LAGRANGIAN APPROACH TO SHEAVES OF VERTEX ALGEBRAS

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Abstract. We explain how sheaves of vertex algebras are related to mathematical structures encoded by a class of Lagrangians. The exposition is focused on two examples: the WZW model and the (1,1)-supersymmetric σ-model. We conclude by showing how to construct a family of vertex algebras with base the Barannikov-Kontsevich moduli space thus furnishing the B-model moduli for Witten's half-twisted model.

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partially supported by the National Science Foundation.

Introduction

More than anything else, the present notes are a report on what we have been able to make out of the recent papers by Kapustin and Witten [Kap,W4]. Even after the tremendous effort [QFS], much of mathematical literature treating various aspects of string theory and related topics is conspicuously lacking any mention of the Lagrangian, an object that is at the heart of a physical theory. We would like to make precise the relation of sheaves of vertex algebras [MSV] to a Lagrangian field theory.

What vertex algebras and Lagrangians have in common is that both produce infinite dimensional Lie algebras. If V is a vertex algebra, then, in particular, it is a vector space with multiplications ${}_{(n)}$, $n \in \mathbb{Z}$, and a derivation T. The corresponding Lie algebra is

$$\mathrm{Lie}(V) = (V/T(V), {}_{(0)}\,). \tag{0.1}$$

We will be concerned with the class of vertex algebras, or rather of sheaves thereof, introduced in [MSV]. To make our life easier, we will, first, mostly consider their quasiclassical limits, i.e., the corresponding vertex Poisson algebras and, second, work in the C^∞-setting. This class comprises vertex algebra analogues of sheaves of symbols of differential operators, and their natural habitat is different versions of ∞-jet spaces. To give an example, let Σ' be a 1-dimensional real manifold and consider $J^\infty(T^*M_{\Sigma'})$, the space of ∞-jets of sections of the trivial bundle

$$T^*M_\Sigma = T^*M \times \Sigma' \to \Sigma'.$$

We will find it convenient to work with families of such jet-spaces, to be denoted $J^\infty(T^*M_{\Sigma/\Sigma''})$, that are are naturally and similarly attached to a "time" fibration $\tau : \Sigma \to \Sigma''$ with fiber Σ'.

The push-forward of the structure sheaf $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ on M_Σ carries a structure of a sheaf of vertex Poisson algebras, a vertex analogue of the Poisson algebra of functions on T^*M. This sheaf is natural in that the assignment $M \to \mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ is functorial in M.

Such sheaves of *symbols of vertex differential operators*, SVDO, can be defined axiomatically and classified [GMS1]. A simple quiasiclassical version of the classification obtained in [GMS1] shows that locally (on M) all SVDOs are isomorphic and the set of isomorphisms classes is identified with $H^3(M, \mathbb{R})$. In particular, for each closed 3-form H, an SVDO $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})} \dot{+} H$ is defined. To each such SVDO construction (0.1) attaches a sheaf of Lie algebras on M

$$\mathrm{Lie}(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})} \dot{+} H) = (\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})} \dot{+} H)/T(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})} \dot{+} H). \tag{0.2}$$

One of the more interesting examples arises when M is a compact simple Lie group. In this case the family of isomorphism classes of SVDOs is 1-dimensional and we denote the representatives by $SD_{G,k}$, $k \in \mathbb{C}$. If we let $\mathfrak{g}$ be the corresponding Lie algebra and $V(\mathfrak{g})_k$ the corresponding vertex Poisson algebra, then there arise two Poisson-commuting subalgebras [F,FP,AG,GMS2]

$$V(\mathfrak{g})_k \overset{j_l}{\hookrightarrow} \Gamma(G, SD_{G,k}) \overset{j_r}{\hookleftarrow} V(\mathfrak{g})_{-k}. \tag{0.3}$$

engendered by the left/right translations of G by itself. This implies the existence of 2 copies of the affine Lie algebra (especially when Σ' is a circle)

$$\hat{\mathfrak{g}}_k \overset{j_l}{\hookrightarrow} \Gamma(G, \mathrm{Lie}(SD_{G,k})) \overset{j_r}{\hookleftarrow} \hat{\mathfrak{g}}_{-k}. \tag{0.3}$$

The case of $k \neq 0$ is exceptional; in this case the $\hat{\mathfrak{g}}_k \times \hat{\mathfrak{g}}_{-k}$-module structure of the SVDO has a form reminiscent of the WZW

$$\Gamma(G, \mathrm{Lie}(SD_{G,k})) = \bigoplus_{\lambda} V_{\lambda,k} \otimes V_{\lambda^*,-k}. \tag{0.4}$$

On the Lagrangian side one deals with a somewhat different jet-space, $J^\infty(M_\Sigma)$, where, recall, Σ is 2-dimensional. Defined on it there is a sheaf of variational bicomplexes, $(\Omega^{\bullet,\bullet}_{J^\infty(M_\Sigma)}, \delta, d)$. An action, S, is defined to be

$$S \in \Gamma(J^\infty(M_\Sigma), \Omega^{2,n}_{J^\infty(M_\Sigma)}/d\Omega^{1,n}_{J^\infty(M_\Sigma)}),\ n = \mathrm{dim} M. \tag{0.5}$$

and can be represented by a collection

$$\{L^j \in \Omega^{2,n}_{J^\infty(M_\Sigma)}\} \tag{0.6}$$

of locally defined Lagrangians, equal to each other modulo d-exact terms on double intersections. Action S also produces a Lie algebra, $\mathcal{I}_S$, the algebra of integrals of motion arising by virtue of the Nöther theorem. Let us relate this algebra to (0.2).

Each action (0.5) defines a space, Sol_S, often referred to as the solution space, which for our purposes had better be chosen to be the infinite dimensional submanifold of $J^\infty(M_\Sigma)$ defined by the Euler-Lagrange equations. Sol_S and the jet-spaces considered are examples of a *diffiety*, the notion introduced by A.M.Vinogradov [V].

Attached to each action is what is usually called a variational 2-form on Sol_S, ω_S. Generally speaking, it is not a form but a global section

$$\omega_S \in \Gamma(Sol_S, \Omega^{1,2}_{Sol_S}/d(\Omega^{0,2}_{Sol_S})) \tag{0.7}$$

annihilated by δ.

Special diffiety properties allow to attach to each such form, not necessarily coming from a Lagrangian, a Lie algebra structure on a certain subsheaf of "functions" on Sol_S. We call it a functional pre-symplectic structure and denote the corresponding sheaf of Lie algebras by $\mathcal{H}^{\omega_S}_{Sol_S}$. One has

$$\mathcal{I}_S \subset \Gamma(Sol_S, \mathcal{H}^{\omega_S}_{Sol_S}). \tag{0.8}$$

We have come to the point. For a class of Lagrangians, comprising those that are convex and of order 1, Sol_S is diffeomorphic to $J^\infty(TM_{\Sigma/\Sigma''})$ and there is a version of the Legendre transform

$$g : J^\infty(TM_{\Sigma/\Sigma''}) \to J^\infty(T^*M_{\Sigma/\Sigma''})$$

that delivers a Lie algebra sheaf isomorphism

$$g^\# : \mathcal{H}^{\omega_S}_{Sol_S} \xrightarrow{\sim} g^{-1}\mathrm{Lie}(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}) \tag{0.9}$$

in the case of a single globally defined Lagrangian.

Classification of SVDOs is also reflected in the Lagrangian world: given a globally defined Lagrangian L and a closed 3-form H, known as an H-flux, one defines, following e.g. [GHR,W1], a collection of Lagrangians L^H, such as in (0.6), so that there is an isomorphism

$$g^{\#}:\mathcal{H}^{\omega_S}_{Sol_S}\stackrel{\sim}{\longrightarrow} g^{-1}\mathrm{Lie}(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}\dot{+}H), \tag{0.10}$$

hence an embedding

$$\mathcal{I}_S\subset\Gamma(M,\mathrm{Lie}(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}). \tag{0.11}$$

As an illustration, let us relate the WZW model [W1, GW] to $SD_{G,k}$. It is conformally invariant meaning that 2 copies, left and right moving, of the Virasoro algebra are among the integrals of motion $\mathcal{I}_{WZW}$. Both are embedded into $\mathrm{Lie}(SD_{G,k})$ by virtue of (0.11). Precisely when the level, k, is non-zero, they coincide with the Virasoro algebra defined via the Sugawara construction inside the corresponding copies of the affine Lie algebra, see (0.3). The existence of the left/right moving Virasoro algebra allows to define the left/right moving subalgebra of $\mathrm{Lie}(SD_{G,k})$, or indeed, the left/right moving subalgebra of $SD_{G,k}$. Again, precisely when $k\neq 0$, the spaces of global sections of these coincide with copies of the affine Poisson vertex algebra $j_l(V(\mathfrak{g})_k)$ or $j_k(V(\mathfrak{g})_{-k})$ resp., see (0.3). This is an easy consequence of decomposition (0.4).

A disclaimer is in order: no attempt at originality is made. But it is also true that we failed to find an exposition of this material suitable for our purposes. Our main source of fact and inspiration was [Di], Ch.19; see also [DF,V,Z]. Needless to say, much of what has just been discussed is contained in one form or another in [BD], e.g. sect. 2.3.20, 3.9, but note that the meaning is somewhat different: we work in the C^∞-setting and our constructions are not necessarily chiral. In fact, one of our wishes was to understand "left and right movers", whose ubiquity in physics literature bedevils some of us in the mathematics community.

All of the above has a more or less straightforward superanalogue. As an example, we analyze the (1,1)-supersymmetric σ-model arising on a Riemannian manifold M. It is similarly governed by a super SVDO, Ω_M^{poiss}, which is a quasiclassical limit of the C^∞-version of the chiral de Rham complex [MSV]. The Lie superalgebra of integrals of motion contains two copies of the N=1 superconformal algebra, and we write explicit formulas for their embeddings into $\Gamma(M,\mathrm{Lie}(\Omega_M^{poiss}))$.

If, in addition, M is a Kähler manifold, then this symmetry algebra is enlarged to include 2 copies of the N=2 superconformal algebra, a remarkable fact known since [Z,A-GF]. If so, a quadruple of operators, $Q^{\bullet,\bullet}$, $\bullet=\pm$, arises, appropriate combinations of which are differentials on Ω_M^{poiss}. The 3 cohomology sheaves,

$$H_{Q^{--}+Q^{++}}(\Omega_M^{poiss}),\ H_{Q^{--}+Q^{+-}}(\Omega_M^{poiss}),\ H_{Q^{--}}(\Omega_M^{poiss}). \tag{0.12}$$

are versions of the quasiclassical limit of Witten's A-, B- and half-twisted models [W2]. Their cohomology can be computed using versions of the de Rham complex and $\bar\partial$ resolution; the result is $H^*(M,\mathbb{C})$, $H^*(M,\Lambda^*\mathcal{T}_M)$, and $H^*(M,\Omega_M^{poiss,an})$ resp., where the latter is a purely holomorphic version of Ω_M^{poiss}, the quasiclassical limit of the chiral de Rham complex [MSV].

Note that in the present situation the left/right moving subalgebra can also be defined by analogy with the WZW case considered above. Of the 3 cohomology vertex Poisson algebras, $H^*(M,\mathbb{C})$, $H^*(M,\Lambda^*\mathcal{T}_M)$, and $H^*(M,\Omega_M^{poiss,an})$, the last one is an infinite dimensional subquotient of the left moving algebra and is often referred to as Witten's chiral algebra [W3,W4]. The first two are also appropriate subquotients but are finite dimensional and of topological nature.

The construction of the sheaf complexes used, such as $(\Omega_M^{poiss}, Q^{--}+Q^{++})$, $(\Omega_M^{poiss}, Q^{--}+Q^{+-})$, $(\Omega_M^{poiss}, Q^{--})$, is easily quantized to produce the complexes of vertex algebra sheaves, $(\Omega_M^{vert}, Q^{--}+Q^{++})$, $(\Omega_M^{vert}, Q^{--}+Q^{+-})$, $(\Omega_M^{vert}, Q^{--})$. The sheaf cohomology of the first two remains the same, that of the 3rd is quite different and equals the cohomology of the chiral de Rham complex. When formulated in the physics language, the relation of this quantization to the genuine supersymmetric model is that the latter equals the former "perturbatively"; this is the main result of Kapustin [Kap]. It seems, however, that the emphasis made in [Kap] on the "infinite metric limit" is somewhat disingenuous: the above supersymmetries depend on a genuine metric and its Kähler form..

By skillfully applying the techniques of SUSY vertex algebras [HK], Ben-Zvi, Heluani, and Szczesny have recently solved [B-ZHS] a harder problem of finding quantum versions of the above mentioned embeddings of the various superconformal algebras in the chiral de Rham complex. Our discussion seems to indicate that their quantization is related to physics (apart from non-perturbative effects) in about the same way the known quantization of (0.3) [FP,F,AG,GMS2] is related to the quantum WZW: it mixes the chiral and anti-chiral sectors. At this point we can only ask, following [FP], if there is a physical model of interest whose chiral algebra is as in [B-ZHS].

The differential Q^{--} of the (quantum) complex $(\Omega_M^{vert}, Q^{--})$ can be deformed. One of the ways to think about the complex $(\Omega_M^{vert}, Q^{--})$ is that it is a vertex algebra version of the the $\bar{\partial}$-resolution of the algebra of polyvector fields. We conclude by showing that the Barannikov-Kontsevich construction [BK] has a vertex algebra analogue: we define a family of vertex algebras with base the Barannikov-Kontsevich moduli space $\mathcal{M}_{\mathrm{BK}}$ by assigning to each $t\in\mathcal{M}_{\mathrm{BK}}$ an element $Q_t^{--}\in\Gamma(M,\Omega_M^{vert})$ and a vertex algebra $H_{Q_t^{--}}(\Gamma(M,\Omega_M^{vert}))$.

