ALGEBRAIC DEFORMATIONS OF TORIC VARIETIES I. GENERAL CONSTRUCTIONS

LUCIO CIRIO, GIOVANNI LANDI, AND RICHARD J. SZABO

Abstract. We construct and study noncommutative deformations of toric varieties by combining techniques from toric geometry, isospectral deformations, and noncommutative geometry in braided monoidal categories. Our approach utilizes the same fan structure of the variety but deforms the underlying embedded algebraic torus. We develop a sheaf theory using techniques from noncommutative algebraic geometry. The cases of projective varieties are studied in detail, and several explicit examples are worked out, including new noncommutative deformations of Grassmann and flag varieties. Our constructions set up the basic ingredients for thorough study of instantons on noncommutative toric varieties, which will be the subject of the sequel to this paper.

Contents

Date: January 2010 HWM–09–14 , EMPG–09–22 .

Introduction

This paper is the first part of a series of articles in which we define and study a class of noncommutative toric varieties, and construct instantons thereon. Our approach is inspired by the theory of isospectral deformations [13] and a construction due to Ingalls [23]. We expand and elaborate on some of the constructions introduced in the latter paper using techniques from noncommutative geometry in braided monoidal categories. We start with a noncommutative deformation of an algebraic torus and use this to deform every toric variety on which the torus acts. This is done in a fashion that does not alter the combinatorial fan data describing the toric variety.

Part of the motivation for our construction comes from enumerative geometry and attempts to provide physical interpretations of enumerative invariants of toric varieties. In [24, 10], it is argued that the computation of Donaldson–Thomas invariants of a toric threefold X can be reduced to the problem of locally enumerating noncommutative instantons on each open patch of X, and then assembling the local contributions into a global quantity using the gluing rules of toric geometry. This heuristic construction works because noncommutative deformations of $\mathbb{C}^3$ are simple enough to explicitly construct instantons thereon, but the construction utilizes commutative toric geometry techniques to glue together quantities which are locally constructed using methods of noncommutative geometry. In the the present paper we define a precise notion of "noncommutative toric variety" which leads to a more global picture of their noncommutative geometry and of the construction of instantons thereon. Although our main interest lies in the construction of noncommutative instantons, the requisite building blocks turn out to be rather technically involved and lengthy. Thus the present paper is a (partly expository) systematic development of the general machinery required. The treatment of instanton counting on these varieties is defered to a sequel [12].

Another motivation for our constructions comes from string geometry. Chiral fermion fields on a quantum curve can be embedded in string theory as an intersecting D-brane configuration together with a B-field [16]. Mathematically, this system is described by a holonomic $\mathcal{D}$-module. In some instances, the category of $\mathcal{D}$-modules is in correspondence with the category of modules on a noncommutative variety, of which some of our constructions furnish explicit examples and give precise realizations of the noncommutative geometry alluded to in [16]. The simplest example of such a correspondence is between right ideals of the algebra of differential operators on the affine line and line bundles over a certain noncommutative deformation of the projective plane $\mathbb{CP}^2$ [6]. The classification of bundles on noncommutative $\mathbb{CP}^2$ is related to the construction of instantons on a noncommutative $\mathbb{R}^4$ [25].

From a mathematical perspective, our general construction produces new examples of noncommutative varieties. In particular, by considering noncommutative deformations of projective toric varieties, we give new examples of noncommutative grassmannians, and more generally flag varieties. We use techniques of noncommutative algebraic geometry to develop a sheaf theory for our varieties. Our treatment of flag varieties includes a noncommutative twistor theory, while our development of sheaf theory also produces sheaves of differential forms, all of which are instrumental in the analysis of instantons [12]. An alternative approach to noncommutative toric varieties can be found in [8].

The organisation of this paper is as follows. In §1 we review the various algebraic constructions that we need, in particular the Hopf cocycle twisting procedure which will allow us to construct our deformations within a braided categorical framework. This framework will be utilized throughout the paper as a systematic means of deforming not only the varieties involved, but also geometric objects defined thereon.

In §2 we apply this twisting procedure to define a noncommutative deformation of the complex algebraic torus $(\mathbb{C}^\times)^n$, which extends the standard (real) noncommutative torus and is the basic building block for all constructions in this paper. We use this to construct a twist deformation of the algebraic group $\mathrm{GL}(n)$, which requires a suitable notion of quantum determinant. We give a new description of these noncommutative determinants. We also work out the related braided exterior algebras of noncommutative minors. These ingredients are used in the description of the noncommutative geometry of Grassmann and flag varieties.

In §3 we use the noncommutative algebraic torus to give a general definition of noncommutative toric varieties, using their combinatorial description in terms of fan data. Only the algebras of characters are deformed, not their group structure, and hence our noncommutative toric varieties are described by the same fan data. We illustrate the construction through several explicit examples.

In §4 we construct categories of quasi-coherent sheaves on generic noncommutative toric varieties, and establish basic properties of them paralleling the commutative case. We provide an explicit categorical description of sheaves which are equivariant with respect to the toric action, and a relationship between ideal sheaves and invariant subschemes of the noncommutative variety. These aspects are crucial ingredients for the enumeration of instantons that will be constructed in [12]. We also build sheaves of differential forms.

In §5 we turn to the special case of deformations of projective toric varieties, for which various constructions can be made very explicit. We demonstrate that our local definition of noncommutative deformations of complex projective spaces $\mathbb{CP}^n$ is equivalent to a "global" description which is a special instance of the noncommutative weighted projective spaces considered in [5]. We use these spaces to define noncommutative Grassmann and flag varieties as noncommutative quadrics in projective space, through suitable deformations of Plücker embeddings. We study the embedding relations in detail and derive conditions for the embeddings into noncommutative projective space to exist.

Finally, in §6 we describe in detail the properties of the categories of quasi-coherent sheaves on our noncommutative projective varieties, some of which are consequences of the general theory developed in [5]. We also study in detail the tautological bundles and sheaves of differential forms on our noncommutative grassmannians. The general framework presented in this section will lie at the heart of our construction of noncommutative instantons and their twistor description in [12].

Acknowledgments. We thank Simon Brain, Ugo Bruzzo, Brian Dolan and Michel Dubois-Violette for helpful discussions. The work of RJS was supported in part by grant ST/G000514/1 "String Theory Scotland" from the UK Science and Technology Facilities Council.

1. Algebraic preliminaries

This section summarizes the algebraic constructions which will be used throughout this paper and its sequel [12]. We present a general framework for working with the symmetries of the noncommutative varieties that we shall encounter later on. We also recall some notions from the localization theory for noncommutative algebras.

1.1. **Twist deformations of symmetries.**

Let $\mathcal{H}$ be a Hopf algebra over $\mathbb{C}$ with coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, counit $\varepsilon : \mathcal{H} \to \mathbb{C}$, and antipode $S : \mathcal{H} \to \mathcal{H}$. We will make use of the conventional Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ (with implicit summation) and

$$(\mathrm{id} \otimes \Delta)\, \Delta(h) = (\Delta \otimes \mathrm{id})\, \Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \ .$$

Definition 1.1. *Let $\mathcal{H} \otimes A \to A$, $h \otimes a \mapsto h \triangleright a$ be a left action of the Hopf algebra $\mathcal{H}$ on a unital algebra A with product $\mu : A \otimes A \to A$. The action is said to be* covariant *if the compatibility conditions*

$$h \triangleright \mu(a \otimes b) = \mu\big(\Delta(h) \triangleright (a \otimes b)\big) := \mu\big((h_{(1)} \triangleright a) \otimes (h_{(2)} \triangleright b)\big) \ , \qquad h \triangleright 1 = \varepsilon(h)\, 1 \tag{1.2}$$

hold for all $h \in \mathcal{H}$ and $a, b \in A$. In this case A is called a left $\mathcal{H}$-module algebra.

Similarly, a left action $\triangleright$ of the Hopf algebra $\mathcal{H}$ on a coalgebra (C, δ, ϵ) is said to be covariant, making the latter a *left $\mathcal{H}$-module coalgebra*, if the compatibility conditions

$$\delta(h \triangleright c) = \Delta(h) \triangleright \delta(c) := \big(h_{(1)} \triangleright c_{(1)}\big) \otimes \big(h_{(2)} \triangleright c_{(2)}\big) \ , \qquad \epsilon(h \triangleright c) = \varepsilon(h)\, \epsilon(c)$$

hold for all $h \in \mathcal{H}$ and $c \in C$, with the notation $\delta(c) = c_{(1)} \otimes c_{(2)}$.

The Hopf algebra $\mathcal{H}$ is itself an $\mathcal{H}$-module algebra with respect to the left adjoint action $h \triangleright^{\mathrm{ad}} g = \mathrm{ad}_h(g) := h_{(1)}\, g\, S(h_{(2)})$ for $h, g \in \mathcal{H}$. We recall next how to produce new Hopf algebra structures on $\mathcal{H}$ by deforming the original one using two-cocycles of $\mathcal{H}$.

Definition 1.3. *An element $F \in \mathcal{H} \otimes \mathcal{H}$ is called a* Drinfel'd twist element *for $\mathcal{H}$ if it has the following properties:*

(1) *F is invertible;*
(2) *F is counital:* $(\mathrm{id} \otimes \varepsilon)(F) = (\varepsilon \otimes \mathrm{id})(F) = 1$*; and*

(3) *F obeys the cocycle condition:* $(1\otimes F)\,(\mathrm{id}\otimes\Delta)(F)=(F\otimes 1)\,(\Delta\otimes\mathrm{id})(F)$.

In the category of left $\mathcal{H}$-modules, a Drinfel'd twist in the Hopf algebra $\mathcal{H}$ generates a deformation of the product $\mu : A\otimes A\to A$ on every algebra object A. Similarly, the twist can be used to deform the coproduct $\delta : C\to C\otimes C$ on every coalgebra object C. The results are $\mathcal{H}$-module algebras or coalgebras respectively. In the present paper we shall concentrate on the algebra cases.

Theorem 1.4. (1) *A Drinfel'd twist element $F=F^{(1)}\otimes F^{(2)}\in\mathcal{H}\otimes\mathcal{H}$ defines a twisted Hopf algebra structure $\mathcal{H}_F$ with the same multiplication and counit as $\mathcal{H}$, but with new coproduct and antipode given for $h\in\mathcal{H}$ by*

$$\Delta_F(h)=F\,\Delta(h)\,F^{-1}\ ,\qquad S_F(h)=U_F\,S(h)\,U_F^{-1} \tag{1.5}$$

where $U_F=F^{(1)}\,S\big(F^{(2)}\big)$.

(2) *If A is a left $\mathcal{H}$-module algebra, the deformed product*

$$a\star_F b:=\mu\big(F^{-1}\triangleright(a\otimes b)\big) \tag{1.6}$$

for $a,b\in A$ makes $A_F=(A,\star_F)$ into a left $\mathcal{H}_F$-module algebra with respect to the same action of $\mathcal{H}$.

There are analogous results for right actions. If A is an $\mathcal{H}$-module algebra, then the collection of left $\mathcal{H}$-invariant elements ${}^{\mathcal{H}}A$ forms an ideal of A in which the product associated to a Drinfel'd twist for $\mathcal{H}$ by Theorem 1.4 coincides with the undeformed product [11].

In general, the deformation of the $\mathcal{H}$-module algebra structure of $\mathcal{H}$ itself provided by Theorem 1.4 need not be compatible with the Hopf algebra structure of $\mathcal{H}$, because generically one has $\Delta(h\star_F g)\neq\Delta(h)\star_F\Delta(g)$. In order to obtain a deformation of both the underlying variety of $\mathcal{H}$ *and* the quantum group associated to $\mathcal{H}$, we use a dual framework dealing with coactions.

Definition 1.7. *Let $\Phi : A\to A\otimes\mathcal{H}$, $\Phi(a)=a_{(0)}\otimes a_{(1)}$ be a right coaction of the Hopf algebra $\mathcal{H}$ on a unital algebra A with product $\mu : A\otimes A\to A$. The coaction is said to be* covariant *if the linear map Φ is a unital algebra morphism,*

$$\Phi\big(\mu(a\otimes b)\big)=\mu\big(a_{(0)}\otimes b_{(0)}\big)\otimes a_{(1)}\,b_{(1)}\ ,\qquad \Phi(1)=1\otimes 1\ , \tag{1.8}$$

for all $a,b\in A$. In this case A is called a right $\mathcal{H}$-comodule algebra.

The initial coproduct Δ of $\mathcal{H}$ defines a right coaction of the Hopf algebra $\mathcal{H}$ on itself, and it makes $\mathcal{H}$ into an $\mathcal{H}$-comodule algebra. For dually paired Hopf algebras $\mathcal{H}$ and $\mathcal{F}$, with nondegenerate pairing $\langle -,-\rangle : \mathcal{H}\times\mathcal{F}\to\mathbb{C}$, to a right coaction of $\mathcal{F}$ on (an algebra, a coalgebra, etc.) A there corresponds a left action of $\mathcal{H}$ on A. Thus, e.g., a right $\mathcal{F}$-comodule algebra is a left $\mathcal{H}$-module algebra. The left regular action of $\mathcal{H}$ on $\mathcal{F}$:

$$h\triangleright\alpha=\alpha_{(1)}\,\big\langle h\,,\,\alpha_{(2)}\big\rangle \tag{1.9}$$

for $h\in\mathcal{H}$ and $\alpha\in\mathcal{F}$, is a covariant action which makes $\mathcal{F}$ into a left $\mathcal{H}$-module algebra.

Definition 1.10. *A linear map $F^\vee : \mathcal{H}\otimes\mathcal{H}\to\mathbb{C}$ is called a* dual Drinfel'd twist element *for $\mathcal{H}$ if it has the following properties for all $f,g,h\in\mathcal{H}$:*

(1) *$F^\vee$ is convolution-invertible: There exists a linear map $F^{\vee\,-1} : \mathcal{H}\otimes\mathcal{H}\to\mathbb{C}$ such that*

$$F^\vee\big(f_{(1)}\otimes g_{(1)}\big)\,F^{\vee\,-1}\big(f_{(2)}\otimes g_{(2)}\big)=F^{\vee\,-1}\big(f_{(1)}\otimes g_{(1)}\big)\,F^\vee\big(f_{(2)}\otimes g_{(2)}\big)=\varepsilon(f)\,\varepsilon(g)\ ;$$

(2) $F^\vee$ *is unital:* $F^\vee(f\otimes 1)=F^\vee(1\otimes f)=\varepsilon(f)$*; and*
(3) $F^\vee$ *obeys the cycle condition:*

$$F^\vee\big(f_{(1)}\otimes g_{(1)}\big)\,F^\vee\big(f_{(2)}\,g_{(2)}\otimes h\big)=F^\vee\big(g_{(1)}\otimes h_{(1)}\big)\,F^\vee\big(f\otimes g_{(2)}\,h_{(2)}\big)\ .$$

Theorem 1.11. (1) *A dual Drinfel'd twist element $F^\vee$ for $\mathcal{H}$ defines a twisted Hopf algebra structure $\mathcal{H}^{F^\vee}$ with the same coproduct and counit as $\mathcal{H}$, but with new algebra structure and antipode given for $g,h\in\mathcal{H}$ by*

$$\begin{aligned} g\times_{F^\vee}h &= F^\vee\big(g_{(1)}\otimes h_{(1)}\big)\,\big(g_{(2)}\cdot h_{(2)}\big)\,F^{\vee\,-1}\big(g_{(3)}\otimes h_{(3)}\big)\ , \\ S^{F^\vee}(g) &= U^{F^\vee}\big(g_{(1)}\big)\,S\big(g_{(2)}\big)\,U^{F^\vee\,-1}\big(g_{(3)}\big) \end{aligned} \tag{1.12}$$

where $U^{F^\vee}(g)=F^\vee(g_{(1)}\otimes S(g_{(2)}))$.
(2) *If A is a right $\mathcal{H}$-comodule algebra, the deformed product*

$$a\star^{F^\vee}b:=\mu\big(a_{(0)}\otimes b_{(0)}\,\big)\,F^{\vee\,-1}\big(a_{(1)}\otimes b_{(1)}\big) \tag{1.13}$$

for $a,b\in A$ makes $A^{F^\vee}=(A,\star^{F^\vee})$ into a right $\mathcal{H}^{F^\vee}$-comodule algebra.

The proof of Theorem 1.11 can be found in [32]. Again, there is an analogous result for left coactions. If the two Hopf algebras $\mathcal{H}$ and $\mathcal{F}$ are dually paired, then to any twist element $F=F^{(1)}\otimes F^{(2)}\in\mathcal{H}\otimes\mathcal{H}$ there is a canonically associated dual twist element $F^\vee:\mathcal{F}\otimes\mathcal{F}\to\mathbb{C}$ defined by

$$F^\vee(\alpha\otimes\beta)=\big\langle F\,,\,\alpha\otimes\beta\big\rangle:=\big\langle F^{(1)}\,,\,\alpha\big\rangle\,\big\langle F^{(2)}\,,\,\beta\big\rangle \tag{1.14}$$

for $\alpha,\beta\in\mathcal{F}$. Every time an $\mathcal{H}$-module algebra is also an $\mathcal{F}$-comodule algebra (i.e. the action determines a coaction of the dual Hopf algebra) any deformation obtained using the twist F of $\mathcal{H}$ can be equivalently described using the dual twist $F^\vee$ of $\mathcal{F}$ defined by (1.14). However, the dual twist element depends only on the pairing, without any reference to an action of $\mathcal{F}$.

In our main examples, we will use this Hopf algebraic approach as a means of deforming the algebra of functions on a variety acted upon by a group. Given a Lie group G, the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G is a Hopf algebra over $\mathbb{C}$. This Hopf algebra has coproduct given on primitive elements $x\in\mathfrak{g}$ by $\Delta(x)=1\otimes x+x\otimes 1$, counit by $\varepsilon(x)=0$, and antipode by $S(x)=-x$. The adjoint action of $\mathcal{H}$ on itself extends the usual adjoint action of Lie algebra elements $x\in\mathfrak{g}$. When the group G acts on a space X, the algebra of functions on X is a $\mathfrak{U}(\mathfrak{g})$-module algebra.

Let $\mathcal{F}=\mathrm{Fun}(G)$ be the algebra generated by commuting matrix elements g_{ij} in finite-dimensional representations of G, with $i,j=1,\dots,\dim(G)$. Let $g_{ij}(P)\in\mathbb{C}$ denote their evaluations on group elements $P\in G$. The commutative algebra $\mathcal{F}$ is a Hopf algebra with coproduct given by $\Delta_\vee(g_{ij})=\sum_k g_{ik}\otimes g_{kj}$, i.e. the transpose of the map given by matrix multiplication, antipode $S_\vee(g_{ij})(P)=g_{ij}(P^{-1})$ for $P\in G$, and counit $\varepsilon_\vee(g_{ij})=\delta_{ij}$. The Hopf algebra $\mathcal{F}$ is dual to the enveloping Hopf algebra $\mathcal{H}$, with dual pairing $\langle h,g\rangle=h(g)(1)$ the evaluation at the identity of the bi-invariant differential operator on G associated to $h\in\mathcal{H}$ acting on the function $g\in\mathcal{F}$. When the group G acts on a space X, the algebra of functions on X is a $\mathrm{Fun}(G)$-comodule algebra.

As we will need below formal power series in some parameters θ, we will rather need to work in the quantum enveloping algebra $\mathcal{H}=\mathfrak{U}(\mathfrak{g})[[\theta]]$, the θ-adic completion of $\mathfrak{U}(\mathfrak{g})$.

1.2. **Braided monoidal categories of Hopf-module algebras.** A useful unifying framework in which to analyse our noncommutative deformations is provided by braided monoidal categories, wherein the noncommutativity is completely encoded in a braiding of a category whose objects are commutative varieties.

Definition 1.15. *A* braided monoidal (or quasitensor) category $(\mathscr{C}, \otimes, \Psi)$ *is a monoidal category* $(\mathscr{C}, \otimes)$ *with a natural equivalence between the two functors* $\otimes, \otimes^{\rm op} : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ *given by functorial isomorphisms*

$$\Psi_{V,W} \, : \, V \otimes W \ \longrightarrow \ W \otimes V \tag{1.16}$$

for all objects V, W *of* $\mathscr{C}$*, obeying hexagon relations which express compatibility of* Ψ *with the associativity structure of the tensor product* $\otimes$ *(see e.g.* [32, Fig. 9.4]*). The operators (1.16) are called* braiding morphisms*. If in addition* $\Psi^2 = \mathrm{id}$*, the category* $(\mathscr{C}, \otimes, \Psi)$ *is said to be a* symmetric (or tensor) category*.*

Our interest in braided monoidal categories stems from the category of Hopf-modules introduced in §1.1. We shall denote by ${}_{\mathcal{H}}\mathscr{M}$ the (sub)category of Hopf-module algebras. An algebra map $A \xrightarrow{\sigma} B$ is a morphism of the category ${}_{\mathcal{H}}\mathscr{M}$ if and only if it fits into the commutative diagram

$$\begin{tikzcd} \mathcal{H} \otimes A \arrow[r, "\mathrm{id}\otimes\sigma"] \arrow[d] & \mathcal{H} \otimes B \arrow[d] \\ A \arrow[r, "\sigma"'] & B \end{tikzcd}$$

where the vertical arrows are the $\mathcal{H}$-actions, i.e. σ is an $\mathcal{H}$-equivariant map.

On the tensor product of two Hopf-module algebras $A \otimes B$ we will consider the action of the Hopf algebra $\mathcal{H}$ defined by

$$\Delta(h) \triangleright (a \otimes b) = \big(h_{(1)} \triangleright a\big) \otimes \big(h_{(2)} \triangleright b\big) \tag{1.17}$$

for all $a \in A$, $b \in B$, and $h \in \mathcal{H}$. Both the algebra structure of $A \otimes B$ and the braiding in the category are determined by a quasitriangular structure of $\mathcal{H}$, i.e. an invertible $\mathcal{R}$-matrix $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ in $\mathcal{H} \otimes \mathcal{H}$ obeying

$$\tau \circ \Delta(h) = \mathcal{R} \, \Delta(h) \, \mathcal{R}^{-1}$$

and

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(1)} \otimes \big(\mathcal{R}^{(2)}\big)^2 \ , \qquad (\mathrm{id} \otimes \Delta)\mathcal{R} = \big(\mathcal{R}^{(1)}\big)^2 \otimes \mathcal{R}^{(2)} \otimes \mathcal{R}^{(2)}$$

where $\tau : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is the flip map which interchanges the two factors of $\mathcal{H}$. See [32] for proofs of the following results.

Proposition 1.18. *If* $(\mathcal{H}, \mathcal{R})$ *is a quasitriangular Hopf algebra, then the category of left* $\mathcal{H}$*-module algebras* ${}_{\mathcal{H}}\mathscr{M}$ *is a braided monoidal category with braiding morphism*

$$\Psi_{A,B}(a \otimes b) = \big(\mathcal{R}^{(2)} \triangleright b\big) \otimes \big(\mathcal{R}^{(1)} \triangleright a\big) \tag{1.19}$$

for all $a \in A$ *and* $b \in B$*.*

When the Hopf algebra is triangular, i.e. $\mathcal{R}^{-1} = \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)}$, or $\tau \circ \mathcal{R}^{-1} = \mathcal{R}$, the category ${}_{\mathcal{H}}\mathscr{M}$ is symmetric, i.e. the braiding in (1.19) squares to the identity: $\Psi^2 = \mathrm{id}$. If in addition $\mathcal{H}$ is cocommutative, like the classical enveloping algebras $\mathfrak{U}(\mathfrak{g})$, then the $\mathcal{R}$-matrix can be taken to be $\mathcal{R} = 1 \otimes 1$ and the braiding morphism is given by the flip morphism τ, where $\tau_{A,B} : A \otimes B \to B \otimes A$ interchanges the factors as $\tau_{A,B}(a \otimes b) = b \otimes a$.

In this case, the ordinary tensor algebra structure of $A \otimes B$ is compatible with the action of $\mathcal{H}$, i.e. $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (a_1\, a_2) \otimes (b_1\, b_2)$. In the general case, the algebra structure on $A \otimes B$ which is acted upon covariantly by $\mathcal{H}$ depends on the quasitriangular structure.

Proposition 1.20. *If $(\mathcal{H}, \mathcal{R})$ is a quasitriangular Hopf algebra and A, B are $\mathcal{H}$-module algebras, then the* braided tensor product $A \widehat{\otimes} B$ *is the vector space $A \otimes B$ endowed with the product*

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (a_1 \otimes 1)\, \Psi_{B,A}(b_1 \otimes a_2)\, (1 \otimes b_2) = a_1 \left(\mathcal{R}^{(2)} \triangleright a_2\right) \otimes \left(\mathcal{R}^{(1)} \triangleright b_1\right) b_2 \ . \tag{1.21}$$

With this product $A \widehat{\otimes} B$ is an $\mathcal{H}$-module algebra.

In a braided monoidal category of algebras it is natural to relate the notion of commutativity to the braiding morphism. The usual definition of commutativity of an algebra A may be expressed as the invariance of the multiplication $\mu : A \otimes A \to A$ under the flip morphism $\tau_{A,A} : A \otimes A \to A \otimes A$, i.e. $\mu \circ \tau_{A,A} = \mu$. In a braided monoidal category $(\mathscr{C}, \otimes, \Psi)$ it is natural to replace τ, which is not necessarily a morphism in the category, by the braiding morphism Ψ. This motivates the following definition.

Definition 1.22. *An algebra object A in the category ${}_{\mathcal{H}}\mathscr{M}$ is* braided commutative *if its multiplication map $\mu : A \otimes A \to A$ is invariant with respect to the braiding morphism $\Psi_{A,A} : A \otimes A \to A \otimes A$ as*

$$\mu \circ \Psi_{A,A} = \mu \qquad \text{or} \qquad a\, b = \left(\mathcal{R}^{(2)} \triangleright b\right) \left(\mathcal{R}^{(1)} \triangleright a\right) \ , \tag{1.23}$$

for every $a, b \in A$.

If A is an object in the category ${}_{\mathcal{H}}\mathscr{M}$, and A_F is the twisted Hopf-module algebra defined by a Drinfel'd twist element $F = F^{(1)} \otimes F^{(2)} \in \mathcal{H} \otimes \mathcal{H}$ as in Theorem 1.4, then the braiding morphism Ψ_F and tensor product $\widehat{\otimes}_F$ on the category ${}_{\mathcal{H}_F}\mathscr{M}$ are defined as in Propositions 1.18 and 1.20 with respect to the twist deformed quasitriangular structure

$$\mathcal{R}_F = \left(F^{(2)} \otimes F^{(1)}\right) \mathcal{R}\, F^{-1} \ .$$

There is a natural equivalence between braided monoidal categories of left Hopf-module algebras defined by the functor

$$\mathscr{F}_F \ : \ \left({}_{\mathcal{H}}\mathscr{M} \, , \, \widehat{\otimes} \, , \, \Psi\right) \ \longrightarrow \ \left({}_{\mathcal{H}_F}\mathscr{M} \, , \, \widehat{\otimes}_F \, , \, \Psi_F\right)$$

which acts as the identity on objects and morphisms of ${}_{\mathcal{H}}\mathscr{M}$ [26, Thm. XV.3.5], the nontriviality being contained in what happens to the braided monoidal structure. This functorial isomorphism implies that any $\mathcal{H}$-covariant construction in the category ${}_{\mathcal{H}}\mathscr{M}$ of $\mathcal{H}$-module algebras has a twisted analog in the category ${}_{\mathcal{H}_F}\mathscr{M}$ of $\mathcal{H}_F$-module algebras.

1.3. **Ore localization.** Given a commutative unital algebra over $\mathbb{C}$ which is a domain, one usually localizes with respect to a subset which is closed under multiplication. For noncommutative algebras, the existence of the localization is guaranteed, for example, by an additional Ore condition on the subset.

Definition 1.24. *Let A be a noncommutative unital algebra over $\mathbb{C}$. A* left denominator set *in A is a subset $S \subset A$ such that for all $a \in A$ and for all $s, t \in S$ the following conditions hold:*

1. *S is closed under multiplication: $s\, t \in S$;*
2. *S satisfies the left permutable Ore condition: $(S \cdot a) \cap (A \cdot s) \neq 0$; and*
3. *S is left invertible: If $a\, s = 0$ then there exists $u \in S$ such that $u\, a = 0$.*

The last condition in Definition 1.24 is automatically satisfied when A is a domain. A completely analogous definition gives the notion of a right denominator set. Given a left denominator set $S \subset A$, one defines the localization algebra $A[S^{-1}] = S^{-1} \cdot A$ as the set of equivalence classes in $S \times A$ generated by the equivalence relation $(s_1, a_1) \sim (s_2, a_2)$ if and only if there exists $t \in S$ such that $(s_1\, a_2 - s_2\, a_1)\, t = 0$. As usual, one regards the equivalence class $[(s, a)]$ as the "fraction" $s^{-1}\, a$, and defines an algebra structure on these equivalence classes as follows. For the addition, the Ore condition applied to s_1 and s_2 means that there are elements $\tilde{s} \in S$ and $\tilde{a} \in A$ such that $\tilde{s}\, s_1 = \tilde{a}\, s_2$. We can thus define

$$s_1^{-1}\, a_1 + s_2^{-1}\, a_2 := (\tilde{s}\, s_1)^{-1}\, (\tilde{s}\, a_1 + \tilde{a}\, a_2)\ . \tag{1.25}$$

It is not difficult to prove that this definition does not depend on the choice of representatives for the equivalence classes. For the multiplication, we use the Ore condition on s_2 and a_1 to introduce elements $\tilde{s}\,' \in S$ and $\tilde{a}\,' \in A$ such that $\tilde{a}\,' s_2 = \tilde{s}\,' a_1$. We then define

$$\big(s_1^{-1}\, a_1\big) \cdot \big(s_2^{-1}\, a_2\big) := \big(\tilde{s}\,' s_1\big)^{-1}\, \big(\tilde{a}\,' a_2\big)\ , \tag{1.26}$$

and again this definition does not depend on the choice of representatives for the equivalence classes. Geometrically, the localization $A \hookrightarrow A[S^{-1}]$ corresponds to deleting the locus specified by the vanishing of elements of S in the variety dual to A.

2. Algebraic torus deformations

This paper systematically combines constructions from toric geometry and the theory of isospectral deformations. Isospectral deformations produce noncommutative geometries by using the isometric action of a real n-dimensional torus $\mathbb{T}^n$ on a riemannian (spin) manifold and its noncommutative deformation $\mathbb{T}^n_\theta$ [13]. We will extend these constructions to actions of the algebraic torus $(\mathbb{C}^\times)^n$, in order to obtain an analogous deformation of toric algebraic varieties. In this section we spell out the various algebraic constructions behind these deformations. Throughout this paper an implicit sum over repeated upper and lower indices is always understood.

2.1. **The noncommutative algebraic torus.** The definition of the noncommutative real torus essentially relies on harmonic analysis and a choice of homomorphism of groups between the space of characters and the torus itself. This procedure may be easily extended to a generic locally compact abelian Lie group G. We are ultimately interested in the case $G = (\mathbb{C}^\times)^n$. Let $A(G) \subset C^\infty(G)$ be the commutative algebra of a class of functions on G with a suitable growth condition "at infinity". The Fourier transform on G provides a decomposition of every function $f \in A(G)$ over a basis of functions $\{\chi_p\}_{p\in\widehat{G}}$ labelled by the group of characters of G, i.e. its Pontrjagin dual $\widehat{G} = \mathrm{Hom}_{\mathbb{C}}(G, \mathbb{C}^\times)$. For every $p \in \widehat{G}$, we set χ_p to be the function on G defined by $\chi_p(g) = \langle p, g\rangle$, for $g \in G$, where $\langle -, -\rangle : \widehat{G} \times G \to \mathbb{C}^\times$ is the pairing between G and $\widehat{G}$. This defines the Fourier components $\widehat{f} : \widehat{G} \to \mathbb{C}$ of $f \in A(G)$ as

$$\widehat{f}(p) = \int_G f(g)\, \overline{\chi_p}(g)\ \mathrm{d}g$$

where $p \in \widehat{G}$ and $\mathrm{d}g$ denotes the bi-invariant Haar measure of G. Using L^2-orthonormality of the characters, the inverse Fourier transformation is given by

$$f(g) = \int_{\widehat{G}} \widehat{f}(p)\, \chi_p(g)\ \mathrm{d}p$$

with dp the bi-invariant Haar measure of $\widehat{G}$.

In order to define a noncommutative associative product on $A(G)$ it is enough to describe it on the G-eigenbasis $\{\chi_p\}_{p\in\widehat{G}}$ and then extend it to $A(G)$ via the Fourier transform. Given a homomorphism of groups $\Theta:\widehat{G}\to G$, we set

$$\chi_p \star_\Theta \chi_q := \chi_p \cdot \big(\Theta(p)\triangleright \chi_q\big) = \big\langle q\,,\,\Theta(p)\big\rangle\ \chi_{p+q}$$

for $p,q\in\widehat{G}$. Here the symbol $\triangleright$ denotes the (left) action of the group G on $A(G)$. Using the Fourier transformation this extends to a product on functions $f,f'\in A(G)$:

$$(f\star_\Theta f'\,)(g) = \int_{\widehat{G}\times\widehat{G}} \widehat{f}(p)\,\widehat{f'}(q)\,\chi_{p+q}(g)\,\big\langle q\,,\,\Theta(p)\big\rangle\ \mathrm{d}p\ \mathrm{d}q\ .$$

The vector space $A(G)$ with this product defines a noncommutative associative algebra denoted $A_\Theta(G)$.

