $L_i \otimes M_j$, of $f^*(\nabla \otimes \nabla')_\tau$ by $\nabla_i \otimes \nabla_j'$. Then one has

$$\sum_{p \geq 0} f^{-1} c_p(E \otimes F, \nabla \otimes \nabla') \cdot t^p = \prod_{i,j} \left(1 + \left[(L_i, \nabla_i) + (M_j, \nabla_j') \right] \cdot t\right)$$

$$\sum_{p \geq 0} f^{-1} c_p(\overset{k}{\Lambda} E, \overset{k}{\Lambda} \nabla) \cdot t^p = \prod_{1 \leq i_1 < \ldots < i_k \leq \text{rank } E} \left(1 + \left[(L_{i_1}, \nabla_{i_1}) + \ldots + (L_{i_k} \nabla_{i_k}) \right] \cdot t\right) ..$$

$$c_p(E, \nabla) = (-1)^p \, c_p(E^\vee, \nabla^\vee) .$$

(2.23) <u>One summarizes the previous statements for standard flat bundles.</u>

<u>Theorem</u>. <u>Let</u> E <u>be a flat bundle on</u> X <u>with an integrable connection</u> ∇. <u>There are classes</u> $c_p(E, \nabla) \in H^{2p}(X, \mathbb{Z}(p) \longrightarrow \mathbb{C})$

whose images in $H^{2p}(X,\mathbb{Z}(p)_{\mathcal{D}})$ are the classes $c_p^{\mathcal{D}}(E)$, whose images in $H^2(X,\mathbb{Z}(p))$ are the Chern classes $c_p^{top}(E)$. They are functorial and additive. The class $c_1(E,\nabla)$ is the isomorphism class of $(\overset{r}{\Lambda}E,\overset{r}{\Lambda}\nabla)$ in $H^2(X,\mathbb{Z}(1) \longrightarrow \mathbb{C})$. Moreover $c_p(E,\nabla)$ is torsion for $p \geq 2$ as soon as E has a flat splitting by rank one bundles (and X is projective).

§ 3. Logarithmic theory.

Let D be a normal crossing divisor on X and $j : X-D \longrightarrow X$ be the open embedding. One may perform the whole theory of § 2 for bundles E with integrable τ_0-connections ∇ with logarithmic poles along D .

The set-up is the following.

One has a commutative diagram

$$\begin{array}{ccc} \Omega^{\bullet} & \xrightarrow{\tau_0} & A^{\bullet} \\ \downarrow & & \downarrow \\ \Omega^{\bullet}\langle D\rangle & \xrightarrow[\tau_{0,D}]{} & A_D^{\bullet} \end{array}$$

with τ_0 as in (2.1) and $\mathcal{O}_P = A_D^0$, A_D^k is a quotient bundle of $\Omega^k\langle D\rangle$. One defines

$$\mathbb{Z}(p)_{\mathcal{D},\tau_0} = \mathbb{Z}(p) \longrightarrow A_D^{\bullet} \; , \; \mathbb{Z}(p)_{\mathcal{D},D,\tau_0} = \mathbb{Z}(p) \longrightarrow A_D^0 \longrightarrow \ldots \longrightarrow A_D^{p-1} \; ,$$

$$a_p : \mathbb{Z}(p)_{D,\tau_0} \longrightarrow \mathbb{Z}(p)_{\mathcal{D},D,\tau_0} \; .$$

Theorem. They are classes $c_{p,D}(E,\nabla) \in H^{2p}(X,\mathbb{Z}(p)_{D,\tau_0})$ which are functorial and additive such that $a_p c_{p,D}(E,\nabla) = \tau_0 c_p^{\mathcal{D}}(E)$ in $H^{2p}(X,\mathbb{Z}(p)_{\mathcal{D},D,\tau_0})$. For standard logarithmic connections, one has $c_{p,D}(E,\nabla) \in H^{2p}(X,\mathbb{Z}(p) \longrightarrow Rj_*\mathbb{C})$, and $a_p c_{p,D}(E)$ is the image of $c_p^{\mathcal{D}}(E)$ in $H^{2p}(X,\mathbb{Z}(p) \longrightarrow \mathcal{O} \longrightarrow \ldots \longrightarrow \Omega^{p-1}\langle D\rangle)$.

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