The conformal weight zero subspace of the family $t\to H_{Q_t^{--}}(\Gamma(M,\Omega_M^{vert}))$ encodes precisely the Frobenius manifold introduced in [BK]. Our construction amounts to defining a morphism of the deformation functor of the Lie algebra of polyvector fields to that of $\Gamma(M,\mathrm{Lie}(\Omega_M^{vert}))$.

This furnishes the B-model moduli for Witten's half-twisted model. The instanton effects seem to be out of reach, to us; see, however, an intriguing sentence in [W4] and [FL] for an interesting new approach based on marginal operators.

Acknowledgments. The author thanks V.Gorbounov, A.Kapustin, and B.Khesin for illuminating discussions. Parts of this work were done while the author was visiting the Fields Institute, IHES, Max-Planck-Institut in Bonn, and Erwin Schrödinger Institut in Vienna. It is a pleasure to acknowledge the support, hospitality, and stimulating atmosphere of these institutions.

1. Diffieties and Homotopy Pre-symplectic Structures

The geometry of jet-spaces is a huge and familiar topic, see sources such as [BD, Di, Ol, V]. The purpose of this section is to introduce a "universal" sheaf of Lie algebras $\mathcal{H}^{can}$, which contains the algebra of symmetries of a class of Lagrangians, see Lemma 1.6.8.1.

1.1. The jets.

Assume given a d-dimensional C^∞-manifold Σ, the "world sheet", and a smooth fiber bundle

$$\tau : \Sigma \rightarrow \Sigma'' \tag{1.1.1}$$

with base Σ'', an open subset of $\mathbb{R}$, and fiber Σ', a $(d-1)$-dimensional manifold. This is the minimal requirement; most of the time, we will have the Cartesian product

$$\Sigma' \times \Sigma'' \tag{1.1.2}$$

with a fixed coordinate (" intrinsic time")

$$\tau = \sigma^0 : \ \Sigma'' \rightarrow \mathbb{R}, \tag{1.1.3}$$

on Σ'' and étale coordinates

$$\sigma = (\sigma^1, ..., \sigma^{d-1}) : \ \mathbb{R}^{d-1} \rightarrow \Sigma' \tag{1.1.4}$$

on Σ', so that (1.1.4) is a universal cover – étale because we would like to include the case of a torus; furthermore, we will be mostly interested in $d = 1$.

Let M, "space-time", be an an n-dimensional C^∞-manifold and $M_\Sigma = M \times \Sigma$. There arises a fiber bundle $M_\Sigma \rightarrow \Sigma$, and we denote by $J^k(M_\Sigma)$ the space of k-jets of its sections.

Each $J^k(M_\Sigma)$ is a finite dimensional C^∞-manifold, and the natural projections

$$\pi_{k,l} : J^k(M_\Sigma) \longrightarrow J^l(M_\Sigma), \ k \geq l$$

organize the collection $\{J^k(M_\Sigma), k \geq 0\}$ in a projective system. The space of ∞-jets, $J^\infty(M_\Sigma)$, is the projective limit, $\lim_{\leftarrow} J^k(M_\Sigma)$. Its sheaf of smooth functions is the direct limit of those on $J^k(M_\Sigma)$, ≥ 0. Let $\mathcal{O}_{J^k(M_\Sigma)}$ be the sheaf of all smooth functions on $J^k(M_\Sigma)$ that are polynomials in positive order jets and define

$$\mathcal{O}_{J^\infty(M_\Sigma)} = \lim_{\rightarrow} \mathcal{O}_{J^k(M_\Sigma)}.$$

The fiber bundle $J^\infty(M_\Sigma) \rightarrow \Sigma$ carries the well-known flat connection, or equivalently, $\mathcal{O}_{J^\infty(M_\Sigma)}$ is a sheaf of D_Σ-algebras, where D_Σ is the sheaf of differential operators on Σ. Denote by

$$\rho : \mathcal{T}_\Sigma \longrightarrow \mathcal{T}_{J^\infty(M_\Sigma)} \tag{1.1.5}$$

the corresponding morphism of the sheaves of vector fields. We will often refer to this situation by calling $J^\infty(M_\Sigma) \rightarrow \Sigma$ a D_Σ-manifold, thus mimicking [BD].

In particular, attached to any tangent vector $\xi \in T_t\Sigma$, there is a tangent vector $\rho(\xi) \in T_{(x,t)}(J^\infty(M_\Sigma))$. Hence there arises an integrable d-dimensional distribution

$$M_\Sigma \ni (x,t) \mapsto \ \text{span}\{\hat{\xi}, \ \xi \in T_t\Sigma\} \subset T_{(x,t)}J^\infty(M_\Sigma)$$

known as the *Cartan distribution.*

$J^\infty(M_\Sigma)$ is a simple example of what A.M.Vinogradov calls a *diffiety*. Since the Cartan distribution is an important structure ingredient, by an infinitesimal automorphism of $J^\infty(M_\Sigma)$ one means a *contact* vector field [V], i.e., a vector field that preserves the Cartan distribution. Locally defined contact vector fields form a Lie algebra subsheaf $\mathcal{C}_{J^\infty(M_\Sigma)} \subset \mathcal{T}_{J^\infty(M_\Sigma)}$. Call a contact vector field *evolutionary* if it is tangent to the fibers of the projection $J^\infty(M_\Sigma) \to \Sigma$ and let $\mathrm{Evol}(J^\infty(M_\Sigma))$ denote the sheaf of all evolutionary vector fields. Of course, $\mathrm{Evol}(J^\infty(M_\Sigma)) \subset \mathcal{C}_{J^\infty(M_\Sigma)} \subset \mathcal{T}_{J^\infty(M_\Sigma)}$ are embeddings of Lie algebra sheaves.

All of this admits a relative version: if one defines $J^\infty(M_{\Sigma/\Sigma''})$ to be the space of jets of sections $\Sigma \to M_\Sigma$ in the direction of fibers of the bundle $\Sigma \to \Sigma''$, then the definitions of the connection

$$\rho/\Sigma'' : \mathcal{T}_{\Sigma/\Sigma''} \to \mathcal{T}_{J^\infty(M_{\Sigma/\Sigma''})}, \tag{1.1.6}$$

(where $\mathcal{T}_{\Sigma/\Sigma''}$ is the sheaf of vector fields on Σ tangent to the fibers of the projection $\Sigma \to \Sigma''$), Cartan distribution, sheaf of evolutionary vector fields $\mathrm{Evol}(J^\infty(M_{\Sigma/\Sigma''}))$, etc., are immediate. Note that since $\rho/\Sigma''(\mathcal{T}_{\Sigma/\Sigma''}) \subset \rho(\mathcal{T}_\Sigma)$,

$$\mathrm{Evol}(J^\infty(M_{\Sigma/\Sigma''})) \subset \mathrm{Evol}(J^\infty(M_\Sigma)). \tag{1.1.7}$$

From now, unless otherwise stated, we will be working over base S, S being either Σ'' or a point.

If (1.1.2-4) are valid, then the fibers of the projection $J^\infty(M_\Sigma) \to \Sigma$ are canonically identified, and

$$J^\infty(M_\Sigma) \xrightarrow{\sim} J^\infty_\Sigma(M) \times \Sigma,\ J^\infty(M_{\Sigma/\Sigma''}) \xrightarrow{\sim} J^\infty_{\Sigma/\Sigma''}(M) \times \Sigma \tag{1.1.8}$$

for some infinite dimensional manifolds, $J^\infty_\Sigma(M)$ and $J^\infty_{\Sigma/\Sigma''}(M)$, whose definition is easy to reconstruct from 1.2 below.

1.2. Local Formulas

As an illustration, and for future use, let us show what all of this means in terms of local coordinates. Let $x^1, ..., x^n$ be local coordinates on M, those on Σ being defined by (1.1.2-4). For a multi-index $(m) = (m_0, ..., m_{d-1})$, let

$$x^j_{(m)} = \partial^{m_0}_{\sigma^0} \cdots \partial^{m_{d-1}}_{\sigma^{d-1}} x^j,$$

where $\partial_{\sigma^m} = \partial/\partial\sigma^m$ and x^j is regarded, formally, as a function of $\sigma^0, ..., \sigma^{d-1}$. Then

$$\{\sigma^i, x^j_{(m)} :\ 0 \le i \le d-1, 1 \le j \le n, (m) \in \mathbb{Z}^d_+\}$$

are local coordinates on $J^\infty(M_\Sigma)$

$$\{\sigma^i, x^j_{(m)} :\ 0 \le i \le d-1, 1 \le j \le n, (m) \in \mathbb{Z}^d_+, m_0 = 0\}$$

are local coordinates on $J^\infty(M_{\Sigma/\Sigma''})$, and sections of $\mathcal{O}_{J^\infty(M_{\Sigma/S})}$ are smooth functions in σ^i, x^j and polynomials in $x^j_{(m)}$, $(m) \neq 0$, with $m_0 = 0$ if $S = \Sigma''$.

Let $\delta/\delta x^j_{(m)}$ denote the vertical vector field $\partial/\partial x^j_{(m)} \in \mathcal{T}_{J^\infty(M_\Sigma)}$. Morphism (1.1.5) is defined by

$$\rho(\partial_{\sigma^i}) = \partial_{\sigma^i} + \sum_{j=1}^n \sum_{(m)\in\mathbb{Z}^d_+} x^j_{(m+e_i)} \frac{\delta}{\delta x^j_{(m)}}, \tag{1.2.1}$$

where $e_i=(0,...,0,1,0,...,0)$, 1 appearing at the i-th position.

Evolution vector fields are in 1-1 correspondence with n-tuples of functions $F^1,...,F^n\in\mathcal{O}_{J^\infty(M_\Sigma)}$ (called the characteristic of a vector field) and are written

$$\xi=\sum_{j=1}^{n}\left(F^j\frac{\delta}{\delta x^j}+\sum_{(m)\neq 0}\left(\rho(\partial_{\sigma^0})^{m_1}\cdots\rho(\partial_{\sigma^{m_{d-1}}})^{m_{d-1}}F^j\right)\frac{\delta}{\delta x^j_{(m)}}\right) \tag{1.2.2}$$

The relative analogs of (1.2.1,2) are obvious.

1.3. De Rham Complex.

Reflecting the product structure of M_Σ, the de Rham complex on $J^\infty(M_{\Sigma/S})$ is bi-graded:

$$\Omega^p_{J^\infty(M_{\Sigma/S})}=\oplus_{i+j=p}\Omega^{i,j}_{J^\infty(M_{\Sigma/S})}.$$

It carries 2 anti-commuting differentials:

$$\delta:\Omega^{i,j}_{J^\infty(M_\Sigma)}\rightarrow\Omega^{i,j+1}_{J^\infty(M_\Sigma)},\ d_{\rho/S}:\Omega^{i,j}_{J^\infty(M_{\Sigma/S})}\rightarrow\Omega^{i+1,j}_{J^\infty(M_{\Sigma/S})},\ [\delta,d_{\rho/S}]=0,$$

defined as follows. The space, or rather the sheaf of spaces, $\Omega^{*,*}_{J^\infty(M_{\Sigma/S})}$ is naturally an $\mathcal{O}_\Sigma$-module, and δ is the vertical de Rham differential, i.e., the one that is $\mathcal{O}_\Sigma$-linear. The flat connection ρ gives rise to a differential, $d_{\rho/S}$, on $\Omega^{*,0}_{J^\infty(M_{\Sigma/S})}$ in the standard manner. For example, in terms of local cordinates,

$$d_{\rho/S}x^j_{(m)}=\sum_{i=\epsilon}^{d-1}x^j_{(m)+e_i}d\sigma^i, \tag{1.3.1}$$

where $\epsilon=0$ if $S=\emptyset$ and $\epsilon=1$ if $S=\Sigma''$.

Then the condition $[\delta,d_{\rho/S}]$ allows one to extend $d_{\rho/S}$ to the entire $\Omega^{*,*}_{J^\infty(M_{\Sigma/S})}$ unambiguously. Thus

$$\delta F(\sigma,x_{(m)})=\frac{\partial F(\sigma,x_{(m)})}{\partial x_{(m)}}\delta x_{(m)};\ \delta d\sigma=0, \tag{1.3.2}$$

$$d_{\rho/S}\delta x^j_{(m)}=-\delta d_{\rho/S}x^j_{(m)}=-\sum_{i=\epsilon}^{d-1}\delta x^j_{(m)+e_i}\wedge d\sigma^i, \tag{1.3.3}$$

There is a mapping of bi-complexes

$$\Omega^{*,*}_{J^\infty(M_\Sigma)}\rightarrow\Omega^{*,*}_{J^\infty(M_{\Sigma/\Sigma''})}, \tag{1.3.4}$$

that sends a form to its restriction to the fibers of the composite projection

$$J^\infty(M_\Sigma)\rightarrow\Sigma\xrightarrow{\tau}\Sigma''. \tag{1.3.5}$$

As a practical matter, (1.3.4) amounts to

$$d\tau\mapsto 0. \tag{1.3.6}$$

Let ι_ξ be the operator of contraction with a vector field ξ. A straightforward computation proves the following.