Example 2.1. *Let $G=V$ be a locally compact abelian vector Lie group of (real) dimension n. Then $\widehat{G}\cong V^*=\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$. By choosing an $\mathbb{R}$-basis of V, there are isomorphisms $V\cong\mathbb{R}^n$ and $V^*\cong\mathbb{R}^n$. In this case the homomorphism Θ may be taken to be a linear endomorphism on V defined by a real skew-symmetric $n\times n$ matrix $\theta\in\bigwedge^2 V$, and we get the Moyal product on $\mathbb{R}^n$.*

Example 2.2. *Let $G=V/L$ with V as in Example 2.1 and $L\subset V$ a lattice of maximal rank n. Then $\widehat{G}\cong L^*=\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$. Upon choosing a $\mathbb{Z}$-basis for L, there are isomorphisms $L\cong\mathbb{Z}^n$, $L^*\cong\mathbb{Z}^n$ and $G\cong\mathbb{T}^n$. In this case we put $\Theta(p)=\exp(\frac{\mathrm{i}}{2}\,\theta\cdot p)$ for $p\in L^*$ with θ again a real skew-symmetric $n\times n$ matrix, and we obtain the noncommutative torus $\mathbb{T}^n_\theta$.*

When $G=T$ is an algebraic torus of (complex) dimension n over $\mathbb{C}$, we proceed as follows. Let L be a lattice of rank n. Let $L^*=\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ be the dual lattice and denote the canonical pairing between the lattices by $\langle -,-\rangle : L^*\times L\to\mathbb{Z}$. The dual lattice is the group of characters $\{\chi_p\}_{p\in L^*}$ which provide a basis of T-eigenfunctions on the algebraic torus $T=L\otimes_{\mathbb{Z}}\mathbb{C}^\times$, i.e. one has $\widehat{G}=L^*\cong\mathrm{Hom}_{\mathbb{C}}(T,\mathbb{C}^\times)$. Thus $L\cong\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^\times,T)$ is the lattice of one-parameter subgroups of T. Pick a $\mathbb{Z}$-basis $e_1,\dots,e_n$ of L, with corresponding dual basis $e_1^*,\dots,e_n^*$ for L^*. Then there is an isomorphism $T\cong(\mathbb{C}^\times)^n$. Set $p=\sum_i p_i\,e_i^*\in L^*$ and $t=\sum_i e_i\otimes t_i\in T$. Then the characters are given by

$$\chi_p(t) = t^p := t_1^{p_1}\cdots t_n^{p_n}\ . \tag{2.3}$$

The Fourier components in this case are given by

$$\widehat{f}(p) = \int_T f(t)\,\overline{t}^{\,p}\ \mathrm{d}^\times t \tag{2.4}$$

with respect to the T-invariant measure $\mathrm{d}^\times t=(\mathrm{d}t\ \mathrm{d}\overline{t}\,)/|t|^2$. Using the discrete measure on the Pontrjagin dual $\widehat{T}=L^*$, every function $f:T\to\mathbb{C}$ with suitable growth "at infinity" can be written in terms of its Fourier components via the Laurent power series expansion

$$f(t) = \sum_{p\in L^*} \widehat{f}(p)\ t^p\ .$$

The space $\mathbb{C}\chi_p$ is the *eigenspace* for the T-action corresponding to the character given by $\langle p,-\rangle : T\to\mathbb{C}^\times$ in $\mathrm{Hom}_{\mathbb{C}}(T,\mathbb{C}^\times)\cong L^*$. Thus the L^*-grading gives precisely the eigenspace decompositions of algebraic objects, dual to T-invariant geometric objects.

The homomorphism $\Theta : L^* \to T$ is defined by a *complex* skew-symmetric $n \times n$ matrix θ via the usual relation $\Theta(p) = \exp(\frac{\mathrm{i}}{2}\,\theta \cdot p)$. The real part of θ again describes the deformation of the compact real torus $\mathbb{T}^n \subset (\mathbb{C}^\times)^n$, while the imaginary part applies to the "dilatation" part given by $(\mathbb{R}^+)^n$, according to the polar decomposition

$$(\mathbb{C}^\times)^n = (\mathbb{R}^+)^n \times \mathbb{T}^n \cong \mathbb{R}^n \times \mathbb{T}^n \ .$$

In this way we may think of the deformation of $(\mathbb{C}^\times)^n$ as a simultaneous and independent deformation of $\mathbb{R}^n$ and $\mathbb{T}^n$ as given in Example 2.1 and Example 2.2. However, for concrete computations this prescription is not very useful, because the Moyal deformation affects $\log|t|$ for elements $t \in (\mathbb{C}^\times)^n$ and thus leads to rather involved commutation relations. The transformation (2.4) with this decomposition of $(\mathbb{C}^\times)^n$ is the Fourier transform with respect to the real torus and the Mellin transform with respect to $(\mathbb{R}^+)^n$.

As an algebraic variety, the torus $(\mathbb{C}^\times)^n$ is dual to the Laurent polynomial algebra in n variables $\mathbb{C}(t_1, \dots, t_n) := \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. The monomials in this coordinate algebra are the functions labelled by the characters $\chi_p(t) = t^p$ that we introduced in (2.3). The deformation of the product between such functions may be written explicitly as

$$z^p \star_\theta w^q = \exp\big(\tfrac{\mathrm{i}}{2}\,p_i\,\theta^{ij}\,q_j\big)\ z^p \cdot w^q \tag{2.5}$$

where $z = \sum_i e_i \otimes z_i$, $w = \sum_i e_i \otimes w_i \in T$, and $p, q \in L^*$. The product (2.5) is extended linearly to all of $\mathbb{C}(t_1, \dots, t_n)$.

Definition 2.6. *The vector space $A(T) = \mathbb{C}(t_1, \dots, t_n)$ with the product $\star_\theta$ is called the* quantum Laurent algebra $A_\theta(T) = \mathbb{C}_\theta(t_1, \dots, t_n)$ *and its elements are called* quantum Laurent polynomials. *It is dual to a noncommutative variety denoted $(\mathbb{C}^\times_\theta)^n$.*

Remember that θ is a complex matrix. As we show explicitly in §2.2, the regular action of the group T on itself extends to an action on $(\mathbb{C}^\times_\theta)^n$. In particular, T acts by algebra automorphisms with respect to the product $\star_\theta$.

2.2. **Twisted toric actions.** Using the Hopf algebraic approach described in §1.1, we can alternatively define the quantum Laurent algebra by twisting the (quantum) enveloping algebra $\mathcal{H}$ of the algebraic torus group T. This is simply the polynomial algebra in n commuting elements H_i, the infinitesimal generators of the group. In fact we rather need formal power series in some parameters θ, but we will abuse notation by simply writing $\mathcal{H} = \mathcal{H}[[\theta]]$, while always implicitly understanding a θ-adic completion of $\mathcal{H}$.

As twisting element we take the abelian Drinfel'd twist

$$F = F_\theta := \exp\big(-\tfrac{\mathrm{i}}{2}\,\theta^{ij}\,H_i \otimes H_j\big) \ . \tag{2.7}$$

The infinitesimal action of T on characters is given by $H_i \triangleright \chi_p = \langle p, e_i \rangle\,\chi_p$ for $p \in L^*$. Then formula (1.6) for $a = z^p$ and $b = w^q$ monomials in the algebra $A(T) = \mathbb{C}(t_1, \dots, t_n)$ coincides exactly with (2.5).

On the other hand, in this case $\mathcal{H} = \mathcal{H}_\theta := \mathcal{H}_{F_\theta}$ as Hopf algebras. Since the Lie algebra of T is abelian, the coproduct $\Delta_\theta := \Delta_{F_\theta}$ of $\mathcal{H}_\theta$ computed from (1.5) is unaffected by the deformation and is given on generators by

$$\Delta_\theta(H_i) = \Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \ .$$

The antipode defined in (1.5) is also unaffected by the deformation, $S_{F_\theta} = S$, as is always the case with Drinfel'd twist elements of the form (2.7) [11]. Indeed, one shows that the element $U_{F_\theta} = F_\theta^{(1)}\, S\big(F_\theta^{(2)}\big)$ in this case is the identity by computing its n-th order term for any $n>0$ in a formal power series expansion in θ. This term is proportional to

$$\theta^{i_1j_1}\cdots\theta^{i_nj_n}\, H_{i_1}\cdots H_{i_n}\, S(H_{j_1}\cdots H_{j_n}) = (-1)^n\,\theta^{i_1j_1}\cdots\theta^{i_nj_n}\, H_{i_1}\cdots H_{i_n}\, H_{j_1}\cdots H_{j_n} = 0\ ,$$

and the vanishing follows from $\theta^{ij} = -\theta^{ji}$ and $H_i\, H_j = H_j\, H_i$ for each $i,j=1,\dots,n$. Thus $\mathcal{H} = \mathcal{H}_\theta$ as a Hopf algebra, and the deformed algebra $A_\theta(T)$ is also an $\mathcal{H}$-module algebra with respect to the same (undeformed) toric action. In this case the deformation of the triangular structure $\mathcal{R} = 1\otimes 1$ of $\mathcal{H}$ by the twist element (2.7) gives the twisted $\mathcal{R}$-matrix

$$\mathcal{R}_{F_\theta} = F_\theta^{-1}\,(1\otimes 1)\,F_\theta^{-1} = F_\theta^{-2}\ , \tag{2.8}$$

so that the twisted enveloping algebra $\mathcal{H}_\theta$ is triangular, $\tau\circ\mathcal{R}_{F_\theta}^{-1} = \mathcal{R}_{F_\theta}$, but no longer cocommutative, resulting in a nontrivial, albeit symmetric, braiding in the category ${}_{\mathcal{H}_\theta}\mathscr{M}$.

The coproduct on the algebra of functions $A(T)$ on the torus T is given on character elements $\chi_p : T\to\mathbb{C}^\times$, $p\in L^*$, by

$$\Delta_\vee(\chi_p) = \chi_p\otimes\chi_p\ , \tag{2.9}$$

while the antipode is the inverse $S_\vee(\chi_p) = \chi_p^{-1}$ in $\mathbb{C}^\times$. For this undeformed case, the dual pairing between generators H_i of T and the character algebra $A(T)$ is provided by the evaluation of the Lie derivative L_{H_i} with respect to the invariant vector field associated to H_i; in particular for the characters one finds:

$$\langle H_i,\chi_p\rangle := L_{H_i}(\chi_p)(1) = p_i\ .$$

Using the Drinfel'd twist (2.7) and its dual twist element $F^\vee = F^\theta$ defined by (1.14), from Theorem 1.11 we obtain the twisted Hopf algebra $\mathrm{Fun}^\theta(T)$ with deformed product on characters given by

$$\begin{aligned}
\chi_p\times_\theta\chi_q &= F^\theta(\chi_p\otimes\chi_q)\,(\chi_p\cdot\chi_q)\,F^{\theta\ -1}(\chi_p\otimes\chi_q)\\
&= \big\langle F_\theta\,,\,\chi_p\otimes\chi_q\big\rangle\,(\chi_p\cdot\chi_q)\,\big\langle F_\theta^{-1}\,,\,\chi_p\otimes\chi_q\big\rangle\\
&= \exp\big(-\tfrac{\mathrm{i}}{2}\,p_i\,\theta^{ij}\,q_j\big)\,(\chi_p\cdot\chi_q)\,\exp\big(\tfrac{\mathrm{i}}{2}\,p_i\,\theta^{ij}\,q_j\big) \ = \ \chi_p\cdot\chi_q\ ,
\end{aligned}$$

which coincides with the undeformed product on the character algebra. The antipode is also unaffected by the deformation, $S_\vee^{F^\theta}(\chi_p) = S_\vee(\chi_p)$, as can be checked directly by using (1.12), or by using duality and the fact that the antipode in $\mathcal{H}_\theta$ is unchanged by the deformation in this case. Thus the quantum group symmetry underlying the quantum Laurent algebra also coincides with the classical (undeformed) toric symmetry.

2.3. **The noncommutative variety** $\mathrm{GL}_\theta(n)$**.** Some of our constructions will rely on a noncommutative $(\mathbb{C}^\times)^n$ deformation of the general linear group $\mathrm{GL}(n)$ over $\mathbb{C}$. The deformation is realized using the action of the algebraic torus by a (dual) Drinfel'd twist on the algebra of functions $\mathcal{F}_n := \mathrm{Fun}(\mathrm{GL}(n))$ on $\mathrm{GL}(n)$, as described in §1.1, which depends on an $n\times n$ skew-symmetric complex matrix θ. The Hopf algebra $\mathcal{F}_n$ is dual to the enveloping Hopf algebra $\mathcal{H}^n = \mathfrak{U}(\mathfrak{gl}(n))$. The left regular action of $\mathcal{H}^n$ on $\mathcal{F}_n$, defined in general in (1.9), is a covariant action which makes $\mathcal{F}_n$ into a left $\mathcal{H}^n$-module algebra. There is an analogous right regular covariant action of $\mathcal{H}^n$ on $\mathcal{F}_n$ which makes $\mathcal{F}_n$ into a right $\mathcal{H}^n$-module algebra.

The deformation of $\mathrm{GL}(n)$ which we use in the following is the only one which deforms $\mathcal{F}_n$ as a Hopf algebra, and also as an $\mathcal{H}^n$-*bimodule* algebra. Within the context of §1.1 and §1.2, it would be more natural to consider $\mathcal{F}_n$ as a left $\mathcal{H}^n$-module algebra via either the left regular action or the left adjoint action, or by their right acting versions. For our purposes this is undesirable as it introduces an asymmetry between row and column operations on matrix elements considered in the following. The deformation we use is compatible with the Hopf algebra structure, which is instrumental in some of our later constructions of differential forms, and moreover it is the one that is compatible with the embeddings we will consider into noncommutative projective spaces.

We first twist the standard Hopf algebra structure of $\mathcal{H}^n$ to obtain $\mathcal{H}^n_\theta$, using the twist element (2.7), where the H_i are the generators of the Lie algebra of the diagonally embedded maximal torus $(\mathbb{C}^\times)^n \subset \mathrm{GL}(n)$. Let $\{E_{ij}\}_{i,j=1,\dots,n}$ be the standard basis of $\mathfrak{gl}(n)$, with matrix elements $(E_{ij})_{kl} = \delta_{ik}\,\delta_{jl}$ and $H_i = E_{ii}$, and the commutation relations

$$[E_{ij}, E_{kl}] = E_{il}\,\delta_{jk} - E_{kj}\,\delta_{il} \ , \qquad [H_k, E_{ij}] = E_{ij}\,\big(\delta_{ki} - \delta_{kj}\big) \ .$$

These are used to compute the twisted coproduct $\Delta_\theta := \Delta_{F_\theta}$ as in (1.5). A straightforward computation, along the lines of [11], yields

$$\Delta_\theta(E_{ij}) = E_{ij} \otimes \lambda_{ij}^{-1} + \lambda_{ij} \otimes E_{ij}$$

with the group-like element λ_{ij} defined by

$$\lambda_{ij} = \exp\big(\tfrac{\mathrm{i}}{2}\,\theta^{kl}\,(\delta_{ik} - \delta_{jk})\,H_l\big) \ .$$

As expected, the generators H_i of the twist have undeformed coproduct.

By the general discussion of §1.1, in order to obtain a deformation of $\mathcal{F}_n$ which preserves the quantum group structure, we use the Drinfel'd twist $F^\vee = F^\theta$ defined as in (1.14), which is dual to the initial twist (2.7). As in §2.2 we compute the pairings

$$\langle H_k, g_{ij}\rangle = H_k(g_{ij})(1) = L_{H_k}(g_{ij})(1) = g_{ij}(H_k) = \delta_{ik}\,\delta_{jk} \ ,$$

with the generators g_{ij} of the algebra $\mathcal{F}_n$. Using Theorem (1.11) we then obtain the twisted Hopf algebra $\mathcal{F}^\theta_n$ still generated by elements g_{ij}, but now with noncommutative relations between them given by

$$\begin{aligned}
g_{ij} \times_\theta g_{kl} \ &= \sum_{m,p,r,s=1}^n F^\theta(g_{ir} \otimes g_{ks})\,(g_{rm} \cdot g_{sp})\,F^{\theta\,-1}(g_{mj} \otimes g_{pl}) \\
&= \sum_{m,p,r,s=1}^n \big\langle F_\theta\ ,\ g_{ir} \otimes g_{ks}\big\rangle\,(g_{rm} \cdot g_{sp})\,\big\langle F_\theta^{-1}\ ,\ g_{mj} \otimes g_{pl}\big\rangle \\
&= \sum_{m,p,r,s=1}^n q_{ki}\,\delta_{ir}\,\delta_{ks}\,(g_{rm} \cdot g_{sp})\,q_{mp}\,\delta_{mj}\,\delta_{pl} \ = \ q_{ki}\,q_{jl}\,(g_{ij} \cdot g_{kl}) \ ,
\end{aligned} \tag{2.10}$$

where

$$q_{ij} := \exp\big(\tfrac{\mathrm{i}}{2}\,\theta^{ij}\big) \ .$$

Introducing coefficients

$$Q_{ij\,;\,kl} = q_{ki}\,q_{jl} = q_{ik}^{-1}\,q_{jl} \ , \qquad Q^2_{ij\,;\,kl} = q^2_{ki}\,q^2_{jl} \tag{2.11}$$

we write the commutation rule for the deformed product as

$$g_{ij} \times_\theta g_{kl} = Q^2_{ij\,;\,kl}\ g_{kl} \times_\theta g_{ij} \ . \tag{2.12}$$

As usual, the coproduct $\Delta_\vee$ and the counit $\varepsilon_\vee$ are left unchanged. On the other hand, the commutativity of the generators H_i implies, as in §2.2, that the antipode $S_\vee^{F^\theta}(g_{ij}) = S_\vee(g_{ij})$ is unaltered as well.

Definition 2.13. *The noncommutative Hopf algebra* $\mathcal{F}_n^\theta = (\mathcal{F}_n, \times_\theta, \Delta_\vee, \varepsilon_\vee, S_\vee)$ *is called the* algebraic torus deformation quantum group of $\mathrm{GL}(n)$. *It is dual to a noncommutative variety denoted* $\mathrm{GL}_\theta(n)$.

A proper definition of the variety $\mathrm{GL}_\theta(n)$ involves the notion of noncommutative determinant; we will return to this point in detail in §2.4.

Remark 2.14. *This formalism may also be adapted to define noncommutative rectangular* $d \times n$ *matrix algebras, with* $d < n$*, as the* $\mathbb{C}$*-subalgebra of* $\mathcal{F}_n^\theta$ *generated by* g_{ij} *with* $i \leq d$*. There is a* $\mathbb{C}$*-algebra retraction of* $\mathcal{F}_n^\theta$ *onto this subalgebra whose kernel is generated by* g_{ij} *with* $i > d$*, and hence the subalgebra is isomorphic to* $\mathcal{F}_n^\theta \,/\, \langle g_{ij} \rangle_{i>d}$.

In the sequel we will drop the product notation $\times_\theta$ for simplicity. The Hopf algebra $\mathcal{F}_n^\theta$ is dually paired with $\mathcal{H}_\theta^n$ under the same pairing which links the untwisted algebras. The left $\mathcal{H}_\theta^n$-module structure of $\mathcal{F}_n^\theta$ is given by (1.9) and is straightforwardly computed to get

$$\begin{aligned} E_{ij} \triangleright g_{kl} &= g_{kl}^{(1)} \left\langle E_{ij} \,,\, g_{kl}^{(2)} \right\rangle \\ &= \sum_{m=1}^n g_{km} \left\langle E_{ij} \,,\, g_{ml} \right\rangle \\ &= \sum_{m=1}^n g_{km}\, g_{ml}(E_{ij}) \\ &= \sum_{m=1}^n g_{km}\, \delta_{mi}\, \delta_{jl} \;=\; \delta_{jl}\, g_{ki} \ . \end{aligned}$$

2.4. **Quantum determinants.** The coordinate algebra of the noncommutative variety $\mathrm{GL}_\theta(n)$ should be properly defined as the Ore localization of the noncommutative algebra generated by arbitrary matrix units with respect to an invertible and permutable element $\det_\theta$, the determinant element. If we consider the elements at the crossings of rows i, j and columns k, l of a given matrix, then the determinant of this 2×2 sub-matrix is classically given by $g_{ik}\, g_{jl} - g_{jk}\, g_{il}$. In order to get a well-defined element of $\mathcal{F}_n^\theta$, we put in front of every monomial in the matrix elements g_{ij} a suitable element of the deformation matrix. For example, in front of $g_{ik}\, g_{jl}$ we write $Q_{jl\,;\,ik}$, so that the determinant of the minor above is $Q_{jl\,;\,ik}\ g_{ik}\, g_{jl} - Q_{il\,;\,jk}\ g_{jk}\, g_{il}$. This is well-defined because if we choose to write the determinant using a different ordering of the monomials, then we get the same element of $\mathcal{F}_n^\theta$ thanks to the relations (2.12) which imply

$$Q_{jl\,;\,ik}\ g_{ik}\, g_{jl} = Q_{ik\,;\,jl}\ g_{jl}\, g_{ik} \ .$$

For a generic $n \times n$ matrix we can define the determinant by adapting the usual Laplace expansion in minors, with respect to either rows or columns, or the Leibniz formula which expresses it as a linear combination of products $\prod_i g_{i\,\sigma(i)}$ or $\prod_i g_{\sigma(i)\,i}$ as σ runs through the symmetric group S_n weighted by its sign. Using the above rule for the coefficients in front of every monomial to pull out a factor $Q_{lj\,;\,ki}$ for every pair $g_{ki}\, g_{lj}$ appearing in

$\prod_i \, g_{i\,\sigma(i)}$, we define

$$
\begin{aligned}
\det{}_\theta \ &:= \ \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \ \Big(\prod_{j=1}^{n-1} \ \prod_{i=1}^{n-j} Q_{i+j\,\sigma(i+j)\,;\,j\,\sigma(j)} \Big) \, g_{1\,\sigma(1)} \cdots g_{n\,\sigma(n)} \\
&= \ \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \ \Big(\prod_{j=1}^{n-1} \ \prod_{i=1}^{n-j} Q_{\sigma(i+j)\,i+j\,;\,\sigma(j)\,j} \Big) \, g_{\sigma(1)\,1} \cdots g_{\sigma(n)\,n} \ . \qquad (2.15)
\end{aligned}
$$

This element corresponds to a mapping of S_n into the braid group B_n on n strands, as we shall see below.

The formula (2.15) may be rewritten in a more succinct way by using the fact that the classical Leibniz formula can be expressed in terms of the totally antisymmetric Levi–Civita symbol ϵ as

$$
\epsilon^{i_1\cdots i_n} \, g_{1\,i_1} \cdots g_{n\,i_n} = \frac{1}{n!} \, \epsilon^{j_1\cdots j_n} \, \epsilon^{i_1\cdots i_n} \, g_{j_1\,i_1} \cdots g_{j_n\,i_n} \ .
$$

In the noncommutative case, we introduce a θ-deformed Levi–Civita symbol ϵ_θ which satisfies braided antisymmetry rules. Since the row and column indices in (2.11) and (2.12) behave differently, we actually require two different symbols $\epsilon_\theta^{(r)}$, which refers to row indices, and $\epsilon_\theta^{(c)}$, which refers to column indices. In this way we may absorb the Q-dependent coefficients of (2.15), consistently with the braided antisymmetry. Explicitly,

$$
\begin{aligned}
\epsilon_\theta^{i_1\cdots i_n\,(r)} \ &= \ \mathrm{sgn}(i_1\cdots i_n) \prod_{k=1}^{n-1} \ \prod_{r=1}^{n-k} Q_{r+k\,i_{r+k}\,;\,k\,i_k} \ , \\
\epsilon_\theta^{j_1\cdots j_n\,(c)} \ &= \ \mathrm{sgn}(j_1\cdots j_n) \prod_{k=1}^{n-1} \ \prod_{r=1}^{n-k} Q_{j_{r+k}\,r+k\,;\,j_k\,k} \ .
\end{aligned}
$$

They obey the alternating rules

$$
\begin{aligned}
\epsilon_\theta^{i_1\cdots\, i_\alpha \cdots i_\beta\, \cdots i_n\,(r)} \ &= \ -\, q^2_{i_\alpha i_\beta} \ \epsilon_\theta^{i_1\cdots\, i_\beta \cdots i_\alpha\, \cdots i_n\,(r)} \ , \\
\epsilon_\theta^{j_1\cdots\, j_\alpha \cdots j_\beta\, \cdots j_n\,(c)} \ &= \ -\, q^2_{j_\beta j_\alpha} \ \epsilon_\theta^{j_1\cdots\, j_\beta \cdots j_\alpha\, \cdots j_n\,(c)} \ . \qquad (2.16)
\end{aligned}
$$

For example, for $n=2$ we define $\epsilon_\theta^{12\,(c)} = q_{21}$ and $\epsilon_\theta^{21\,(c)} = -q_{12}$, and the sole braided antisymmetry relation $\epsilon_\theta^{12\,(c)} = -q_{21}^2 \ \epsilon_\theta^{21\,(c)}$ is satisfied. Similarly, we put $\epsilon_\theta^{12\,(r)} = q_{12}$. In this sense $\epsilon_\theta^{(r)}$ may be thought of as the inverse of the symbol $\epsilon_\theta^{(c)}$.

Definition 2.17. *The* quantum determinant *is the element of $\mathcal{F}_n^\theta$ given by*

$$
\det{}_\theta = \frac{1}{n!} \, \epsilon_\theta^{i_1\cdots i_n\,(r)} \, \epsilon_\theta^{j_1\cdots j_n\,(c)} \ g_{i_1 j_1} \cdots g_{i_n j_n} \ . \qquad (2.18)
$$

Theorem 2.19. *The element* $\det_\theta$ *is a T-eigenvector which is left and right permutable in $\mathcal{F}_n^\theta$.*

Proof: The first statement follows from an elementary calculation using the coproduct $\Delta_\theta(H_i)$ of §2.2 and the $(\mathbb{C}^\times)^n$-action $H_i \triangleright g_{kl} = \delta_{il}\, g_{kl}$. For the second statement, note that since every monomial occuring in $\det_\theta$ is of the form $\prod_i \, g_{i\,\sigma(i)}$ for some permutation σ in

S_n, every row and column index appears exactly once. By (2.12), commuting a generic element g_{kl} from right to left in such a monomial picks up the coefficient

$$\prod_{i=1}^{n} Q^2_{i\,\sigma(i)\,;\,kl} = \prod_{i=1}^{n} q^2_{ki}\, q^2_{\sigma(i)\,l} \ .$$

It follows that

$$(\det_\theta)\, g_{kl} = \Big(\prod_{i=1}^{n} Q^2_{ii\,;\,kl} \Big)\, g_{kl}\, (\det_\theta)$$

for all $k,l = 1,\dots,n$, and hence $(\det_\theta)\,\mathcal{F}_n^\theta = \mathcal{F}_n^\theta\,(\det_\theta)$. ∎

Corollary 2.20. *The set of non-negative powers of* $\det_\theta$ *is a left and right denominator set in* $\mathcal{F}_n^\theta$*.*

Corollary 2.21. *The element* $\det_\theta$ *is central in* $\mathcal{F}_n^\theta$ *if and only if*

$$\sum_{k=1}^{n} \theta^{ki} = \sum_{k=1}^{n} \theta^{kj} \qquad (\mathrm{mod}\ 2\pi)$$

for all $i,j = 1,\dots,n$*.*

Although our deformation of the general linear group lies in the class of deformations considered in [3], our definition of quantum determinant is different, though it satisfies the same formal properties. The element (2.18) originates from the braiding of the category of Hopf-module algebras described in §1.2, in the enveloping algebra approach, since this captures pairwise noncommutativity relations in a deformed exterior algebra. Consider the Hopf algebra $\mathcal{H}_\theta^n$ dual to $\mathcal{F}_n^\theta$. The θ-deformed exterior algebra of degree d for an element V in the category ${}_{\mathcal{H}_\theta^n}\mathscr{M}$ of $\mathcal{H}_\theta^n$-module algebras is defined as

$$\textstyle\bigwedge_\theta^d V := V^{\otimes d} \,\big/\, \big\langle v_1 \otimes v_2 + \Psi_\theta(v_1 \otimes v_2) \big\rangle_{v_1,v_2\in V} \ , \tag{2.22}$$

where $\Psi_\theta := \Psi_{F_\theta} = \tau \circ F_\theta^{-2}$ is the braiding morphism of the category. For $\theta = 0$ we recover the usual flip operator $\Psi_0 = \tau$ and the exterior algebra $\bigwedge^d V$. For $\theta \neq 0$ we obtain a braided skew-symmetric algebra $\bigwedge_\theta^d V$, which is spanned by the collection of minors of order $d \le n$ in elements of V when n is the number of generators of V. For this, consider two multi-indices $I = (i_1 \cdots i_d)$ and $J = (j_1 \cdots j_d)$ which label the rows and columns of a given minor, and define the determinant Λ^{IJ} of this sub-matrix as

$$\Lambda^{IJ} = \frac{1}{d!}\, \epsilon_\theta^{i_1\cdots i_d\,(r)}\, \epsilon_\theta^{j_1\cdots j_d\,(c)}\, g_{i_1 j_1} \cdots g_{i_d j_d} \tag{2.23}$$

where the symbols ϵ_θ satisfy alternating rules derived from (2.22). Here the $\mathcal{H}_\theta^n$-module structure of $\mathrm{GL}(n) \cong \mathrm{GL}(V)$ is induced from the $\mathcal{H}_\theta^n$-module structure of V and of its dual V^*. When this $\mathcal{H}_\theta^n$-module structure induces the noncommutative product (2.12) among the entries of elements of $\mathrm{GL}(V)$, the alternating properties of the deformed Levi–Civita symbols coincide with those of (2.16).

Remark 2.24. *Our definition (2.22) of exterior algebra is equivalent to the standard definition of an exterior algebra in a braided monoidal category* [37] *(see also* [27, §13.2.2]*), written in the symmetric case. In this construction one takes the quotient of the tensor algebra by the kernel of the antisymmetrizer. A slightly different, but somewhat simpler, definition involves the quotient by the ideal generated by the kernel of the antisymmetrizer in degree two, which coincides with the morphism* $\mathrm{id} - \Psi_\theta$ [27, p. 512]*. This agrees with* *our definition (2.22), since we work in a symmetric category with* $\Psi_\theta^2 = \mathrm{id}$*, and so the kernel of the antisymmetrizer* $\mathrm{id} - \Psi_\theta$ *coincides with the image of the symmetrizer* $\mathrm{id} + \Psi_\theta$*.*

For later use we work out the explicit commutation rules between any two $d\times d$ and $d'\times d'$ minors Λ^{IJ} and $\Lambda^{I'J'}$ for the case $V = \mathcal{F}_n^\theta$, regarded as the coordinate algebra $\mathcal{A}(\mathrm{GL}_\theta(n))$ of the noncommutative variety $\mathrm{GL}_\theta(n)$, with $|I| = |J| = d$ and $|I'| = |J'| = d'$. One has

$$\begin{aligned}
\Lambda^{IJ}\,\Lambda^{I'J'} &= \epsilon_\theta^{i_1\cdots i_d\,(r)}\,\epsilon_\theta^{j_1\cdots j_d\,(c)}\,\epsilon_\theta^{i'_1\cdots i'_{d'}\,(r)}\,\epsilon_\theta^{j'_1\cdots j'_{d'}\,(c)}\,\big(g_{i_1j_1}\cdots g_{i_dj_d}\big)\,\big(g_{i'_1j'_1}\cdots g_{i'_{d'}j'_{d'}}\big) \\
&= \Big(\prod_{\alpha=1}^{d}\prod_{\alpha'=1}^{d'} Q^2_{i_\alpha j_\alpha\,;\,i'_{\alpha'}j'_{\alpha'}}\Big) \\
&\quad\times \epsilon_\theta^{i_1\cdots i_d\,(r)}\,\epsilon_\theta^{j_1\cdots j_d\,(c)}\,\epsilon_\theta^{i'_1\cdots i'_{d'}\,(r)}\,\epsilon_\theta^{j'_1\cdots j'_{d'}\,(c)}\,\big(g_{i'_1j'_1}\cdots g_{i'_{d'}j'_{d'}}\big)\,\big(g_{i_1j_1}\cdots g_{i_dj_d}\big) \\
&= \Big(\prod_{\alpha=1}^{d}\prod_{\alpha'=1}^{d'} Q^2_{i_\alpha j_\alpha\,;\,i'_{\alpha'}j'_{\alpha'}}\Big)\,\Lambda^{I'J'}\,\Lambda^{IJ}\ .
\end{aligned}$$

Introducing the coefficient

$$R_{IJ\,;\,I'J'} = \prod_{\alpha=1}^{d}\prod_{\alpha'=1}^{d'} Q_{i_\alpha j_\alpha\,;\,i'_{\alpha'}j'_{\alpha'}} \tag{2.25}$$

we have the commutation relations

$$\Lambda^{IJ}\,\Lambda^{I'J'} = R^2_{IJ\,;\,I'J'}\,\Lambda^{I'J'}\,\Lambda^{IJ}\ . \tag{2.26}$$

In particular, this shows that the minors of order d generate a subalgebra.