1.3.1. Lemma. *A vertical vector field ξ on $J^\infty(M_{\Sigma/S})$ is evolutionary iff*

$$[\iota_\xi, d_{\rho/S}] = 0. \tag{1.3.7}$$

1.3.2. Corollary. *If ξ is evolutionary, then*

$$[d_{\rho/S}, \mathrm{Lie}_\xi] = 0. \tag{1.3.8}$$

Indeed,

$$[d_{\rho/S}, \mathrm{Lie}_\xi] = [d_{\rho/S}, [\delta, \iota_\xi]] = -[\delta, [d_{\rho/S}, \iota_\xi]] = 0.$$

1.4. Differential Equations.

Let $\mathcal{J} \subset \mathcal{O}_{J^\infty(M_\Sigma)}$ be a sheaf of ideals closed under the connection ρ. Let

$$Sol \subset J^\infty(M_\Sigma)$$

be the zero locus of this ideal. If some regularity conditions hold, then his submanifold delivers another example of a diffiety. For example, one can, and we will, assume that $\mathcal{J}$ is locally pseudo-Cauchy-Kovalevskaya, i.e., there is a distinguished coordinate on Σ, say, τ, and for any point in M there is a coordinate system $x^1, ..., x^n$ s.t. the ideal is generated, around the pre-image of this point on the jet space, by the functions $E^1, ..., E^n$ satisfying

$$E^j = \rho(\partial_\tau)^{l_j} x^j + \cdots, \tag{1.4.1}$$

where $\cdots$ stand for the terms that do not involve jets of x's of degree $\geq l_j$ in the direction of ∂_τ.

Here are some of the structure properties Sol shares with the ambient jet space:

Sol is fibered over Σ, hence over Σ'' via $\Sigma \xrightarrow{\tau} \Sigma''$;

the algebra of functions $\mathcal{O}_{Sol}$ is a D_Σ-algebra (because the flat connection preserves Sol), hence a $D_{\Sigma/\Sigma''}$-algebra, where $D_{\Sigma/\Sigma''}$ is the subalgebra of D_Σ that commutes with $\tau^{-1}\mathcal{O}_{\Sigma''}$; we will write $Sol_{\Sigma''}$ if we wish to emphasize the $D_{\Sigma/\Sigma''}$-algebra structure;

the de Rham complex $\Omega^*_{Sol/S}$ is bi-graded and carries two commuting differentials, δ, the vertical differential, and $d_{\rho/S}$, the $D_{\Sigma/S}$-module differential.

If (1.4.1) is valid, then solving the equation $E^j = 0$ for $\rho(\partial_\tau)^{l_j} x^j$ one sees that Sol looks like ∞-jets in the direction of the fiber of the bundle $\Sigma \xrightarrow{\tau} \Sigma''$ to something finite dimensional. In particular, if $l_1 = l_2 = \cdots = l_n = 2$, then

$$Sol \xrightarrow{\sim} J^\infty(TM_{\Sigma/\Sigma''}), \tag{1.4.2}$$

as $D_{\Sigma/\Sigma''}$-manifolds.

Any evolutionary vector field on the ambient jet space that preserves the ideal $\mathcal{J}$ descends to a vector field on Sol, which still satisfies (1.3.7). We emulate this situation by making the following definition.

1.4.1. Definition. *A vertical vector field ξ on Sol is called evolutionary (relative to S) if*

$$[\iota_\xi, d_{\rho/S}] = 0. \tag{1.4.3}$$

1.4.2. Lemma. *If ξ is evolutionary, then*

$$[d_{\rho/S}, \mathrm{Lie}_\xi] = 0. \tag{1.4.4}$$

1.4.3. Let $\mathrm{Evol}(Sol)_S$ denote the sheaf of all evolutionary vector fields on Sol relative to S. The identity $[\mathrm{Lie}_\xi, \iota_\eta] = \iota_{[\xi,\eta]}$ combined with (1.4.3,4) implies that $\mathrm{Evol}(Sol)_S$ is a Lie algebra.

Note that, cf. (1.1.7),

$$\mathrm{Evol}(Sol) \subset \mathrm{Evol}(Sol)_{\Sigma''}. \tag{1.4.5}$$

1.5. A functional pre-symplectic structure.

A symplectic form, that is, a non-degenerate closed 2-form gives rise to a Poisson algebra structure on the structure sheaf of a manifold. A degenerate closed 2-form similarly gives rise to a Poisson algebra structure on a certain, *admissible*, subalgebra of the structure sheaf. This subalgebra consists of functions constant along the leaves of the foliation tangent to the kernel of the form [Fad]. We would like to explain, in the spirit of [DF,Di], that in the case of a diffiety, such as Sol, a pre-symplectic structure gives rise to a Poisson structure, which may be just as good for all practical purposes as the symplectic one. This notion is a rather straightforward geometric version of Dorfman's symplectic operator [D].

1.5.1. The following is a list of standard symplectic geometry notions adjusted to the case where vector fields are replaced with evolutionary vector fields and equalities are valid up to $d_{\rho/S}$-exact terms.

From now on $\mathcal{M}$ is a diffiety, such as $Sol \subset J^\infty(M_\Sigma)$ or $J^\infty(M_{\Sigma/S})$.

The relation $[\delta, d_{\rho/S}] = 0$ implies that δ descends to a differential

$$\bar\delta : \bar\Omega^{\bullet,\bullet}_{\mathcal{M}/S} \to \bar\Omega^{\bullet,\bullet+1}_{\mathcal{M}/S}, \text{ where } \bar\Omega^{\bullet,\bullet}_{\mathcal{M}/S} = \Omega^{\bullet,\bullet}_{\mathcal{M}/S}/d_{\rho/S}\Omega^{\bullet-1,\bullet}_{\mathcal{M}/S}. \tag{1.5.1}$$

Elements of the quotient complex $\bar\Omega^{\bullet,\bullet}_{\mathcal{M}/S}$ are often referred to as *functional forms* [Ol].

We will call $\bar\omega \in H^0(\mathcal{M}, \bar\Omega^{d-1,2}_{\mathcal{M}/S})$ a *functional pre-symplectic form* if

$$\bar\delta\bar\omega = 0. \tag{1.5.2}$$

Note that if ξ is an evolutionary vector field, then Lie_ξ and ι_ξ are well-defined operators acting on the quotient complex $\bar\Omega^{\bullet,\bullet}_{\mathcal{M}/S}$ thanks to (1.4.3-4).

An evolutionary vector field $\xi \in \mathrm{Evol}(\mathcal{M})_{\Sigma/S}$ is called *Hamiltonian* if

$$\mathrm{Lie}_\xi\bar\omega = 0. \tag{1.5.3}$$

Call a functional form $\bar F\in\bar\Omega^{d-1,0}_{\mathcal{M}/S}$ *admissible* relative to $\bar\omega$ if there is an evolutionary vector field ξ_F such that

$$\bar\delta\bar F=\iota_{\xi_{\bar F}}\bar\omega. \tag{1.5.4}$$

Note that (1.5.4) implies that $\xi_{\bar F}$ is Hamiltonian, which prompts one to think of $\bar F$ as a Hamiltonian associated to $\xi_{\bar F}$. Hence the following bit of notation: let $\mathcal{H}^{\bar\omega}_{\mathcal{M}/S}$ be the sheaf of all functional $(d-1,0)$-forms that are admissible relative to $\bar\omega$.

Note that (1.5.4) implies that ξ is Hamiltonian and that although an admissible $\bar F$ does not determine $\xi_{\bar F}$ uniquely it does so up to the kernel of $\bar\omega$: any two such vector fields $\xi_{\bar F},\eta_{\bar F}$ satisfy

$$\iota_{\xi_{\bar F}-\eta_{\bar F}}\bar\omega=0. \tag{1.5.5}$$

For any admissible $\bar F,\bar G$ define the bracket

$$\{\bar F,\bar G\}=\xi_{\bar F}\bar G. \tag{1.5.6}$$

Equivalently,

$$\{\bar F,\bar G\}=\iota_{\xi_{\bar F}}\bar\delta\bar G=\iota_{\xi_{\bar F}}\iota_{\xi_{\bar G}}\bar\omega, \tag{1.5.7}$$

which shows, by virtue of (1.5.5), that (1.5.6) is independent of the choice of $\xi_{\bar F}$.

Next,

$$\{\bar F,\bar G\}\in\mathcal{H}^{\bar\omega}_{\mathcal{M}/S}. \tag{1.5.8}$$

In fact,

$$\bar\delta\{\bar F,\bar G\}=\iota_{[\xi_{\bar F},\xi_{\bar G}]}\bar\omega, \tag{1.5.9}$$

as the following computation (based on a repeated use of (1.5.2-3))shows:

$$\bar\delta\{\bar F,\bar G\}=\bar\delta(\xi G)=\bar\delta\left(\iota_{\xi_{\bar F}}\iota_{\xi_{\bar G}}\bar\omega\right)=\mathrm{Lie}_{\xi_{\bar F}}\iota_{\xi_{\bar G}}\bar\omega-\iota_{\xi_{\bar F}}\mathrm{Lie}_{xi_{\bar G}}\bar\omega+\iota_{\xi_{\bar F}}\iota_{\xi_{\bar G}}\bar\delta\bar\omega=$$
$$\iota_{[\xi_{\bar F},\xi_{\bar G}]}\bar\omega+\iota_{\xi_{\bar G}}\mathrm{Lie}_{\xi_{\bar F}}\bar\omega=\iota_{[\xi_{\bar F},\xi_{\bar G}]}\bar\omega.$$

Therefore, we obtain the map

$$\{.,.\}:\mathcal{H}^{\bar\omega}_{\mathcal{M}/S}\times\mathcal{H}^{\bar\omega}_{\mathcal{M}/S}\to\mathcal{H}^{\bar\omega}_{\mathcal{M}/S},\ (\bar F,\bar G)\mapsto\{\bar F,\bar G\}=\xi_{\bar F}\bar G. \tag{1.5.10}$$

1.5.2. Proposition.
Map (1.5.10) makes $\mathcal{H}^{\bar\omega}_{\mathcal{M}/S}$ into a sheaf of Lie algebras.

Proof. The antisymmetry of (1.5.10) is an immediate consequence of (1.5.7). The Jacobi identity is proved as follows:

$$\{\{F,G\},H\}=[\xi,\eta]H=\xi\eta H-\eta\xi H=\{F,\{G,H\}\}-\{G,\{F,H\}\},$$

where the first equality is a consequence (1.5.9).

1.5.3. The absolute and relative versions of this construction can be compared. Indeed, by virtue of (1.4.5), morphism of bi-complexes (1.3.4) induces a morphism of the Lie algebra sheaves

$$\mathcal{H}^{\bar\omega}_{\mathcal{M}}\to\mathcal{H}^{\bar\omega}_{\mathcal{M}/\Sigma''}, \tag{1.5.11}$$

which in terms of local coordinates amounts to

$$d\tau\mapsto 0, \tag{1.5.12}$$

cf. (1.3.6).

1.5.4. Example 1: canonical commutation relations. Replace M as the target space with T^*M and consider $J^\infty(T^*M_{\Sigma/\Sigma''})$ with $d = dim\Sigma = 2$. There arises the projection

$$\pi:\ J^\infty(T^*M_{\Sigma/\Sigma''}) \to T^*M_{\Sigma/\Sigma''}. \qquad (1.5.13)$$

Let ω^o be the canonical symplectic form on T^*M and $\omega = \omega^o \wedge d\sigma$; the latter is a $(1,2)$-form on $T^*M_{\Sigma/\Sigma''}$ – we are taking advantage of coordinates (1.1.2-4). There arises then $\pi^*\omega$, the pull-back of ω onto $J^\infty(T^*M_{\Sigma/\Sigma''})$ under (1.5.13). Let us now compute $\mathcal{H}^{\overline{\pi^*\omega}}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$.

1.5.4.1. Lemma.

$$\mathcal{H}^{\overline{\pi^*\omega}}_{J^\infty(T^*M_{\Sigma/\Sigma''})} = \bar{\Omega}^{1,0}_{J^\infty(T^*M_{\Sigma/\Sigma''})}. \qquad (1.5.13)$$

Informally speaking, this lemma says that in this case any function is admissible,

hence $\overline{\pi^*\omega}$ is as good as symplectic.

To prove (1.5.13), note that, ω^o being non-degenerate, any section of $\Omega^{1,1}_{T^*M_{\Sigma/\Sigma''}}$ can be written as $\iota_{\xi^o}\omega$ for some vector field ξ^o on T^*M_Σ. Pulling back on $J^\infty(T^*M_{\Sigma/\Sigma''})$ one sees that likewise any section of $\pi^*\Omega^{1,1}_{T^*M_{\Sigma/\Sigma''}}$ can be written as $\iota_{\xi^o}\pi^*\omega$, where ξ^o is now a vector field on $J^\infty(T^*M_{\Sigma/\Sigma''})$, which is locally a linear combination of $\delta/\delta x^i$, x^i being local coordinates on T^*M. Thinking of ξ^o as a characteristic, one can prolong it to an evolutionary vector field ξ, as in (1.2.2), and thus obtain

$$\pi^*\Omega^{1,1}_{T^*M_{\Sigma/\Sigma''}} = \{\iota_\xi\pi^*\omega,\ \xi \in \mathrm{Evol}(J^\infty(T^*M_{\Sigma/\Sigma''}))\} \qquad (1.5.14).$$

Now observe that $\Omega^{1,1}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ is generated, as a $D_{\Sigma/\Sigma''}$-module, by $\pi^*\Omega^{1,1}_{T^*M_{\Sigma/\Sigma''}}$. Hence (1.5.14) holds true for the entire $\Omega^{1,1}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ modulo $d_{\rho/\Sigma''}$-exact terms, i.e.,

$$\Omega^{1,1}_{J^\infty(T^*M_{\Sigma/\Sigma''})} = \{\iota_\xi\pi^*\omega + d_{\rho/\Sigma''}\beta,\ \xi \in \mathrm{Evol}(J^\infty(T^*M_{\Sigma/\Sigma''})), \beta \in \Omega^{0,1}_{J^\infty(T^*M_{\Sigma/\Sigma''})}\} \qquad (1.5.15)$$

and (1.5.13) follows.

Because of its importance, the sheaf of Lie algebras arising in this way will be denoted thus

$$\mathcal{H}^{can} \stackrel{\text{def}}{=} \mathcal{H}^{\overline{\pi^*\omega}}_{J^\infty(T^*M_{\Sigma/\Sigma''})} \qquad (1.5.16)$$

Computationally, the gist of our discussion is as follows. The algebra of functions on the cotangent bundle with the canonical Poisson bracket is a Lie subalgebra of $\mathcal{H}^{can}$:

$$\pi^\#:\ \left(\pi^{-1}\mathcal{O}_{T^*M_{\Sigma/\Sigma''}}d\sigma, \{.,.\}_{T^*M}\right) \hookrightarrow \mathcal{H}^{can}, \qquad (1.5.17)$$

and the rest of the Lie algebra structure is determined by

$$\{\overline{Fd\sigma}, \overline{GHd\sigma}\} = \overline{\{Fd\sigma, Gd\sigma\}H} + \overline{G\{Fd\sigma, Hd\sigma\}}, \qquad (1.5.18a)$$

$$\{\overline{Fd\sigma}, \overline{G\rho(\partial_\sigma)Hd\sigma}\} = \overline{\{Fd\sigma, Gd\sigma\}\rho(\partial_\sigma)H} + \overline{G\rho(\partial_\sigma)\{Fd\sigma, Hd\sigma\}}, \qquad (1.5.18b)$$

because an evolutionary vector field is a derivation commuting with $\rho(\partial_\sigma)$, see e.g.(1.4.4.).

To see what all of this means, let us compute some brackets.Let F, G be functions on $M_{\Sigma/\Sigma''}$, ξ, η vector fields on M_Σ vertical w.r.t. $M \to \Sigma$, which we regard as fiberwise linear functions on T^*M_Σ. Then,

$$\{\overline{Fd\sigma}, \overline{Gd\sigma}\} = 0, \tag{1.5.19a}$$

$$\{\overline{\xi d\sigma}, \overline{Gd\sigma}\} = \overline{\xi G d\sigma}, \tag{1.5.19b}$$

$$\{\overline{\xi d\sigma}, \overline{\eta d\sigma}\} = \overline{[\xi, \eta] d\sigma}. \tag{1.5.19c}$$

The first instance of the bracket jet nature manifesting itself is as follows. If $F_i dx^i$ is a 1-form on M,then $\alpha = F_i(x)\rho(\partial_\sigma)x^i$ is a well-defined $(0,0)$-form on $J^\infty(T^*M_\Sigma)$. Having thus embedded Ω^1_M into $\Omega^{0,0}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$, one uses (1.5.19a-b) to obtain

$$\{\overline{\xi d\sigma}, \overline{\alpha d\sigma}\} = \overline{(\mathrm{Lie}_\xi \alpha) d\sigma}, \tag{1.5.19d}$$

if ξ does not depend on σ explicitly, and

$$\{\overline{\xi d\sigma}, \overline{\alpha d\sigma}\} = \overline{(\mathrm{Lie}_\xi \alpha + \iota_{\partial_\sigma \xi}\alpha) d\sigma} \tag{1.5.19e}$$

in general, where $\mathrm{Lie}_\xi \alpha$ is the Lie derivative of α along ξ.

Formulas (1.5.19a–d), without functions explicitly depending on τ, σ, are a familiar definition of the Lie algebra associated with the *Courant algebroid* on $TM \oplus T^*M$. The idea that the Courant algebroid has infinite dimensional nature apparently goes back to I.Dorfman [Dor]. It was revived recently, in a slightly different context, by P.Bressler [Bre].

Note that identities (1.5.19a,b) seem to incorporate the Leibnitz identity, which they do not, because $\mathcal{H}^{can}$ is not an associative algebra. It is, however, a quotient of $\Omega^{1,0}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$, and the latter is. In fact, $\Omega^{1,0}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ is a sheaf of vertex Poisson algebras, and its quotient $\mathcal{H}^{can}$ is the canonically associated to it sheaf of Lie algebras, see Proposition 2.7.3 below.

1.5.5. Example 2: the solution space of an order 2 system in the pseudo-Cauchy-Kovalevskaya form. Let us place ourselves in the situation of 1.4 and let Sol satisfy (1.4.2). Then

$$Sol_{\Sigma''} \xrightarrow{\sim} J^\infty(TM_{\Sigma/\Sigma''}).$$

The latter does not carry any canonical 2-form, but let us fix a diffeomorphism

$$g : TM \xrightarrow{\sim} T^*M, \tag{1.5.20}$$

which in practice is most often defined by a metric on M. It is lifted, uniquely, to a diffeomorphism of $D_{\Sigma/S}$-manifolds (for any base S)

$$g : J^\infty(TM_{\Sigma/S}) \xrightarrow{\sim} J^\infty(T^*M_{\Sigma/S}),$$

hence a diffeomorphism

$$g : Sol \xrightarrow{\sim} J^\infty(T^*M_{\Sigma/\Sigma''}),$$

and a sheaf isomorphism

$$g^\# : \Omega^{\bullet,\bullet}_{Sol/\Sigma''} \xrightarrow{\sim} g^{-1}\Omega^{\bullet,\bullet}_{J^\infty(T^*M_\Sigma)/\Sigma''}, \tag{1.5.21}$$

where g^{-1} stands for the inverse image in the category of sheaves.

Let $g^*\omega$ be the symplectic form on TM obtained by pulling back the canonical symplectic form ω on T^*M. We have arrived at

1.5.5.1. Lemma. *Mapping (1.5.21) descends to an isomorphism of Lie algebra sheaves*

$$g^{\#}:\mathcal{H}^{\overline{g^{*}\omega}}_{Sol/\Sigma''}\stackrel{\sim}{\longrightarrow} g^{-1}\mathcal{H}^{can}.$$

1.6. Calculus of variations and integrals of motion.

Calculus of variations is the principal source of the brackets discussed in 1.4-5.

1.6.1. An action A is a global section, cf. (1.5.1),

$$A\in\Gamma(J^{\infty}(M_{\Sigma}),\bar{\Omega}^{d,0}_{J^{\infty}(M_{\Sigma})}). \tag{1.6.0}$$

It can be represented by a Lagrangian which is a collection of sections

$$L=\{L^{(i)}\in\Gamma(U_i,\Omega^{d,0}_{J^{\infty}(M_{\Sigma})})\text{ s.t. }L^{(j)}-L^{(i)}\in\mathrm{Im}d_{\rho}\text{ on }U_j\cap U_i\}, \tag{1.6.1a}$$

determined up to a transformation

$$L^{(i)}\mapsto L^{(i)}+d_{\rho}\beta^{(i)}, \tag{1.6.1b}$$

where $\{U_i\}$ is an open covering of $J^{\infty}(M_{\Sigma})$.

Choosing local coordinates one observes that

$$\delta L^{(i)}=-d_{\rho}\gamma^{(i)}+E^{(i)}_{j}\delta x^{j}, \tag{1.6.2}$$

for some $\gamma^{(i)}\in\Omega^{d-1,1}_{J^{\infty}(M_{\Sigma})}$, known as a variational 1-form, and some $E_j\in\Omega^{d,0}_{J^{\infty}(M_{\Sigma})}$.

Since representation (1.6.2) is unique [Di], and transformation (1.6.1b) leaves $E^{(i)}_j$ unaffected (because $[\delta,d_{\rho}]=0$, see 1.3), associated to the action A there arises the sheaf of Euler-Lagrange ideals

$$\mathcal{J}_L=<D_{\Sigma}E_1,D_{\Sigma}E_2,...,D_{\Sigma}E_n>\subset\mathcal{O}_{J^{\infty}(M_{\Sigma})}. \tag{1.6.3}$$

Let $Sol_L\subset J^{\infty}(M_{\Sigma})$ be the corresponding zero locus. We will assume that $\mathcal{J}_L$ is of pseudo Cauchy-Kovalevskaya type, and usually (1.4.2) will hold.

The variational 1-form $\gamma^{(i)}$ is not quite uniquely defined, but due to the well-known acyclicity theorem [T,Di], locally it is determined up to a d_{ρ}-exact term. Therefore, *thr variational 2-form*

$$\omega^{(i)}=\delta\gamma^{(i)}.$$

unambiguously defines a section of the quotient sheaf $\bar{\Omega}^{d-1,2}_{J^{\infty}(M_{\Sigma})}$ over U_i. Since transformation (1.6.1b) leaves it invariant, there arises

$$\bar{\omega}_L\stackrel{\text{def}}{=}\{U_i\mapsto\omega^{(i)}\}\in\Gamma(J^{\infty}(M_{\Sigma}),\bar{\Omega}^{d-1,2}_{J^{\infty}(M_{\Sigma})}). \tag{1.6.4}$$

By construction, $\bar{\omega}_L$ satisfies (1.5.2); hence on Sol_L there arises the sheaf of Lie algebras $\mathcal{H}^{\omega_L}_{Sol}$, see Proposition 1.5.2. Our task now is to detect inside it a subalgebra of integrals of motion. As we have seen already, the nature of the argument tends to be purely local, and until further notice it will be assumed that

$$L\in\Gamma(J^{\infty}(M_{\Sigma}),\Omega^{d,0}_{J^{\infty}(M_{\Sigma})}).$$

1.6.2. A *symmetry* of L is an evolutionary vector field ξ s.t.

$$\mathrm{Lie}_\xi L = d_\rho \alpha_\xi, \tag{1.6.5}$$

for some $\alpha_\xi \in \Omega^{d-1,0}_{J^\infty(M_\Sigma)}$. Denote by Sym_L the set of all symmetries of L; it is naturally a Lie algebra.

It is easy to derive from (1.6.5) that any $\xi \in \mathrm{Sym}_L$ preserves $\mathcal{J}_L$, see [Di], hence defines a vector field on Sol_L, to be denoted $\bar{\xi}$. Let $\overline{\mathrm{Sym}}_L$ be the Lie algebra of all such vector fields.

An *integral of motion* of L is an $\bar{F} \in \Gamma(J^\infty(M_\Sigma), \bar{\Omega}^{d-1,0}_{J^\infty(M_\Sigma)})$ s.t.

$$d_\rho \bar{F}|_{Sol_L} = 0. \tag{1.6.6}$$

Let

$$\mathcal{I}_L = \{\bar{F} \in \Gamma(J^\infty(M_\Sigma), \bar{\Omega}^{d-1,0}_{J^\infty(M_\Sigma)}) \text{ s.t. } d_\rho \bar{F}|_{Sol_L} = 0\} \tag{1.6.7a}$$

and

$$\tilde{\mathcal{I}}_L = \mathcal{I}_L|_{Sol_L} \subset \Gamma(Sol_L, \bar{\Omega}^{d-1,0}_{Sol_L}). \tag{1.6.7b}$$

If ξ is a symmetry of L with characteristic $\{Q^j\}$, see (1.2.2), then the computation

$$d_\rho \alpha_\xi = \iota_\xi \delta L \overset{(1.6.2)}{=} -\iota_\xi d_\rho \gamma + E_j Q^j = d_\rho \iota_\xi \gamma + E_j Q^j \tag{1.6.8}$$

shows that $\alpha_\xi - \iota_\xi \gamma$ is an integral of motion.

The form α_ξ being determined by ξ up to a d_ρ-exact term, (1.6.8) defines a map

$$\mathrm{Sym}_L \to \mathcal{I}_L. \tag{1.6.9}$$

It is easy to see, cf. [Di], that since $\mathcal{J}_L$ is of pseudo Cauchy-Kovalevskaya type, the kernel of (1.6.9) consists of vector fields vanishing on Sol_L, hence the maps

$$\overline{\mathrm{Sym}}_L \to \mathcal{I}_L. \tag{1.6.10}$$

and

$$\overline{\mathrm{Sym}}_L \to \tilde{\mathcal{I}}_L. \tag{1.6.11}$$

1.6.3. Nöther's Theorem. ([Di,Ol]) *Map (1.6.10) is a surjection, map (1.6.11) is an isomorphism.*

Therefore, $\tilde{\mathcal{I}}_L$ inherits a Lie algebra structure from $\overline{\mathrm{Sym}}_L$. Let us now show that this Lie algebra structure is consistent with that on the sheaf $\mathcal{H}^{\bar{\omega}_L}$.

1.6.4. Lemma. *If $\tilde{\mathcal{I}}_L$ is a Lie subalgebra of $\Gamma(Sol_L, \mathcal{H}^{\bar{\omega}_L}_{Sol_L})$ such that (1.6.11) is a Lie algebra isomorphism.*

It is this Lie algebra that is often referred to as the algebra of integrals of motion or current algebra.

Proof. It is known, see e.g. [Di] 19.6.17 or [DF] Proposition 2.76, that if ξ is a symmetry of L such that (1.6.5) holds, then, upon restriction to Sol_L,

$$\mathrm{Lie}_\xi \gamma = \delta \alpha_\xi + d_\rho \beta$$

for some β. An application of δ to both sides of this equality shows that ξ is homotopy Hamiltonian, see (1.5.2). The corresponding integral of motion $F_\xi = \alpha_\xi - \iota_\xi\gamma$ is admissible because

$$\delta F_\xi = \delta\alpha_\xi - \delta\iota_\xi\gamma = \delta\alpha_\xi - \mathrm{Lie}_\xi\gamma + \iota_\xi\delta\gamma = -d_\rho\beta + \iota_\xi\omega_L.$$

Hence $\tilde{\mathcal{I}}_L \subset \Gamma(Sol_L, \mathcal{H}^{\bar{\omega}_L}_{Sol_L})$. Furthermore, the line above shows that modulo d_ρ

$$\bar{\delta}\bar{F}_\xi = \iota_\xi\bar{\omega}_L.$$

Hence the bracket of two integrals of motion induced by the Lie algebra structure on $\mathcal{H}^{\bar{\omega}_L}$, see (1.5.10), is as follows

$$\{\bar{F}_\xi, \bar{F}_\eta\} = \xi\bar{F}_\eta,$$

which is also an integral of motion, because due to (1.4.4),

$$d_\rho\xi G = \xi d_\rho G = 0.$$

The corresponding symmetry of L is, of course, $[\xi, \eta]$.