Another useful identity concerns how minors behave when we choose two multi-indices which differ only by transposition on a pair of indices. Consider a pair of multi-indices of the form $J = (j_1\cdots j_\alpha\cdots j_\beta\cdots j_d)$ and $J^{t_{\alpha\beta}} = (j_1\cdots j_\beta\cdots j_\alpha\cdots j_d)$. From (2.23) it is straightforward to obtain the alternating relations

$$\Lambda^{IJ} = -\,q^2_{j_\alpha j_\beta}\,\Lambda^{IJ^{t_{\alpha\beta}}}\ , \tag{2.27}$$

which can be further generalized to arbitrary permutations.

3. Noncommutative toric varieties

The strategy of (toric) isospectral deformations is that once we have a noncommutative deformation of the torus we can deform every space acted upon by it. For riemannian manifolds the isospectral condition means restricting to isometric actions. Using the algebraic torus $T \cong (\mathbb{C}^\times)^n$ and its deformation constructed in §2.1, we will now proceed to deform toric algebraic varieties. Our approach makes use of and extends a construction due to Ingalls [23].

3.1. **Noncommutative deformations of toric varieties.**

Toric varieties X may be described in several equivalent ways. As complex varieties they come with an open embedding of an algebraic torus, which is dense in X. In this picture their geometry is encoded by combinatorial data, a fan, that describes the way in which $(\mathbb{C}^\times)^n$ acts on X. As symplectic manifolds they come with a hamiltonian action of a real torus. The corresponding moment map, whose image is a convex polytope, provides the information about the structure of X. Noncommutative deformations of toric varieties in the symplectic framework are defined in [8]. In this paper we will use the fan picture. For a more exhaustive introduction to toric varieties, along with further definitions and terminology, see e.g. [15, 20].

Definition 3.1. *A* toric variety *X of dimension n is an irreducible algebraic variety over $\mathbb{C}$ which contains $(\mathbb{C}^\times)^n$ as a Zariski open subset and the regular action of $(\mathbb{C}^\times)^n$ on itself extends to an action on the whole of X.*

Basic examples are the affine planes $\mathbb{C}^n$, the projective spaces $\mathbb{CP}^n$, the grassmannians $\mathbb{G}\mathrm{r}(d;n)$, and the weighted projective spaces $\mathbb{CP}^n[a_0,a_1,\dots,a_n]$. In the following we will denote by $L_\mathbb{R} = L\otimes_\mathbb{Z}\mathbb{R}\cong\mathbb{R}^n$ the real vector space obtained from a lattice L. Its dual vector space is $L^*_\mathbb{R} = L^*\otimes_\mathbb{Z}\mathbb{R}\cong(\mathbb{R}^n)^*$.

Definition 3.2. *A* rational polyhedral cone *$\sigma\subset L_\mathbb{R}$ is a cone $\sigma=\mathbb{R}^+v_1\oplus\cdots\oplus\mathbb{R}^+v_s$ generated by finitely many elements $v_1,\dots,v_s\in L$. It is* strongly convex *if it does not contain any real line, $\sigma\cap(-\sigma)=0$.*

Definition 3.3. *For every strongly convex rational polyhedral cone $\sigma\subset L_\mathbb{R}$ of dimension n we define the* dual cone

$$\sigma^\vee = \big\{ m\in L^*_\mathbb{R} \ \big| \ \langle m,u\rangle \geq 0 \quad \forall u\in\sigma \big\} \ .$$

Given a strongly convex rational polyhedral cone σ, we will now show how to construct a normal affine toric variety $U[\sigma]$. The set $\sigma^\vee\cap L^*$ is a finitely generated semigroup under addition. Let $(m_1,\dots,m_l)$ be the generators of this semigroup, so that

$$\sigma^\vee\cap L^* \cong \mathbb{Z}^+m_1\oplus\cdots\oplus\mathbb{Z}^+m_l \ .$$

Note that in general $\sigma^\vee$ is not strongly convex, so $l\geq n$. To each $m_a=\sum_i\,(m_a)_i\,e_i^*$ we associate a Laurent monomial in $\mathbb{C}(t_1,\dots,t_n)$ by the assignment $m_a\mapsto t^{m_a}=t_1^{(m_a)_1}\cdots t_n^{(m_a)_n}$. The product between two such elements is given by the corresponding sum of characters, $t^{m_a}\cdot t^{m_b}:=t^{m_a+m_b}$. Thus the generators of $\sigma^\vee\cap L^*$ span a subalgebra of $\mathbb{C}(t_1,\dots,t_n)$ which we denote by $\mathbb{C}[\sigma]$. The affine toric variety $U[\sigma]$ is defined to be the spectrum of $\mathbb{C}[\sigma]$, i.e. $\mathbb{C}[\sigma]$ is the coordinate algebra of $U[\sigma]$. Note that the inclusion $0\hookrightarrow\sigma$ induces an embedding of the torus $T=U[0]$ as a dense open subset of $U[\sigma]$.

The variety $U[\sigma]$ may also be described as an embedding in the complex plane $\mathbb{C}^l$. If $\sigma^\vee\cap L^*$ has l generators, consider the polynomial algebra $\mathbb{C}[x_1,\dots,x_l]$ (one variable x_a for each m_a). Recall that the generators m_a are l rational vectors in $L^*_\mathbb{R}$, so there are exactly $l-n$ linear relations among them. Then we may quotient the algebra $\mathbb{C}[x_1,\dots,x_l]$ by the ideal generated by the $l-n$ relations among the vectors m_a, realized as multiplicative relations among the variables x_a. If we denote the subspace generated by these relations as $R[m_a]\subset\mathbb{C}[x_1,\dots,x_l]$, then we get a realization of $U[\sigma]$ as the spectrum of the quotient algebra $\mathbb{C}[\sigma]=\mathbb{C}[x_1,\dots,x_l]/\langle R[m_a]\rangle$.

We obtain generic toric varieties by gluing together affine toric varieties. This has a corresponding picture in terms of cones.

Definition 3.4. *Given a cone $\sigma\subset L_\mathbb{R}$, a* face *$\tau\subset\sigma$ is a subset of the form $\tau=\sigma\cap m^\perp$ for some $m\in\sigma^\vee$, where $m^\perp:=\{u\in L_\mathbb{R}\ |\ \langle m,u\rangle=0\}$.*

Definition 3.5. *A* fan *$\Sigma\subset L_\mathbb{R}$ is a non-empty finite collection of strongly convex rational polyhedral cones in $L_\mathbb{R}$ satisfying the following conditions:*

(1) *If* $\sigma \in \Sigma$ *and* τ *is a face of* σ*, then* $\tau \in \Sigma$*; and*
(2) *If* $\sigma, \tau \in \Sigma$*, then the intersection* $\sigma \cap \tau$ *is a face of both* σ *and* τ*.*

To a fan Σ in $L_{\mathbb{R}}$ we associate a toric variety $X = X[\Sigma]$. The cones $\sigma \in \Sigma$ correspond to the open affine subvarieties $U[\sigma] \subset X[\Sigma]$, and $U[\sigma]$ and $U[\tau]$ are glued together along their common open subset $U[\sigma \cap \tau] = U[\sigma] \cap U[\tau]$. Various properties of $X[\Sigma]$, such as smoothness and compactness, may be stated entirely in terms of the fan structure Σ (see e.g. [20] for details).

Our definition of noncommutative toric varieties will involve a multi-parameter deformation $X[\Sigma] \to X_\theta[\Sigma]$ which makes use of the same fan structure Σ, deforming only the product structure of the coordinate algebra of every strongly convex rational polyhedral cone of Σ. We have already defined the quantum Laurent algebra $\mathbb{C}_\theta(t_1, \dots, t_n)$, the coordinate algebra of the noncommutative algebraic torus $(\mathbb{C}_\theta^\times)^n$. Since the undeformed torus $(\mathbb{C}^\times)^n$ is densely contained in every toric variety $X[\Sigma] = \bigcup_{\sigma \in \Sigma} U[\sigma]$, we expect to have morphisms between the noncommutative algebras corresponding to the noncommutative varieties $X_\theta[\Sigma]$ and $\mathbb{C}_\theta(t_1, \dots, t_n)$.

We begin by defining noncommutative affine toric varieties. They are associated to a strongly convex rational polyhedral cone $\sigma \subset L_{\mathbb{R}}$, just as in the commutative case. However, now we use the complex skew-symmetric matrix θ to define a noncommutative product in the algebra $\mathbb{C}[\sigma]$, according to the group character relation given by

$$\chi_p \star_\theta \chi_q = \exp\left(\tfrac{\mathrm{i}}{2}\, p_i\, \theta^{ij}\, q_j\right)\, \chi_{p+q} \ .$$

Thus if $(m_1, \dots, m_l)$ are the generators of the semigroup $\sigma^\vee \cap L^*$ and t^{m_a} are the associated Laurent monomials, then the algebra $\mathbb{C}_\theta[\sigma]$ is defined to be the subalgebra of $\mathbb{C}_\theta(t_1, \dots, t_n)$ generated by $\{t^{m_a}\}$ with product

$$t^{m_a} \star_\theta t^{m_b} := \exp\left(\tfrac{\mathrm{i}}{2}\, (m_a)_i\, \theta^{ij}\, (m_b)_j\right)\, t^{m_a + m_b} \ .$$

This may be regarded as a deformation of the algebra generated by the characters, but, we stress once again, not of their group structure. It is for this reason that we will describe noncommutative toric varieties by using the same fan of the corresponding commutative varieties. The noncommutative affine variety corresponding to the algebra $\mathbb{C}_\theta[\sigma]$ is denoted $U_\theta[\sigma]$. It is a multi-parameter deformation of $U[\sigma]$.

Proposition 3.6. *The action of the torus* T *on* $(\mathbb{C}_\theta^\times)^n$ *restricts to a faithful torus action* Φ *on* $U_\theta[\sigma]$*, which is dually a map* $\Phi : T \to \mathrm{Aut}(\mathbb{C}_\theta[\sigma])$*.*

Proof: On generators of the algebra $\mathbb{C}_\theta[\sigma]$ of the form $t^{m_a} = \big(t_1^{(m_a)_1}, \dots, t_n^{(m_a)_n}\big)$ with $m_a \in \sigma^\vee \cap L^*$ and $a = 1, \dots, l$, the action of $\tau = (\tau_1, \dots, \tau_n) \in T$ is given by

$$\Phi_\tau(t^{m_a}) = \big(t_1^{(m_a)_1}\ \tau_1, \dots, t_n^{(m_a)_n}\ \tau_n\big) \ .$$

The corresponding infinitesimal action of the torus generator H_i is then

$$H_i \triangleright t^{m_a} = (m_a)_i\ t^{m_a} \ ,$$

i.e. multiplication by the coefficient $(m_a)_i$, the i-th component of m_a. If the action is not faithful, there is at least one index i with corresponding generator H_i acting trivially and for this i one would have $(m_a)_i = 0$ for every a, i.e. the generators of the dual cone would have vanishing i-th component. But this would mean that every vector along the i-th component has negative pairing with elements of the cone σ, which contradicts the assumption that σ is strongly convex. ■

This toric action really parallels the undeformed situation: strongly convex cones σ represent the affine toric varieties $U[\sigma]$ that are glued together to get the full toric variety X; and in each of them the torus is embedded and acts freely (the usual extension of the action of the torus on itself). In other words, the $U[\sigma]$'s are open affine toric subvarieties of X, so they carry a faithful action of the torus.

Recall that the L^*-grading gives precisely the eigenspace decompositions of algebraic objects, dual to T-invariant geometric objects. In particular, since the torus T acts on $\mathbb{C}_\theta[\sigma]$ by $\mathbb{C}$-algebra automorphisms for each $\sigma \in \Sigma$, the algebra $\mathbb{C}_\theta[\sigma]$ is spanned by T-eigenvectors for which the corresponding eigenvalues are rational. This yields a vector space decomposition

$$\mathbb{C}_\theta[\sigma] = \bigoplus_{p\in L^*} \mathbb{C}_\theta[\sigma]^p \ , \tag{3.7}$$

where $\mathbb{C}_\theta[\sigma]^p$ denotes the eigenspace of $\mathbb{C}_\theta[\sigma]$ labelled by the character $p \in L^*$, and $\mathbb{C}_\theta[\sigma]^p \star_\theta \mathbb{C}_\theta[\sigma]^q \subset \mathbb{C}_\theta[\sigma]^{p+q}$ for all $p, q \in L^*$, since T acts by automorphisms. Thus we get a grading of $\mathbb{C}_\theta[\sigma]$ by the free abelian group of characters L^*, such that the homogeneous elements are the T-eigenvectors in $\mathbb{C}_\theta[\sigma]$.

We have seen how affine toric varieties may also be regarded as subvarieties of complex planes $\mathbb{C}^l$, via the quotient algebra $\mathbb{C}[\sigma] = \mathbb{C}[x_1, \dots, x_l]/\langle R[m_a]\rangle$. An analogous realization is possible for noncommutative affine toric varieties. Remembering that in general $l \geq n$, the noncommutative deformation of the polynomial algebra $\mathbb{C}[x_1, \dots, x_l]$ is obtained from the multiplicative relations between the monomials t^{m_a}. If we denote $\check\theta_{ab} := (m_a)_i\, \theta^{ij}\, (m_b)_j$ with $a, b = 1, \dots, l$, $i, j = 1, \dots, n$ and $\check q_{ab} = \exp(\frac{\mathrm{i}}{2}\, \check\theta_{ab})$, then the relation between Laurent monomials becomes

$$t^{m_a} \star_{\check\theta} t^{m_b} := \check q_{ab}\ t^{m_a+m_b} \ . \tag{3.8}$$

As a consequence, the generators of the algebra of the affine variety obey

$$\check q_{ba}\ x_a \star_{\check\theta} x_b = \check q_{ab}\ x_b \star_{\check\theta} x_a$$

or equivalently

$$x_a \star_{\check\theta} x_b = \big(\check q_{ab}\big)^2\ x_b \star_{\check\theta} x_a \ . \tag{3.9}$$

The relations (3.9) define the l-dimensional noncommutative complex plane with coordinate algebra $\mathbb{C}_{\check\theta}[x_1, \dots, x_l]$, which is a special instance of the general class of quantum affine spaces considered by Manin [34].

The $l-n$ linear relations among the generators of the dual cone $\{m_a\}$ are now expressed in the character algebra. These relations can always be brought to the form

$$\sum_{a=1}^{l} (p_{s,a} - r_{s,a})\ m_a = 0 \ ,$$

for $s = 1, \dots, l-n$, with non-negative integer coefficients $p_{s,a}$, $r_{s,a}$. For each s, one obtains from (3.8) the additional relation

$$x_1^{p_{s,1}} \star_{\check\theta} \cdots \star_{\check\theta} x_l^{p_{s,l}} = \Big(\prod_{1\leq a<b\leq l} \big(\check q_{ab}\big)^{p_{s,a}\, p_{s,b} - r_{s,a}\, r_{s,b}} \Big)\ x_1^{r_{s,1}} \star_{\check\theta} \cdots \star_{\check\theta} x_l^{r_{s,l}} \ . \tag{3.10}$$

The subspace of relations (3.10) is denoted $R_{\check\theta}[m_a]$. It is a multi-parameter deformation of the subspace $R[m_a]$, which generates a two-sided ideal in $\mathbb{C}_{\check\theta}[x_1, \dots, x_l]$. Thus we may realize $U_\theta[\sigma]$ either as the noncommutative algebra $\mathbb{C}_\theta[\sigma]$ or as the quotient algebra $\mathbb{C}_{\check\theta}[x_1,\dots,x_l]/\langle R_{\check\theta}[m_a]\rangle$.

We obtain generic noncommutative toric varieties $X_\theta[\Sigma]$ by gluing together noncommutative affine toric varieties. If σ and σ' are two cones in the fan Σ which intersect along the face $\tau=\sigma\cap\sigma'$, then there are canonical morphisms between the associated noncommutative algebras $\mathbb{C}_\theta[\sigma]\to\mathbb{C}_\theta[\tau]$ and $\mathbb{C}_\theta[\sigma'\,]\to\mathbb{C}_\theta[\tau]$ induced by the inclusions $\tau\hookrightarrow\sigma$ and $\tau\hookrightarrow\sigma'$. The images of these morphisms in $\mathbb{C}_\theta[\tau]$ are related by an equivariant algebra automorphism which plays the role of a "coordinate transition function" between $U_\theta[\sigma]$ and $U_\theta[\sigma'\,]$, and may be described explicitly as follows. On $\tau^\vee\cap L^*$ there is a complete set of relations of the form

$$\sum_{a=1}^{l}\big(u_a-v_a\big)\,m_a+\sum_{a'=1}^{l'}\big(u'_{a'}-v'_{a'}\big)\,m'_{a'}=0$$

among the generators $\{m_a\}_{a=1}^{l}$ and $\{m'_{a'}\}_{a'=1}^{l'}$ of the dual semigroups of σ and σ', with non-negative integers u_a,v_a and $u'_{a'},v'_{a'}$. For each of these relations, the generators x_a and $x'_{a'}$ of the algebras $\mathbb{C}_\theta[\sigma]$ and $\mathbb{C}_\theta[\sigma'\,]$ are identified in $\mathbb{C}_\theta[\tau]$ through the relation

$$\begin{aligned}
&x_1^{u_1}\star_{\check\theta}\cdots\star_{\check\theta}x_l^{u_l}\star_{\check\theta^\circ}x_1'^{\,u_1'}\star_{\check\theta'}\cdots\star_{\check\theta'}x_{l'}'^{\,u_{l'}'}\\
&\qquad=\Big(\prod_{1\le a<b\le l}\big(\check q_{ab}\big)^{u_a\,u_b-v_a\,v_b}\Big)\Big(\prod_{1\le a'<b'\le l'}\big(\check q'_{a'b'}\big)^{u'_{a'}\,u'_{b'}-v'_{a'}\,v'_{b'}}\Big)\\
&\qquad\quad\times\Big(\prod_{a=1}^{l}\prod_{a'=1}^{l'}\big(\check q^\circ_{aa'}\big)^{u_a\,u'_{a'}-v_a\,v'_{a'}}\Big)\,x_1^{v_1}\star_{\check\theta}\cdots\star_{\check\theta}x_l^{v_l}\star_{\check\theta^\circ}x_1'^{\,v_1'}\star_{\check\theta'}\cdots\star_{\check\theta'}x_{l'}'^{\,v_{l'}'}
\end{aligned}$$

where $\check\theta^\circ_{aa'}:=(m_a)_i\,\theta^{ij}\,(m'_{a'})_j$ and $\check q^\circ_{aa'}=\exp(\frac{\mathrm{i}}{2}\,\check\theta^\circ_{aa'})$, while $\check\theta'_{a'b'}:=(m'_{a'})_i\,\theta^{ij}\,(m'_{b'})_j$ and $\check q'_{a'b'}=\exp(\frac{\mathrm{i}}{2}\,\check\theta'_{a'b'})$, together with the commutation relations

$$x_a\star_{\check\theta^\circ}x'_{a'}=\big(\check q^\circ_{aa'}\big)^2\,x'_{a'}\star_{\check\theta^\circ}x_a\ .$$

Since each algebra $\mathbb{C}_\theta[\sigma]$ for $\sigma\in\Sigma$ is a subalgebra of $\mathbb{C}_\theta(t_1,\dots,t_n)$, there is a morphism $(\mathbb{C}_\theta^\times)^n\hookrightarrow X_\theta[\Sigma]$. This also means that the intersection of the algebras $\mathbb{C}_\theta[\sigma]$ is well-defined, and the "algebra of functions" $\mathcal{A}\big(X_\theta[\Sigma]\big)$ on $X_\theta[\Sigma]$ can be represented via the exact sequence

$$0\longrightarrow\mathcal{A}\big(X_\theta[\Sigma]\big)\longrightarrow\prod_{\sigma\in\Sigma}\mathbb{C}_\theta[\sigma]\longrightarrow\prod_{\sigma,\sigma'\in\Sigma}\mathbb{C}_\theta[\sigma\cap\sigma'\,]\ ,\tag{3.11}$$

with the gluing automorphisms above. By Proposition 3.6, the toric actions on $U_\theta[\sigma]$ for $\sigma\in\Sigma$ all agree, and hence combine to give an action of T on $X_\theta[\Sigma]$.

Remark 3.12. *Toric isospectral deformations can be shown to be strict deformation quantizations in the sense of Rieffel* [35]*. It is an open question if our deformation, which may be thought of as generated by $\mathbb{C}^n$ instead of Rieffel's $\mathbb{R}^n$, satisfies a similar property.*

In the remainder of this section we will work out some explicit examples of noncommutative deformations of toric varieties. We set $q_{ij}:=\exp\big(\frac{\mathrm{i}}{2}\,\theta_{ij}\big)$ for $i<j$. It may be regarded as a form $q\in\bigwedge^2T\cong(\mathbb{C}^\times)^{n\,(n-1)/2}$ with $q_{ij}=q(e_i^*,e_j^*)=\langle e_i^*,\Theta(e_j^*)\rangle$, or equivalently as a map $q\in\mathrm{Hom}_{\mathbb{Z}}(\bigwedge^2L^*,\mathbb{C}^\times)$. When $n=2$ we write $q:=\exp\big(\frac{\mathrm{i}}{2}\,\theta\big)$ with $\theta=\theta^{12}=-\theta^{21}\in\mathbb{C}$. In the following we omit the star product symbol $\star_\theta$ from the notation for brevity.

3.2. **Algebraic Moyal plane and $\mathcal{D}$-modules.** The simplest toric variety (besides T itself) is the n-dimensional complex plane $\mathbb{C}^n$. Let us start from the embedding of the commutative torus $(\mathbb{C}^\times)^n \hookrightarrow \mathbb{C}^n$ given by the log map

$$t_i \longmapsto z_i = \log t_i \ , \qquad i = 1, \dots, n \ ,$$

so that the toric action on $\mathbb{C}^n$ is $\lambda_i \triangleright z_j = z_j + \delta_{ij} \log \lambda_j$ for a set of generators $\lambda_1, \dots, \lambda_n$ of the $(\mathbb{C}^\times)^n$-action. Consider the multi-parameter deformation $(\mathbb{C}^\times_\theta)^n$ of the torus defined by the quantum Laurent algebra $\mathbb{C}_\theta(t_1, \dots, t_n)$ with generators t_i and relations

$$t_i\, t_j = q_{ij}^2 \ t_j\, t_i \ .$$

An application of the Baker–Campbell–Hausdorff formula shows that the corresponding elements z_i obey the commutator relations

$$[z_i, z_j] = \mathrm{i}\, \theta_{ij} \ .$$

The algebra of polynomial functions $\mathbb{C}_\theta[z_1, \dots, z_n]$ over $\mathbb{C}$ generated by z_i, $i = 1, \dots, n$ subject to these relations is dual to a noncommutative affine variety that we call the *algebraic Moyal plane* $\mathbb{C}^n_\theta$. This algebra can be identified with the d-th Weyl algebra $\mathcal{D}(\mathbb{C}^d)$ of polynomial differential operators on the complex space $\mathbb{C}^d$, with $d = \lfloor \frac{n}{2} \rfloor$, whose projective modules furnish basic examples of $\mathcal{D}$-modules. For $n = 4$, this is the same as the noncommutative variety $\mathbb{C}^4_\hbar$ defined in [25, §3.4]. All algebras $\mathbb{C}_\theta[z_1, \dots, z_n]$ for $\theta = (\theta_{ij})$ nondegenerate are isomorphic, and hence the varieties $\mathbb{C}^n_\theta$ are the same for all nondegenerate θ. More generally, $\mathbb{C}^n_\theta$ and $\mathbb{C}^n_{\theta'}$ are isomorphic if and only if the matrices θ and θ' have the same rank.

3.3. **Noncommutative projective plane.** The toric geometry of the projective plane $\mathbb{CP}^2$ can be described by a fan Σ of the lattice $L \cong \mathbb{Z}^2$ of characters for the action of the algebraic torus $T = L \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong (\mathbb{C}^\times)^2$ on $\mathbb{CP}^2$. Let e_1, e_2 be a basis of L. Set $v_1 = e_1$, $v_2 = e_2$ and $v_3 = -e_1 - e_2$. These vectors generate the three one-dimensional cones $\tau_i = \mathbb{R}^+ v_i$ of Σ. The three maximal cones of Σ are generated by pairs of these vectors as

$$\sigma_i = \mathbb{R}^+ v_{i+1} \oplus \mathbb{R}^+ v_{i+2} \ , \qquad i = 1, 2, 3$$

(with the labels read mod 3) with $\sigma_i \cap \sigma_{i+1} = \tau_{i+2}$ and $\sigma_i \cap \sigma_j = 0$ otherwise. The corresponding open affine subvarieties $U[\sigma_i]$ generate an open cover of $X[\Sigma] = \mathbb{CP}^2$. The zero cone is the triple overlap $\sigma_1 \cap \sigma_2 \cap \sigma_3 = 0$.

We now go through the maximal cones and write out the relations among the generators of the subalgebra $\mathbb{C}_\theta[\sigma_i] \subset \mathbb{C}_\theta(t_1, t_2)$. There are no relations $R[m_a]$ in this case, as each dual cone $\sigma_i^\vee$ is strongly convex and hence the generators of $\sigma_i^\vee \cap L^*$ are independent.

(1) The generators of the semigroup $\sigma_3^\vee \cap L^*$ are $m_1 = e_1^*$ and $m_2 = e_2^*$. In this case $\check\theta = \theta$ and the algebra $\mathbb{C}_\theta[\sigma_3] = \mathbb{C}_\theta[x_1, x_2]$ is generated by the elements $x_a = t^{m_a} = t_a$, $a = 1, 2$ with the relations

$$x_1\, x_2 = q^2 \ x_2\, x_1 \ . \tag{3.13}$$

(2) The semigroup $\sigma_2^\vee \cap L^*$ is generated by $m_1 = -e_1^*$ and $m_2 = e_2^* - e_1^*$. In this case $\check\theta = -\theta$, and $\mathbb{C}_\theta[\sigma_2]$ is generated by elements $x_1 = t^{m_1} = t_1^{-1}$, $x_2 = t^{m_2} = t_1^{-1}\, t_2$ satisfying the relation

$$x_1\, x_2 = q^{-2} \ x_2\, x_1$$

which is (3.13) after sending $q \to q^{-1}$.

(3) The semigroup $\sigma_1^\vee \cap L^*$ is generated by $m_1 = e_1^* - e_2^*$ and $m_2 = -e_2^*$. In this case $\check{\theta} = \theta$, and $\mathbb{C}_\theta[\sigma_1]$ is generated by elements $x_1 = t^{m_1} = t_1\, t_2^{-1}$, $x_2 = t^{m_2} = t_2^{-1}$, again with the relations (3.13).

All three varieties $U_\theta[\sigma_i] \cong \mathbb{C}^2_\theta$ are thus copies of the two-dimensional algebraic Moyal plane.

We now glue the noncommutative affine toric varieties together. Consider, for example, the face $\tau_1 = \sigma_3 \cap \sigma_1$. The semigroup $\tau_1^\vee \cap L^*$ is generated by $m_1 = e_1^*$, $m_2 = e_2^*$ and $m_3 = -e_2^* = -m_2$. The generators of the subalgebra $\mathbb{C}_\theta[\tau_1] = \mathbb{C}_\theta[t_1, t_2, t_2^{-1}]$ are the elements $y_1 = t_1$, $y_2 = t_2$ and $y_3 = t_2^{-1}$ with the relations

$$y_1\, y_2 = q^2\ y_2\, y_1\ , \qquad y_1\, y_3 = q^{-2}\ y_3\, y_1\ , \qquad y_2\, y_3 = 1 = y_3\, y_2\ . \tag{3.14}$$

Recalling that $\mathbb{C}_\theta[\sigma_1] = \mathbb{C}_\theta[t_1, t_2]$ and $\mathbb{C}_\theta[\sigma_3] = \mathbb{C}_\theta[t_1\, t_2^{-1}, t_2^{-1}]$, it follows that the algebra morphisms $\mathbb{C}_\theta[\sigma_1] \to \mathbb{C}_\theta[\tau_1]$ and $\mathbb{C}_\theta[\sigma_3] \to \mathbb{C}_\theta[\tau_1]$ are both natural inclusions of subalgebras. Moreover, as subalgebras of $\mathbb{C}_\theta[\tau_1]$, there is a natural algebra automorphism $\mathbb{C}_\theta[\sigma_1] \to \mathbb{C}_\theta[\sigma_3]$ defined on generators by $(t_1, t_2) \mapsto (t_1\, t_2^{-1}, t_2^{-1})$. The other faces are similarly treated, and the noncommutative toric geometry of $\mathbb{CP}^2_\theta = X_\theta[\Sigma]$ can thus be assembled into a diagram of gluing morphisms

$$\begin{tikzcd}
 & \mathbb{C}_\theta\big[t_1^{-1}\,,\,(t_1\, t_2^{-1})\,,\,(t_1\, t_2^{-1})^{-1}\big] \arrow[d] & \\
\mathbb{C}_\theta\big[t_1^{-1}\,,\,t_1^{-1}\, t_2\big] \arrow[ur] \arrow[r] \arrow[d] & \mathbb{C}_\theta(t_1, t_2) & \mathbb{C}_\theta\big[t_1\, t_2^{-1}\,,\,t_2^{-1}\big] \arrow[ul] \arrow[l] \arrow[d] \\
\mathbb{C}_\theta\big[t_1\,,\,t_1^{-1}\,,\,t_2\big] \arrow[ur] & \mathbb{C}_\theta[t_1, t_2] \arrow[l] \arrow[u] \arrow[r] & \mathbb{C}_\theta\big[t_1\,,\,t_2\,,\,t_2^{-1}\big] \arrow[ul]
\end{tikzcd}$$

The noncommutative affine variety associated to the zero cone is the spectrum of the full deformed character algebra $\mathbb{C}_\theta[0] = \mathbb{C}_\theta(t_1, t_2)$, corresponding to the open embedding of the noncommutative algebraic torus.

3.4. **Noncommutative orbifold.** We can also deform singular toric varieties in our formalism. For illustration, let us consider the quotient singularity $\mathbb{C}^2/\mathbb{Z}_2$, where the cyclic group $\mathbb{Z}_2$ is generated by the action $(z_1, z_2) \mapsto (-z_1, -z_2)$ for $(z_1, z_2) \in \mathbb{C}^2$. The quotient can be described as the locus of the equation $x\, y - z^2 = 0$ in $\mathbb{C}^3$. The fan Σ of the lattice $L \cong \mathbb{Z}^2$ consists of a single cone $\sigma = \mathbb{R}^+ v_1 \oplus \mathbb{R}^+ v_2$, where $v_1 = e_1$ and $v_2 = e_1 + 2e_2$. The semigroup $\sigma^\vee \cap L^*$ is generated by $m_1 = 2e_1^* - e_2^*$, $m_2 = e_2^*$ and $m_3 = e_1^*$, so that $R[m_a]$ is generated by the single relation $m_1 + m_2 = 2m_3$. The coordinate algebra $\mathbb{C}_\theta[t_1, t_2]^{\mathbb{Z}_2}$ of the noncommutative affine variety $X_\theta[\Sigma] = U_\theta[\sigma]$ is thus generated by $x = t_1^2\, t_2^{-1}$, $y = t_2$ and $z = t_1$ with the relations

$$x\, y = q^4\ y\, x\ , \qquad x\, z = q^2\ z\, x\ , \qquad y\, z = q^{-2}\ z\, y$$

and

$$x\, y - q^2\ z^2 = 0\ .$$

The blow-up of the quotient singularity $\mathbb{C}^2/\mathbb{Z}_2$ is the total space of the canonical holomorphic line bundle $\mathcal{O}_{\mathbb{CP}^1}(-2) \to \mathbb{CP}^1$, which defines the non-singular Hirzebruch–Jung resolution isomorphic to the toric A_1 ALE space. It is obtained by adding the vector $v_0 = e_1 + e_2$ to the fan Σ above. There are now two maximal cones $\sigma_+ = \mathbb{R}^+ v_1 \oplus \mathbb{R}^+ v_0$ and $\sigma_- = \mathbb{R}^+ v_0 \oplus \mathbb{R}^+ v_2$, with dual semigroups generated by $m_1^\pm = \pm\, e_1^*$ and $m_2^\pm = e_2^* \mp e_1^*$, respectively. The coordinate algebras of the noncommutative affine toric varieties $U_\theta[\sigma_\pm]$ are generated respectively by elements $u_\pm = t_1^{\pm 1}$, $v_\pm = t_1^{\mp 1}\, t_2$ subject to the relations

$$u_\pm\, v_\pm = q^{\pm 2}\ v_\pm\, u_\pm \ ,$$

and hence $U_\theta[\sigma_\pm] \cong \mathbb{C}^2_\theta$. The dual semigroup of the one-dimensional cone given by the intersection $\tau = \sigma_+ \cap \sigma_- = \mathbb{R}^+ v_0$ is generated by $m_1 = e_1^*$ and $m_2 = e_1^* - e_2^*$, together with $m_3 = e_2^* - e_1^* = -m_2$. The generators of $\mathbb{C}_\theta[\tau]$ are the elements $y_1 = t_1$, $y_2 = t_1^{-1}\, t_2$ and $y_3 = t_1\, t_2^{-1}$ with the relations (3.14). The noncommutative algebraic torus deformation of the resolution is thus described by the diagram of gluing morphisms given by

$$\begin{tikzcd} & \mathbb{C}_\theta\big[t_1\,,\,(t_1^{-1}\,t_2)\,,\,(t_1^{-1}\,t_2)^{-1}\big] \arrow[d] & \\ \mathbb{C}_\theta\big[t_1\,,\,t_1^{-1}\,t_2\big] \arrow[ur] \arrow[r] & \mathbb{C}_\theta(t_1,t_2) & \mathbb{C}_\theta\big[t_1^{-1}\,,\,t_1\,t_2\big] \arrow[ul] \arrow[l] \end{tikzcd}$$

where the first diagonal arrow is the natural subalgebra inclusion and the second diagonal arrow is inclusion after the algebra automorphism $\mathbb{C}_\theta[\sigma_+] \to \mathbb{C}_\theta[\sigma_-]$ given by sending $(t_1,t_2) \mapsto (t_1^{-1},t_2)$.