1.6.5. Let us now drop the requirement that L be globally defined. The exposition above has to be altered a little. An evolutionary vector field is a symmetry of L, see (1.6.1), if it is of each $L^{(i)}$:

$$\mathrm{Lie}_\xi L^{(i)} = d_\rho\alpha_\xi^{(i)}.$$

There may arise discrepancies $\alpha_\xi^{(i)} - \alpha_\xi^{(j)}$ on double intersections $U_i \cap U_j$, but (1.6.1a)and (1.4.4) ensure that they are d_ρ-exact. Therefore, while the collection

$$\{F_\xi^{(i)} = \alpha_\xi^{(i)} - \iota_\xi\gamma^{(i)}\}$$

does not define a global section of $\Omega^{d-1,0}_{J^\infty(M_\Sigma)}$, taken modulo d_ρ it defines a global section of $\bar{\Omega}^{d-1,0}_{J^\infty(M_\Sigma)}$. The rest of the discussion in 1.6.2-4 goes through unchanged, and we obtain

1.6.6. Corollary. *Lemma 1.6.4 holds true for any Lagrangian (1.6.1a,b).*

Along with $\mathcal{H}^{\bar{\omega}_L}_{Sol}$, there is its relative version, $\mathcal{H}^{\bar{\omega}_L}_{Sol/\Sigma''}$ and the Lie algebra sheaf morphism

$$\mathcal{H}^{\bar{\omega}_L}_{Sol} \to \mathcal{H}^{\bar{\omega}_L}_{Sol/\Sigma''}$$

defined in (1.5.11), which seems to be neither surjection nor injection, generally speaking.

1.6.7. Lemma. *If Sol_L satisfies (1.4.2), then the composition*

$$\tilde{\mathcal{I}}_L \hookrightarrow \Gamma(Sol, \mathcal{H}^{\omega_L}_{Sol}) \to \Gamma(Sol, \mathcal{H}^{\omega_L}_{Sol/\Sigma''})$$

is an injection.

Proof. Assume that $\tilde{F} \in \tilde{\mathcal{I}}_L$ is annihilated by the composite map. This means that if $F \in \mathcal{I}_L$ is a representative of $\tilde{F}$, then $F = F^o \wedge d\tau$, and $d_{\rho/\Sigma''}F^o = 0$. Due to the Takens acyclicity theorem [T] (applicable thanks to (1.4.2)), $F^o = d_{\rho/\Sigma''}G$ for some G. Therefore, $F = \pm d_\rho(Gd\tau)$ and $\tilde{F} = 0$, as desired.

Now we would like to explain that for an important class of Lagrangians, the sheaf $\mathcal{H}^{\bar{\omega}_L}_{Sol/\Sigma''}$ is isomorphic to the canonical $\mathcal{H}^{can}$ defined, see (1.5.16), and exhibit some concrete Lie algebras of integrals of motion.

1.6.8. *Order 1 Lagrangians and the Legendre transform.* Let us assume that the Lagrangian L depends only on 1-jets of the coordinates x^j. If we let

$$L = \tilde{L} d\sigma^0 \wedge \cdots \wedge d\sigma^{d-1},$$

then (1.6.2) becomes

$$\begin{aligned}\delta L = &- d_\rho \left((-1)^{p+1} \left(\frac{\partial \tilde{L}}{\partial(\partial_{\sigma^p} x^j)} \right) \delta x^j \wedge d\sigma^0 \wedge \cdots \wedge \widehat{d\sigma^p} \wedge \cdots \wedge d\sigma^{d-1} \right) \\ &+ \left(\frac{\partial \tilde{L}}{\partial x^j} - \partial_{\sigma^p} \left(\frac{\partial \tilde{L}}{\partial(\partial_{\sigma^p} x^j)} \right) \right) \delta x^j \wedge d\sigma^0 \wedge \cdots \wedge d\sigma^{d-1},\end{aligned} \tag{1.6.12}$$

where $\widehat{\ }$ means that the term is omitted and summation w.r.t. repeated indices is assumed.

Assume now that on Σ there is a distinguished coordinate, say $\tau = \sigma^0$, such that L is a convex function of jets of coordinates in the τ-direction. It follows then that Sol_L satisfies (1.4.2). Applying (1.3.4) to γ we obtain

$$\gamma' := \gamma|_{d\tau=0} = \left(\frac{\partial \tilde{L}}{\partial(\partial_\tau x^j)} \right) \delta x^j \wedge d\sigma^1 \wedge \cdots \wedge d\sigma^{d-1}.$$

Note that, as a function of $\partial_\tau x^j$, $\tilde{L}$ is canonically a function on the tangent space TM. It follows that γ' is unambiguously a 1-form on TM.

The convexity of L implies that the *Legendre transform*

$$d_{TM}\tilde{L} : TM \to T^*M \tag{1.6.13}$$

is a diffeomorphism. A moment's thought shows that γ' is the pull-back of the canonical 1-form on T^*M w.r.t. $d_{TM}\tilde{L}$, which places us in the situation of Lemma 1.5.5.1. In a coordinate form, we have: if x^j are coordinates on M, $x_j = \partial/\partial x^j$ are fiberwise linear functions on T^*M, then

$$(d_{TM}\tilde{L})^\#(x^j) = x^j, \ (d_{TM}\tilde{L})^\#(x_j) = \frac{\partial \tilde{L}}{\partial(\partial_\tau x^j)},$$

and

$$\begin{aligned}\gamma' &= (d_{TM}\tilde{L})^\# \left(x_j \delta x^j \wedge d\sigma^1 \wedge \cdots \wedge d\sigma^{d-1} \right), \\ \omega'_L = (d_{TM}\tilde{L})^\# (\delta\gamma') &= (d_{TM}\tilde{L})^\# \left(\delta x_j \wedge \delta x^j \wedge d\sigma^1 \wedge \cdots \wedge d\sigma^{d-1} \right),\end{aligned} \tag{1.6.14}$$

are the pull-backs of the canonical degenerate symplectic form. Hence Lemmas 1.5.5.1 and 1.6.7 specialized to the present situation read as follows.

1.6.8.1. Lemma. *If L depends only on the 1-jets of coordinates and is convex, then in the case where $d = 2$, then there are the following Lie algebra (sheaf) morphisms*

$$\mathcal{H}^{\bar{\omega}_L}_{Sol/\Sigma''} \xrightarrow{\sim} (d_{TM}\tilde{L})^{-1}\mathcal{H}^{can}, \ \tilde{\mathcal{I}}_L \hookrightarrow \Gamma(M, \mathcal{H}^{\bar{\omega}_L}_{Sol/\Sigma''}) \xrightarrow{\sim} \Gamma(M, \mathcal{H}^{can}).$$

1.6.8.2. This lemma explains the universality of $\mathcal{H}^{can}$. One can argue, therefore, that the Lie algebra content of the "theory" is independent of the Lagrangian. What captures the properties of an individual Lagrangian is the subalgebra of integrals of motion. For example, if L is independent of τ, the intrinsic time, then $\rho(\partial_\tau)$ is a symmetry of L, and (1.6.9) produces the corresponding integral of motion as follows: since

$$\rho(\partial_\tau)L = d_\rho(\tilde{L}d\sigma^1 \wedge \cdots \wedge d\sigma^{d-1}),$$

the corresponding integral of motion, upon restriction to the fibers of $Sol_L \to \Sigma''$, becomes

$$H_{\rho(\partial_\tau)} = \alpha_{\rho(\partial_\tau)} - \iota_{\rho(\partial_\tau)}\gamma' = \left(\tilde{L} - \left(\frac{\partial \tilde{L}}{\partial(\partial_\tau x^j)}\right)\partial_\tau x^j\right) d\sigma^1 \wedge \cdots \wedge d\sigma^{d-1}, \tag{1.6.15}$$

which is the familiar energy function, of course.

1.6.9. *Bosonic string, left/right movers, and a rudiment of generalized geometry.* Let M be a Riemannian manifold with metric $(.,.)$, Σ be 2-dimensional with coordinates τ and σ. By definition, a point in $J^1(M_\Sigma)$ is a triple $(t, x, \partial x)$, where $t \in \Sigma$, $x \in M$, and ∂x is a linear map

$$\partial x : T_t\Sigma \to T_xM,\ \xi \mapsto \partial_\xi x.$$

This makes sense out of the symbol $(\partial_\xi x, \partial_\eta x)$ as a function on $J^1(M_\Sigma)$. The following

$$L = \frac{1}{2}\left(\left(\partial_\sigma - \partial_\tau\right)x, \left(\partial_\sigma + \partial_\tau\right)x\right) d\sigma \wedge d\tau \tag{1.6.16}$$

is then a well-defined Lagrangian, the celebrated *σ-model Lagrangian.* In terms of local coordinates $x^1, ..., x^n$ s.t. $(.,.) = g_{ij}dx^idx^j$ it looks as follows:

$$L = \frac{1}{2}\left(g_{ij}\partial_\sigma x^i\partial_\sigma x^j - g_{ij}\partial_\tau x^i\partial_\tau x^j\right) d\sigma \wedge d\tau.$$

A direct computation shows that

$$\begin{aligned} \delta L = &- d_\rho\left(\left(\partial_\tau x, \delta x\right) d\sigma - \left(\partial_\sigma x, \delta x\right) d\tau\right) \\ &+ \left(\nabla_{\partial_\tau x}\partial_\tau x - \nabla_{\partial_\sigma x}\partial_\sigma x\right) d\sigma \wedge d\tau, \end{aligned} \tag{1.6.17}$$

where $\nabla_{\partial_\bullet x}$ is the value of the Levi-Civita connection on $\partial_\bullet x$. It is clear that L satisfies all the conditions of Lemma 1.6.8.1.

The Lagrangian being independent of τ and σ, associated to $\rho(\partial_\tau)$ and $\rho(\partial_\tau)$ there arise two integrals of motion, energy and momentum, and any linear combination thereof. But much more is true. In fact, any vector field of the type

$$\text{either } \xi^- = \frac{1}{2}f(\sigma - \tau)\rho(\partial_\sigma - \partial_\tau) \text{ or } \xi^+ = \frac{1}{2}f(\sigma + \tau)\rho(\partial_\sigma + \partial_\tau) \tag{1.6.18}$$

is a symmetry of L. Indeed, precisely because $(\partial_\sigma \pm \partial_\tau)(\sigma \mp \tau) = 0$, one has

$$\begin{aligned} \xi^- L &= d_\rho \left(\frac{1}{4} f(\sigma - \tau)\left((\partial_\sigma - \partial_\tau) x, (\partial_\sigma + \partial_\tau) x \right) (d\sigma + d\tau) \right), \\ \xi^+ L &= -d_\rho \left(\frac{1}{4} f(\sigma + \tau)\left((\partial_\sigma - \partial_\tau) x, (\partial_\sigma + \partial_\tau) x \right) (d\sigma - d\tau) \right) \end{aligned} \tag{1.6.19}$$

Using (1.6.9) and Lemma 1.6.7 one obtains the corresponding integrals of motion, inside $\mathcal{H}^{\bar{\omega}_L}_{Sol_L/\Sigma''}$,

$$\begin{aligned} F_{\xi^-} &= \frac{1}{4} f(\sigma - \tau)\left((\partial_\sigma - \partial_\tau) x, (\partial_\sigma - \partial_\tau) x \right) d\sigma, \\ F_{\xi^+} &= -\frac{1}{4} f(\sigma + \tau)\left((\partial_\sigma + \partial_\tau) x, (\partial_\sigma + \partial_\tau) x \right) d\sigma. \end{aligned} \tag{1.6.20}$$

Upon Legendre transform (1.6.13), which in terms of local coordinates is this

$$x_i = g_{i\alpha}\partial_\tau x^\alpha,\ \partial_\tau x^i = g^{i\alpha} x_\alpha,$$

formulas (1.6.20) become

$$\begin{aligned} F_{\xi^-} &= f(\sigma - \tau)\left(\frac{1}{4} g^{ij} x_i x_j + \frac{1}{4} g_{ij}\partial_\sigma x^i \partial_\sigma x^j - \frac{1}{2} x_j \partial_\sigma x^j \right) d\sigma, \\ F_{\xi^+} &= f(\sigma + \tau)\left(-\frac{1}{4} g^{ij} x_i x_j - \frac{1}{4} g_{ij}\partial_\sigma x^i \partial_\sigma x^j - \frac{1}{2} x_j \partial_\sigma x^j \right) d\sigma, \end{aligned} \tag{1.6.21}$$

and this computes the image of $F_{\xi^\pm}$ under the composite map of Lemma 1.6.8.1. Let

$$\mathcal{V}ir^\pm = \mathrm{span}\{\overline{F_{\xi^\pm}}\} \subset \Gamma(M, \mathcal{H}^{can}). \tag{1.6.22}$$

All of this means that the space of global sections of the sheaf of Lie algebras $\mathcal{H}^{can}$ contains 2 commuting copies of the Lie algebra of vector fields on Σ. In the case where $\Sigma = S^1 \times \Sigma''$, each is the centerless Virasoro algebra, hence the notation. In view of canonical commutation relations discussed in 1.5.5, formulas (1.6.21) are 2 bozonizations of the Virasoro algebra – in the quasiclassical limit.