3.5. **Noncommutative conifold.** Consider the threefold ordinary double point, or conifold singularity, defined by the locus of the equation $x\,y - z\,w = 0$ in $\mathbb{C}^4$. Its fan Σ in $L \cong \mathbb{Z}^3$ consists of a single maximal cone σ generated by $w_1 = e_1$, $w_2 = e_2$, $w_3 = e_1 + e_3$ and $w_4 = e_2 + e_3$. The dual cone $\sigma^\vee \cap L$ is generated by $m_1 = e_1$, $m_2 = e_2$, $m_3 = e_3$ and $m_4 = e_1 + e_2 - e_3$, so that $m_1 + m_2 = m_3 + m_4$. The generators of the coordinate algebra of the noncommutative conifold $X_\theta[\Sigma] = U_\theta[\sigma]$ are thus the elements $x = t_1$, $y = t_2$, $z = t_3$ and $w = t_1\, t_2\, t_3^{-1}$ subject to the relations

$$\begin{aligned} x\,y &= q_{12}^2\ y\,x\ , \qquad & x\,z &= q_{13}^2\ z\,x\ , \qquad & x\,w &= q_{12}^2\, q_{13}^{-2}\ w\,x\ , \\ y\,z &= q_{23}^2\ z\,y\ , \qquad & y\,w &= q_{12}^{-2}\, q_{23}^{-2}\ w\,y\ , \qquad & z\,w &= q_{13}^2\, q_{23}^2\ w\,z\ , \end{aligned}$$

and

$$x\,y - q_{12}^2\, q_{13}^{-2}\, q_{23}^{-2}\ z\,w = 0\ ,$$

where as before $q_{ij} = \exp(\frac{\mathrm{i}}{2}\,\theta_{ij})$ and $\theta = (\theta_{ij})$ is the 3×3 matrix describing the deformation of the embedded algebraic torus.

The crepant resolution of the conifold singularity is the total space of the rank two holomorphic bundle $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1) \to \mathbb{CP}^1$, which is a non-singular toric Calabi–Yau threefold. The fan Σ of the lattice $L \cong \mathbb{Z}^3$ is defined by the vectors $v_1 = e_1 + e_2 + e_3$, $v_2 = e_1 + e_3$, $v_3 = e_1$ and $v_4 = e_1 + e_2$, the maximal cones $\sigma_1 = \mathbb{R}^+ v_1 \oplus \mathbb{R}^+ v_2 \oplus \mathbb{R}^+ v_3$ and $\sigma_2 = \mathbb{R}^+ v_1 \oplus \mathbb{R}^+ v_3 \oplus \mathbb{R}^+ v_4$, and their overlap $\tau = \sigma_1 \cap \sigma_2 = \mathbb{R}^+ v_1 \oplus \mathbb{R}^+ v_3$.

(1) $\sigma_1^\vee \cap L^*$ is generated by $m_1 = e_2^*$, $m_2 = e_3^* - e_2^*$ and $m_3 = e_1^* - e_3^*$, so $\mathbb{C}_\theta[\sigma_1]$ is generated by $x = t_2$, $y = t_2^{-1}\, t_3$ and $z = t_1\, t_3^{-1}$ with the relations

$$x\,y = q_{23}^2\ y\,x\ , \qquad x\,z = q_{12}^{-2}\, q_{23}^{-2}\ z\,x\ , \qquad y\,z = q_{12}^{-2}\ z\,y\ .$$

(2) $\sigma_2^\vee \cap L^*$ is generated by $m_1 = e_3^*$, $m_2 = e_1^* - e_2^*$ and $m_3 = e_2^* - e_3^*$, so $\mathbb{C}_\theta[\sigma_2]$ is generated by $x = t_3$, $y = t_1\, t_2^{-1}$ and $z = t_2\, t_3^{-1}$ with the relations

$$x\,y = q_{13}^{-2}\, q_{23}^2\ y\,x\ , \qquad x\,z = q_{23}^{-2}\ z\,x\ , \qquad y\,z = q_{12}^2\, q_{13}^{-2}\, q_{23}^2\ z\,y\ .$$

(3) $\tau^\vee \cap L^*$ is generated by $m_1 = e_2^*$, $m_2 = e_1^* - e_2^*$ and $m_3 = e_2^* + e_3^*$, together with $m_4 = -e_2^* - e_3^* = -m_3$, so $\mathbb{C}_\theta[\tau]$ is generated by $y_1 = t_2$, $y_2 = t_1\, t_2^{-1}$, $y_3 = t_2\, t_3$ and $y_4 = t_2^{-1}\, t_3^{-1}$ with the relations

$$y_1\, y_2 = q_{12}^{-2}\ y_2\, y_1\ , \qquad y_1\, y_3 = q_{23}^{2}\ y_3\, y_1\ , \qquad y_1\, y_4 = q_{23}^{-2}\ y_4\, y_1\ ,$$
$$y_2\, y_3 = q_{12}^{2}\, q_{13}^{2}\, q_{23}^{-2}\ y_3\, y_2\ , \qquad y_2\, y_4 = q_{12}^{-2}\, q_{13}^{-2}\, q_{23}^{2}\ y_4\, y_2\ , \qquad y_3\, y_4 = q_{23}^{2}\ y_4\, y_3\ .$$

The noncommutative toric geometry is described by the diagram of gluing morphisms

$$\begin{tikzcd}[column sep=small]
& \mathbb{C}_\theta\big[t_2\, ,\, t_1\, t_2^{-1}\, ,\, (t_2\, t_3)\, ,\, (t_2\, t_3)^{-1}\big] \arrow[d] & \\
\mathbb{C}_\theta\big[t_2\, ,\, t_2^{-1}\, t_3\, ,\, t_1\, t_3^{-1}\big] \arrow[ur] \arrow[r] & \mathbb{C}_\theta(t_1, t_2, t_3) & \mathbb{C}_\theta\big[t_3\, ,\, t_1\, t_2^{-1}\, ,\, t_2\, t_3^{-1}\big] \arrow[ul] \arrow[l]
\end{tikzcd}$$

where the second diagonal arrow is the subalgebra inclusion and the first diagonal arrow is inclusion after the automorphism sending $t_3 \mapsto t_1$. Note the similarity with the gluing morphisms of the quotient singularity blow-up of §3.4.

4. Sheaves on noncommutative toric varieties

In this section we develop a sheaf theory on noncommutative toric varieties, following [23]. The idea is that the "topology" of the noncommutative space $X_\theta = X_\theta[\Sigma]$ is given by the cones in the fan Σ (the toric open sets in the topology of X_θ). The assignment $\sigma \mapsto \mathbb{C}_\theta[\sigma]$ of the noncommutative algebra $\mathbb{C}_\theta[\sigma]$ to every cone $\sigma \in \Sigma$ is viewed as the *structure sheaf* $\mathcal{O}_{X_\theta}$ of the noncommutative toric variety X_θ.

4.1. **Quasi-coherent sheaves.** We begin with the following elementary result.

Lemma 4.1. *For each cone $\sigma \in \Sigma$, the algebra $\mathbb{C}_\theta[\sigma]$ is a noetherian domain.*

<u>Proof</u>: Since the quantum Laurent algebra $\mathbb{C}_\theta(t_1, \dots, t_n)$ is a domain, so is the subalgebra $\mathbb{C}_\theta[\sigma]$. As the algebra $\mathbb{C}_{\check\theta}[x_1, \dots, x_l]$ is an iterated polynomial algebra over $\mathbb{C}$, all of its ideals are finitely-generated, i.e. it is noetherian. Since $\mathbb{C}_\theta[\sigma]$ can be realized as a quotient algebra of the noncommutative polynomial algebra $\mathbb{C}_{\check\theta}[x_1, \dots, x_l]$, it is also noetherian. ∎

We use the category $\mathsf{Open}(X_\theta)$ of toric open sets to define the category of sheaves on the variety $X_\theta = X_\theta[\Sigma]$. We call a set of inclusions $(\sigma_i \hookrightarrow \sigma)_{i\in I}$ of cones a *covering* if $\sigma = \bigcup_{i\in I} \sigma_i$. Then $\mathsf{Open}(X_\theta)$ always contains a sufficiently fine open cover. The category $\mathsf{Open}(X_\theta)$ with the data of coverings forms a Grothendieck topology on X_θ.

Proposition 4.2. *The association $\sigma \mapsto \mathbb{C}_\theta[\sigma]$ defines a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_{X_\theta}$ on $\mathsf{Open}(X_\theta)$.*

<u>Proof</u>: Let $(\sigma_i \hookrightarrow \sigma)_{i\in I}$ be a covering, i.e. $\sigma = \bigcup_{i\in I} \sigma_i$. Then $\mathbb{C}_\theta[\sigma] = \bigcap_{i\in I} \mathbb{C}_\theta[\sigma_i]$, where the intersection is well-defined since each algebra $\mathbb{C}_\theta[\sigma_i]$ is contained in $\mathbb{C}_\theta(t_1, \dots, t_n)$. Thus, as in (3.11), the sequence

$$0 \longrightarrow \mathbb{C}_\theta[\sigma] \xrightarrow{p} \prod_{i\in I} \mathbb{C}_\theta[\sigma_i] \xrightarrow{q} \prod_{i,j\in I} \mathbb{C}_\theta[\sigma_i \cap \sigma_j] \tag{4.3}$$

is exact, and the result follows. ∎

We now define $\mathsf{mod}(X_\theta)$ to be the category of sheaves of right $\mathcal{O}_{X_\theta}$-modules on $\mathsf{Open}(X_\theta)$. If Σ consists of a single cone σ, i.e. $X_\theta[\Sigma] = U_\theta[\sigma]$ is an affine variety, then

$$\mathsf{mod}\big(U_\theta[\sigma]\big) \cong \mathsf{mod}\big(\mathbb{C}_\theta[\sigma]\big) \tag{4.4}$$

coincides with the category of right $\mathbb{C}_\theta[\sigma]$-modules. We denote by $\widetilde{M}$ the sheaf associated to a module M under the isomorphism (4.4). A sheaf of right $\mathcal{O}_{X_\theta}$-modules is called *quasi-coherent* if its restriction to each affine open set $U_\theta[\sigma]$ is of the form $\widetilde{M}$ for some right $\mathbb{C}_\theta[\sigma]$-module M. It is called *coherent* if M is finitely-generated.

Let $\mathsf{coh}(X_\theta)$ denote the category of quasi-coherent sheaves of right $\mathcal{O}_{X_\theta}$-modules. Given a cone σ in Σ, we write $\mathsf{coh}(\sigma)$ for the category of right $\mathbb{C}_\theta[\sigma]$-modules. There are restriction functors

$$j_\sigma^\bullet \, : \, \mathsf{coh}(X_\theta) \ \longrightarrow \ \mathsf{coh}(\sigma) \tag{4.5}$$

for each open inclusion $j_\sigma : U[\sigma] \hookrightarrow X[\Sigma]$. Let $\mathsf{tor}(\sigma)$ be the full Serre subcategory of $\mathsf{coh}(X_\theta)$ generated by objects E such that $j_\sigma^\bullet(E) = 0$. In [23, Prop. 4.3] the following fundamental result is proven.

Proposition 4.6. *Let σ be a cone in Σ. Then the restriction functor (4.5) is exact, and there is a natural equivalence of categories*

$$\mathsf{coh}(X_\theta) \big/ \mathsf{tor}(\sigma) \ \cong \ \mathsf{coh}(\sigma) \ .$$

Each cone σ in the fan Σ gives a toric open set of $X_\theta[\Sigma]$. We will use Proposition 4.6 to reduce geometric problems in the category $\mathsf{coh}(X_\theta)$ to algebraic problems in the algebra $\mathbb{C}_\theta[\sigma]$ via the localization functors $j_\sigma^\bullet$. This gives an explicit description of the quotient category. The objects of $\mathsf{coh}(\sigma)$ are the same as those of $\mathsf{coh}(X_\theta)$, and we write E_σ for the object in $\mathsf{coh}(\sigma)$ corresponding to a sheaf E. The morphisms are given by

$$\mathrm{Hom}_{\mathsf{coh}(\sigma)}(E_\sigma, F_\sigma) = \varinjlim_{E'} \ \mathrm{Hom}_{\mathsf{coh}(X_\theta)}(E', F) \ ,$$

where the inductive limit is taken over all subsheaves $E' \subset E$ with $j_\sigma^\bullet(E/E') = 0$.

For any pair of sheaves $E, F \in \mathsf{coh}(X_\theta)$, let $\mathrm{Ext}^p(E,F)$ be the p-th derived functor of the Hom-functor $\mathrm{Hom}(E,F) = \mathrm{Hom}_{\mathsf{coh}(X_\theta)}(E,F)$. For a sheaf $E \in \mathsf{coh}(X_\theta)$, we define

$$H^p(X_\theta, E) := \mathrm{Ext}^p(\mathcal{O}_{X_\theta}, E) \ .$$

Definition 4.7. (1) *A coherent sheaf $\mathcal{E} \in \mathsf{coh}(X_\theta)$ is called* locally free *or a* bundle *if each $\mathcal{E}_\sigma$, $\sigma \in \Sigma$ corresponds to a free module $\mathbb{C}_\theta[\sigma]^{\oplus r}$ for some $r \in \mathbb{N}$. The integer r is called the* rank *of $\mathcal{E}$.*

(2) *A coherent sheaf $E \in \mathsf{coh}(X_\theta)$ is called* torsion free *if each E_σ, $\sigma \in \Sigma$ has no finite-dimensional submodules, or equivalently if it admits an embedding $E \hookrightarrow \mathcal{E}$ into a locally free sheaf $\mathcal{E}$. The* rank *of E is the rank of $\mathcal{E}$ minus the rank of $\mathcal{E}/E$.*

4.2. **Equivariant sheaves.** Recall from §3.1 that for each $\sigma \in \Sigma$ there is a grading (3.7) of the algebra $\mathbb{C}_\theta[\sigma]$ by the free abelian group of characters L^*, the homogeneous elements in the decomposition being identified with the eigenvectors of the T-action on $\mathbb{C}_\theta[\sigma]$. To get a similar eigenspace decomposition on right $\mathbb{C}_\theta[\sigma]$-modules, we need to lift the T-action. We denote with $\mathsf{mod}^{\mathcal{H}_\theta}\big(\mathbb{C}_\theta[\sigma]\big)$ the subcategory of the category $\mathsf{mod}(\mathbb{C}_\theta[\sigma])$ made of left T-equivariant right $\mathbb{C}_\theta[\sigma]$-modules. There is a left action of the Hopf algebra $\mathcal{H}_\theta$ on elements $M \in \mathsf{mod}^{\mathcal{H}_\theta}\big(\mathbb{C}_\theta[\sigma]\big)$ which is compatible with the $\mathcal{H}_\theta$-action on $\mathbb{C}_\theta[\sigma]$.

This means that $h \triangleright (M \cdot a) = (h_{(1)} \triangleright M) \cdot (h_{(2)} \triangleright a)$ for $h \in \mathcal{H}_\theta$, a right $\mathbb{C}_\theta[\sigma]$-module M, and $a \in \mathbb{C}_\theta[\sigma]$ (with the usual notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for the coproduct). Objects of $\mathsf{mod}^{\mathcal{H}_\theta}\big(\mathbb{C}_\theta[\sigma]\big)$ admit then an L^*-graded T-eigenspace decomposition $M = \bigoplus_{p\in L^*} M^p$ such that $M^p \cdot \mathbb{C}_\theta[\sigma]^q \subset M^{p+q}$ for all $p, q \in L^*$, and $t^{m_a} \triangleright M^p \subset M^{m_a+p}$ for all $p \in L^*$ and for $m_a \in \sigma^\vee \cap L^*$. This also means that the category of right $\mathbb{C}_\theta[\sigma]$-modules carrying a compatible left $\mathcal{H}_\theta$-action is naturally a braided monoidal category of left $\mathcal{H}_\theta$-modules. Via the braiding morphism Ψ_θ, we can deform the category ${}_{\mathcal{H}}\mathscr{M}$ as described in §1.2, and there is a functorial equivalence between the categories $\mathsf{mod}^{\mathcal{H}}\big(\mathbb{C}[\sigma]\big)$ and $\mathsf{mod}^{\mathcal{H}_\theta}\big(\mathbb{C}_\theta[\sigma]\big)$.

This construction extends to give a left $\mathcal{H}_\theta$-action on the category $\mathsf{coh}(X_\theta)$ and T-equivariant sheaves on $\mathsf{Open}(X_\theta)$, i.e. the subcategory $\mathsf{coh}^{\mathcal{H}_\theta}(X_\theta)$ of coherent sheaves $E \in \mathsf{coh}(X_\theta)$ with a compatible T-action, which decompose as direct sums

$$E = \bigoplus_{p\in L^*} E^p$$

of T-eigensheaves E^p of $\mathcal{O}_{X_\theta}$-modules. If E is locally free, then each summand E^p is also locally free. There is a functorial equivalence between the categories $\mathsf{coh}^{\mathcal{H}}(X)$ and $\mathsf{coh}^{\mathcal{H}_\theta}(X_\theta)$.

4.3. **Invariant subschemes and ideal sheaves.** In applications to instanton counting problems, which will be presented in [12], one is faced with the task of classifying the fixed points of the natural torus action on the category $\mathsf{coh}(X_\theta)$ obtained by lifting the action of T on X_θ as described in §4.2. In the classical case, one uses the orbit decomposition theorem [20] asserting that the toric variety $X[\Sigma]$ is a disjoint union over the orbits O_σ of the T-action on X, which are in bijective correspondence with the cones $\sigma \in \Sigma$. One has $\dim_{\mathbb{C}}(\sigma) + \dim_{\mathbb{C}}(O_\sigma) = n$, and $O_\sigma \subset \overline{O_\tau}$ if and only if τ is a face of σ. In particular, the fixed points of the torus action, i.e. the closed T-orbits, correspond to the maximal cones in the fan Σ, while $O_0 = T$. We will now show that these orbits are somewhat more easily classified in the noncommutative case, in the sense that they arise as the generic T-invariant subvarieties in X_θ.

In analogy with the classical setting, we have the notion of a "noncommutative scheme".

Definition 4.8. *A* closed subscheme *of X_θ is a full subcategory $Y_\theta \subseteq \mathsf{coh}(X_\theta)$ whose inclusion functor $i_\bullet$ has a right-adjoint $i^!$ and a left-adjoint $i^\bullet$.*

Definition 4.9. *An* ideal sheaf *on $\mathsf{Open}(X_\theta)$ is a coherent sheaf $I \in \mathsf{coh}(X_\theta)$ whose restriction to each affine open set $U_\theta[\sigma]$ is a two-sided ideal I_σ of the algebra $\mathbb{C}_\theta[\sigma]$.*

For each cone $\sigma \in \Sigma$, it follows from Lemma 4.1 that every torsion free module of rank one in $\mathsf{coh}(\sigma) = \mathsf{mod}(\mathbb{C}_\theta[\sigma])$ is the image of a right ideal of $\mathbb{C}_\theta[\sigma]$. Hence an ideal sheaf $I \in \mathsf{coh}(X_\theta)$ can be regarded as a torsion free sheaf of rank one on $\mathsf{Open}(X_\theta)$. Moreover, the category of sheaves of right $\mathcal{O}_{X_\theta}/I$-modules determines a closed subscheme Y_θ of X_θ. The following result describes to what extent this correspondence fails to be a bijection (generalizing the commutative case; see e.g. [14, §3]).

Theorem 4.10. *There is a bijective correspondence between closed subschemes of X_θ and ideal sheaves I on $\mathsf{Open}(X_\theta)$ such that $I_\sigma \star_\theta \mathbb{C}_\theta[\sigma \cap \sigma'\,] = I_{\sigma'} \star_\theta \mathbb{C}_\theta[\sigma \cap \sigma'\,]$ on overlaps $U_\theta[\sigma \cap \sigma'\,]$.*

<u>Proof</u>: Let $i_\bullet$ be the inclusion of a subcategory in $\mathsf{coh}(X_\theta)$ corresponding to a closed subscheme Y_θ, with left adjoint functor $i^\bullet$. Then the map $Y_\theta \to Y_\theta$, $M \mapsto i_\bullet\, i^\bullet(M)$ is

surjective. Fix a cone $\sigma \in \Sigma$, and suppose that $M \in \mathsf{tor}(\sigma)$, i.e. $j_\sigma^\bullet(M) = 0$. Since the restriction functor $j_\sigma^\bullet$ is exact, the map $j_\sigma^\bullet(M) \to j_\sigma^\bullet\, i_\bullet\, i^\bullet(M)$ is also surjective, and hence by Proposition 4.6 the functor $i_\bullet\, i^\bullet$ acts on the category $\mathsf{coh}(\sigma)$. It follows [23, Prop. 4.5] that $\mathbb{C}_\theta[\sigma] \to i_\bullet\, i^\bullet(\mathbb{C}_\theta[\sigma])$ is a surjective bimodule morphism, whose kernel is the desired two-sided ideal I_σ. Conversely, given an ideal sheaf I on $\mathsf{Open}(X_\theta)$ with the stated property, we define the functor $i^\bullet$ by mapping the module M over $\mathbb{C}_\theta[\sigma]$ to $M/M \star_\theta I_\sigma \in \mathsf{mod}(\mathbb{C}_\theta[\sigma]/I_\sigma)$. ■

If σ is a cone in the fan Σ, and $\tau \in \Sigma$ is a face of σ, define $I_\sigma(\tau)$ to be the kernel of the algebra morphism $\mathbb{C}_\theta[\sigma] \to \mathbb{C}_\theta[\tau]$. Then

$$I_\sigma(\tau) = \bigoplus_{m \notin \tau^\vee \cap L^*} \mathbb{C}\chi_m \tag{4.11}$$

is an ideal in $\mathbb{C}_\theta[\sigma]$, and hence each face $\tau \subset \sigma$ canonically determines a closed subscheme of X_θ. The cone point of a strongly convex cone σ is a distinguished torus fixed point of $U[\sigma]$. It follows that for any given face $\tau \hookrightarrow \sigma$, there is a natural morphism $\mathbb{C}_\theta[\sigma] \to \mathbb{C}_\theta[\tau]$ dual to inclusion of an orbit closure.

Definition 4.12. *A closed subscheme Y_θ is* irreducible *if each inclusion of a full subcategory $Y_\theta \subset W_\theta \cup Z_\theta$ implies $Y_\theta \subset W_\theta$ or $Y_\theta \subset Z_\theta$, where W_θ, Z_θ are closed subschemes of X_θ and $W_\theta \cup Z_\theta$ is the full subcategory of $\mathsf{coh}(X_\theta)$ whose objects M are extensions*

$$0 \longrightarrow \omega \longrightarrow M \longrightarrow \zeta \longrightarrow 0 ,$$

of objects ω and ζ of W_θ and Z_θ respectively.

The union operation $\cup$ in Definition 4.12 corresponds to the product of ideals in each algebra $\mathbb{C}_\theta[\sigma]$, $\sigma \in \Sigma$ [23, Prop. 4.5] for the correspondence in Theorem 4.10. It follows that irreducible subschemes give prime ideals on each open affine set $U_\theta[\sigma]$ under the correspondence of this theorem. For a subset $S \subset L_{\mathbb{R}}^*$, we denote

$$S^\perp := \big\{ v \in L_{\mathbb{R}} \bigm| \langle u, v\rangle = 0 \quad \forall u \in S \big\} ,$$

and for a $\mathbb{C}$-algebra A we denote by $\mathrm{Spec}(A)$ the *spectrum* of A, i.e. the set of prime ideals equipped with the Zariski topology. Recall from Definition 3.3 that $\sigma^\vee$ denotes the cone dual to σ.

The following characterization of the irreducible subschemes of X_θ is proven in [23, Thm. 6.8].

Proposition 4.13. *There is a natural bijection between the set of irreducible subschemes of $X_\theta(\Sigma)$ and the disjoint union $\bigsqcup_{\sigma\in\Sigma} \mathrm{Spec}\big(\mathbb{C}_\theta[(\sigma^\perp)^\vee]\big)$.*

For θ sufficiently generic, the only subschemes of X_θ are dual to closed T-orbits and to all points of one-dimensional torus orbits [23, §6.2]. To better understand this point, notice that if J is any ideal of the algebra $\mathbb{C}_\theta[\sigma]$ for $\sigma \in \Sigma$, the intersection, $\bigcap_{t\in T} t \triangleright J$, of the T-orbit of J is the largest torus invariant ideal of $\mathbb{C}_\theta[\sigma]$ contained in J. In particular, it is a T-invariant prime ideal for every $J \in \mathrm{Spec}(\mathbb{C}_\theta[\sigma])$. The T-strata partition the space of prime ideals $\mathrm{Spec}(\mathbb{C}_\theta[\sigma])$ into a disjoint union over T-invariant prime ideals.

Proposition 4.14. *For each cone $\sigma \in \Sigma$ and for every T-invariant prime ideal I in $\mathbb{C}_\theta[\sigma]$, the T-stratum $\{J \in \mathrm{Spec}(\mathbb{C}_\theta[\sigma]) \mid \bigcap_{t\in T} t \triangleright J = I\}$ is a single T-orbit.*

Proof: This follows by Lemma 4.1 and [22, Thm. 6.8], which imply that the torus T acts transitively on the T-strata of prime ideals in $\mathbb{C}_\theta[\sigma]$. ■

Proposition 4.15. *There is a natural bijection between the sets of T-equivariant ideal sheaves on $\mathsf{Open}(X_\theta)$, satisfying the conditions of Theorem 4.10, and L^*-graded subschemes of $X_\theta[\Sigma]$.*

Proof: Let Y_θ be a closed subscheme of X_θ, defined by an ideal sheaf I according to Theorem 4.10. Then Y_θ is invariant under the torus action if and only if the action of T on the category $\mathsf{coh}(X_\theta)$ induces an action on Y_θ. Suppose first that $X_\theta[\Sigma] = U_\theta[\sigma]$ is affine. Then this invariance is equivalent to the requirement that there is a commutative diagram

$$\begin{tikzcd} \mathbb{C}_\theta[\sigma]\times T \arrow[r, "\Phi"] \arrow[d] & \mathbb{C}_\theta[\sigma] \arrow[d] \\ I_\sigma \times T \arrow[r, "\Phi|_{I_\sigma\times T}"'] & I_\sigma \end{tikzcd}$$

where Φ is the right covariant action of T on $\mathbb{C}_\theta[\sigma]$ constructed in Proposition 3.6, I_σ is a two-sided ideal in $\mathbb{C}_\theta[\sigma]$, and the vertical morphisms are restrictions. This is true if and only if $\Phi_\tau(I_\sigma)\subset I_\sigma$, for all $\tau=(\tau_1,\dots,\tau_n)\in T$. It follows that if $\sum_a \alpha_a\, t^{m_a}$ is in I_σ, with $m_a\in\sigma^\vee\cap L^*\subset L^*$ for $a=1,\dots,l$, then $\sum_a \alpha_a\,\big(t_1^{(m_a)_1}\,\tau_1,\dots,t_n^{(m_a)_n}\,\tau_n\big)$ is also in I_σ, and so $\alpha_a\,t^{m_a}\in I_\sigma$ for every $a=1,\dots,l$. Thus I_σ is an L^*-graded ideal of $\mathbb{C}_\theta[\sigma]$. If we now write $I_\sigma=\bigoplus_{p\in S}\mathbb{C}\chi_p$ for some subset $S\subset\sigma^\vee\cap L^*$, then the condition for I_σ to be an ideal in $\mathbb{C}_\theta[\sigma]$ is equivalent to the requirement that for all $m_a\in\sigma^\vee\cap L^*$ and $p\in S$, one has $m_a+p\in S$. Hence I_σ is T-equivariant. The global statement for general $X_\theta[\Sigma]$ now follows by gluing these equivalences together. ■

Remark 4.16. *For $\sigma\in\Sigma$, the T-invariant ideal I_σ of the algebra $\mathbb{C}_\theta[\sigma]$ appearing in the proof of Proposition 4.15 is generated by elements of the form t^{m_a} for $m_a\in\sigma^\vee\cap L^*$, i.e. I_σ is a* monomial ideal. *Moreover, I_σ is prime if and only if $(\sigma^\vee\cap L^*)\setminus S$ is a sub-semigroup of $\sigma^\vee\cap L^*$. It follows that the irreducible invariant subschemes of $U_\theta[\sigma]$ are in bijective correspondence with the faces τ of σ, such that the corresponding monomial ideal is given by (4.11).*

For fixed $\sigma\in\Sigma$, let $L_\sigma=L\cap\sigma$ and let $p:L\to L(\sigma):=L/L_\sigma$ be the canonical projection. Then $L(\sigma)^*=L^*\cap\sigma^\perp$. The homomorphism $\Theta:L^*\to T$ naturally restricts to the sublattice $L(\sigma)^*\subset L^*$. Let $p_{\mathbb{R}}=p\otimes\mathbb{R}$. Then the collection of cones $p_{\mathbb{R}}(\tau)$, where $\tau\in\Sigma$ is a cone for which σ is a face of τ, form a fan $\Sigma(\sigma)$ in $L(\sigma)\otimes_{\mathbb{Z}}\mathbb{R}$. Set $V_\theta(\sigma)=X_\theta[\Sigma(\sigma)]$. By Theorem 4.10, the projection $\Sigma\to\Sigma(\sigma)$ shows that $V_\theta(\sigma)$ defines a closed subscheme of $X_\theta=X_\theta[\Sigma]$.

Example 4.17. *Suppose that σ is the maximal cone of Σ generated by the basis $e_1,\dots,e_n$ of the lattice $L\cong\mathbb{Z}^n$, with dual basis $e_1^*,\dots,e_n^*$. Then the corresponding noncommutative affine variety is the algebraic Moyal plane $U_\theta[\sigma]\cong\mathbb{C}_\theta^n$, i.e. $\mathbb{C}_\theta[\sigma]=\mathbb{C}_\theta[t_1,\dots,t_n]$ where $t_i=t^{e_i^*}$. Let τ be a face of σ generated by $\{e_i\}_{i\in N}$ for some subset $N\subset\{1,\dots,n\}$. Then $V_\theta(\tau)$ is defined by the monomial ideal $\langle\,t_i\,\rangle_{i\in N}$ in $\mathbb{C}_\theta[t_1,\dots,t_n]$.*

4.4. **Kähler differential forms.** We will now construct sheaves of noncommutative differential forms. We start by recalling some definitions and properties of Kähler differentials. We then show how the general construction behaves under a Drinfel'd twist using the braided monoidal category theory of §1.2. This formalism may be used to define sheaves of Kähler differentials over generic noncommutative toric varieties $X_\theta = X_\theta[\Sigma]$.

The general framework we need from the theory of Kähler differentials describes derivations of a unital $\mathbb{C}$-algebra (A, μ) into an A-bimodule M, i.e. $\mathbb{C}$-linear maps $D : A \to M$ obeying the Leibniz rule $D(a\,b) = (Da)\,b + a\,(Db)$ for every $a, b \in A$.