This prompts the following definitions:

1.6.9.1. Definition.

(i) Denote by $\mathcal{H}^{\bar{\omega}_L,+}_{Sol_L/\Sigma''}$ *the centralizer of* $\mathcal{V}ir^-$ *in* $\mathcal{H}^{\bar{\omega}_L}_{Sol_L/\Sigma''}$ *and call it the right moving algebra.*

(ii) Denote by $\mathcal{H}^{\bar{\omega}_L,-}_{Sol_L/\Sigma''}$ *the centralizer of* $\mathcal{V}ir^+$ *in* $\mathcal{H}^{\bar{\omega}_L}_{Sol_L/\Sigma''}$ *and call it the left moving algebra.*

We will present a computation of left/right moving algebra in the context of the WZW model in sect. 2.9.2. Let us also note that $\mathcal{H}^{\bar{\omega}_L}_{Sol_L/\Sigma''}$ contains yet another Virasoro algebra – the sum of the first two, which upon restriction to $\{\tau = 0\}$ becomes

$$\begin{aligned} \mathcal{V}ir^o &= \mathrm{span}\{\overline{F_{\xi^+}} + \overline{F_{\xi^-}}\} \\ &= \mathrm{span}\{\overline{f(\sigma) x_j \partial_\sigma x^j d\sigma}\}. \end{aligned} \tag{1.6.23}$$

Bosonization (1.6.23) is much simpler than (1.6.21) and was thoroughly investigated in [MSV,GMS1], but the corresponding Virasoro algebra is neither right nor left moving.

1.6.9.2. *Generalized geometry interpretation.*

Formulas (1.6.18) admit a nice, Lagrangian free, interpretation in the spirit of Hitchin's "generalized geometry", [G]. The idea of generalized geometry is that the tangent bundle of a manifold must be consistently replaced with the direct sum of the tangent and cotangent bundles. From this point of view, a metric on M is a reduction of the structure group of $TM \oplus T^*M$ from $SO(n,n)$ to $SO(n,0)\times SO(0,n)$. Letting $\{e_i\}$, $\{e^j\}$ be a pair of relatively dual bases of the $SO(n,0)$-subbundle and letting $\{f_i\}$, $\{f^j\}$ the same for the $SO(0,n)$-subbundle, one can form 2 invariantly defined tensors, e^ie_i and f^if_i. Noticing that x_i, in (1.6.18), is naturally identified with ∂_{x^i}, $\partial_\sigma x^j$ with dx^i, one concludes that $\mathcal{V}ir^+$ is generated by e^ie_i and $\mathcal{V}ir^-$ by $-f^if_i$

To talk about these and other issues coherently, one must change gears and introduce vertex Poisson algebras.

2. Vertex Poisson Algebras

Our presentation of this well-known topic, see e.g. [FB-Z], will be a little different in the following respects. First of all, we will fix an associative commutative $\mathbb{C}$-algebra B to be the ground ring for all linear algebra constructions of this section.

Second of all, we will let $\mathfrak{g} = \mathrm{Der}B$ and demand that all the structures be $\mathfrak{g}$-equivariant. These assumptions are intended to handle functions of τ and σ should they appear. Therefore, two examples to be kept in mind are these:

$$B = C^\infty(\Sigma), \mathfrak{g} = \mathcal{T}_\Sigma(\Sigma) \text{ or } B = \mathbb{C}, \mathfrak{g} = 0. \tag{2.1}$$

The case at hand, where $M_\Sigma = M\times\Sigma$, is rather special, and we could have avoided including B and $\mathfrak{g}$ as part of data (which is customary in works on vertex algebras), but we decided against it. That the natural setting for what follows is equivariant was pointed out by Beilinson and Drinfeld [BD,3.9].

2.1. Definition. A $\mathfrak{g}$-equivariant vertex Poisson algebra is a collection $(V, T, _{(n)}, \mathfrak{g}; n \geq -1)$, where V is a B-module,

$$T : V \to V$$

is a B-linear map, and

$$_{(n)} : V \otimes V \to V,\ a_{(n)}b = 0 \text{ if } n >> 0$$

is a family of B-bilinear multiplications, such that the following axioms hold:

I. The triple $(V, T, _{(-1)})$ is a commutative associative algebra with derivation T.

II. The collection $(V, T, _{(n)}; n \geq 0)$ is a vertex Lie algebra, i.e., the following holds:

II.1 skew-commutativity

$$a_{(n)}b = (-1)^{n+1}\sum_{j=0}^{\infty} \frac{)(-1)^j}{j!} T^j(b_{(n+j)}a),$$

II.2. Jacobi identity

$$a_{(m)}b_{(n)}c - b_{(n)}a_{(m)}c = \sum_{j=0}^{\infty}\binom{m}{j}(a_{(j)}b)_{(n+m-j)}c,$$

II.3. properties of T:

$$(Ta)_{(n)} = [T, a_{(n)}] = -na_{(n-1)}.$$

III. Leibnitz identity: for any $n \geq 0$, $a_{(n)}$ is a derivation of $_{(-1)}$.

IV. $\mathfrak{g}$-equivariance: V is a $\mathfrak{g}$-module, and the maps $_{(n)}$ and T are $\mathfrak{g}$-module morphisms.

In addition, we will always be assuming that a vertex Poisson algebra $(V, T, _{(n)}; n \geq -1)$ is $\mathbb{Z}_+$-graded, i.e.,

$$V = \bigoplus_{n=0}^{\infty} V_n,\ T(V_n) \subset V_{n+1},\ \mathfrak{g}(V_n) \subset V_n,\ V_{m(j)}V_{(n)} \subset V_{(m+n-j-1)}. \tag{2.1.1}$$

We will unburden the notation by letting V stand for $(V, T, _{(n)}, \mathfrak{g}; n \geq -1)$ when this does not lead to confusion and by suppressing $_{(-1)}$ so that ab stands for $a_{(-1)}$. We will also tend to drop the adjective "equivariant" whenever doing so seems appropriate.

Note that if $m = n = 0$, then **II.2** becomes

$$a_{(0)}b_{(0)}c - b_{(0)}a_{(0)}c = \left(a_{(0)}b\right)_{(0)} c, \tag{2.1.2}$$

which is the usual Jacobi identity for $(V, _{(0)})$. Anticommutativity fails, but **II.1** ensures that it holds up to $T(...)$. This almost proves the following important

2.2. Lemma. *If V is a vertex Poisson algebra, then $T(V) \subset V$ is a 2-sided ideal w.r.t. $_{(0)}$, and $(V/T(V), _{(0)})$ is a Lie algebra.*

2.3. Tensor products

The simplest example of a vertex Poisson algebra is a commutative associative algebra V with derivation T. Defining $a_{(-1)}$ to be multiplication by a and letting $a_{(n)} = 0$ if $n \geq 0$. makes V into a vertex Poisson algebra with $T = \partial$.

If $(V_{1}, _{(n)} T_1)$ and $(V_{2}, _{(n)}, T_2)$ are two vertex Poisson algebras, then $V_1 \otimes V_2$ carries at least two vertex Poisson algebra structures. First of all, one can simply regard $V_1 \otimes V_2$ as an extension of scalars whereby $V_1 \otimes V_2$ becomes a vertex Poisson algebra over V_1 with derivation $T_{(2)}$ and multiplications coming from V_2.

Second of all, one can define $T = T_1 + T_2$ and

$$(a \otimes b)_{(n)} = \begin{cases} a_{(-1)}b_{(-1)} \text{ if } n = -1 \\ \sum_{i=0}^{\infty} \frac{1}{i!}\left(\left(T_1^i a\right)_{(-1)} b_{(n+i)} + a_{(n+i)} \left(T_1^i b\right)_{(-1)}\right) \text{ if } n \geq 0 \end{cases} \tag{2.3.1}$$

If, in addition, V_1 is of the type we started with, i.e., if $(V_1)_{(n)}(V_1) = 0$ for all $n \geq 0$, then (2.3.1) is simplified as follows

$$(a \otimes b)_{(n)} = \begin{cases} a_{(-1)}b_{(-1)} \text{ if } n = -1 \\ \sum_{i=0}^{\infty} \frac{1}{i!}\left(\left(T_1^i a\right)_{(-1)} b_{(n+i)}\right) \text{ if } n \geq 0 \end{cases} \tag{2.3.2}$$

In a sense, the second version is a twist of the first by derivation $T_1 \in \mathrm{Der}(V_1)$. In the context of equivariant vertex Poisson algebras this can be generalized as follows.

If $(V,_{(n)}, T)$ is an equivariant vertex Poisson algebra over B and $\xi \in \mathfrak{g}$, then letting

$$a_{(n)_\xi} = \sum_{i=0}^{\infty} \frac{1}{i!} \left(\xi^i a\right)_{(n+i)} \tag{2.3.3}$$

defines a vertex Poisson algebra $(V,_{(n)_\xi}, T+\xi)$. We will refer to this construction as the ξ-twist. Note that the ξ-twist reduces the constants from B to the algebra of ξ-invariants,B^ξ.

2.4. From vertex Poisson algebras to Courant algebroids

The Poisson vertex algebra structure on $V = \oplus_{n=0}^{\infty} V_n$ defines on the subspace $V_0 + V_1$ the following operations:

$$_{(-1)} : V_0 \otimes V_0 \to V_0, \tag{2.4.1a}$$
$$_{(-1)} : V_0 \otimes V_1 \to V_1,\ V_1 \otimes V_0 \to V_1, \tag{2.4.1b}$$
$$_{(0)} : V_1 \otimes V_0 \to V_0,\ V_0 \otimes V_1 \to V_0, \tag{2.4.1c}$$
$$_{(0)} : V_1 \otimes V_1 \to V_1, \tag{2.4.1d}$$
$$_{(1)} : V_1 \otimes V_1 \to V_0, \tag{2.4.1e}$$
$$T : V_0 \to V_1, \tag{2.4.1f}$$

all the other operations either not preserving the subspace $V_0 + V_1$ or being zero due to condition (2.1.1).

Vertex Poisson algebra axioms imply that (2.4.1a-f) satisfy certain conditions; e.g., (2.4.1a) is such that $(V_0,_{(-1)})$ is an associative commutative B-algebra, and (2.4.1b) is such that V_1 is a V_0-module. In [GMS1], these conditions were written down explicitly and made into an axiomatic definition of a vertex algebroid – in a more complicated, quantum, situation. It is a nice observation due to Bressler [Bre] that under some non-degeneracy assumptions a quasiclassical limit of a vertex algebroid is an exact Courant V_0-algebroid; e.g. (2.4.1d) is the Dorfman barcket [Dor,G] on V_1. Therefore, the assignment $V \mapsto \left(V_0 \oplus V_1,\ T,_{(-1)},_{(0)},_{(1)}\right)$ defines a functor from a subcategory of vertex Poisson algebras to the category of exact Courant V_0-algebroids. This functor is actually an equivalence of categories, and a classification of exact Courant algebroids furnishes that of a subclass of vertex Poisson algebras. For the future use, and for the reader's convenience – after all the present situation is somewhat different – let us now reproduce the essence of this argument.

2.4.1. We have seen already that the pair $(V_0,_{(-1)})$ is an associative commutative B-algebra. Let $A = V_0$. The entire V, hence V_1, is an A-module and $A_{(n)}A = 0$ if $n \geq 0$.

By virtue of Axiom I, the map

$$T : A \to AT(A) \subset V_1 \text{ is a } B\text{-derivation}, \tag{2.4.2}$$

i.e., $T(ab) = aT(b) + bT(a)$ and $T(B) = 0$. Therefore, $AT(A)$ is a quotient of the module of relative Kähler differentials, $\Omega_{A/B}$.

It is easy to see that

$$(AT(A))_{(n)}A = (AT(A))_{(n)}(AT(A)) = 0,\ n \geq 0. \tag{2.4.3}$$

Assumption 1. Let $(A; T : A \to AT(A))$ *be isomorphic to* $(A; d : A \to \Omega_{A/B})$*.*

There arises an exact sequence of A-modules

$$0 \to \Omega_{A/B} \to V_1 \to V_1/\Omega_{A/B} \to 0. \tag{2.4.4}$$

Let $\mathcal{T} = V_1/\Omega_{A/B}$. It is an A-module and a Lie algebra w.r.t. the operation ${}_{(0)}$, by virtue of Lemma 2.2. Furthermore, the map

$$_{(0)} : \mathcal{T} \otimes A \to A \tag{2.4.5}$$

is well defined and gives A a $\mathcal{T}_{A/B}$-module structure (see (2.4.3)) compatible with the A-module structure in that $(a\xi)_{(0)}b = a(\xi_{(0)}b)$.

For each $\tau \in \mathcal{T}$, $\tau_{(0)} \in \mathrm{End}(A)$ is actually a B-derivation of A, and this defines a Lie algebra homomorphism over A

$$\mathcal{T} \to \mathrm{Der}_B(A), \tag{2.4.6}$$

All of this can be summarized by saying that $\mathcal{T}$ is an A-algebroid Lie.

Assumption 2. Morphism (2.4.6) is an isomorphism.

The map

$$_{(0)} : \mathcal{T} \otimes \Omega_{A/B} \to \Omega_{A/B} \tag{2.4.7}$$

arising by virtue of (2.4.3) equals the Lie derivative:

$$\xi_{(0)}\omega = \mathrm{Lie}_\xi \omega, \tag{2.4.8}$$

cf. (1.5.19d). (Indeed, $\xi_{(0)}(aTb) = (\xi_{(0)}a)Tb + a(T\xi_{(0)}b)$.)

Next, again thanks to (2.4.3), there arises the map

$$_{(1)} : \mathcal{T} \otimes \Omega_{A/B} \to A. \tag{2.4.9}$$

It is the natural pairing of vector fields and forms:

$$\xi_{(1)}\omega = \iota_\xi \omega. \tag{2.4.10}$$

(Indeed, $\xi_{(1)}(aTb) = (\xi_{(1)}a)Tb + a(\xi_{(1)}Tb) = a(\xi_{(0)}b)$, where axioms II.3 and III are used.)

This determines all of (2.3.1a-f) that makes sense on the graded object $A \oplus (\mathcal{T} \oplus \Omega_{A/B})$. To continue our analysis we need to make the following

Assumption 3. Let sequence (2.4.4) be splitting.

Let us fix a splitting

$$s : \mathcal{T} \to V_1. \tag{2.4.11}$$

Then there arise the following two maps:

$$_{(1)_s} : \mathcal{T} \otimes \mathcal{T} \to A, \tag{2.4.12}$$

$$_{(0)_s} : \mathcal{T} \otimes \mathcal{T} \to \Omega_{A/B}, \tag{2.4.13}$$

where (2.4.12) is the restriction of ${}_{(1)}$ to $s(\mathcal{T})$, and (2.4.13) is the composition of the restriction of ${}_{(0)}$ to $s(\mathcal{T})$ with the projection $V_1 \to \Omega_{A/B} = V_1/s(\mathcal{T})$. These two maps determine all of (2.4.1a-f).

The map ${}_{(1)_s}$ is, in fact, a symmetric A-bilinear form on $\mathcal{T}$. By varying the splitting s it can killed. Indeed, letting $h(.,.) ={}_{(1)_s}$, we obtain, for any $\xi\in\mathcal{T}$, an A-linear form $h(\xi,.)\in\Omega_{A/B}$. Replacing s with s_h defined to be

$$s_h(\xi)=s(\xi)-\frac{1}{2}h(\xi,.)$$

we get ${}_{(1)_{s_h}}=0$.

Therefore, we can, and usually will, assume that

$$V_1=\mathcal{T}\oplus\Omega_{A/B} \tag{2.4.14}$$

and

$${}_{(1)}:\left(\mathcal{T}\oplus\Omega_{A/B}\right)\otimes\left(\mathcal{T}\oplus\Omega_{A/B}\right)\to A \tag{2.4.15}$$

is the canonical pairing $(\xi+\omega)_{(1)}(\xi'+\omega')=\iota_\xi\omega'+\iota_{\xi'}\omega$, cf. (2.4.10).

Sometimes the following version of (2.4.15) will be used: let h be a symmetric A-bilinear form on $\mathcal{T}$ and define

$$\begin{gathered}{}_{(1)}:\left(\mathcal{T}\oplus\Omega_{A/B}\right)\otimes\left(\mathcal{T}\oplus\Omega_{A/B}\right)\to A\\ (\xi+\omega)_{(1)}(\xi'+\omega')=\iota_\xi\omega'+\iota_{\xi'}\omega+h(\xi,\xi').\end{gathered} \tag{2.4.15$_h$}$$

2.4.2. Therefore, all moduli, if any, come from ${}_{(0)_s}$. A short computation shows that it is A-linear. Furthermore, axiom IV implies that

$${}_{(0)_s}\in\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{T}\otimes\mathcal{T},\Omega_{A/B}\right). \tag{2.4.16}$$

Hence ${}_{(0)_s}$ can be considered as an A-trilinear $\mathfrak{g}$-invariant functional on $\mathcal{T}$, and as such it will be denoted by H:

$${}_{(0)_s}\approx H\in\left(\Omega_{A/B}^{\otimes 3}\right)^{\mathfrak{g}}. \tag{2.4.17}$$

Skew-commutativity II.1 implies that it is anti-commutative in the first 2 variables: $H(\xi,\eta,.)=-H(\eta,\xi,.)$.

Jacobi identity II.2 applied to $[\xi_{(1)},\eta_{(0)}](\zeta)$, $\xi,\eta,\zeta\in s(\mathcal{T}_A)$, shows that, in fact, $H(.,.,.)$, is totally anti-commutative, hence belongs to $\left(\Omega^3_{A/B}\right)^{\mathfrak{g}}$.

Jacobi identity II.2 applied to $[\xi_{(0)},\eta_{(0)}](\zeta)$, $\xi,\eta,\zeta\in s(\mathcal{T}_A)$, shows that H is closed, i.e.,

$$H\in\left(\Omega^{3,cl}_{A/B}\right)^{\mathfrak{g}}. \tag{2.4.18}$$

Conditions (2.4.14-15 or 15_h) do not determine the splitting s; they are respected by the shearing transformation

$$\mathcal{T}\ni\xi\mapsto\xi+\iota_\xi\alpha \text{ for a fixed } \alpha\in\left(\Omega^2_{A/B}\right)^{\mathfrak{g}} \qquad (2.4.19).$$

The effect of this transformation on H is

$$H\mapsto H+d_{DR}\alpha. \tag{2.4.20}$$

2.4.3. Checking the various properties of maps (2.4.1a-f) derived in 2.3.1 against the definition of an exact Courant A-algebroid [LWX] (especially in the form proposed in [Bre]) shows the following. If Assumptions 1–3 hold, then the equivariant Poisson vertex algebra structure on V defines an equivariant exact Courant

A-algebroid structure on $\mathcal{T} \oplus \Omega_A$ such that

$$_{(0)}: (\mathcal{T} \oplus \Omega_A) \otimes (\mathcal{T} \oplus \Omega_A) \to \mathcal{T} \oplus \Omega_A$$

is the Dorfman [Dor,G] bracket,

$$_{(1)}: (\mathcal{T} \oplus \Omega_A) \otimes (\mathcal{T} \oplus \Omega_A) \to A$$

is the symmetric pairing, and (2.4.6) is the anchor.

The discussion in 2.4.2 practically proves (see [Bre, GMS1] for a complete analysis) that the category of exact equivariant Courant A-algebroids is an $\left(\Omega^{3,cl}_{A/B}\right)^{\mathfrak{g}}$-torsor. Indeed, if $\mathcal{C}$ is one such algebroid and $H \in \left(\Omega^{3,cl}_{A/B}\right)^{\mathfrak{g}}$, then the H-twisted Courant algebroid

$$\mathcal{C} \dot{+} H \text{ is defined by replacing } {}_{(0)_s} \text{ with } {}_{(0)_s} + H \tag{2.4.21}$$

A "canonical" Courant algebroid $\mathcal{C}_0$ can be chosen by letting the only "unknown" operation ${}_{(0)_s}$ be zero: define

$$\mathcal{C}_0 \text{ to be s.t. (2.3.14,15) hold and } {}_{(0)_s} = 0. \tag{2.4.22}$$

This identifies the category of equivariant exact Courant A-algebroids with $\left(\Omega^{3,cl}_{A/B}\right)^{\mathfrak{g}}$ s.t.

$$\left(\Omega^{3,cl}_{A/B}\right)^{\mathfrak{g}} \ni H \mapsto \mathcal{C}_H = \mathcal{C}_0 \dot{+} H. \tag{2.4.23}$$

The effect of shear (2.4.19) on H recorded in (2.4.20) implies the following description of morphisms

$$\mathrm{Mor}(\mathcal{C}, \mathcal{C} \dot{+} H) = \{\alpha \in \left(\Omega^2_{A/B}\right)^{\mathfrak{g}} \text{ s.t. } d_{DR}\alpha = H\} \tag{2.4.24}$$

and automorphisms

$$\mathrm{Aut}(\mathcal{C}) = \left(\Omega^{2,cl}_{A/B}\right)^{\mathfrak{g}}. \tag{2.4.25}$$

In particular, the set of isomorphism classes of exact Courant A-algebroids is identified with the $\mathfrak{g}$-invariant de Rham cohomology group,

$$\left(\Omega^{3,cl}_{A/B}\right)^{\mathfrak{g}} / d_{DR}\left(\Omega^2_{A/B}\right)^{\mathfrak{g}}. \tag{2.4.26}$$

2.5. Symbols of vertex differential operators

Let Σ be an open subset of a $\mathbb{R}^d$, U of $\mathbb{R}^n$ and $U_\Sigma = U \times \Sigma$. Define $B = C^\infty(\Sigma)$, $\mathfrak{g} = \mathcal{T}_{\mathbb{R}^d}(\Sigma)$. Identify $\mathfrak{g}$ with the subalgebra of horizontal vector fields on U_Σ, thereby making $C^\infty(U_\Sigma)$ into a $\mathfrak{g}$-module. These are the prerequisites to the definition of a $\mathcal{T}_{\mathbb{R}^d}(\Sigma)$-equivariant vertex Poisson algebra over B.

2.5.1. Definition. *Call V an algebra of symbols of vertex differential operators, SVDO for short, if*

(i) $V_0 = C^\infty(U_\Sigma)$, V_1 is a $\mathcal{T}_{\mathbb{R}^d}(\Sigma)$-equivariant exact Courant $C^\infty(U_\Sigma)$-algebroid over $B = C^\infty(\Sigma)$,

(ii) V is generated as an associative commutative algebra with derivation T by $V_0 \oplus V_1$.

The discussion in 2.4.3 means that we have obtained a functor, say $\mathcal{F}$, from the category of SVDOs to the category of equivariant exact Courant $C^\infty(U_\Sigma)$-algebroids:

$$\mathcal{F} : \{\text{SVDOs}\} \to \{\text{Courant algebroids}\}. \tag{2.5.1}$$

2.5.2. Theorem. ([GMS1, Bre]). *This functor is an equivalence of categories.*

To be precise, [GMS1, Bre] only construct $\mathcal{F}^*$, the left adjoint to $\mathcal{F}$, but a simple representation-theoretic argument shows that the "vertex envelope", $\mathcal{F}^*(\mathcal{C})$, is simple.

2.6. A sheaf-theoretic version

All of this can be spread over manifolds. The geometric prerequisite is a fiber bundle

$$\pi : M_\Sigma \to \Sigma \tag{2.6.1a}$$

with a flat connection

$$\nabla : \mathcal{T}_\Sigma \to \mathcal{T}_{M_\Sigma} \tag{2.6.1b}$$

A sheaf of SVDOs, $\mathcal{V}$, over M_Σ is a sheaf of vector spaces s.t. the space of sections $\mathcal{V}(U)$ is an SVDO for each open $U \subset M_\Sigma$ with

$$\mathcal{V}(U)_0 = \mathcal{O}_{M_\Sigma}(U),\ B(U) = \pi^*\mathcal{O}_\Sigma(\pi U),\ \mathfrak{g} = \mathcal{T}_\Sigma(\pi U), \tag{2.6.2}$$

and equivariant structure determined by ∇.

The condition that $\mathcal{V}(U)_0 = \mathcal{O}_{M_\Sigma}(U)$ implies that $\mathcal{V}$ is automatically a sheaf of $\mathcal{O}_{M_\Sigma}(U)$-modules. It follows from (2.4.6) that the next homogeneous component, $\mathcal{V}_1$ is an extension of vertical vector fields by relative 1-forms:

$$0 \to \Omega_{M_\Sigma/\Sigma} \to \mathcal{V}_1 \to \mathcal{T}_{M_\Sigma/\Sigma} \to 0. \tag{2.6.3}$$

As to the existence of such sheaves, they are plentiful locally: for any sufficiently small open $U \subset M_\Sigma$, the category of such sheaves over U is an $\left(\Omega^{3,cl}_{M_\Sigma/\Sigma}\right)^\nabla(U)$-torsor, as follows from (2.4.23). If $\mathcal{V}_U$ is one such sheaf and $H \in \left(\Omega^{3,cl}_{M_\Sigma/\Sigma}\right)^\nabla(U)$, then

$$\mathrm{Mor}(\mathcal{V}_U, \mathcal{V}_U \dot{+} H) = \{\alpha \in \left(\Omega^2_{M_\Sigma/\Sigma}\right)^\nabla(U) \text{ s.t. } d_{DR}\alpha = H\}, \tag{2.6.4}$$

cf. (2.4.24).

Technically, (2.6.4) means that there is a gerbe, in particular, a sheaf of categories, of SVDOs bound by the sheaf complex

$$0 \to \left(\Omega^2_{M_\Sigma/\Sigma}\right)^\nabla(U) \stackrel{d_{DR}}{\to} \left(\Omega^{3,cl}_{M_\Sigma/\Sigma}\right)^\nabla(U) \to 0.$$

so that the categories over sufficiently small U are equivalent to that of SVDOs with $V_0 = \mathcal{O}_{M_\Sigma}(U)$.

A priori there may be no single sheaf of SVDOs on the entire M; an obstruction to its existence is a certain canonical characteristic class lying in

$$H^2(M_\Sigma, \left(\Omega^2_{M_\Sigma/\Sigma}\right)^\nabla \to \left(\Omega^{3,cl}_{M_\Sigma/\Sigma}\right)^\nabla).$$

At this point let us return to the concrete situation of interest to us, where $M_\Sigma = M\times\Sigma$ and ∇ is the horizontal connection. If so, the above discussion is simplified in that the sheaves $\left(\Omega^\bullet_{M_\Sigma/\Sigma}\right)^\nabla$ can be replaced with $\Omega^\bullet_M$. For example, the obstruction becomes a class lying in

$$H^2(M, \Omega^2_M\to\Omega^{3,cl}_M).$$

This class vanishes; the obstruction (equal to the 1st Pontryagin class) computed in [GMS1], see also [Bre], is a purely quantum phenomenon, and in any case, an example of such sheaf will be exhibited shortly.

Furthermore, (2.4.24-26) imply that the set of isomorphism classes of such sheaves is an $H^1(M,\Omega^2_M\to\Omega^{3,cl}_M)$-torsor, and the group of automorphisms of any such sheaf is isomorphic to $H^0(M,\Omega^2_M\to\Omega^{3,cl}_M)\xrightarrow{\sim} H^0(M,\Omega^{2,cl}_M)$.