The universal algebra of derivations over A is realized by the A-bimodule

$$\Omega^1_{A,un} = I_A := \ker(\mu : A \otimes A \to A)$$

which is a two-sided ideal of the algebra $A\otimes A$ generated by elements of the form $a\otimes 1-1\otimes a$ with $a \in A$, and differential given by $\mathrm{d}a := a\otimes 1-1\otimes a$. The universal property means that every derivation $D : A \to M$ factors through $\Omega^1_{A,un}$ by a unique morphism of A-bimodules $\phi_D : \Omega^1_{A,un} \to M$ with $D = \phi_D \circ \mathrm{d}$. The morphism ϕ_D is defined by

$$\phi_D\big(a_1\,(\mathrm{d}a)\,a_2\big) := a_1\,D(a)\,a_2\ . \tag{4.18}$$

The construction of $\Omega^1_{A,un}$ respects the inclusion of subalgebras, i.e. $\Omega^1_{A',un} = \ker(\mu|_{A'\otimes A'}) = \ker(\mu) \cap (A' \otimes A'\,)$ for any subalgebra $A' \subset A$.

For the Kähler differential forms one is interested (see e.g. [30, §1.3]) in derivations with values in a symmetric A-bimodule M (i.e. $a\,m = m\,a$ for all $a \in A$ and $m \in M$). Since for all $a, a_1 \in A$ one has

$$a_1\,(\mathrm{d}a) - (\mathrm{d}a)\,a_1 = (a_1 \otimes 1 - 1 \otimes a_1)\,(a \otimes 1 - 1 \otimes a) \ \in \ I_A^2\ , \tag{4.19}$$

the A-bimodule of symmetric differential forms is $I_A/I_A^2 =: \Omega^1_A$, which can be shown to be universal.

We will begin by defining bimodules $\Omega^1_\theta[\sigma]$ of noncommutative Kähler differentials on noncommutative affine varieties for each cone $\sigma \in \Sigma$, and then show that the assignment $\sigma \mapsto \Omega^1_\theta[\sigma]$ defines a sheaf on $\mathsf{Open}(X_\theta)$. Each affine open set $U_\theta[\sigma]$ of a noncommutative toric variety $X_\theta[\Sigma]$ has noncommutative coordinate algebra $\mathbb{C}_\theta[\sigma]$ which is a Drinfel'd twist deformation of the classical coordinate algebra, coming from the algebraic torus action. The construction of Kähler differential forms on noncommutative affine toric varieties follows from the general theory of Kähler differentials for twisted Hopf-module algebras, and the natural setting for the construction is the functorial framework of §1.2. When the noncommutative algebra is a deformation of a commutative algebra induced by a Drinfel'd twist, we can functorially interpret each step in the general construction described above as a deformation of the corresponding commutative construction.

Indeed, if A is an object in the braided monoidal category ${}_{\mathcal{H}}\mathscr{M}$, the A-bimodule of universal one-forms $\Omega^1_{A,un}$ is naturally an $\mathcal{H}$-module algebra with $\mathcal{H}$-action

$$h \triangleright \mathrm{d}a := \mathrm{d}(h \triangleright a)\ .$$

This is the universal covariant differential calculus, in the sense of Woronowicz [37], and it has a natural noncommutative deformation in the category ${}_{\mathcal{H}_F}\mathscr{M}$ of twisted Hopf-module algebras. If A_F is a twisted Hopf-module algebra defined by a Drinfel'd twist element $F \in \mathcal{H} \otimes \mathcal{H}$ as in Theorem 1.4, then the bimodule $\Omega^1_{A_F,un}$ is defined as before to be the kernel of the multiplication map $\mu_F = \mu \circ \big(F^{-1} \triangleright\big) : A_F \otimes A_F \to A_F$. Higher degree differential forms may be introduced via the $\mathbb{N}_0$-graded braided exterior algebra of one-forms

$$\Omega^\bullet_{A_F,un} = \textstyle\bigwedge^\bullet_F \Omega^1_{A_F,un} := T\big(\Omega^1_{A_F,un}\big) \,/\, \big\langle \omega\otimes\eta + \Psi_F(\omega\otimes\eta)\big\rangle_{\omega,\eta\in\Omega^1_{A_F,un}} ,$$

where $T(\Omega^1_{A_F,un}) = \bigoplus_{n\geq 0} (\Omega^1_{A_F,un})^{\otimes_{A_F} n}$ is the tensor algebra of covariant twisted differential one-forms with $(\Omega^1_{A_F,un})^0 := A_F$, and Ψ_F is the braiding morphism on the category ${}_{\mathcal{H}_F}\mathscr{M}$ defined as in Proposition 1.18 with the twist deformed $\mathcal{R}$-matrix $\mathcal{R}_F$. This algebra coincides with the twist deformation of the Hopf-module algebra $\Omega^\bullet_{A,un}$, with the action of the twist F extended to the whole of $T(\Omega^1_{A,un})$ by

$$F \triangleright (\omega_1\otimes\cdots\otimes\omega_n) = \big(F^{(1)}\triangleright(\omega_1\otimes\cdots\otimes\omega_k)\big) \,\otimes\, \big(F^{(2)}\triangleright(\omega_{k+1}\otimes\cdots\otimes\omega_n)\big) \ .$$

The choice of k here is irrelevant thanks to the associativity of the tensor product, and $F^{(1)}$ and $F^{(2)}$ act by iterating the formula (1.2) for covariant actions on $\mathcal{H}$-module algebras.

The A_F-bimodule structure of $\Omega^1_{A_F,un}$ is then deformed according to the deformation of the associative product in A_F as

$$a_1 \blacktriangleright_F (\mathrm{d}a) \blacktriangleleft_F a_2 := a_1 \star_F (a\otimes 1 - 1\otimes a) \star_F a_2 \ .$$

It agrees with the usual deformation induced in the category,

$$a_1 \blacktriangleright_F (\mathrm{d}a) = \alpha\big(F^{-1}\triangleright(a_1\otimes\mathrm{d}a)\big) \ , \qquad (\mathrm{d}a) \blacktriangleleft_F a_2 = \alpha\big(F^{-1}\triangleright(\mathrm{d}a\otimes a_2)\big) \ ,$$

where $\alpha : A\otimes\Omega^1_{A,un}\otimes A\to A$ denotes the action of A on $\Omega^1_{A,un}$. Then the differential d of the untwisted differential calculus is still a derivation of the deformed product $\star_F$, as expected by general twisting theory [33]. It naturally extends to the braided exterior algebra $\Omega^\bullet_{A_F,un}$ as a graded derivation of degree one by defining

$$\mathrm{d}(\gamma_1\otimes\gamma_2) := (\mathrm{d}\gamma_1)\otimes\gamma_2 + (-1)^{\deg(\gamma_1)}\,\gamma_1\otimes(\mathrm{d}\gamma_2)$$

for homogeneous differential forms $\gamma_1,\gamma_2\in\Omega^\bullet_{A_F,un}$.

The notion of symmetric bimodule has a braided analog by demanding that the left and right module morphisms $\lambda_F : A_F\otimes\Omega^1_{A_F,un}\to\Omega^1_{A_F}$ and $\rho_F : \Omega^1_{A_F,un}\otimes A_F\to\Omega^1_{A_F}$ are related by the braiding morphism of ${}_{\mathcal{H}_F}\mathscr{M}$.

Definition 4.20. *Let A_F be an $\mathcal{H}_F$-module algebra, and let $\Psi=\Psi_F$ be the braiding morphism of Proposition 1.18. An A_F-bimodule M in the category ${}_{\mathcal{H}_F}\mathscr{M}$ is said to be* braided symmetric *if one of the following two conditions is satisfied:*

(1) $\lambda_F = \rho_F\circ\Psi_{A_F,M}$; *or*
(2) $\rho_F = \lambda_F\circ\Psi_{M,A_F}$.

The two conditions in Definition 4.20 are not equivalent unless the category itself is symmetric, i.e. $\Psi^2 = \mathrm{id}$. This is the case, for example, for Drinfel'd twists of triangular Hopf algebras such as the ones we are dealing with in this paper. In the non-symmetric case they are not compatible with each other, so there are two distinct and inequivalent notions of braided symmetric bimodule structure that one can choose from.

We want to show that a natural quotient $I_{A_F}/I^2_{A_F}$ is the universal braided symmetric A_F-bimodule for braided commutative algebras in (twisted) braided monoidal categories ${}_{\mathcal{H}_F}\mathscr{M}$, with universality understood in the same sense as the untwisted A-bimodule Ω^1_A. Then we can define noncommutative differential forms via the usual deformation in the category of Hopf-module algebras, and this definition is compatible with the construction of universal differential forms in braided monoidal categories.

Proposition 4.21. *Let A be a commutative $\mathcal{H}$-module algebra, and F a Drinfel'd twist element for a triangular Hopf algebra $\mathcal{H}$. Let $I_{A_F} = \ker(\mu_F : A_F \otimes A_F \to A_F)$, and consider the quotient $\Omega^1_{A_F} = I_{A_F}/I^2_{A_F}$. Then $(\Omega^1_{A_F}, \mathrm{d})$ is the universal algebra of derivations over A_F with values in a braided symmetric A_F-bimodule.*

Proof: We will prove this by direct computation for the twisted Hopf algebra of §2.2. The general result is just another example of the generic functorial equivalence between ${}_{\mathcal{H}}\mathscr{M}$ and ${}_{\mathcal{H}_F}\mathscr{M}$ discussed in §1.2. We will denote $A_\theta := A_{F_\theta}$, etc. Given a simple tensor $a \otimes \omega \in A_\theta \otimes \Omega^1_{A_\theta}$ with $a \in A_\theta$ and ω the class of $w \otimes 1 - 1 \otimes w$, $w \in A_\theta$, we will compare the quantity $(\lambda_\theta - \rho_\theta \circ \Psi_{A_\theta, \Omega^1_{A_\theta}})(a \otimes \omega)$ with $(a \otimes 1 - 1 \otimes a) \star_\theta (w \otimes 1 - 1 \otimes w) \in I^2_{A_\theta}$.

On the one hand, one computes

$$\begin{aligned}(a \otimes 1 - 1 \otimes a) \star_\theta (w \otimes 1 - 1 \otimes w) &= a \star_\theta w \otimes 1 - a \otimes w + 1 \otimes a \star_\theta w \\ &\quad - \sum_{n=0}^{\infty} \frac{\mathrm{i}^{\,n}}{n!}\, \theta^{i_1 j_1} \cdots \theta^{i_n j_n} \big(H_{j_1} \cdots H_{j_n} \triangleright w\big) \otimes \big(H_{i_1} \cdots H_{i_n} \triangleright a\big) \ .\end{aligned}$$

On the other hand, one has

$$\lambda_\theta(a \otimes \omega) = a \star_\theta (w \otimes 1 - 1 \otimes w) = a \star_\theta w \otimes 1 - a \otimes w \ ,$$

while

$$\begin{aligned}\rho_\theta \circ \Psi_{A_\theta, \Omega^1_{A_\theta}}(a \otimes \omega) = \sum_{n=0}^{\infty} \frac{\mathrm{i}^{\,n}}{n!}\, \theta^{i_1 j_1} \cdots \theta^{i_n j_n} \Big(&\big(H_{j_1} \cdots H_{j_n} \triangleright w\big) \otimes \big(H_{i_1} \cdots H_{i_n} \triangleright a\big) \\ &- 1 \otimes \big(H_{j_1} \cdots H_{j_n} \triangleright w\big) \star_\theta \big(H_{i_1} \cdots H_{i_n} \triangleright a\big) \Big) \ .\end{aligned}$$

It remains to show that the second formal power series in this last equation is equal to $1 \otimes a \star_\theta w$. This follows from the equality $a_1 \star_\theta a_2 = \mu(F_\theta \triangleright (a_2 \otimes a_1))$ [11, Lem. 1.16].

Universality follows by the same argument of the undeformed case, i.e. by the formula (4.18) now understood in the twisted setting. ■

We can now apply this construction of Kähler differentials for noncommutative algebras with product induced by a Drinfel'd twist to each affine open set in a toric variety $X[\Sigma]$. Starting from a strongly convex rational polyhedral cone $\sigma \in \Sigma$, we form the noncommutative coordinate algebra $\mathbb{C}_\theta[\sigma]$ as in §3.1 and define the $\mathbb{C}_\theta[\sigma]$-bimodule of Kähler differentials $\Omega^1_\theta[\sigma] = \Omega^1_{\mathbb{C}_\theta[\sigma]}$ as above. To show that this construction defines a sheaf of noncommutative differential forms on a generic noncommutative toric variety X_θ, as we did for the structure sheaf $\mathcal{O}_{X_\theta}$ in Proposition 4.2, we have to show that these local definitions glue together in such a way that they satisfy the sheaf axioms.

Proposition 4.22. *The noncommutative differential forms $\sigma \mapsto \Omega^1_\theta[\sigma]$ define a coherent sheaf of $\mathcal{O}_{X_\theta}$-bimodule algebras $\Omega^1_{X_\theta}$ on $\mathsf{Open}(X_\theta)$.*

Proof: We will show that for each affine covering $(\sigma_i \hookrightarrow \sigma)_{i \in I}$ there is an exact sequence

$$0 \longrightarrow \Omega^1_\theta[\sigma] \longrightarrow \prod_{i \in I} \Omega^1_\theta[\sigma_i] \longrightarrow \prod_{i,j \in I} \Omega^1_\theta[\sigma_i \cap \sigma_j] \ . \tag{4.23}$$

Exactness of (4.23) is proved by using the exactness of the corresponding sequence (4.3) of coordinate algebras. For brevity, we use the shorthand notation

$$A_i = \mathbb{C}_\theta[\sigma_i] \ , \qquad A = \mathbb{C}_\theta[\sigma] = \bigcap_{i\in I} A_i \ , \qquad A_{ij} = \mathbb{C}_\theta[\sigma_i \cap \sigma_j] \ ,$$

and let μ_A denote the product map of A. Let $I_A = \ker(\mu_A)$ with canonical inclusion denoted by $\imath_A : I_A \to A \otimes A$.

Consider the commutative diagram of sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "p"] & \prod_{i\in I} A_i \arrow[r, "q"] & \prod_{i,j\in I} A_{ij} \\
0 \arrow[r] & A\otimes A \arrow[r, "p_1"] \arrow[u, "\mu_A"] & \prod_{i\in I} A_i\otimes A_i \arrow[r, "q_1"] \arrow[u, "\mu_{A_i}"] & \prod_{i,j\in I} A_{ij}\otimes A_{ij} \arrow[u, "\mu_{A_{ij}}"] \\
0 \arrow[r] & I_A \arrow[r, "p_2"] \arrow[u, "\imath_A"] & \prod_{i\in I} I_{A_i} \arrow[r, "q_2"] \arrow[u, "\imath_{A_i}"] & \prod_{i,j\in I} I_{A_{ij}} \arrow[u, "\imath_{A_{ij}}"] \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

where $p_1 = p\otimes p$, $p_2 = p_1|_{I_A}$ and similarly for q_1, q_2. All columns are exact. The exactness of the middle row thus follows from the exactness of the top row. Then the exactness of the bottom row is proven with standard homological algebra. The map p_2 is injective due to the injectivity of the maps $\imath_A$, p_1 and $\imath_{A_i}$, because if there exists $0 \neq \omega \in I_A$ such that $p_2(\omega) = 0$ then $p_1(\imath_A(\omega)) \neq 0$ but $p_1(\imath_A(\omega)) = \imath_{A_i}(p_2(\omega)) = 0$. The composition $q_2 \circ p_2$ is zero, since if there exists $\omega \in I_A$ such that $q_2(p_2(\omega)) \neq 0$ then further composing with $\imath_{A_{ij}}$ gives a non-zero element in $\prod_{i,j\in I} A_{ij} \otimes A_{ij}$, while $q_1(p_1(\imath_A(\omega))) = 0$. Finally, we show that $\mathrm{im}(p_2) = \ker(q_2)$. Let $\beta \in \ker(q_2)$ and consider its lift $b = \imath_{A_i}(\beta)$. One has $q_1(b) = 0$ since $\imath_{A_{ij}}(q_2(\beta)) = 0$, so there exists $b' \in A \otimes A$ such that $p_1(b'\,) = b$. But $p(\mu_A(b'\,)) = 0$ since $\mu_{A_i}(b) = 0$, so there exists $\beta' \in I_A$ such that $\imath_A(\beta'\,) = b'$ and $p_2(\beta'\,) = \beta$. This completes the proof for universal differential forms (the third row).

For braided-symmetric differential forms, we further consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & I_A^2 \arrow[r, "\bar{p}_2"] \arrow[d, "\jmath_A"] & \prod_{i\in I} I_{A_i}^2 \arrow[r, "\bar{q}_2"] \arrow[d, "\jmath_{A_i}"] & \prod_{i,j\in I} I_{A_{ij}}^2 \arrow[d, "\jmath_{A_{ij}}"] \\
0 \arrow[r] & I_A \arrow[r, "p_2"] \arrow[d, "\pi_A"] & \prod_{i\in I} I_{A_i} \arrow[r, "q_2"] \arrow[d, "\pi_{A_i}"] & \prod_{i,j\in I} I_{A_{ij}} \arrow[d, "\pi_{A_{ij}}"] \\
0 \arrow[r] & \Omega_A^1 \arrow[r, "\widetilde{p}_2"] \arrow[d] & \prod_{i\in I} \Omega_{A_i}^1 \arrow[r, "\widetilde{q}_2"] \arrow[d] & \prod_{i,j\in I} \Omega_{A_{ij}}^1 \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

where $\jmath_A$ is the inclusion $I_A^2 \hookrightarrow I_A$ and π_A is the projection $I_A \to I_A/I_A^2$, while we set $\bar{p}_2 = p_2|_{I_A^2}$, $\widetilde{p}_2 = p_2|_{I_A/I_A^2}$ and similarly for $\bar{q}_2$, $\widetilde{q}_2$. Again all columns are exact, and the exactness of the bottom row follows from the exactness of the top and middle rows, as one can check directly by using the same homological algebra we employed above. It follows that the noncommutative differential forms define a sheaf $\Omega^1_{X_\theta}$ on $\mathsf{Open}(X_\theta)$.

The fact that this sheaf is coherent follows from the construction of $\Omega^1_{X_\theta}$. Since the construction of Kähler differentials commutes with the localization functors $j_\sigma^\bullet$ of §4.1 (see e.g. [14, §3] and [31, Thm. 1.2.1]), for each affine open set $U_\theta[\sigma]$ there is an isomorphism of sheaves $j_\sigma^\bullet\big(\Omega^1_{X_\theta}\big) \cong \Omega^1_\theta[\sigma]$ over $U_\theta[\sigma]$. For any finitely generated algebra A the A-bimodule of Kähler differentials Ω^1_A is a finitely generated module over A, since if $a_1, \dots, a_n$ are the generators of A then Ω^1_A is generated by $\mathrm{d}a_1, \dots, \mathrm{d}a_n$ as an A-bimodule. ■

5. Noncommutative projective varieties

In this section we will specialize to the noncommutative projective spaces $X_\theta = \mathbb{CP}^n_\theta$. The example $n = 2$ was treated in detail in §3.3. These classes of examples admit a more "global" description of their noncommutative toric geometry which reduces after Ore localization to the local description of $\mathbb{CP}^n_\theta$ provided by the noncommutative affine open sets $U_\theta[\sigma]$. Moreover, they may be used to define noncommutative deformations of projective varieties via restriction from $\mathbb{CP}^n_\theta$. In the remainder of this paper we will omit the star product symbols $\star_\theta$ for brevity.

5.1. **Noncommutative projective spaces $\mathbb{CP}^n_\theta$.**

The construction in §3.3 for $\mathbb{CP}^2$ generalizes straightforwardly to the higher-dimensional projective spaces $\mathbb{CP}^n$, $n > 2$, regarded as a toric variety $X[\Sigma]$ generated by a fan Σ of the lattice $L \cong \mathbb{Z}^n$ of characters of the torus $T = L \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong (\mathbb{C}^\times)^n$. Choose a basis $e_1, \dots, e_n$ of L. Set $v_i = e_i$ for $i = 1, \dots, n$ and $v_{n+1} = -e_1 - \cdots - e_n$, which generate the one-dimensional cones $\tau_i = \mathbb{R}^+ v_i$ of Σ. The $n+1$ maximal cones of Σ are labelled by the missing generator and are given by

$$\sigma_i = \mathbb{R}^+ v_{i+1} \oplus \cdots \oplus \mathbb{R}^+ v_{i+n} \ , \qquad i = 1, \dots, n+1 \ ,$$

with indices understood mod $n+1$ and $\sigma_i \cap \sigma_{i+k} = \mathbb{R}^+ v_{i+k+1} \oplus \cdots \oplus \mathbb{R}^+ v_{i+n}$ a maximal cone of $\mathbb{CP}^{n-k} \hookrightarrow \mathbb{CP}^n$. There are of course many other overlaps, and hence cones, in this instance.

Again there are no relations and $\mathbb{C}[\sigma] = \mathbb{C}[x_1, \dots, x_n]$ for each maximal cone.

(1) The generators of the semigroup $\sigma_{n+1}^\vee \cap L^*$ are $m_i = e_i^*$ for $i = 1, \dots, n$. The subalgebra $\mathbb{C}_\theta[\sigma_{n+1}] \subset \mathbb{C}_\theta(t_1, \dots, t_n)$ is generated by the elements $x_i = t^{m_i} = t_i$ subject to the relations

$$x_i\, x_j = q_{ij}^2\ x_j\, x_i \ , \qquad i < j \ , \tag{5.1}$$

and hence $U_\theta[\sigma_{n+1}] \cong \mathbb{C}_\theta^n$.

(2) For $1 \le k \le n$, the semigroup $\sigma_k^\vee \cap L^*$ is generated by $m_i = e_i^* - e_k^*$ for $i \ne k$ and $m_k = -e_k^*$. The subalgebra $\mathbb{C}_\theta[\sigma_k]$ in this case is generated by elements $x_i = t_i\, t_k^{-1}$, $i \ne k$ and $x_k = t_k^{-1}$ with relations

$$\begin{aligned} x_i\, x_k &= q_{ki}^2\ x_k\, x_i \ , \qquad i \ne k \ , \\ x_i\, x_j &= q_{ij}^2\, q_{ik}^2\, q_{jk}^2\ x_j\, x_i \ , \qquad k \ne i < j \ . \end{aligned} \tag{5.2}$$

The faces can be treated analogously to the $n = 2$ case.

5.2. **Homogeneous coordinate algebras.** We will now show that there is a noncommutative homogeneous coordinate algebra for the noncommutative projective spaces $\mathbb{CP}_\theta^n$, with a local description given by noncommutative Ore localization which is equivalent to that of the noncommutative affine open sets $U_\theta[\sigma]$. For this, we consider an embedding $(\mathbb{C}_\theta^\times)^n \hookrightarrow (\mathbb{C}_{\tilde\theta}^\times)^{\tilde n}$, $\tilde n > n$, where the $\tilde n \times \tilde n$ skew-symmetric complex matrix $\tilde\theta$ depends on both the $n \times n$ skew-symmetric complex matrix θ and the embedding as follows. Given isomorphisms $(\mathbb{C}^\times)^n \cong L \otimes_{\mathbb{Z}} \mathbb{C}^\times$ and $(\mathbb{C}^\times)^{\tilde n} \cong \tilde L \otimes_{\mathbb{Z}} \mathbb{C}^\times$, an embedding of an n-torus into an $\tilde n$-torus is given by an injective group homomorphism $\psi : L \to \tilde L$, which upon fixing bases for the lattices $L \cong \mathbb{Z}^n$ and $\tilde L \cong \mathbb{Z}^{\tilde n}$ is represented by an $\tilde n \times n$ matrix $\psi = (\psi_{ij})$ with entries in $\mathbb{Z}$ and rank n. The k-th column of ψ gives the components of the image of the k-th basis element $e_k \in L$ with respect to the basis $\tilde e_1, \dots, \tilde e_{\tilde n}$ of the lattice $\tilde L$. Given $t = \sum_i e_i \otimes t_i$ in $(\mathbb{C}^\times)^n$ and $w = \sum_j \tilde e_j \otimes w_j$ in $(\mathbb{C}^\times)^{\tilde n}$, the generic embedding is thus a map $t_i \mapsto \prod_j (w_j)^{\psi_{ji}}$. We define $\tilde\theta = \psi\, \theta\, \psi^\top$, which is a skew-symmetric matrix of rank n that describes a noncommutative deformation $(\mathbb{C}_{\tilde\theta}^\times)^{\tilde n}$ induced by the embedded noncommutative algebraic torus $(\mathbb{C}_\theta^\times)^n$.

For explicit computations it is convenient to choose specific bases for the lattices L and $\tilde L$ in which the embedding ψ is induced by the diagonal embedding of $(\mathbb{C}^\times)^n$ in $\mathrm{GL}(\tilde n)$. This corresponds to block forms of the matrices ψ and $\tilde\theta \in \mathrm{Mat}(\tilde n, \mathbb{C})$ given by

$$\psi = \begin{pmatrix} \mathbb{1}_{n\times n} \\ 0 \end{pmatrix} \ , \qquad \tilde\theta = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \ .$$

For the problem at hand, we take $\tilde n = n+1$. Then the corresponding algebraic Moyal plane $\mathbb{C}_{\tilde\theta}^{n+1}$ is defined by the graded polynomial algebra $\mathbb{C}_{\tilde\theta}[w_1, \dots, w_{n+1}]$ in $n+1$ generators w_i, $i = 1, \dots, n+1$ of degree 1 with the quadratic relations

$$\begin{aligned} w_{n+1}\, w_i &= w_i\, w_{n+1} \ , \qquad i = 1, \dots, n \ , \\ w_i\, w_j &= q_{ij}^2\ w_j\, w_i \ , \qquad i, j = 1, \dots, n \ . \end{aligned} \tag{5.3}$$

This algebra is called the homogeneous coordinate algebra $\mathcal{A} = \mathcal{A}(\mathbb{CP}^n_\theta)$ of the noncommutative toric variety $\mathbb{CP}^n_\theta$. It is a special instance of the noncommutative weighted projective spaces defined in [5, §2.2]. For $n = 2$, it is the same as the noncommutative variety $\mathbb{P}^2_{q,\hbar=0}$ defined in [25, §9], which is an Artin–Schelter regular algebra of global homological dimension three [1] in the classification of noncommutative deformations of the projective plane. The grading on $\mathcal{A}$ is by the usual polynomial degree and one has

$$\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}_k \ ,$$

with $\mathcal{A}_0 = \mathbb{C}$ and $\mathcal{A}_k = \bigoplus_{i_1+\cdots+i_{n+1}=k} \mathbb{C} w_1^{i_1} \cdots w_{n+1}^{i_{n+1}}$ for $k > 0$. There is a natural action of the torus $T = (\mathbb{C}^\times)^n$ defined via the embedding above by $H_i \triangleright w_j = \delta_{ij}\, w_i$ for $i, j = 1, \dots, n$ and $H_i \triangleright w_{n+1} = 0$. The toric automorphisms can be viewed as $\mathbb{C}$-algebra automorphisms $\mathcal{A} \to \mathcal{A} \otimes \mathcal{F}^{\tilde{\theta}}_{n+1}$ when $\mathcal{A}$ is made into a right comodule algebra over the Hopf algebra $\mathcal{F}^{\tilde{\theta}}_{n+1}$. Indeed, on $\mathcal{A}$ there is a natural action of $\mathrm{GL}(n+1)$ and the $(\mathbb{C}^\times)^n$ torus action can be recovered as the restriction of this $\mathrm{GL}(n+1)$-action with respect to the embedding of $(\mathbb{C}^\times)^n$ in $(\mathbb{C}^\times)^{n+1}$ described above.

It is straightforward to verify that each monomial w_i generates a left (and right) denominator set in $\mathcal{A}$. Let $\mathcal{A}[w_i^{-1}]$ be the left Ore localization of $\mathcal{A}$ with respect to w_i. Since w_i is homogeneous of degree 1, the algebra $\mathcal{A}[w_i^{-1}]$ is also $\mathbb{N}_0$-graded. Elements of degree 0 in $\mathcal{A}[w_i^{-1}]$ form a subalgebra which we denote by $\mathcal{A}[w_i^{-1}]_0$.

Theorem 5.4. *For each maximal cone $\sigma_i \in \Sigma$, $i = 1, \dots, n+1$, there is a natural T-equivariant isomorphism of noncommutative algebras $\mathbb{C}_\theta[\sigma_i] \cong \mathcal{A}\big[w_i^{-1}\big]_0$.*

Proof: From §5.1 it follows that the noncommutative affine variety $U_\theta[\sigma_i] = \mathbb{C}^n_{\check{\theta}}$ is dual to the noncommutative polynomial algebra $\mathbb{C}_\theta[\sigma_i] = \mathbb{C}_{\check{\theta}}[x_1, \dots, x_n]$, with the relations (5.1) and (5.2). On the other hand, the degree 0 subalgebra of $\mathcal{A}[w_i^{-1}]$ is generated by the elements $y_k = w_i^{-1}\, w_k$ for $k = 1, \dots, n+1$, $k \neq i$. An elementary calculation using the relations (5.3) in $\mathcal{A}$ and the multiplication rule (1.26) in the localized algebra then shows that the desired isomorphism is defined by sending $x_k \mapsto y_k$ for $k \neq i$ and $x_i \mapsto y_{n+1}$. The action of T on $\mathcal{A}$ naturally extends to an action by $\mathbb{C}$-algebra automorphisms on the localizations $\mathcal{A}[w_i^{-1}]$, which is consistent with the action defined in Proposition 3.6 in this case. ■

Theorem 5.4 identifies $U_\theta[\sigma_i]$ with the open subset $\{w_i \neq 0\}$ in $\mathbb{C}^{n+1}_{\tilde{\theta}}$. It includes the self-dual cone $\sigma_{n+1} = \sigma^\vee_{n+1}$, for which $\check{\theta} = \theta$ and the localization is made with respect to the central element w_{n+1} of $\mathcal{A}$. If $I \subset \mathcal{A}$ is a graded two-sided ideal generated by a set of homogeneous polynomials $f_1, \dots, f_m \in \mathbb{C}_{\tilde{\theta}}[w_1, \dots, w_{n+1}]$, then the quotient algebra $\mathcal{A}_I := \mathcal{A}/I$ is identified as the coordinate algebra of a *noncommutative projective variety*. The projection $\pi_I : \mathcal{A} \to \mathcal{A}_I$ can be regarded as the dual of a closed embedding given by $X_\theta(I) \hookrightarrow \mathbb{CP}^n_\theta$, identified with the common zero locus in $\mathbb{C}^{n+1}_{\tilde{\theta}}$ given by the set of relations $\{f_1 = 0, \dots, f_m = 0\}$. Its homogeneous coordinate algebra $\pi_I(\mathbb{C}_{\tilde{\theta}}[w_1, \dots, w_{n+1}])$ has relations (5.3) and $f_1 = 0, \dots, f_m = 0$. It is also graded, $\mathcal{A}_I = \bigoplus_{k\geq 0} (\mathcal{A}_I)_k$, with $(\mathcal{A}_I)_0 = \mathbb{C}$ and $\dim_{\mathbb{C}}(\mathcal{A}_I)_k < \infty$ for all $k \geq 0$. The torus action on $\mathcal{A}$ naturally restricts to $\mathcal{A}_I$.

In the remainder of this section we will look at some explicit examples, which among other things will illustrate that in general certain additional algebraic constraints must be imposed on the noncommutative ambient space $\mathbb{CP}^n_\theta$.

5.3. **Noncommutative grassmannians** $\mathbb{G}\mathrm{r}_\theta(d;n)$**.** Using our noncommutative deformation of the general linear group $\mathrm{GL}(n)$ from §2.3, we will now construct a noncommutative deformation of the Grassmann variety $\mathbb{G}\mathrm{r}(d;n) \cong \mathbb{G}\mathrm{r}(d;V)$, $d \le n$ of d-dimensional subspaces of an n-dimensional complex vector space V. For this, we will derive a suitable noncommutative version of Plücker equations in $\mathcal{A}(\mathbb{CP}^N_\Theta)$ for $N = \binom{n}{d} - 1$, yielding a noncommutative projective variety $\mathbb{G}\mathrm{r}_\theta(d;n)$ whose homogeneous coordinate algebra is a graded quadratic algebra with (2.22) as the space of generators. The Drinfel'd twist via the $n \times n$ skew-symmetric complex matrix θ induces constraints on the form of the $N \times N$ matrix Θ which realizes the noncommutativity relations in the projective space in which we embed the grassmannian. We will find these constraints, whence showing that in general it is not possible to go in the opposite direction, i.e. there are noncommutative projective spaces $\mathbb{CP}^N_\Theta$ which do not admit any such embedding due to the form of their deformation matrix Θ.

There is a rich literature on quantum or noncommutative deformations of grassmannians (see e.g. [28, 36, 21, 19, 25]), mostly relying on q-deformations of matrices, so our noncommutative relations are somewhat different and easier to deal with. This is because in our construction the minors of a noncommutative matrix still close to a noncommutative algebra and in §2.4 we have explicitly computed their noncommutativity relations; these will be the noncommutativity relations of the homogeneous coordinate algebra generators of the noncommutative projective space $\mathbb{CP}^N_\Theta$. Here we shall follow [28] to define the noncommutative deformation of Plücker equations, or Young symmetry relations, which is an approach to noncommutative grassmannians based on quasideterminants [21].

Classically, the Plücker embedding $\mathrm{Pl} : \mathbb{G}\mathrm{r}(d;n) \cong \mathbb{G}\mathrm{r}(d;V) \to \mathbb{P}(\bigwedge^d V) \cong \mathbb{CP}^N$, with $\dim_{\mathbb{C}}(V) = n$ and $N = \binom{n}{d} - 1$, is defined as follows: a $d \times n$ matrix Λ of maximal rank, representing an element in $\mathbb{G}\mathrm{r}(d,n)$ by associating to Λ the subspace of V spanned by the rows of Λ, is mapped into the $\binom{n}{d}$-tuple $(\dots, \Lambda^J, \dots)$ where each component is a $d \times d$ minor of Λ. In the notation of §2.4, the row multi-index is always $I = (1\,2 \cdots d)$ so we label minors by the column multi-index J alone. Plücker equations in $\mathbb{CP}^N$ express the condition on points of the projective space to belong to the image of this embedding. Each Plücker coordinate can be viewed as a section of a certain ample line bundle over $\mathbb{G}\mathrm{r}(d;n)$, and the collection of such sections defines an embedding of $\mathbb{G}\mathrm{r}(d;n)$ into $\mathbb{CP}^N$.

Let us fix some notation. For $1 \le r \le d$, denote with $I = (i_1 \cdots i_{d+r})$ a $(d+r)$ multi-index, with J a $(d-r)$ multi-index, and with $\Xi = (i_{\xi_1} \cdots i_{\xi_r})$ an r multi-index. Then by $I \setminus \Xi$ we mean the multi-index $(i_1 \cdots \hat{i}_{\xi_1} \cdots \hat{i}_{\xi_r} \cdots i_{d+r})$ with the hats indicating omitted indices, and by $A \cup B$ the multi-index $(a_1 \cdots a_k\, b_1 \cdots b_s)$ when $|A| = k$ and $|B| = s$. Finally, we will use the short-hand notation $\epsilon^A = \epsilon^{a_1 \cdots a_k}$. One way to express the Plücker equations is through the following result [28].

Proposition 5.5. *A point $x \in \mathbb{CP}^N \cong \mathbb{P}(\bigwedge^d V)$ belongs to the image of the Plücker map $\mathrm{Pl}(\mathbb{G}\mathrm{r}(d;V))$ if and only if for all $1 \le r \le d$, and for all choices of multi-indices I and J, the homogeneous coordinates of x, expressed as $d \times d$ minors Λ^K of $d \times n$ matrices, satisfy*

$$\sum_{\Xi \subset I\,:\,|\Xi| = r} \epsilon^{(I \setminus \Xi) \cup \Xi}\, \Lambda^{I \setminus \Xi}\, \Lambda^{\Xi \cup J} = 0 \ . \tag{5.6}$$

Note that each equation (5.6) is quadratic in the homogeneous coordinates of the projective space and has as many terms as the number of submulti-indices of I with cardinality r. The total number of equations is quite large as there is one for each choice of the integer r, and of the multi-indices I and J. One shows [28, Prop. 13] that all relations with $r \geq 1$ are generated from those at $r = 1$.

Let us now turn to the noncommutative setting. In §2.4 we have defined minors for matrices in the homogeneous coordinate algebra of $\mathrm{GL}_\theta(n) \cong \mathrm{GL}_\theta(V)$, where V is an $\mathcal{H}_\theta^n$-module of dimension n. An element of the homogeneous coordinate algebra of the noncommutative grassmannian $\mathbb{G}\mathrm{r}_\theta(d;n) \cong \mathbb{G}\mathrm{r}_\theta(d;V)$ is defined as an element in $\mathbb{P}(\bigwedge_\theta^d V)$, obtained by taking the θ-deformed exterior product of d rows of a matrix in $\mathcal{A}(\mathrm{GL}_\theta(V))$ (and quotienting by the appropriate equivalence relation). The Plücker maps still make sense. We take a noncommutative $d \times n$ matrix representing an element of $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))$ and send it into the $\binom{n}{d}$-tuple of its minors. Then we need to find the noncommutativity relations between the minors, seen now as homogeneous coordinates in $\mathcal{A}(\mathbb{CP}_\Theta^N)$ with $N = \binom{n}{d} - 1$, as well as noncommutative Plücker relations between them.

From (2.26) with $|J| = |J'| = d$ representing two different minors we have

$$\Lambda^J \, \Lambda^{J'} = \Big(\prod_{\alpha,\beta=1}^{d} q^2_{j_\alpha j'_\beta} \Big) \, \Lambda^{J'} \, \Lambda^J \ . \tag{5.7}$$

This implies that the $N \times N$ noncommutativity matrix Θ of the projective space containing the embedding of $\mathbb{G}\mathrm{r}_\theta(d;n)$ is completely determined (mod 2π) from the $n \times n$ noncommutativity matrix θ of the grassmannian as

$$\Theta^{JJ'} = \sum_{\alpha,\beta=1}^{d} \theta^{j_\alpha j'_\beta} \ . \tag{5.8}$$

These relations mean that while given θ there is always one and only one noncommutative projective space $\mathbb{CP}_\Theta^N$ in which the grassmannian $\mathbb{G}\mathrm{r}_\theta(d;n)$ embeds, the converse is in general not true. One can always find a noncommutative projective space for which there is no compatible noncommutativity matrix θ parametrizing a grassmannian $\mathbb{G}\mathrm{r}_\theta(d;n)$ which would embed into it. The necessary and sufficient conditions for such an embedding to exist are given by (5.8). Note that if we instead chose to use ordered column multi-indices, we would again obtain noncommutative relations among the minors which agree with those in $\mathbb{CP}_\Theta^N$, now with a minus sign on the right-hand side of (5.8).

Given the noncommutative relations between generators of the projective space, the next step is to exhibit noncommutative Plücker relations. They generate an ideal in the homogeneous coordinate algebra $\mathcal{A}(\mathbb{CP}_\Theta^N)$ of the projective space, and we will *define* the noncommutative quotient algebra to be the homogeneous coordinate algebra $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))$ of the (embedding of the) noncommutative grassmannian. The natural noncommutative version of (5.6) is obtained by taking into account the braided antisymmetry of the minors, so that using the same notation as before the Plücker relations now read

$$\sum_{\Xi \subset I \,:\, |\Xi| = r} \epsilon_\theta^{(I \setminus \Xi) \cup \Xi \, (c)} \, \Lambda^{I \setminus \Xi} \, \Lambda^{\Xi \cup J} = 0 \ . \tag{5.9}$$

By these definitions, one has $\mathbb{G}\mathrm{r}_\theta(1;n) = (\mathbb{CP}_\theta^{n-1})^*$. Since $\dim_{\mathbb{C}}(\mathbb{G}\mathrm{r}(d;n)) = d\,(n-d)$, the $n \times n$ matrix θ, which deforms the maximal torus of $\mathrm{GL}(n)$, should be expressed in terms of the $(\mathbb{C}^\times)^{d\,(n-d)}$-action on the grassmannian through a suitable embedding, analogous to those described in §5.2. We will return to this point in §6.4.

Remark 5.10. *We have not found a complete and general proof that the noncommutative Plücker equations (5.9) can be reduced to the case $r = 1$, as in the undeformed situation, though it is true for every example we have worked out. For q-deformations, this is implied by* [28, Prop. 13].

The classical Plücker relations (5.6) contain trivial identities when $I \cap J \neq \emptyset$, together with "true" Plücker equations. The same situation arises in the noncommutative case, but now the "trivial" identities encode the noncommutativity and alternating relations of the noncommutative minors. In fact, in certain instances it seems that starting from (5.9), one can derive all relations necessary to describe the noncommutative Grassmann variety, i.e. the "true" Plücker equations as well as the noncommutativity relations between the generators of $\mathcal{A}(\mathbb{CP}^N_\Theta)$ in (5.7) and the alternating property (2.27). Again we will return to this point in more generality below.

5.4. **Noncommutative flag varieties** $\mathbb{F}\mathrm{l}_\theta(d_1, \dots, d_r; n)$**.** We will now generalize the constructions of §5.3 to flag varieties. Classically, given an n-dimensional complex vector space V and a sequence of positive integers $\gamma = (\gamma_1, \dots, \gamma_{r+1})$ with $1 \leq r \leq n-1$ which is a partition of n, i.e. a Young diagram, we consider an increasing chain of nested vector subspaces of V,

$$0 = V_0 \ \subsetneq\ V_1 \ \subsetneq\ V_2 \ \subsetneq\ \cdots \ \subsetneq\ V_{r+1} = V \ ,$$

such that $\gamma_i = \dim_{\mathbb{C}}(V_i) - \dim_{\mathbb{C}}(V_{i-1})$ for $i = 1, \dots, r+1$. The corresponding flag variety $\mathbb{F}\mathrm{l}(\gamma; V) \cong \mathbb{F}\mathrm{l}(\gamma; n)$ is the moduli space of chains (or "flags") associated to the sequence $\gamma = (\gamma_1, \dots, \gamma_{r+1})$. Two typical examples are the complete flag varieties with partition $\gamma = (1, \dots, 1)$ (n times), i.e. the sequences of subspaces where $\dim_{\mathbb{C}}(V_i) = i$ for $i = 1, \dots, n$, and the grassmannians $\mathbb{G}\mathrm{r}(d; n)$ which are here recovered from the two-term partitions $\gamma = (d, n-d)$.

By choosing a basis in V, the flag varieties $\mathbb{F}\mathrm{l}(\gamma; V)$ can also be represented as spaces of equivalence classes of matrices in the reductive algebraic group $\mathrm{GL}(n)$. We represent a chain of subspaces by a matrix whose rows are the basis vectors of each subspace, and notice that the part of $\mathrm{GL}(n)$ which acts trivially on such a representation is given by block upper (or lower) triangular matrices, with $r+1$ diagonal blocks of dimensions $\gamma_1, \dots, \gamma_{r+1}$. These matrices form a subgroup of $\mathrm{GL}(n)$ denoted P_γ. It is a parabolic group, and the flag variety may be realized as the homogeneous space $\mathbb{F}\mathrm{l}(\gamma; n) = \mathrm{GL}(n)/P_\gamma$ with associated principal bundle

$$P_\gamma \ \hookrightarrow\ \mathrm{GL}(n) \ \longrightarrow\ \mathbb{F}\mathrm{l}(\gamma; n) \ . \tag{5.11}$$

The Borel subgroup of $\mathrm{GL}(n)$ is the parabolic group P_γ associated with $\gamma = (1, \dots, 1)$ representing the complete flag, i.e. the group of upper (or lower) triangular matrices, and we will denote it by B_n. Since B_n is the minimal parabolic subgroup of $\mathrm{GL}(n)$, each flag variety $\mathbb{F}\mathrm{l}(\gamma; n)$ is the total space of a canonical fibration over the corresponding complete flag variety with fibre P_γ/B_n given by

$$P_\gamma \,/\, B_n \ \hookrightarrow\ \mathrm{GL}(n) \,/\, P_\gamma \ \xrightarrow{\ \pi\ }\ \mathrm{GL}(n) \,/\, B_n \ .$$

We shall describe the Plücker embedding of flag varieties into projective spaces, in a similar way as in the case of grassmannians. This involves the minors of the $n \times n$ matrix representing each flag. Set $d_i = \sum_{a \leq i} \gamma_a = \dim_{\mathbb{C}}(V_i)$ for $i = 1, \dots, r+1$. Given a point in

$\mathbb{F}\mathrm{l}(\gamma;n)$ represented by an equivalence class $[A]$ in $\mathrm{GL}(n)/P_\gamma$, there is a natural Plücker map $\mathrm{Pl}_i : \mathbb{F}\mathrm{l}(\gamma;n) \to \mathbb{CP}^{N_i}$, with $N_i = \binom{n}{d_i} - 1$ for each i, where the image is the $\binom{n}{d_i}$-tuple of all minors of A obtained from the first d_i rows. Hence each minor is labelled by a multi-index representing the d_i columns involved while the rows are always given by the standard ordered multi-index $(1\,2\cdots d_i)$. Assembling all of these maps together we get a Plücker embedding

$$\mathrm{Pl} \,:\, \mathbb{F}\mathrm{l}(\gamma;n) \ \longrightarrow\ \mathbb{CP}(\gamma;n) := \mathbb{CP}^{N_1} \times \cdots \times \mathbb{CP}^{N_r} \ , \tag{5.12}$$

where the last factor corresponding to $i = r+1$ gives a trivial contribution since $N_{r+1} = \binom{n}{n} - 1 = 0$. The image of the Plücker map Pl in $\mathbb{CP}(\gamma;n)$ is described by a set of quadratic equations called the Young symmetry relations. With the same notation, a generalization of Proposition 5.5 to flag varieties is given by the following result [29].

Proposition 5.13. *Given a partition γ of n and the Plücker map* Pl *in (5.12), a point x in $\mathbb{CP}(\gamma;n)$ belongs to the image* $\mathrm{Pl}(\mathbb{F}\mathrm{l}(\gamma;n))$ *if and only if for all choices of multi-indices given by $I = (i_1 \cdots i_{d-s})$ and $J = (j_1 \cdots j_{d'+s})$, as subsets of $(1\,2\cdots n)$ for all $s \geq 1$ and for all $d, d' \in \{d_i\}_{i=1,\dots,r+1}$ with $d \leq d'$, the homogeneous coordinates of x, expressed as $d_i \times d_i$ minors of $n \times n$ matrices now of variable size, satisfy the Young symmetry relations*

$$\sum_{\Xi\subseteq J\,:\,|\Xi|=s} \epsilon^{\Xi\cup(J\setminus\Xi)}\, \Lambda^{I\cup\Xi}\, \Lambda^{J\setminus\Xi} = 0 \ . \tag{5.14}$$

We are now ready to construct a noncommutative deformation of flag varieties, generalizing what we did in §5.3 for noncommutative grassmannians. The definition of minors of matrices with noncommuting entries is the same as in (2.23). We now need to handle noncommutative minors of different size, with each size describing a projective space in the cartesian product $\mathbb{CP}(\gamma;n)$, and apply a noncommutative version of the Young symmetry relations (5.14) instead of (5.6). The relations (5.6) essentially describe the relations among minors of fixed size, so they describe the image of the Plücker embedding in each projective space copy (with appropriate dimension) inside $\mathbb{CP}(\gamma;n)$. What (5.14) adds is to express relations between minors of different size, i.e. relations between coordinates of different factors in $\mathbb{CP}(\gamma;n)$.

In this case we use the more general noncommutative relations (2.26) between $d\times d$ and $d'\times d'$ minors of different size, i.e. with multi-indices of different lengths $|I| = |J| = d$ and $|I'\,| = |J'\,| = d'$. The noncommutative Young symmetry relations are expressed by taking into account the braided antisymmetry of the indices representing columns in the minors, i.e. by substituting the Levi–Civita symbol ϵ in (5.14) with the braided symbol $\epsilon_\theta^{(c)}$ of (2.16). Recall that additional symbols ϵ_θ are hidden inside the noncommutative minors entering the equations. In this setting the coordinate algebra of the noncommutative flag variety $\mathbb{F}\mathrm{l}_\theta(\gamma;n) = \mathbb{F}\mathrm{l}_\theta(d_1,\dots,d_r;n)$ is the quotient of the homogeneous coordinate algebra of $\mathbb{CP}_\Theta(\gamma;n)$ by the ideal generated by the noncommutative Young symmetry relations. As we did for noncommutative grassmannians, it is useful to distinguish between the different kinds of equations that are generated by the noncommutative Young symmetry relations. We will divide them into three classes, called alternating equations, structure equations, and Plücker equations.

By *alternating equations* we mean relations like (2.27), i.e. the behaviour of a minor under interchange of two indices inside the multi-index which parametrizes it. These equations are in principle contained in the definition of noncommutative minors, and once we have decided to parametrize coordinates in the projective spaces which are targets for our

Plücker map by ordered multi-indices, they are not to be interpreted as relations between coordinates of these projective spaces. However, in Proposition 5.13 it is convenient to consider unordered multi-indices I and J, since even when I and Ξ are ordered the multi-index $I \cup \Xi$ is in general not ordered, so the Young symmetry relations automatically generate equations with unordered multi-indices. This increases the number of equations in the Young symmetry relations, as it increases the number of ways in which one can choose I and J, exactly by adding relations of alternating type. These are the ones in which $I \cup \Xi$ and $J \setminus \Xi$ differ only by permutations. This can only happen when $d = d'$, and the alternating relations are a particular class of equations where only two terms in the sum over Ξ survive. Thus by including unordered multi-indices, alternating relations arise as a subset of the Young symmetry relations.

By *structure equations* we mean the class of equations where only two terms representing distinct noncommuting coordinates in $\mathcal{A}(\mathbb{CP}_\Theta(\gamma; n))$ survive. They specify the noncommutativity of the target space of the Plücker embedding. In §5.3 we showed that not every noncommutative projective space (of the appropriate dimension) can contain a Plücker embedding of a noncommutative grassmannian, since the noncommutativity matrix Θ of $\mathbb{CP}^N_\Theta$ has to satisfy the constraints (5.8). It is natural to now ask if these structure equations could have been completely deduced from the noncommutative Young symmetry relations, or if they have to be put in by hand when defining the noncommutative projective space of the Plücker embedding. Some straightforward combinatorial considerations show that only a small part of the structure equations for $\mathbb{CP}_\Theta(\gamma; n)$ are a subset of the Young symmetry relations, and all other noncommutativity relations must be introduced independently.

Proposition 5.15. *The only structure equations contained in the noncommutative Young symmetry relations are those within a single factor of the algebra $\mathcal{A}(\mathbb{CP}_\Theta(\gamma; n))$ involving minors whose multi-indices differ in only one index.*

Proof: We look at the conditions needed for an equation of the Young symmetry relations (5.14) to reduce to a two-term equation. For generic $s \geq 1$, one has $|I| = d - s$ and $|J| = d' + s$. Each equation of the Young symmetry relations has $\binom{d'+s}{s}$ terms, the number of choices of Ξ contained in J. To reduce this number to 2 and get a structure equation, I and J must contain some common indices so that when I takes indices from J via Ξ we get a repetition of indices in $I \cup \Xi$, and the corresponding term in the equation vanishes. Denote by k the number of shared indices, i.e. $|I \cap J| = k$. The constraints are $k \leq d - s$ and $d \leq d'$. Under these conditions the number of surviving terms in each equation is given by the number of choices of s indices (those of Ξ) among $d' + s - k$ indices of J (those not shared with I). This number is $\binom{d'+s-k}{s}$, and hence the condition we want is $\binom{d'+s-k}{s} = 2$. This implies that we must have $d' + s - k = 2$ and $s = 1$. Now the constraint $k \leq d - s$ becomes $d' - 1 \leq d - 1$, which together with the constraint $d \leq d'$ forces $d = d'$. Thus structure equations only arise for noncommutative minors of equal size $d = d'$ (i.e. inside a single factor of $\mathcal{A}(\mathbb{CP}_\Theta(\gamma; n))$), and it is not possible to recover any of the structure equations between minors of different size (i.e. between coordinates of different noncommutative projective space factors in $\mathbb{CP}_\Theta(\gamma; n)$). For fixed $d = d'$, these constraints also show that $|J| = d' + s = d + 1$, $|I| = d - s = d - 1$ and $k = d' - 1 = d - 1$. So to obtain structure equations, I must be a subset of J (since $|I| = k = d - 1$), and J is obtained by adding two more indices to those of I. Thus all terms in the corresponding equation are of the form $\Lambda^{I \cup (i)}\, \Lambda^{J \setminus (i)}$, i.e. the two minors involved differ only by one index. ■

The remaining relations involving more than two terms are called *Plücker equations*. They are quadratic in the coordinate algebra generators of the noncommutative projective spaces, and are the ones which genuinely describe the image of the Plücker embedding, i.e. the projection given by $\mathcal{A}(\mathbb{CP}_\Theta(\gamma;n)) \to \mathcal{A}(\mathbb{Fl}_\theta(\gamma;n))$ which realizes $\mathbb{Fl}_\theta(\gamma;n)$ as a noncommutative quadric in $\mathbb{CP}_\Theta(\gamma;n)$. By (2.26) and Proposition 5.13, there are canonical inclusions of homogeneous coordinate algebras

$$p_i\,:\,\mathcal{A}\big(\mathbb{Fl}_\theta(d_1,\dots,\hat{d}_i,\dots,d_r;n)\big)\ \longrightarrow\ \mathcal{A}\big(\mathbb{Fl}_\theta(d_1,\dots,d_r;n)\big)$$

of noncommutative flag varieties for each $i=1,\dots,r$. For generic n, this leads to a web of multiple noncommutative fibrations, which are classically obtained by truncating flags in the obvious way. Furthermore, the additional relations coming from (2.26) are naturally compatible with the structure of the *braided* tensor product of algebras $\mathcal{A}(\mathbb{Gr}_\theta(d_1;n))\,\widehat{\otimes}_\theta\,\cdots\,\widehat{\otimes}_\theta\,\mathcal{A}(\mathbb{Gr}_\theta(d_r;n))$ induced by the braiding morphism Ψ_θ on the category ${}_{\mathcal{H}_\theta^n}\mathscr{M}$ of $\mathcal{H}_\theta^n$-module algebras as explained in §1.2. By definition and Proposition 5.13, the algebra $\mathcal{A}(\mathbb{Fl}_\theta(d_1,\dots,d_r;n))$ may be realized as the quotient algebra of this braided tensor product by the additional relations arising from (5.14), and there is a natural algebra surjection

$$\mathcal{A}\big(\mathbb{Gr}_\theta(d_1;n)\big)\,\widehat{\otimes}_\theta\,\cdots\,\widehat{\otimes}_\theta\,\mathcal{A}\big(\mathbb{Gr}_\theta(d_r;n)\big)\ \longrightarrow\ \mathcal{A}\big(\mathbb{Fl}_\theta(d_1,\dots,d_r;n)\big)\ .$$

6. Geometry of noncommutative projective varieties

We will now develop a more thorough noncommutative sheaf theory and, with the alternative description of §5 in hand, apply it in particular to noncommutative deformations of projective varieties. In this way noncommutative projective varieties inherit many algebraic and geometric properties from $\mathbb{CP}^n_\theta$ by restriction. These properties are described below.

6.1. Cohomology of $\mathbb{CP}^n_\theta$.

We start by summarizing the pertinent cohomological properties of the homogeneous coordinate algebras $\mathcal{A}$. We write $\mathsf{mod}(\mathcal{A})$ for the category of all finitely-generated right $\mathcal{A}$-modules.

Proposition 6.1. *The algebra $\mathcal{A}=\mathcal{A}(\mathbb{CP}^n_\theta)$ is a quadratic algebra whose Koszul dual $\mathcal{A}^!$ is generated by elements $\check{w}_i$, $i=1,\dots,n+1$ with the relations*

$$\begin{aligned} \check{w}_i^2 &= 0\ , && i=1,\dots,n+1\ ,\\ \check{w}_i\,\check{w}_{n+1}+\check{w}_{n+1}\,\check{w}_i &= 0\ , && i=1,\dots,n\ ,\\ \check{w}_i\,\check{w}_j+q_{ij}^2\ \check{w}_j\,\check{w}_i &= 0\ , && i,j=1,\dots,n\ . \end{aligned} \tag{6.2}$$

Proof: The graded algebra $\mathcal{A}=\bigoplus_{k\geq 0}\mathcal{A}_k$ can be identified as $\mathcal{A}=T(\mathcal{A}_1)/\langle R\rangle$, where $T(\mathcal{A}_1)$ is the free tensor algebra generated by the $n+1$-dimensional vector space $\mathcal{A}_1$ and $\langle R\rangle$ is the two-sided ideal generated by the $\frac{n}{2}\,(n+1)$-dimensional subspace R of $\mathcal{A}_1\otimes\mathcal{A}_1$ spanned by the quadratic relations (5.3). Thus $\mathcal{A}$ is a quadratic algebra. Its Koszul dual is the algebra $\mathcal{A}^! = T(\mathcal{A}_1^*)/\langle R^\perp\rangle$ defined by taking the basis $\check{w}_i$ of $\mathcal{A}^*$ dual to w_i and quotienting by the annihilator of the relations $R\subset\mathcal{A}_1\otimes\mathcal{A}_1$, i.e. the subspace $R^\perp$ of the tensor product $\mathcal{A}_1^*\otimes\mathcal{A}_1^*=(\mathcal{A}_1\otimes\mathcal{A}_1)^*$ consisting of elements q such that $q(r)=0$ for any $r\in R$. A direct calculation using (5.3) then gives the relations (6.2). ■

The dual algebra $\mathcal{A}^! = \bigoplus_{k\geq 0} \mathcal{A}^!_k$ is a deformation of the exterior algebra of $\mathcal{A}^*$, graded again by polynomial degree. It is a special case of the graded DG-algebras defined in [5, §2.6]. In the category ${}_{\mathcal{H}_{\tilde\theta}}\mathscr{M}$ of $\mathcal{H}_{\tilde\theta}$-modules, there are isomorphisms

$$\mathcal{A}^!_k \ \cong\ \textstyle\bigwedge^k_{\tilde\theta} \mathcal{A}^*_1 \ .$$

There is a canonical identification $(\mathcal{A}^!)^! = \mathcal{A}$. With $R \subset \mathcal{A}_1 \otimes \mathcal{A}_1$ the space of quadratic relations (5.3), the dual vector spaces $(\mathcal{A}^!_k)^*$ of $\mathcal{A}^!_k$ are given by

$$\begin{aligned} \big(\mathcal{A}^!_0\big)^* &= \mathcal{A}_0 \ = \ \mathbb{C} \ , \\ \big(\mathcal{A}^!_1\big)^* &= \mathcal{A}_1 \ , \\ \big(\mathcal{A}^!_k\big)^* &= \bigcap_{l=0}^{k-2} \mathcal{A}_1^{\otimes l} \otimes R \otimes \mathcal{A}_1^{\otimes (k-2-l)} \ , \qquad k\geq 2 \ . \end{aligned}$$

It follows that $(\mathcal{A}^!_k)^* \subset \mathcal{A}_1^{\otimes k}$ for all $k \in \mathbb{N}_0$.

By considering $\mathcal{A}$ as a right $\mathcal{A}$-module, one defines the (right) Koszul complex $\mathcal{K}^\bullet(\mathcal{A})$ as the sequence of homomorphisms of (free) right $\mathcal{A}$-modules given by [34, 9]

$$\cdots \ \xrightarrow{\ \mathrm{d}\ } \ \big(\mathcal{A}^!_{k+1}\big)^* \otimes \mathcal{A} \ \xrightarrow{\ \mathrm{d}\ } \ \big(\mathcal{A}^!_{k}\big)^* \otimes \mathcal{A} \ \xrightarrow{\ \mathrm{d}\ } \ \cdots \ \xrightarrow{\ \mathrm{d}\ } \ \big(\mathcal{A}^!_{1}\big)^* \otimes \mathcal{A} \ \xrightarrow{\ \mathrm{d}\ } \ \mathcal{A} \ \longrightarrow \ 0 \ ,$$

where the differential $\mathrm{d} : \big(\mathcal{A}^!_{k+1}\big)^* \otimes \mathcal{A} \to \big(\mathcal{A}^!_{k}\big)^* \otimes \mathcal{A}$ is induced by the map

$$(a_1 \otimes \cdots \otimes a_{k+1}) \otimes a \ \longmapsto \ (a_1 \otimes \cdots \otimes a_k) \otimes a_{k+1}\, a$$

on $\mathcal{A}_1^{\otimes (k+1)} \otimes \mathcal{A} \to \mathcal{A}_1^{\otimes k} \otimes \mathcal{A}$. Since $(\mathcal{A}^!_k)^* \subset \mathcal{A}_1^{\otimes (k-2)} \otimes R$ for $k \geq 2$, one has $\mathrm{d}^2 = 0$, and hence $\mathcal{K}^\bullet(\mathcal{A})$ is a differential chain complex. It is a special case of the complex defined in [5, eq. (2.8)]. By considering $\mathcal{A}$ as a left $\mathcal{A}$-module and exchanging factors, one also defines a left Koszul complex of (free) left $\mathcal{A}$-modules. Exactness of the right Koszul complex is equivalent to exactness of the left Koszul complex.

One use we will make of the Koszul complex is in establishing some crucial "smoothness" properties of our algebras. The presentation of $\mathcal{A}$ by generators and relations is equivalent [1] to exactness of the sequence of right $\mathcal{A}$-modules given by

$$R \otimes \mathcal{A} \ \longrightarrow \ \mathcal{A}_1 \otimes \mathcal{A} \ \xrightarrow{\ \mu_{\mathcal{A}}\ } \ \mathcal{A} \ \xrightarrow{\ \varepsilon\ } \ \mathcal{A}_0 \ \longrightarrow \ 0 \ ,$$

where $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the product of $\mathcal{A}$, ε is the projection onto $\mathcal{A}_0 = \mathbb{C}$ (which defines a counit on $\mathcal{A}$), and the first arrow is multiplication (in $\mathcal{A}$) by the matrix of relations in $\mathcal{A}_1$. This exact sequence extends as a minimal free resolution of the trivial right $\mathcal{A}$-module $\mathcal{A}_0 = \mathbb{C}$ given by

$$0 \ \longrightarrow \ E_d \otimes \mathcal{A} \ \longrightarrow \ \cdots \ \longrightarrow \ E_1 \otimes \mathcal{A} \ \longrightarrow \ \mathcal{A} \ \longrightarrow \ \mathcal{A}_0 \ \longrightarrow \ 0 \tag{6.3}$$

with $E_1 = \mathcal{A}_1$ and $E_2 = R$. The integer d is the "global homological dimension" gl-dim$(\mathcal{A})$ of the algebra $\mathcal{A}$ [4], and is computed explicitly below (and shown to be finite) for the case at hand.

By applying the functor $\mathrm{Hom}_{\mathsf{mod}(\mathcal{A})}(\mathcal{A}, -)$ to the chain complex of (free) right $\mathcal{A}$-modules defined by (6.3), one obtains a cochain complex of left $\mathcal{A}$-modules whose cohomology is denoted by $\mathrm{Ext}^\bullet_{\mathsf{mod}(\mathcal{A})}(\mathcal{A}_0, \mathcal{A})$. We will show below that $\mathrm{Ext}^k_{\mathsf{mod}(\mathcal{A})}(\mathcal{A}_0, \mathcal{A}) = \delta_{k,d}\, \mathbb{C}$. This means that the algebra $\mathcal{A}$ is "Gorenstein" and that the cochain complex defines a minimal projective resolution of the trivial left $\mathcal{A}$-module $\mathcal{A}_0$. Together with (6.3) this implies the isomorphisms

$$E^*_k = \mathrm{Ext}^k_{\mathsf{mod}(\mathcal{A})}(\mathcal{A}_0, \mathcal{A}_0) \cong E_{d-k}$$

of vector spaces for $k = 0, 1, \dots, d$. Thus the Gorenstein property is a variant of Poincaré duality for the noncommutative toric variety $\mathbb{CP}^n_\theta$.

Recall that a graded algebra $\mathfrak{A} = \bigoplus_{k\geq 0} \mathfrak{A}_k$ is a Frobenious algebra of index m if: (i) $\mathfrak{A}_k = 0$ in all degrees $k > m$; (ii) $\mathfrak{A}_m \cong \mathfrak{A}_0 = \mathbb{C}$; and (iii) the multiplication map $\mathfrak{A}_k \otimes \mathfrak{A}_{m-k} \to \mathfrak{A}_m$ gives a non-degenerate pairing for each $k = 0, 1, \dots, m$. Furthermore, a quadratic algebra $\mathfrak{A}$ is a Koszul algebra if its (right) Koszul complex $\mathcal{K}^\bullet(\mathfrak{A})$ is acyclic in positive degrees, i.e. $H^p(\mathcal{K}^\bullet(\mathfrak{A})) = 0$ for all $p \geq 1$.

Proposition 6.4. *The homogeneous coordinate algebra $\mathcal{A} = \mathcal{A}(\mathbb{CP}^n_\theta)$ enjoys the following properties:*

(1) $\mathcal{A}^!$ *is a Frobenius algebra of index* $n+1$*;*
(2) $\mathcal{A}$ *is a noetherian domain; and*
(3) $\mathcal{A}$ *is a Koszul algebra.*

Proof: (1) The first two properties of a Frobenius algebra of index $n+1$ are immediate from the defining relations (6.2). The third property is essentially a consequence of the functorial equivalence of §1.2. We note first that the result is easily seen to be true for $\theta = 0$ (wherein $\mathcal{A}^! = \mathcal{A}(\mathbb{CP}^n)^!$ is the exterior algebra of an ordinary polynomial algebra). We can view the pairing

$$\mathcal{A}(\mathbb{CP}^n_\theta)^!_k \otimes \mathcal{A}(\mathbb{CP}^n_\theta)^!_{n+1-k} \longrightarrow \mathcal{A}(\mathbb{CP}^n_\theta)^!_{n+1}$$

as a family (parametrized by $\theta = (\theta_{ij})$) of pairings $\mathcal{A}(\mathbb{CP}^n)^!_k \otimes \mathcal{A}(\mathbb{CP}^n)^!_{n+1-k} \to \mathcal{A}(\mathbb{CP}^n)^!_{n+1}$. Thus the result is true for θ sufficiently close to the zero matrix. Since the algebras $\mathcal{A}(\mathbb{CP}^n_\theta)^!$ and $\mathcal{A}(\mathbb{CP}^n_{g\cdot\theta})^!$ are isomorphic for any $g \in \mathrm{GL}(n)$, it follows that $\mathcal{A}(\mathbb{CP}^n_\theta)^!$ is a Frobenius algebra for any θ.

(2) This follows as in the proof of Lemma 4.1.

(3) To show that the quadratic algebra $\mathcal{A}$ is Koszul, we compute the Hilbert–Poincaré series of $\mathcal{A}$ which from the definition of the grading in §5.2 is given by the formal power series

$$H_{\mathcal{A}}(s) := \sum_{k=0}^\infty \dim_{\mathbb{C}}(\mathcal{A}_k)\ s^k = \sum_{k=0}^\infty\ \sum_{i_1,\dots,i_{n+1}=0}^\infty \delta_{k,i_1+\dots+i_{n+1}}\ s^k = \Big(\frac{1}{1-s}\Big)^{n+1} ,$$

independently of the noncommutativity parameter matrix θ. On the other hand, using the Frobenius property and the relations (6.2) we can compute the series

$$H_{\mathcal{A}^!}(-s) = \sum_{k=0}^\infty \dim_{\mathbb{C}}\big(\mathcal{A}^!_k\big)\ (-s)^k = \sum_{k=0}^{n+1} \binom{n+1}{k} (-s)^k = (1-s)^{n+1} ,$$

hence $H_{\mathcal{A}}(s)\, H_{\mathcal{A}^!}(-s) = 1$ in $\mathbb{Z}[[s]]$ and the result now follows by [7, Thm. 2.11.1]. ■

Algebras of finite global homological dimension with the Gorenstein property are called regular [17]. The following result is a corollary of [5, Prop. 2.6].

Corollary 6.5. *The quadratic algebra $\mathcal{A}$ is a regular algebra of global homological dimension $d = \text{gl-dim}(\mathcal{A}) = n+1$.*

Proof: This follows similarly to [6, Prop. 7.2.3]. The global homological dimension of $\mathcal{A}$ equals the length of the minimal projective resolution for $\mathcal{A}_0 = \mathbb{C}$. Since by point (3) of Proposition 6.4 the Koszul complex is exact, it provides such a minimal resolution, and the global homological dimension coincides with the number of non-trivial graded components of the algebra $\mathcal{A}^!$, each of which can be identified as

$$\mathcal{A}^!_k \cong \mathrm{Ext}^k_{\mathsf{mod}(\mathcal{A})}(\mathcal{A}_0, \mathcal{A}_0) \ .$$

By point (1) of Proposition 6.4 the dual algebra $\mathcal{A}^!$ provides a Frobenius resolution, since as an $\mathcal{A}_0$-bimodule one has $\mathcal{A}^!_k \cong \mathcal{A}^!_{n+1} \otimes (\mathcal{A}^!_{n+1-k})^*$ and hence $\mathrm{Ext}^\bullet_{\mathsf{mod}(\mathcal{A})}(\mathcal{A}_0, \mathcal{A})$ coincides with the cohomology of the complex $\mathcal{A}^!_{n+1} \otimes \mathcal{K}^\bullet(\mathcal{A})$ truncated at the rightmost term. Thus the only non-trivial cohomology group is

$$\mathrm{Ext}^{n+1}_{\mathsf{mod}(\mathcal{A})}(\mathcal{A}_0, \mathcal{A}) \cong \mathcal{A}^!_{n+1} \otimes \mathcal{A} \ ,$$

and the Gorenstein property follows. ■

For an algebra $\mathcal{A}$ of polynomial growth (which is the case for the homogeneous coordinate algebra $\mathcal{A} = \mathcal{A}(\mathbb{CP}^n_\theta)$), one has also the notion of Gel'fand–Kirillov dimension

$$\text{GK-dim}(\mathcal{A}) := \liminf_{k\to\infty} \Big\{ \alpha \in \mathbb{R} \ \Big| \ \dim_{\mathbb{C}} \Big(\bigoplus_{l=0}^{k} \mathcal{A}_l \Big) \leq k^\alpha \Big\} \ .$$

When it is finite, combining it with the Gorenstein properties leads to the notion of Artin–Schelter regularity [1].

Proposition 6.6. *The quadratic algebra $\mathcal{A} = \mathcal{A}(\mathbb{CP}^n_\theta)$ is an Artin–Schelter regular algebra of Gel'fand–Kirillov dimension* $\text{GK-dim}(\mathcal{A}) = n + 1$.

Proof: From the definition of the grading on $\mathcal{A} = \bigoplus_{k\geq 0} \mathcal{A}_k$ in §5.2, $\dim_{\mathbb{C}}(\mathcal{A}_k) = p_{n+1}(k)$ is the number of partitions of k into $n+1$ parts. It is a classic result [18] that the function $p_{n+1}(k)$ grows asymptotically like $\frac{1}{(n+1)!}\binom{k-1}{n}$. Then the Stirling expansion shows that the dimension of $\mathcal{A}_k$ grows like k^n for $k \gg 0$, and the result for the dimension follows. In Corollary 6.5 we established the Gorenstein properties; hence the result follows. ■

6.2. **Sheaves on $\mathbb{CP}^n_\theta$.** By Propositions 4.2 and 4.6, and Theorem 5.4, it follows that quasi-coherent sheaves on $\mathsf{Open}(\mathbb{CP}^n_\theta)$ can be identified with objects of the module category $\mathsf{mod}(\mathcal{A})$, with $\mathcal{A} = \mathcal{A}(\mathbb{CP}^n_\theta)$. Let $\mathsf{gr}(\mathcal{A})$ be the category of finitely-generated graded right $\mathcal{A}$-modules $M = \bigoplus_{k\geq 0} M_k$ and degree zero morphisms, and let $\mathsf{tor}(\mathcal{A})$ be the full Serre subcategory of $\mathsf{gr}(\mathcal{A})$ consisting of finite-dimensional graded $\mathcal{A}$-modules M, i.e. $M_k = 0$ for $k \gg 0$. Henceforth, we will identify the category of coherent sheaves on $\mathsf{Open}(\mathbb{CP}^n_\theta)$ with the abelian quotient category $\mathsf{gr}(\mathcal{A})/\mathsf{tor}(\mathcal{A})$, and denote it by $\mathsf{coh}(\mathbb{CP}^n_\theta)$. Let $\pi : \mathsf{gr}(\mathcal{A}) \to \mathsf{coh}(\mathbb{CP}^n_\theta)$ be the canonical projection functor. Under this correspondence, the structure sheaf $\mathcal{O}_{\mathbb{CP}^n_\theta}$ is the image $\pi(\mathcal{A})$ of the homogeneous coordinate algebra itself, regarded as a free right $\mathcal{A}$-module of rank one. If $E = \pi(M)$ where $M \in \mathsf{gr}(\mathcal{A})$ is a graded right $\mathcal{A}$-module, then $M[w_i^{-1}]_0 = (M \otimes_{\mathcal{A}} \mathcal{A}[w_i^{-1}])_0$ is a right $\mathbb{C}_\theta[\sigma_i]$-module for each $i = 1, \dots, n+1$.

On the category $\mathsf{gr}(\mathcal{A})$ there is a natural autoequivalence defined by the degree shift functor $M \mapsto M(1)$, where $M(l)$ is the l-th shift of the graded module $M = \bigoplus_{k\geq 0} M_k$ with degree k component $M(l)_k = M_{l+k}$. For each $k \in \mathbb{Z}$ we define the sheaf

$$\mathcal{O}_{\mathbb{CP}^n_\theta}(k) := \pi\big(\mathcal{A}(k)\big) \ .$$

For any sheaf $E=\pi(M)$ we write $E(k)$ for the sheaf $\pi(M(k))$ in $\mathsf{coh}(\mathbb{CP}^n_\theta)$. Conversely, given a sheaf $E\in\mathsf{coh}(\mathbb{CP}^n_\theta)$, the vector space

$$M=\Gamma(E):=\bigoplus_{k=0}^{\infty}\mathrm{Hom}\big(\mathcal{O}_{\mathbb{CP}^n_\theta}(-k)\,,\,E\big)$$

is a graded right $\mathcal{A}$-module with $\pi(M)=E$ (with the $\mathcal{A}$-module structure given in general by [2, eq. (4.0.3)]).

As in [5, §2.3], sheaves on $\mathsf{Open}(\mathbb{CP}^n_\theta)$ have the following basic cohomological properties.

Proposition 6.7. *Every sheaf $E\in\mathsf{coh}(\mathbb{CP}^n_\theta)$ enjoys the following properties:*

(1) *Ampleness: There exists an epimorphism*

$$\bigoplus_{i=1}^{s}\mathcal{O}_{\mathbb{CP}^n_\theta}(-k_i)\ \longrightarrow\ E\ \longrightarrow\ 0$$

for some positive integers $k_1,\dots,k_s$, and there exists a positive integer k_0 such that $H^p(\mathbb{CP}^n_\theta,E(k))=0$ for all $k\geq k_0$ and $p>0$;
(2) *χ-condition: $\dim_{\mathbb{C}}(H^p(\mathbb{CP}^n_\theta,E))<\infty$ for all $p\geq 0$; and*
(3) *Serre duality: There are natural isomorphisms of complex vector spaces*

$$H^p\big(\mathbb{CP}^n_\theta\,,\,E\big)\cong\mathrm{Ext}^{n-p}\big(E\,,\,\mathcal{O}_{\mathbb{CP}^n_\theta}(-n-1)\big)^*$$

where $(-)^*$ denotes the $\mathbb{C}$-dual.

Proof: This follows from the regularity properties of the algebra $\mathcal{A}$ derived in §6.1, together with [2, Thm. 8.1] (for points (1) and (2)) and [38, Thm. 2.3] (for point (3)). ■

The following result is a special case of [5, Prop. 2.7].

Proposition 6.8. (1) *There are isomorphisms*

$$H^p\big(\mathbb{CP}^n_\theta\,,\,\mathcal{O}_{\mathbb{CP}^n_\theta}(k)\big)=\begin{cases}\mathcal{A}_k & \text{for}\quad p=0\ ,\ k\geq 0\ ,\\ \mathcal{A}^*_{-k-n-1} & \text{for}\quad p=n\ ,\ k\leq -n-1\ ,\\ 0 & \text{otherwise}\ .\end{cases}$$

(2) *The cohomological dimension of the category $\mathsf{coh}(\mathbb{CP}^n_\theta)$ is equal to n, i.e. one has $H^p(\mathbb{CP}^n_\theta,E)=0$ for all $E\in\mathsf{coh}(\mathbb{CP}^n_\theta)$ and $p>n$.*

Proof: This follows from the regularity properties of the homogeneous coordinate algebra $\mathcal{A}$ derived in §6.1, together with the Serre duality of Proposition 6.7 and [2, Thm. 8.1]. ■

Let $\mathsf{gr}_L(\mathcal{A})$ be the abelian category of finitely-generated graded *left* $\mathcal{A}$-modules. We will denote by $\pi_L:\mathsf{gr}_L(\mathcal{A})\to\mathsf{coh}_L(\mathbb{CP}^n_\theta):=\mathsf{gr}_L(\mathcal{A})\,/\,\mathsf{tor}_L(\mathcal{A})$ the corresponding quotient projection. For any sheaf $E\in\mathsf{coh}(\mathbb{CP}^n_\theta)$, the graded space

$$\underline{\mathcal{H}om}\,\big(E\,,\,\mathcal{O}_{\mathbb{CP}^n_\theta}\big)=\pi_L\Big(\bigoplus_{k=0}^{\infty}\mathrm{Hom}\big(E\,,\,\mathcal{O}_{\mathbb{CP}^n_\theta}(k)\big)\Big)$$

has a natural left $\mathcal{A}$-module structure (see [25, §5.3] and [6, §1.1]), and is thus a well-defined object of the abelian category $\mathsf{coh}_L(\mathbb{CP}^n_\theta)$. It is called the *dual sheaf* of E and is denoted $E^\vee$. The internal Hom-functor $\underline{\mathcal{H}\mathrm{om}}\,(-,\mathcal{O}_{\mathbb{CP}^n_\theta})$ is left exact on $\mathsf{coh}(\mathbb{CP}^n_\theta)\to\mathsf{coh}_L(\mathbb{CP}^n_\theta)$ and has corresponding right derived functors $\underline{\mathcal{E}\mathrm{xt}}^p(-,\mathcal{O}_{\mathbb{CP}^n_\theta})$ given by

$$\underline{\mathcal{E}\mathrm{xt}}^p\big(E\,,\,\mathcal{O}_{\mathbb{CP}^n_\theta}\big)=\pi_L\Big(\,\bigoplus_{k=0}^\infty\mathrm{Ext}^p\big(E\,,\,\mathcal{O}_{\mathbb{CP}^n_\theta}(k)\big)\Big)$$

for $p\ge0$. Since $\mathcal{A}$ is a noetherian regular algebra, the functor $\underline{\mathcal{E}\mathrm{xt}}^p(-,\mathcal{O}_{\mathbb{CP}^n_\theta})$ gives an anti-equivalence between the derived categories of $\mathsf{coh}(\mathbb{CP}^n_\theta)$ and $\mathsf{coh}_L(\mathbb{CP}^n_\theta)$ (see [38, §4] and [25, §5.3]). It follows that there are isomorphisms

$$\mathrm{Ext}^p(E,F)\cong\mathrm{Ext}^p_L(F^\vee,E^\vee\,):=\mathrm{Ext}^p_{\mathsf{coh}_L(\mathbb{CP}^n_\theta)}(F^\vee,E^\vee\,)$$

for any $p\ge0$ and for any pair of torsion-free sheaves $E,F\in\mathsf{coh}(\mathbb{CP}^n_\theta)$.

For a sheaf $F\in\mathsf{coh}(\mathbb{CP}^n_\theta)$, there is a functorial isomorphism

$$H^0_L\big(\mathbb{CP}^n_\theta\,,\,\underline{\mathcal{H}\mathrm{om}}\,(F,\mathcal{O}_{\mathbb{CP}^n_\theta})\big)\cong\mathrm{Hom}\big(F\,,\,\mathcal{O}_{\mathbb{CP}^n_\theta}\big)\ ,$$

and also a functorial spectral sequence

$$E_2^{p,q}=H^p_L\big(\mathbb{CP}^n_\theta\,,\,\underline{\mathcal{E}\mathrm{xt}}^q(F,\mathcal{O}_{\mathbb{CP}^n_\theta})\big)\qquad\Longrightarrow\qquad\mathrm{Ext}^\bullet\big(F\,,\,\mathcal{O}_{\mathbb{CP}^n_\theta}\big)\ .$$

The sheaves $\mathcal{O}_{\mathbb{CP}^n_\theta}(k)$, $k\in\mathbb{Z}$ are locally free, with $\underline{\mathcal{H}\mathrm{om}}\,(\mathcal{O}_{\mathbb{CP}^n_\theta}(k),\mathcal{O}_{\mathbb{CP}^n_\theta}(l))=\mathcal{O}_{\mathbb{CP}^n_\theta}(l-k)$ as sheaves of bimodules. More generally, bundles over noncommutative projective varieties may be characterized as follows.

Proposition 6.9. *Let $\mathcal{E}\in\mathsf{coh}(\mathbb{CP}^n_\theta)$ and $M=\Gamma(\mathcal{E})\in\mathsf{gr}(\mathcal{A})$. Then the following statements are equivalent:*

1. *$\mathcal{E}$ is a locally free sheaf;*
2. *$\underline{\mathcal{E}\mathrm{xt}}^p(\mathcal{E},\mathcal{O}_{\mathbb{CP}^n_\theta})=0$ for all $p>0$; and*
3. *$M[w_i^{-1}]_0$ is projective in $\mathsf{coh}(\sigma_i)$ for each $i=1,\dots,n+1$.*

<u>Proof</u>: This is a consequence of Proposition 4.6 and Definition 4.7, together with the functorial equivalence of §1.2, and the fact that the result holds in the commutative case $\theta=0$ [25]. If $\mathcal{E}$ is locally free, then its restrictions $\mathcal{E}_{\sigma_i}$ are direct sums of shifts of $\mathbb{C}_\theta[\sigma_i]$, with

$$\mathbb{C}_\theta[\sigma_i](k):=\big(\mathcal{A}(k)\otimes_{\mathcal{A}}\mathcal{A}[w_i^{-1}]\big)_0\ .$$

Since $\mathrm{Ext}^p_{\mathsf{gr}(\mathcal{A})}(\mathcal{A}(l),\mathcal{A}(k))=0$ for $k>l$ and $p>0$, it follows from the χ-condition of Proposition 6.7 that $\bigoplus_{k\ge0}\mathrm{Ext}^p(E,\mathcal{O}_{\mathbb{CP}^n_\theta}(k))$ is finite-dimensional, and hence one has $\underline{\mathcal{E}\mathrm{xt}}^p(\mathcal{E},\mathcal{O}_{\mathbb{CP}^n_\theta})=0$ for all $p>0$. Conversely, by Serre duality of Proposition 6.7 one has

$$\underline{\mathcal{E}\mathrm{xt}}^p(\mathcal{E},\mathcal{O}_{\mathbb{CP}^n_\theta})\cong\pi_L\Big(\,\bigoplus_{k=0}^\infty H^{n-p}\big(\mathbb{CP}^n_\theta\,,\,\mathcal{E}(-k-n-1)\big)^*\Big)\ ,$$

where the group $H^{n-p}(\mathbb{CP}^n_\theta,\mathcal{E}(-k-n-1))$ coincides with $\mathrm{Ext}^{n-p}(\mathcal{O}_{\mathbb{CP}^n_\theta}(k+n+1),\mathcal{E})$. Hence if $\underline{\mathcal{E}\mathrm{xt}}^p(\mathcal{E},\mathcal{O}_{\mathbb{CP}^n_\theta})=0$ for $p>0$, then by the χ-condition $\mathrm{Ext}^s(\mathcal{O}_{\mathbb{CP}^n_\theta}(k+n+1),\mathcal{E})=0$ for all $0\le s<n$ and $k\gg0$. That $\mathcal{E}$ is locally free now follows again by localization and the corresponding result in the category $\mathsf{gr}(\mathcal{A})$. Finally, if M is projective, then the functor $\mathrm{Hom}_{\mathsf{gr}(\mathcal{A})}(M,-)$ is exact, and hence $\mathrm{Ext}^p_{\mathsf{gr}(\mathcal{A})}(M,\mathcal{A}(k))=0$ for all $p>0$ and $k\ge0$. ■

Example 6.10. *For noncommutative projective varieties we can provide an equivalent global description of the sheaves of differential forms, constructed in §4.4 using Kähler differentials, in terms of Koszul complexes, since by Proposition 6.4 their homogeneous* *coordinate algebras are Koszul algebras. Affine open subsets $U_\theta[\sigma]$ correspond to localizations of the homogeneous coordinate algebra $\mathcal{A} = \mathbb{C}_{\tilde{\theta}}[w_1,\dots,w_{n+1}]$ of $\mathbb{CP}^n_\theta$ according to Theorem 5.4, and the construction of Kähler differentials commutes with Ore localization (see e.g.* [14, §3] *and* [31, Thm. 1.2.1]*). The bimodule of Kähler differentials $\Omega^1_{\mathcal{A}} = I_{\mathcal{A}}/I^2_{\mathcal{A}}$ is defined as in §4.4 via the kernel of the multiplication map $\mu_{\mathcal{A}} : \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$. Using the constructions of §4.4, it is easy to see that $\Omega^1_{\mathcal{A}}$ is isomorphic to the free $\mathcal{A}$-bimodule $\mathcal{A}^{\oplus(n+1)}$. On the other hand, since $\mathcal{A}$ is a Koszul algebra one can define the left (resp. right) $\mathcal{A}$-module $\mathcal{K}_p(\mathcal{A})$ as the cohomology of the left (resp. right) Koszul complex of $\mathcal{A}$ in §6.1 truncated at the p-th term* [25, Def. 4.8]*. For $p=1$, the module $\mathcal{K}_1(\mathcal{A})$ sits in the exact sequence*

$$0 \longrightarrow \mathcal{K}_1(\mathcal{A}) \longrightarrow \big(\mathcal{A}^!_1\big)^* \otimes \mathcal{A} \xrightarrow{\ \mathrm{d}\ } \mathcal{A} \xrightarrow{\ \varepsilon\ } \mathbb{C} \longrightarrow 0$$

so that $\mathcal{K}_1(\mathcal{A}) = \ker(\mathrm{d})$. But here the differential d is exactly $\mu_{\mathcal{A}}$. It follows that there is a natural identification $\Omega^p_{\mathcal{A},un} \cong \mathcal{K}_p(\mathcal{A})$, and so the Koszul description of the sheaves of differential forms coincides with that in terms of Kähler differentials in these cases.

6.3. **Tautological bundles on $\mathbb{G}\mathrm{r}_\theta(d;n)$.** We give some explicit examples of locally free sheaves on the noncommutative Grassmann varieties of §5.3, which further admit straightforward extensions to the general noncommutative flag varieties of §5.4. Recall that in the commutative case the tautological hyperplane bundle (or universal sub-bundle) $\mathcal{S}$ is the vector bundle over $\mathbb{G}\mathrm{r}(d;V)$ such that the fibre over each point $[\Lambda]\in\mathbb{G}\mathrm{r}(d;V)$ is the d-plane $V_\Lambda\subset V$ defined by Λ itself. It sits inside the Euler sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathbb{G}\mathrm{r}(d;V)\times V \longrightarrow \mathcal{Q} \longrightarrow 0\ , \tag{6.11}$$

where $\mathcal{Q}$ is the quotient sub-bundle. To describe the embedding of $\mathcal{S}$ in the trivial bundle $\mathbb{G}\mathrm{r}(d;V)\times V$, we note that, when $\dim_{\mathbb{C}}(V)=n$, a section of $\mathbb{G}\mathrm{r}(d;V)\times V$ is an n-dimensional vector

$$w = \sum_{i=1}^n w_i(\Lambda)\otimes v_i \ \in\ \mathcal{A}\big(\mathbb{G}\mathrm{r}(d;V)\big)\otimes V \tag{6.12}$$

of functions $w_i(\Lambda)$ on $\mathbb{G}\mathrm{r}(d;V)$, where $\{v_i\}_{i=1}^n$ is any basis for V. This defines a section of $\mathcal{S}$ if and only if for each Λ the vector (6.12) belongs to V_Λ.

In that case, if we add the vector w to the $d\times n$ matrix Λ as the $(d+1)$-th row, thus generating a $(d+1)\times n$ matrix, then all the minors of order $d+1$ are zero. Denote by $J=(j_1\cdots j_{d+1})$ an ordered $(d+1)$ multi-index with $j_1<j_2<\cdots<j_{d+1}$, and by $J\setminus j_d$ the order d multi-index with j_d removed. Then, as before, $\Lambda^{J\setminus j_d}$ is the minor of order d in Λ obtained from the columns labelled by $J\setminus j_d$. By expanding the minors with respect to the $(d+1)$-th row w, the requisite condition can be expressed as the equations

$$\epsilon^{j_d\cup(J\setminus j_d)}\, w_{j_d}\, \Lambda^{J\setminus j_d} = 0 \tag{6.13}$$

for every ordered $(d+1)$ multi-index J. A section of the trivial bundle (6.12) is a section of $\mathcal{S}$ if and only if it satisfies (6.13). This is a local description since we have to choose a $d\times n$ matrix Λ to represent a point in $\mathbb{G}\mathrm{r}(d;V)$, and our condition (6.13) is written using the data of this local representative.

To pass to the noncommutative coordinate algebra $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))$, we insert into (6.13) the θ-deformed Levi–Civita symbol. Then $\mathcal{S}_\theta$ is defined to be the subsheaf of elements of the free module $(w_1(\Lambda),\dots,w_n(\Lambda)) \in \mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))^{\oplus n}$ over the noncommutative grassmannian which satisfy the equations

$$\epsilon_\theta^{j_d \cup (J\setminus j_d)\,(c)}\, w_{j_d}\, \Lambda^{J\setminus j_d} = 0 \tag{6.14}$$

for every ordered $(d+1)$ multi-index J, where the minors of order d obey the relations (5.7). We can use the Plücker map to regard the noncommutative minors $\Lambda^{J\setminus j_d}$ as homogeneous coordinates in $\mathbb{P}(\bigwedge_\theta^d V)$. Then the quotient by the graded two-sided ideal generated by the set of homogeneous relations (6.14) defines the projection from the free module $\mathbb{P}(\bigwedge_\theta^d V) \otimes V \to \mathcal{S}_\theta$. In this case we have to consider the restriction of (6.14) to those elements Λ which also satisfy the Young symmetry relations (5.9). This gives the sheaf $\mathcal{S}_\theta$ the natural structure of a graded $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))$-bimodule.

Proposition 6.15. *The sheaf $\mathcal{S}_\theta$ is locally free on $\mathsf{Open}(\mathbb{G}\mathrm{r}_\theta(d;n))$.*

Proof: The geometric description of the embedding of $\mathcal{S}$ in $\mathbb{G}\mathrm{r}(d;V) \times V$ by a projector amounts to taking a section (6.12) and projecting the vector w over the d-plane V_Λ for each $[\Lambda] \in \mathbb{G}\mathrm{r}(d;V)$. To obtain a well-defined projector, we choose an inner product $\langle -,- \rangle_\Lambda$ on the complex vector space V such that the vectors $v_1,\dots,v_d$ which span V_Λ are orthonormal. Then the projection of a vector $w \in V$ over V_Λ is given by $p_\Lambda(w) = \sum_i \langle w, v_i\rangle_\Lambda\, v_i$. This yields a unique idempotent $p : \mathcal{A}(\mathbb{G}\mathrm{r}(d;V)) \otimes V \to \mathcal{A}(\mathbb{G}\mathrm{r}(d;V)) \otimes V$ which maps $w(\Lambda)$ in (6.12) to $p_\Lambda(w(\Lambda))$, with $p^2 = p$, trace equal to d, and $\mathrm{im}(p) = \mathcal{S}$. The matrix representation of p_Λ is given by the $n \times n$ matrix $\Lambda^\top \Lambda$, where for Λ we choose a matrix representative whose d rows are the orthonormal generators of the plane V_Λ so that $\Lambda\,\Lambda^\top = 1$ and $(\Lambda^\top \Lambda)\,(\Lambda^\top \Lambda) = \Lambda^\top \Lambda$. The extension to the noncommutative setting only requires using noncommuting entries in Λ with noncommutative relations in the coordinate algebra $\mathcal{F}_n^\theta$ of $\mathrm{GL}_\theta(n)$, given in §2.3, in a way which is compatible with the projector constraints. The statement now follows by point (3) of Proposition 6.9. ■

Example 6.16. *For $d = 1$, it is easy to see that the equations (6.14) are solved by taking $w_j(\Lambda) = \Lambda^j$ to be the generators of the homogeneous coordinate algebra $\mathcal{A}(\mathbb{CP}_\theta^{n-1})$, and one has a canonical isomorphism of bimodules $\mathcal{S}_\theta \cong \mathcal{O}_{\mathbb{CP}_\theta^{n-1}}(1)$. Alternatively, use Proposition 6.15 to get* $\mathrm{im}(p) \cong \mathcal{A}(\mathbb{CP}_\theta^{n-1})$.

6.4. **Differential forms on $\mathbb{G}\mathrm{r}_\theta(d;n)$.** There is also a useful alternative description of the bundle of Kähler differentials $\Omega^1_{\mathbb{G}\mathrm{r}_\theta(d;n)}$. In the classical case, the tangent bundle over $\mathbb{G}\mathrm{r}(d;V)$ is represented in terms of the Euler sequence (6.11) as the morphism bundle $\mathrm{Hom}(\mathcal{S},\mathcal{Q}) \cong \mathcal{S}^\vee \otimes \mathcal{Q}$, whose fibre spaces are given by $T_{[\Lambda]}\mathbb{G}\mathrm{r}(d;V) = \mathrm{Hom}_\mathbb{C}(V_\Lambda, V/V_\Lambda)$. This description can be transported to the noncommutative setting via the following characterization.

Lemma 6.17. *The total space of the cotangent bundle over the grassmannian $\mathbb{G}\mathrm{r}(d;n)$ is the base of the principal fibration*

$$L_{d,n-d} := \mathrm{GL}(d) \times \mathrm{GL}(n-d) \hookrightarrow \mathrm{GL}(n) \longrightarrow T^*\mathbb{G}\mathrm{r}(d;n)\ .$$

Proof: Let E denote the principal $P_{d,n-d}$-bundle given in (5.11) for $\gamma = (d, n-d)$. Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of $\mathrm{GL}(n)$ and $P_{d,n-d}$, respectively. Then the cotangent bundle can be represented by $T^*\mathbb{G}\mathrm{r}(d;n) = E \times_{\mathrm{Ad}^*(P_{d,n-d})} (\mathfrak{g}/\mathfrak{p})^*$. If $P_{d,n-d}$ is embedded in $\mathrm{GL}(n)$ as the subgroup of upper triangular matrices, then $a \in \mathfrak{g}/\mathfrak{p}$ is represented by a (strictly) block upper triangular matrix. Embed $L_{d,n-d}$ in $\mathrm{GL}(n)$ as the subgroup of block diagonal matrices. Then $L_{d,n-d}$ is the reductive Levi subgroup of $P_{d,n-d}$ and there is a Levi decomposition $P_{d,n-d} = R_{d,n-d} \ltimes_{\mathrm{Ad}(L_{d,n-d})} L_{d,n-d}$, where $R_{d,n-d}$ is the unipotent radical of the parabolic group $P_{d,n-d}$ which is the additive subgroup of $\mathrm{GL}(n)$ represented by block upper $d \times (n-d)$ matrices with respect to this embedding. On $\mathrm{GL}(n)/L_{d,n-d}$ there is still the proper and free left action of $R_{d,n-d}$, and the quotient is our grassmannian

$$R_{d,n-d} \big\backslash \mathrm{GL}(n) \big/ L_{d,n-d} = \mathrm{GL}(n) \big/ P_{d,n-d} = \mathbb{G}\mathrm{r}(d;n) \ .$$

We claim that this principal $R_{d,n-d}$-bundle $F \to \mathbb{G}\mathrm{r}(d;n)$ is isomorphic to the cotangent bundle. For this, we define a bundle map $T^*\mathbb{G}\mathrm{r}(d;n) \to F$, such that on the fibre over the equivalence class of the identity of $\mathrm{GL}(n)$ in $\mathrm{GL}(n)/P_{d,n-d}$ there is an isomorphism $P_{d,n-d} \times_{\mathrm{Ad}^*(P_{d,n-d})} (\mathfrak{g}/\mathfrak{p})^* \to R_{d,n-d}$. With respect to the block embeddings described above, this is given by

$$\Big(\begin{pmatrix} M & A \\ 0 & N \end{pmatrix} , a \Big) \longmapsto \begin{pmatrix} 1 & M\,a\,N^{-1} \\ 0 & 1 \end{pmatrix} \ .$$

Since the two fibrations have the same base space, the bundle map reduces to a morphism between the fibre spaces. Since the base space is homogeneous with respect to the action of $\mathrm{GL}(n)$, the isomorphism on a generic fibre is the conjugation by $\mathrm{GL}(n)$ of the isomorphism over the identity constructed above. ■

We will use Lemma 6.17 to provide a purely algebraic description of the cotangent bundle in terms of coinvariant elements in the Hopf algebra $\mathcal{F}_n$ of $\mathrm{GL}(n)$ with respect to the coaction induced from the subgroup $L_{d,n-d}$. Then we will deform this construction using a Drinfel'd twist, obtaining an alternative description of the bundle of noncommutative Kähler differentials $\Omega^1_{\mathbb{G}\mathrm{r}_\theta(d;n)}$. The algebraic version of the inclusion $L_{d,n-d} \hookrightarrow \mathrm{GL}(n)$ is a surjective algebra homomorphism $\pi^{(\mathcal{L}_{d,n})}$ from $\mathcal{F}_n$ to the Hopf subalgebra $\mathcal{L}_{d,n}$ dual to the subgroup $L_{d,n-d}$. As in §2.3, we denote the generators of $\mathcal{F}_n = \mathrm{Fun}(\mathrm{GL}(n))$ by g_{ij} with $i,j = 1,\dots,n$. The generators of $\mathcal{L}_{d,n} = \mathrm{Fun}(L_{d,n-d})$ are denoted l_{ij} with $1 \le i,j \le d$ and $d+1 \le i,j \le n$. Then the projection homomorphism $\pi^{(\mathcal{L}_{d,n})} : \mathcal{F}_n \to \mathcal{L}_{d,n}$ is given by

$$\pi^{(\mathcal{L}_{d,n})}(g_{ij}) = \begin{cases} l_{ij} \ , & 1 \le i,j \le d \quad \text{and} \quad d+1 \le i,j \le n \ , \\ 0 \ , & \text{otherwise} \ . \end{cases} \tag{6.18}$$

The left coaction ${}^{\mathcal{L}_{d,n}}\Phi : \mathcal{F}_n \to \mathcal{L}_{d,n} \otimes \mathcal{F}_n$ dual to the right multiplicative action of $L_{d,n-d}$ on $\mathrm{GL}(n)$ is the unital algebra morphism given by ${}^{\mathcal{L}_{d,n}}\Phi := \big(\pi^{(\mathcal{L}_{d,n})} \otimes 1\big)\,\Delta_\vee$, or explicitly

$${}^{\mathcal{L}_{d,n}}\Phi(g) = \big(\pi^{(\mathcal{L}_{d,n})} \otimes 1\big)\,\Delta_\vee(g) = \pi^{(\mathcal{L}_{d,n})}(g_{(1)}) \otimes g_{(2)} \ . \tag{6.19}$$

The subalgebra of left coinvariants, defined in the usual way by

$${}^{\mathrm{co}-\mathcal{L}_{d,n}}\mathcal{F}_n = \big\{ g \in \mathcal{F}_n \ \big| \ {}^{\mathcal{L}_{d,n}}\Phi(g) = 1 \otimes g \big\} \ ,$$

gives the algebraic description of the base of the fibration $\mathrm{GL}(n)/L_{d,n-d}$, i.e. the cotangent bundle $T^*\mathbb{G}\mathrm{r}(d;n)$. We use the general strategy to find coinvariants through projector maps [27, Ch. 13].

Proposition 6.20. *A set of generators for* ${}^{\mathrm{co}-\mathcal{L}_{d,n}}\mathcal{F}_n$ *is given by elements*

$$\eta_{ij} := \sum_{k=1}^{d} S_\vee(g_{ik})\, g_{kj} \ , \qquad 1 \le i,j \le n \tag{6.21}$$

and

$$\eta^\perp_{ij} := \sum_{k=d+1}^n S_\vee(g_{ik})\, g_{kj} \ , \qquad 1 \leq i,j \leq n \ . \tag{6.22}$$

<u>Proof</u>: By direct computation one has

$$\begin{aligned} {}^{\mathcal{L}_{d,n}}\Phi\Big(\sum_{k=1}^d S_\vee(g_{ik})\, g_{kj}\Big) &= \big(\pi^{(\mathcal{L}_{d,n})} \otimes 1\big) \Big(\sum_{k=1}^d \ \sum_{m,p=1}^n \big(S_\vee(g_{pk})\, g_{km}\big) \otimes \big(S_\vee(g_{ip})\, g_{mj}\big)\Big) \\ &= \sum_{k,m,p=1}^d \big(S_\vee(l_{pk})\, l_{km}\big) \otimes \big(S_\vee(g_{ip})\, g_{mj}\big) \\ &= \sum_{m,p=1}^d \delta_{pm} \otimes \big(S_\vee(g_{ip})\, g_{mj}\big) \ = \ 1 \otimes \Big(\sum_{p=1}^d S_\vee(g_{ip})\, g_{pj}\Big) \ . \end{aligned}$$

The coinvariance of the second set of generators follows easily from

$$\sum_{k=1}^n S_\vee(g_{ik})\, g_{kj} = \delta_{ij} \ ,$$

since the coinvariants generate a vector space. ■

The generators η_{ij} and $\eta^\perp_{ij} = \delta_{ij} - \eta_{ij}$ are not independent, but are characterized by a set of relations. They can be regarded as entries of $n \times n$ matrices, yielding an algebraic description of the vector bundle with associated principal bundle given in Lemma 6.17.

Proposition 6.23. *The generators η_{ij} (resp. $\eta^\perp_{ij}$) for $i,j = 1,\dots,n$ are the entries of an idempotent matrix η (resp. $\eta^\perp$) with trace equal to d (resp. $n-d$).*

<u>Proof</u>: Again by direct computation one has

$$\begin{aligned} \sum_{m=1}^n \eta_{im}\, \eta_{mj} &= \sum_{m=1}^n \ \sum_{k,p=1}^d S_\vee(g_{ik})\, g_{km}\, S_\vee(g_{mp})\, g_{pj} \\ &= \sum_{k,p=1}^d S_\vee(g_{ik})\, \delta_{kp}\, g_{pj} \\ &= \sum_{k=1}^d S_\vee(g_{ik})\, g_{kj} \ = \ \eta_{ij} \ . \end{aligned}$$

The trace condition is easily computed as

$$\sum_{m=1}^n \eta_{mm} = \sum_{m=1}^n \ \sum_{k=1}^d S_\vee(g_{mk})\, g_{km} = \sum_{k=1}^d \delta_{kk} = d \ .$$

The corresponding results for $\eta^\perp = 1\!\!1_{n\times n} - \eta$ now easily follow. ■

Comparing with Proposition 6.15 and Lemma 6.17, it follows that we can interpret η as the matrix describing the finitely-generated projective $\mathcal{A}(\mathbb{G}\mathrm{r}(d;n))$-module $\mathcal{S} \cong$

$\eta\big(\mathcal{A}(\mathbb{G}\mathrm{r}(d;n))^{\oplus n}\big)$. Recall that there is a canonical isomorphism $\mathbb{G}\mathrm{r}(d;V) \xrightarrow{\approx} \mathbb{G}\mathrm{r}(n-d;V^*)$ of grassmannians given by $V_\Lambda \mapsto (V/V_\Lambda)^*$. Under this isomorphism, the universal quotient bundle $\mathcal{Q}$ on $\mathbb{G}\mathrm{r}(d;V)$ corresponds to the dual of the tautological bundle $\mathcal{S}^\perp$ of rank $n-d$ on the variety $\mathbb{G}\mathrm{r}(n-d;V^*)$. We may then identify $\mathcal{S}^\perp = \eta^\perp\big(\mathcal{A}(\mathbb{G}\mathrm{r}(d;n))^{\oplus n}\big)$, and one has the anticipated isomorphism ${}^{\mathrm{co}-\mathcal{L}_{d,n}}\mathcal{F}_n \cong \mathcal{S} \otimes_{\mathcal{A}(\mathbb{G}\mathrm{r}(d;n))} \mathcal{S}^\perp$ of $\mathcal{A}(\mathbb{G}\mathrm{r}(d;n))$-modules.

We now consider the Drinfel'd twist deformation $\mathcal{F}_n^\theta$ of the coordinate algebra of $\mathrm{GL}(n)$, given in Definition 2.13. This deformation applies to the Hopf subalgebra $\mathcal{L}_{d,n}$ as well. Since we are interested in toric $(\mathbb{C}^\times)^{d\,(n-d)}$ deformations of the variety $\mathbb{G}\mathrm{r}(d;n)$, we consider a deformation $\mathcal{F}_d^{\theta_{(d)}} \otimes \mathcal{F}_{n-d}^{\theta_{(n-d)}}$ of the Hopf algebra $\mathrm{Fun}(L_{d,n-d})$ and use the subgroup inclusion described by the algebra homomorphism (6.18). Then, as explained in §5.2, the $n\times n$ matrix θ is given by $\theta^{ij} = {\theta_{(d)}}^{ij}$ for the block $1 \le i,j \le d$, $\theta^{ij} = {\theta_{(n-d)}}^{ij}$ for the block $d+1 \le i,j \le n$, and $\theta^{ij} = 0$ otherwise. Hence the noncommutative Hopf algebra $\mathcal{L}_{d,n}^\theta$ is also well-defined. The left coaction of $\mathcal{L}_{d,n}^\theta$ on $\mathcal{F}_n^\theta$ is the same as that of (6.19), since the twist does not change the coproduct. In analogy with the undeformed case, we interpret the algebra ${}^{\mathrm{co}-\mathcal{L}_{d,n}^\theta}\mathcal{F}_n^\theta$ of left coinvariants as the algebra of the $(\mathbb{C}^\times)^{d\,(n-d)}$ deformation of the cotangent manifold $T^*\mathbb{G}\mathrm{r}(d;n) = \mathrm{GL}(n)/L_{d,n}$. This identification will be justified below. The algebra ${}^{\mathrm{co}-\mathcal{L}_{d,n}^\theta}\mathcal{F}_n^\theta$ is generated by elements η_{ij} introduced in (6.21) and by elements $\eta_{ij}^\perp$ given in (6.22).

Theorem 6.24. *The noncommutative product in ${}^{\mathrm{co}-\mathcal{L}_{d,n}^\theta}\mathcal{F}_n^\theta$ is described by commutation relations among generators η_{ij} and $\eta_{ij}^\perp$ given by*

$$
\begin{aligned}
\eta_{ij} \times_\theta \eta_{i'j'} &= K^2_{ij\,;\,i'j'}\ \eta_{i'j'} \times_\theta \eta_{ij} \ , \\
\eta^\perp_{ij} \times_\theta \eta^\perp_{i'j'} &= K^2_{ij\,;\,i'j'}\ \eta^\perp_{i'j'} \times_\theta \eta^\perp_{ij} \ , \\
\eta_{ij} \times_\theta \eta^\perp_{i'j'} &= K^2_{ij\,;\,i'j'}\ \eta^\perp_{i'j'} \times_\theta \eta_{ij} \ ,
\end{aligned}
\tag{6.25}
$$

where

$$
K_{ij\,;\,i'j'} = q_{ii'}\, q_{j'i}\, q_{i'j}\, q_{jj'} \ . \tag{6.26}
$$

Proof: We compute the twisted relations between η_{ij} directly from the definition (1.12). For this, we need the quantity $(\mathrm{id} \otimes \Delta_\vee)\,\Delta_\vee(\eta_{ij}) = \eta_{ij}^{(1)} \otimes \eta_{ij}^{(2)} \otimes \eta_{ij}^{(3)}$. Beginning with

$$
\begin{aligned}
\Delta_\vee(\eta_{ij}) &= \sum_{k=1}^d\ \sum_{m,p=1}^n \big(S_\vee(g_{pk}) \otimes S_\vee(g_{ip})\big) \cdot \big(g_{km} \otimes g_{mj}\big) \\
&= \sum_{k=1}^d\ \sum_{m,p=1}^n \big(S_\vee(g_{pk})\, g_{km}\big) \otimes \big(S_\vee(g_{ip})\, g_{mj}\big) \ ,
\end{aligned}
$$

we expand the second factor at the end to get

$$
\eta_{ij}^{(1)} \otimes \eta_{ij}^{(2)} \otimes \eta_{ij}^{(3)} = \sum_{k=1}^d\ \sum_{m,p,r,s=1}^n \big(S_\vee(g_{pk})\, g_{km}\big) \otimes \big(S_\vee(g_{rp})\, g_{ms}\big) \otimes \big(S_\vee(g_{ir})\, g_{sj}\big)
$$

and similarly

$$
\eta_{i'j'}^{(1)} \otimes \eta_{i'j'}^{(2)} \otimes \eta_{i'j'}^{(3)} = \sum_{k'=1}^d\ \sum_{m',p',r',s'=1}^n \big(S_\vee(g_{p'k'})\, g_{k'm'}\big) \otimes \big(S_\vee(g_{r'p'})\, g_{m's'}\big) \otimes \big(S_\vee(g_{i'r'})\, g_{s'j'}\big) \ .
$$

Using these expressions we compute the three terms of the deformed product in (1.12). Starting with

$$F^{\theta}\big(\eta^{(1)}_{ij}\otimes\eta^{(1)}_{i'j'}\big)=\big\langle F_\theta\,,\,\eta^{(1)}_{ij}\otimes\eta^{(1)}_{i'j'}\big\rangle=\big\langle\exp\big(-\tfrac{\mathrm{i}}{2}\,\theta^{ab}\,H_a\otimes H_b\big)\,,\,\eta^{(1)}_{ij}\otimes\eta^{(1)}_{i'j'}\big\rangle$$

and looking at the first order term in θ we compute separately

$$\begin{aligned}\big\langle H_a\,,\,\eta^{(1)}_{ij}\big\rangle &= \sum_{k=1}^{d}\big\langle H_a\,,\,S_\vee(g_{pk})\,g_{km}\big\rangle\\ &= \sum_{k=1}^{d}\big\langle H_a\otimes 1+1\otimes H_a\,,\,S_\vee(g_{pk})\otimes g_{km}\big\rangle\\ &= \sum_{k=1}^{d}\Big(-\langle H_a,g_{pk}\rangle\,\varepsilon_\vee(g_{km})+\varepsilon_\vee\big(S_\vee(g_{pk})\big)\,\langle H_a,g_{km}\rangle\Big)\\ &= \sum_{k=1}^{d}\big(-\,\delta_{ap}\,\delta_{ak}\,\delta_{km}+\delta_{pk}\,\delta_{ak}\,\delta_{am}\big)\;=\;0\;,\end{aligned}$$

where we have used duality to transfer the antipode $S_\vee$ from $\mathcal{F}^\theta_n$ to the enveloping algebra $\mathcal{H}^n_\theta$ in the pairing. An identical calculation shows that $\langle H_b,\eta^{(1)}_{i'j'}\rangle=0$. Only the zeroth order term gives a contribution, so that

$$\begin{aligned}F^{\theta}\big(\eta^{(1)}_{ij}\otimes\eta^{(1)}_{i'j'}\big) &= \big\langle 1\otimes 1\,,\,\eta^{(1)}_{ij}\otimes\eta^{(1)}_{i'j'}\big\rangle\\ &= \sum_{k,k'=1}^{d}\varepsilon_\vee\big(S_\vee(g_{pk})\,g_{km}\big)\,\varepsilon_\vee\big(S_\vee(g_{p'k'})\,g_{k'm'}\big)\;=\;\sum_{k,k'=1}^{d}\delta_{pk}\,\delta_{mk}\,\delta_{p'k'}\,\delta_{m'k'}\;.\end{aligned}$$

The third factor in (1.12) is given by

$$F^{\theta\,-1}\big(\eta^{(3)}_{ij}\otimes\eta^{(3)}_{i'j'}\big)=\big\langle F_\theta^{-1}\,,\,\eta^{(3)}_{ij}\otimes\eta^{(3)}_{i'j'}\big\rangle=\big\langle\exp\big(\tfrac{\mathrm{i}}{2}\,\theta^{bc}\,H_b\otimes H_c\big)\,,\,\eta^{(3)}_{ij}\otimes\eta^{(3)}_{i'j'}\big\rangle\;.$$

Looking at the first order term in θ, we compute separately

$$\begin{aligned}\big\langle H_b\,,\,\eta^{(3)}_{ij}\big\rangle &= \big\langle H_b\,,\,S_\vee(g_{ir})\,g_{sj}\big\rangle\\ &= \big\langle H_b\otimes 1+1\otimes H_b\,,\,S_\vee(g_{ir})\otimes g_{sj}\big\rangle\\ &= -\,\langle H_b,g_{ir}\rangle\,\varepsilon_\vee(g_{sj})+\varepsilon_\vee\big(S_\vee(g_{ir})\big)\,\langle H_b,g_{sj}\rangle\;=\;-\,\delta_{bi}\,\delta_{ri}\,\delta_{sj}+\delta_{bj}\,\delta_{ri}\,\delta_{sj}\;.\end{aligned}$$

An identical calculation shows that $\langle H_c,\eta^{(3)}_{i'j'}\rangle=-\,\delta_{ci'}\,\delta_{r'i'}\,\delta_{s'j'}+\delta_{cj'}\,\delta_{r'i'}\,\delta_{s'j'}$. So the first order term is given by

$$\tfrac{\mathrm{i}}{2}\,\theta^{bc}\,(-\,\delta_{bi}\,\delta_{ri}\,\delta_{sj}+\delta_{bj}\,\delta_{ri}\,\delta_{sj})\,(-\,\delta_{ci'}\,\delta_{r'i'}\,\delta_{s'j'}+\delta_{cj'}\,\delta_{r'i'}\,\delta_{s'j'})$$

and summing over all orders we finally arrive at

$$F^{\theta\,-1}\big(\eta^{(3)}_{ij}\otimes\eta^{(3)}_{i'j'}\big)=q_{ii'}\,q_{j'i}\,q_{i'j}\,q_{jj'}\;\,\delta_{ri}\,\delta_{sj}\,\delta_{r'i'}\,\delta_{s'j'}\;.$$

We are now ready to write the deformed product between generators η_{ij} as

$$
\begin{aligned}
\eta_{ij} \times_\theta \eta_{i'j'} &= F^\theta\big(\eta_{ij}^{(1)} \otimes \eta_{i'j'}^{(1)}\big)\,\big(\eta_{ij}^{(2)} \cdot \eta_{i'j'}^{(2)}\big)\, F^{\theta\,-1}\big(\eta_{ij}^{(3)} \otimes \eta_{i'j'}^{(3)}\big) \\
&= \sum_{m,p,r,s=1}^{n}\ \sum_{m',p',r',s'=1}^{n} \Big(\sum_{k,k'=1}^{d} \delta_{pk}\,\delta_{mk}\,\delta_{p'k'}\,\delta_{m'k'} \Big) \\
&\qquad\qquad \times \big(S_\vee(g_{rp})\, g_{ms}\, S_\vee(g_{r'p'})\, g_{m's'}\big)\, \big(q_{ii'}\, q_{j'i}\, q_{i'j}\, q_{jj'}\, \delta_{ri}\, \delta_{sj}\, \delta_{r'i'}\, \delta_{s'j'}\big) \\
&= q_{ii'}\, q_{j'i}\, q_{i'j}\, q_{jj'}\ \eta_{ij}\, \eta_{i'j'} \ .
\end{aligned}
$$

Computing in exactly the same way the deformed product $\eta_{i'j'} \times_\theta \eta_{ij}$ and comparing the two expressions, we find the first set of relations in (6.25). The remaining relations follow from $\eta_{ij}^\perp = \delta_{ij} - \eta_{ij}$. ■

The noncommutative relations (6.25) are not compatible with the constraints of Proposition 6.23. However, the new generators

$$
\hat\eta_{ij} = q_{ij}^{-1}\, \eta_{ij} \ , \qquad \hat\eta_{ij}^\perp = q_{ij}^{-1}\, \eta_{ij}^\perp
$$

enjoy the same commutation relations (6.25) as well as the orthogonal projector relations of Proposition 6.23. By Proposition 6.15, there is a natural isomorphism $\mathcal{S}_\theta \cong \hat\eta\,\big(\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))^{\oplus n}\big)$ of bundles on $\mathsf{Open}(\mathbb{G}\mathrm{r}_\theta(d;n))$, and we define the orthogonal complement of the tautological bundle $\mathcal{S}_\theta^\perp := \hat\eta^\perp\,\big(\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))^{\oplus n}\big)$. Note that the duality between the bundles $\mathcal{S}_\theta$ and $\mathcal{S}_\theta^\perp$ now also involves interchange of the block matrices $\theta_{(d)}$ and $\theta_{(n-d)}$ above. Denoting by $\mathcal{V}_\theta$ the trivial bimodule $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))\otimes V$, the noncommutative version of the exact sequence (6.11) of bundles is then given by

$$
0 \longrightarrow \big(\mathcal{S}_\theta^\perp\big)^\vee \xrightarrow{(\hat\eta^\perp)^*} \mathcal{V}_\theta \xrightarrow{\hat\eta} \mathcal{S}_\theta \longrightarrow 0 \ , \tag{6.27}
$$

and it follows from Theorem 6.24 that the sheaf of noncommutative differential forms is isomorphic to the braided tensor product

$$
{}^{\mathrm{co}-\mathcal{L}^\theta_{d,n}}\mathcal{F}^\theta_n \ \cong\ \mathcal{S}_\theta\, \widehat{\otimes}_\theta\, \mathcal{S}_\theta^\perp \tag{6.28}
$$

as a bimodule algebra over $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))$ in the category ${}_{\mathcal{H}^n_\theta}\mathscr{M}$.

The geometric meaning of the generators η_{ij} and $\eta_{ij}^\perp$ can be better understood by computing their transformation properties under the action of the torus $T = (\mathbb{C}^\times)^{d\,(n-d)}$.

Proposition 6.29. *The noncommutative fibration ${}^{\mathrm{co}-\mathcal{L}^\theta_{d,n}}\mathcal{F}^\theta_n$ is a T-equivariant bundle with eigenbasis generated by η_{ij}.*

Proof: We show that the generators η_{ij} are T-eigenvectors with respect to the left action of $(\mathbb{C}^\times)^{d\,(n-d)}$ induced by the algebra homomorphism (6.18) and the right coaction $\Phi^{\mathcal{L}_{d,n}} : \mathcal{F}^\theta_n \to \mathcal{F}^\theta_n \otimes \mathcal{L}^\theta_{d,n}$ given by

$$
\Phi^{\mathcal{L}_{d,n}}(g_{ij}) = \big(1 \otimes \pi^{(\mathcal{L}_{d,n})}\big)\, \Delta_\vee(g_{ij}) = g_{ij}^{(1)} \otimes \pi^{(\mathcal{L}_{d,n})}\big(g_{ij}^{(2)}\big) \ . \tag{6.30}
$$

Let H_a (resp. h_a), $a = 1, \dots, n$ be the toric generators in the enveloping algebra of $\mathrm{GL}(n)$ (resp. $L_{d,n-d}$). Dual to $\pi^{(\mathcal{L}_{d,n})}$, there is an injective algebra homomorphism $\iota^{(\mathcal{L}_{d,n})}$ between the corresponding enveloping algebras such that $\iota^{(\mathcal{L}_{d,n})}(h_a) = H_a$. Using results of §2.3, the image under $\iota^{(\mathcal{L}_{d,n})}$ of the left action (1.9) of the enveloping algebra of T dually induced by the the right coaction (6.30) of $\mathcal{L}^\theta_{d,n}$ on $\mathcal{F}^\theta_n$ is then given by

$$\begin{aligned} H_a \triangleright g_{ij} &= g_{ij}^{(1)} \big\langle h_a\,,\,\pi^{(\mathcal{L}_{d,n})}\big(g_{ij}^{(2)}\big)\big\rangle \\ &= \sum_{k=1}^n g_{ik}\,\big\langle \iota^{(\mathcal{L}_{d,n})}(h_a)\,,\,g_{kj}\big\rangle \\ &= \sum_{k=1}^n g_{ik}\,\langle H_a, g_{kj}\rangle \;=\; \delta_{aj}\;g_{ij}\;, \end{aligned} \tag{6.31}$$

where we have used the duality between $\pi^{(\mathcal{L}_{d,n})}$ and $\iota^{(\mathcal{L}_{d,n})}$. Similarly, one computes

$$\begin{aligned} H_a \triangleright S_\vee(g_{ij}) &= \big\langle h_a\,,\,\pi^{(\mathcal{L}_{d,n})}\big(S_\vee(g_{ij})_{(2)}\big)\big\rangle\,S_\vee(g_{ij})_{(1)} \\ &= \big\langle \iota^{(\mathcal{L}_{d,n})}(h_a)\,,\,S_\vee(g_{ij})_{(2)}\big\rangle\,S_\vee(g_{ij})_{(1)} \\ &= \sum_{k=1}^n \big\langle H_a\,,\,S_\vee(g_{ik})\big\rangle\,S_\vee(g_{kj}) \\ &= -\sum_{k=1}^n \langle H_a, g_{ik}\rangle\,S_\vee(g_{kj}) \;=\; -\,\delta_{ai}\;S_\vee(g_{ij})\;. \end{aligned} \tag{6.32}$$

Using (6.31) and (6.32), the left action of H_a on the left coinvariant generators η_{ij} is thus computed to be

$$H_a \triangleright \eta_{ij} = \sum_{k=1}^d \Big(\big(H_a \triangleright S_\vee(g_{ik})\big)\,g_{kj} + S_\vee(g_{ik})\,(H_a \triangleright g_{kj})\Big) = (\delta_{aj} - \delta_{ai})\;\eta_{ij}\;, \tag{6.33}$$

as required. ∎

By (6.33), we notice that the diagonal elements of the matrices η and $\eta^\perp$ are T-invariant. However, in contrast to the deformed products obtained by Drinfel'd twists of Hopf-module algebras (such as those defined in §2.2), they do not span a commutative ideal but rather only a commutative subalgebra, as one easily checks from the relations (6.25).

Example 6.34. *For $d = 1$, one has $\mathbb{G}\mathrm{r}_\theta(1;n) = (\mathbb{CP}^{n-1}_\theta)^*$ with $\theta = \theta_{(n-1)}$, and the Ore localization with respect to the embeddings above identifies the generators η_{ik} with the elements*

$$\tfrac{1}{n}\,y_k = \tfrac{1}{n}\,w_i^{-1}\,w_k$$

generating the degree 0 localized subalgebras of Theorem 5.4, as one readily checks using (6.25). The noncommutative affine subvarieties $U_\theta[\sigma_i]$, $i = 1,\dots,n$ constructed from each maximal cone σ_i in the fan Σ of $\mathbb{CP}^{n-1}$ are thus generated exactly by each row of the matrix η. By Example 6.16 one has a natural isomorphism $\mathcal{S}_\theta \cong \mathcal{O}_{\mathbb{CP}^{n-1}_\theta}(1)$, and in a similar vein $\mathcal{S}^\perp_\theta \cong \mathcal{O}_{\mathbb{CP}^{n-1}_\theta}(-1)$. By tensoring the exact sequence (6.27) from the right with the locally free sheaf $\mathcal{S}^\vee_\theta \cong \mathcal{O}_{\mathbb{CP}^{n-1}_\theta}(-1)$, and by using (6.28) and dualizing, one finds the Euler sequence

$$0 \longrightarrow {}^{\mathrm{co}-\mathcal{L}^\theta_{1,n}}\mathcal{F}^\theta_n \longrightarrow \mathcal{V}^\vee_\theta(-1) \longrightarrow \mathcal{O}_{\mathbb{CP}^{n-1}_\theta} \longrightarrow 0\;,$$

analogous to that of [25, §8.11]. In the commutative case, this sequence is dual to the description of the tangent bundle in terms of the surjective bundle map $\mathcal{O}_{\mathbb{CP}^{n-1}} \otimes V \to$

$\mathcal{O}_{\mathbb{CP}^{n-1}}$ *which evaluates global sections of the hyperplane bundle. The construction above provides a geometrical interpretation for the sequence of Example 6.10 which describes the bundle of Kähler differentials* $\Omega^1_{\mathbb{CP}^{n-1}_\theta}$.

References

[1] M. Artin, W. Schelter: *Graded algebras of global dimension* 3, Adv. Math. 66 (1987), 171–216.

[2] M. Artin, J. J. Zhang: *Noncommutative projective schemes*, Adv. Math. 109 (1994), 228–287.

[3] M. Artin, W. Schelter, J. Tate: *Quantum deformations of* $\mathrm{GL}(n)$, Commun. Pure Appl. Math. 44 (1991), 879–895.

[4] M. Artin, J. Tate, M. Van den Bergh: *Some algebras associated to automorphisms of elliptic curves*, Progr. Math. 86 (1990), 33–85.

[5] D. Auroux, L. Katzarkov, D. Orlov: *Mirror symmetry for weighted projective planes and their noncommutative deformations*, Ann. Math. 167 (2008), 867–943.

[6] V. Baranovsky, V. Ginzburg, A. Kuznetsov: *Quiver varieties and a noncommutative* $\mathbb{P}^2$, Compos. Math. 134 (2002), 283–318.

[7] A. Beilinson, V. Ginzburg, W. Soergel: *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. 9 (1996), 473–527.

[8] M. Bennai, E. H. Saidi: *Toric varieties with NC toric actions: NC type IIA geometry*, Nucl. Phys. B 667 (2004), 587–613.

[9] R. Berger, M. Dubois-Violette, M. Wambst: *Homogeneous algebras*, J. Algebra 261 (2003), 172–185.

[10] M. Cirafici, A. Sinkovics, R. J. Szabo: *Cohomological gauge theory, quiver matrix models and Donaldson–Thomas theory*, Nucl. Phys. B 809 (2009), 452–518.

[11] L. Cirio: *Twisted noncommutative equivariant cohomology: Weil and Cartan models*, preprint arXiv:0706.3602 [math.QA].

[12] L. Cirio, G. Landi, R. J. Szabo: *Algebraic deformations of toric varieties II. Noncommutative instantons*, in preparation.

[13] A. Connes, G. Landi: *Noncommutative manifolds, the instanton algebra and isospectral deformations*, Commun. Math. Phys. 221 (2001), 141–159.

[14] D. A. Cox: *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. 4 (1995), 17–50.

[15] D. A. Cox: *Toric varieties and toric resolutions*, Progr. Math. 181 (2000), 259–284; *What is a toric variety?*, Contemp. Math. 334 (2003), 203–223.

[16] R. Dijkgraaf, L. Hollands, P. Sulkowski: *Quantum curves and* $\mathcal{D}$*-modules*, J. High Energy Phys. 0911 (2009), 047.

[17] M. Dubois-Violette: *Noncommutative coordinate algebras*, preprint arXiv:0810.1114 [math.QA].

[18] P. Erdös, J. Lehner: *The distribution of the number of summands in the partitions of a positive integer*, Duke Math. J. 8 (1941), 335–345.

[19] R. Fioresi: *Quantum deformation of the grassmannian manifold*, J. Algebra 214 (1999), 418–447.

[20] W. Fulton, *Introduction to Toric Varieties* (Princeton University Press, 1993).

[21] I. Gel'fand, S. Gel'fand, V. Retakh, R. L. Wilson: *Quasideterminants*, Adv. Math. 193 (2005), 56–141.

[22] K. R. Goodearl, E. S. Letzter: *The Dixmier–Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras*, Trans. Amer. Math. Soc. 352 (1999), 1381–1403.

[23] C. Ingalls: *Quantum toric varieties*, preprint available at `http://kappa.math.unb.ca/~colin/research/nctoric.pdf`

[24] A. Iqbal, N. A. Nekrasov, A. Okounkov, C. Vafa: *Quantum foam and topological strings*, J. High Energy Phys. 0804 (2008), 011.

[25] A. Kapustin, A. Kuznetsov, D. Orlov: *Noncommutative instantons and twistor transform*, Commun. Math. Phys. 221 (2001), 385–432.

[26] C. Kassel: *Quantum groups*, Graduate Texts in Mathematics 155, (Springer–Verlag, 1995).

[27] A. Klimyk, K. Schmudgen: *Quantum Groups and their Representations* (Springer–Verlag, 1997).

[28] A. Lauve: *Quantum and quasi-Plücker coordinates*, J. Algebra 296 (2006), 440–461.

[29] A. Lauve: *Flag varieties for the yangian* $Y(\mathfrak{gl}(n))$, preprint arXiv:math/0601056 [math.QA].

[30] J. L. Loday: *Cyclic Homology* (Springer–Verlag, 1992).

[31] V. A. Lunts, A. L. Rosenberg: *Differential operators on noncommutative rings*, Sel. Math. New ser. 3 (1997), 335–359.

[32] S. Majid: *Foundations of Quantum Group Theory* (Cambridge University Press, 1995).
[33] S. Majid, R. Oeckl: *Twisting of quantum differentials and the Planck scale Hopf algebra*, Commun. Math. Phys. 205 (1999), 617–655.
[34] Yu. I. Manin: *Some remarks on Koszul algebras and quantum groups*, Ann. Inst. Fourier (Grenoble) 37 (1987), 191–205.
[35] M. Rieffel: *Deformation quantization for actions of* $\mathbb{R}^d$, Mem. Amer. Math. Soc. 506 (1993), 1–93.
[36] E. Taft, J. Towber: *Quantum deformation of flag schemes and Grassmann schemes. I. A q-deformation of the shape algebra for* $\mathrm{GL}(n)$, J. Algebra 142 (1991), 1–36.
[37] S. L. Woronowicz: *Differential calculus on compact matrix pseudogroups (quantum groups)*, Commun. Math. Phys. 122 (1989), 125–170.
[38] A. Yekutieli, J. J. Zhang: *Serre duality for noncommutative projective schemes*, Proc. Amer. Math. Soc. 125 (1997), 697–707.

Max Planck Institute for Mathematics, Vivatgasse 7, 53111 Bonn, Germany

E-mail address: cirio@mpim-bonn.mpg.de

Dipartimento di Matematica e Informatica, Università di Trieste, Via A. Valerio 12/1, I-34127 Trieste, Italy and INFN, Sezione di Trieste, Trieste, Italy

E-mail address: landi@univ.trieste.it

Department of Mathematics, Heriot-Watt University, Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, U.K. and Maxwell Institute for Mathematical Sciences, Edinburgh, U.K.

E-mail address: R.J.Szabo@ma.hw.ac.uk