Note that since the sequence

$$0\to\Omega^{2,cl}_M\to\Omega^2_M\stackrel{d_{DR}}{\to}\Omega^{3,cl}_M\to 0 \tag{2.6.5}$$

is exact, we obtain isomorphisms

$$\begin{aligned} H^1(M,\Omega^2_M\to\Omega^{3,cl}_M)&\xrightarrow{\sim} H^1(M,\Omega^{2,cl}_M),\\ H^0(M,\Omega^2_M\to\Omega^{3,cl}_M)&\xrightarrow{\sim} H^0(M,\Omega^{2,cl}_M). \end{aligned} \tag{2.6.6}$$

The long exact cohomology sequence associated with (2.6.5) implies, in addition, that

$$H^1(M,\Omega^{2,cl}_M)\xrightarrow{\sim} H^0(M,\Omega^3_M)/dH^0(M,\Omega^2_M)\xrightarrow{\sim} H^3(M,\mathbb{R}), \tag{2.6.7}$$

where the last isomorphism is the de Rham theorem. This proves

2.6.1. Proposition. *a) The set of isomorphism classes of sheaves of SVDOs on $M_\Sigma = M\times\Sigma$ with horizontal connection is identified with either of the isomorphic groups $H^1(M,\Omega^{2,cl}_M)$ and $H^3(M,\mathbb{R})$.*

b) If $\mathcal{V}$ is a sheaf of SVDOs over M, then

$$Aut\mathcal{V}\xrightarrow{\sim} H^0(M,\Omega^{2,cl}_M).$$

2.6.2. Here is an explicit construction of identifications a) and b) of Proposition 2.6.1. The presentation of the set of isomorphism classes as $H^1(M,\Omega^{2,cl}_M)$ emphasizes the fact that locally all such sheaves are isomorphic (this is an immediate consequence of (2.4.26)). Indeed, let $\{U_i\}$ be a covering by balls. Let $\mathcal{V}_i$ be the restriction $\mathcal{V}$ to U_i. Then there arise canonical identifications,

$$\phi_{ij}:\mathcal{V}_i|_{U_i\cap U_j}\xrightarrow{\sim}\mathcal{V}_j|_{U_i\cap U_j}, \tag{2.6.8}$$

to be thought of as gluing functions. Let now $\alpha_{ij}\in\Omega^{2,cl}_M(U_i\cap U_j)$ be a Čech cocycle representing $\alpha\in H^1(M,\Omega^{2,cl}_M)$. Regarding α_{ij} as an automorphism of $\mathcal{V}_j|_{U_i\cap U_j}$, define

$$\hat\phi_{ij}\stackrel{\text{def}}{=}\phi_{ij}\dot{+}\alpha_{ij}:\mathcal{V}_i|_{U_i\cap U_j}\xrightarrow{\sim}\mathcal{V}_j|_{U_i\cap U_j}, \tag{2.6.9}$$

to be the composition of ϕ_{ij} and the shear by α_{ij} defined in (2.4.19). The Čech cocycle condition satisfied by $\{\alpha_{ij}\}$ implies that $\hat\phi_{ik}\circ\hat\phi_{kj}\circ\hat\phi_{ji}=\text{id}$ on the triple intersection $U_i \cap U_j \cap U_k$ for any i,j,k. Thus $\hat{\phi}_{ij}$ are gluing functions of a new sheaf of SVDOs, to be denoted $\mathcal{V} \dot{+} \alpha$.

Contrary to this, the presentation of the set of isomorphism classes as $H^3(M,\mathbb{R})$ has nothing to do with gluing functions or even the $\mathcal{O}_M$-module structure. Indeed, for an element of $H^3(M,\mathbb{R})$, pick a global closed 3-form H representing it. By definition (2.4.21), the sheaf $\mathcal{V} \dot{+} H$ is different from $\mathcal{V}$ only in that the operation

$$_{(0)}:\ \mathcal{V}_0 \otimes \mathcal{V}_0 \to \mathcal{V}_0$$

is replaced with $_{(0)} + H$ (and the sheaf $\mathcal{V} \dot{+} d\beta$, β a global 2-form, is canonically isomorphic to $\mathcal{V}$.)

The relation of one point of view to another is as follows. For example, given $\mathcal{V} \dot{+} H$, find a collection $\beta = \{\beta_i \in \Omega^2_M(U_i)\}$ so that $d\beta_i = H|_{U_i}$. Then $d_{\check{C}}(\beta)$ is de Rham-closed and hence is a Čech 1-cocycle with coefficients in $\Omega^{2,cl}_M$. The map

$$H^0(M,\Omega^{3,cl}_M) \ni H \mapsto \beta \mapsto \text{class of } d_{\check{C}}(\beta) \in H^1(M,\Omega^{2,cl}_M) \tag{2.6.10}$$

descends to the inverse of (2.6.7).

Now, $(\mathcal{V} \dot{+} H)|_{U_i} = \mathcal{V}_i$ as vector spaces but not as SVDOs; to obtain an SVDO isomorphism, the shear by β_i is needed:

$$\beta_i:\ \mathcal{V}_i \to (\mathcal{V} \dot{+} H)|_{U_i} \qquad (2.6.11).$$

The effect of this transformation on the gluing functions is as follows:

$$\phi_{ij} \mapsto \phi_{ij} \dot{+} d_{\check{C}}\beta, \tag{2.6.12}$$

cf. (2.4.20), and this delivers the desired isomorphism

$$\mathcal{V} \dot{+} H \xrightarrow{\sim} \mathcal{V} \dot{+} (\text{ class of } d_{\check{C}}\beta). \tag{2.6.13}$$

2.7. A natural sheaf of SVDOs.

Let us attach to any smooth M a sheaf of SVDOs which depends on M functorially. In order to do so, let us place ourselves in the situation where $T^*M_\Sigma = T^*M \times \Sigma$, Σ satisfies (1.1.2-4) and carries, in particular, a distinguished coordinate system, σ and τ.

Taking advantage of (1.1.7), we note that the operator of the jet connection, (1.1.5), splits in the vertical and horizontal components, e.g.,

$$\rho(\partial_\sigma) = \partial^v_\sigma + \partial^h_\sigma, \tag{2.7.1}$$

where the latter stands for the operator of differentiation w.r.t. σ "appearing explicitly".

Let

$$\pi: J^\infty(T^*M_{\Sigma/\Sigma''}) \to M \tag{2.7.2}$$

be the natural projection. There arises the direct image of the structure sheaf $\pi_*\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ which we will take the liberty to denote also by $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ because this is unlikely to cause confusion. Thus, for example, if $U \subset M$ is open, then $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}(U)$ will stand for the space of functions on the jet-space regular over $\pi^{-1}(U)$.

Being a structure sheaf, $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ carries a canonical multiplication. Let us define a grading

$$\begin{gathered}\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}=\bigoplus_{i=0}^{\infty}\mathcal{O}^i_{J^\infty(T^*M_{\Sigma/\Sigma''})}\text{ s.t.}\\ \mathcal{O}^i_{J^\infty(T^*M_{\Sigma/\Sigma''})}\cdot\mathcal{O}^j_{J^\infty(T^*M_{\Sigma/\Sigma''})}\subset\mathcal{O}^{i+j}_{J^\infty(T^*M_{\Sigma/\Sigma''})}\end{gathered}\tag{2.7.3}$$

by requiring that the pull-back of functions on M have degree 0, the pull-back of fiberwise linear functions on T^*M have degree 1, and the operator ∂^v_σ, defined in (2.7.1), have degree 1, i.e., that $\partial^v_\sigma(\mathcal{O}^i_{J^\infty(T^*M_{\Sigma/\Sigma''})})\subset\mathcal{O}^{i+1}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$. Thus, for example,

$$\mathcal{O}^0_{J^\infty(T^*M_{\Sigma/\Sigma''})}=\mathcal{O}_{M_\Sigma},\ \mathcal{O}^1_{J^\infty(T^*M_{\Sigma/\Sigma''})}=\mathcal{T}_{M_\Sigma/\Sigma}\oplus\Omega_{M_\Sigma/\Sigma},\tag{2.7.4}$$

cf. (2.6.3), where $\mathcal{T}_{M_\Sigma/\Sigma}$ is realized inside $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ as the pull-back of fiberwise linear functions on T^*M, and $\Omega_{M_\Sigma/\Sigma}$ is realized as $\mathcal{O}_M\partial^v_\sigma\mathcal{O}_M$, cf. sect. 2.4.1, Assumption 1.

2.7.1. Proposition. *The sheaf $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ carries a unique structure of a sheaf of SVDOs over $B=\mathcal{O}_\Sigma$ such that ${}_{(-1)}$ is the canonical multiplication, $T=\partial^v_\sigma$ (which furnishes (2.4.1a,b,f) in this case), and (2.4.1c-e) take the following form: if $\xi,\xi'\in\mathcal{T}_M$, $\omega,\omega'\in\Omega_M$, then*

$$\begin{gathered}{}_{(0)}:\left(\mathcal{T}_{M_\Sigma/\Sigma}\oplus\Omega_{M_\Sigma/\Sigma}\right)\otimes\mathcal{O}_{M_\Sigma}\to\mathcal{O}_{M_\Sigma},\ \mathcal{O}_{M_\Sigma}\otimes\left(\mathcal{T}_{M_\Sigma/\Sigma}\oplus\Omega_{M_\Sigma/\Sigma}\right)\to\mathcal{O}_{M_\Sigma}\\ (\xi+\omega)_{(0)}F=-F_{(0)}(\xi+\omega)=\xi F,\end{gathered}\tag{2.7.5}$$

$$\begin{gathered}{}_{(0)}:\left(\mathcal{T}_{M_\Sigma/\Sigma}\oplus\Omega_{M_\Sigma/\Sigma}\right)\otimes\left(\mathcal{T}_{M_\Sigma/\Sigma}\oplus\Omega_{M_\Sigma/\Sigma}\right)\to\mathcal{T}_{M_\Sigma/\Sigma}\oplus\Omega_{M_\Sigma/\Sigma},\\ (\xi+\omega)_{(0)}(\xi'+\omega')=[\xi,\xi']+Lie_\xi\omega'-Lie_{\xi'}\omega+\partial^v_\sigma(\iota_{\xi'}\omega)\end{gathered}\tag{2.7.6}$$

$$\begin{gathered}{}_{(1)}:\left(\mathcal{T}_{M_\Sigma/\Sigma}\oplus\Omega_{M_\Sigma/\Sigma}\right)\otimes\left(\mathcal{T}_{M_\Sigma/\Sigma}\oplus\Omega_{M_\Sigma/\Sigma}\right)\to\mathcal{O}_{M_\Sigma},\\ (\xi+\omega)_{(1)}(\xi'+\omega')=\iota_\xi\omega'+\iota_{\xi'}\omega\end{gathered}\tag{2.7.7}$$

Note that (2.7.5–7) restricted to some $U\subset M$ are nothing but the definition of the canonical Courant $C^\infty(U_\Sigma)$-algebroid $\mathcal{C}_0$ of (2.4.21); therefore $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}(U)$ is nothing but $\mathcal{F}^*(\mathcal{C}_0)$, where $\mathcal{F}$ is equivalence of categories (2.5.1).

The vertex Poisson algebra structure of Proposition 2.7.1 is not quite what we need. Being $\mathcal{T}_\Sigma$-equivariant, it is subject to the ξ-twist, see (2.3.3), for any $\xi\in H^0(\Sigma,\mathcal{T}_\Sigma)$.

2.7.2. Definition. *Let $\widehat{\mathcal{O}}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ denote the sheaf $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ with the vertex Poisson algebra structure defined in Proposition 2.7.1 and let $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ denote the latter's ∂^h_σ-twist, see (2.7.1).*

Note that in the case of $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$, the derivation T becomes

$$T=\rho(\partial_\sigma).\tag{2.7.8}$$

In particular, (2.7.6) is changed as follows

$$\begin{aligned}
{}_{(0)}:\left(\mathcal{T}_{M_\Sigma/\Sigma}\oplus\Omega_{M_\Sigma/\Sigma}\right)\otimes\left(\mathcal{T}_{M_\Sigma/\Sigma}\oplus\Omega_{M_\Sigma/\Sigma}\right)&\to\mathcal{T}_{M_\Sigma/\Sigma}\oplus\Omega_{M_\Sigma/\Sigma},\\
(\xi+\omega)_{(0)}(\xi'+\omega')=[\xi,\xi']+\mathrm{Lie}_\xi\omega'-\mathrm{Lie}_{\xi'}\omega&+\rho(\partial_\sigma)\left(\iota_{\xi'}\omega\right),
\end{aligned}\tag{2.7.9}$$

and the operations on $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ are no longer linear over $\mathcal{O}_\Sigma$, only over $\mathcal{O}_{\Sigma''}$.

Let us now relate $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ to the canonical Lie algebra sheaf $\mathcal{H}^{can}$ defined in (1.5.17).

Lemma 2.2 associates with $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ the sheaf of Lie algebras

$$\mathrm{Lie}(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})})=\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}/\rho(\partial_\sigma)\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}.$$

2.7.3. Proposition. *The Lie algebra sheaves $\mathcal{H}^{can}$ and $\mathrm{Lie}(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})})$ are canonically isomorphic.*

Proof. The sheaf isomorphism

$$\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}\xrightarrow{\sim}\Omega^{1,0}_{J^\infty(T^*M_{\Sigma/\Sigma''})},\ F\mapsto Fd\sigma$$

descends to

$$\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}/\rho(\partial_\sigma)\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}\xrightarrow{\sim}\Omega^{1,0}_{J^\infty(T^*M_{\Sigma/\Sigma''})}/d_{\rho/\Sigma}\Omega^{0,0}_{J^\infty(T^*M_{\Sigma/\Sigma''})}.$$

Lemma 1.5.4.1 (and (1.5.16)) identifies the range of this map with $\mathcal{H}^{can}$, and thanks to (2.7.8), the domain of this map is $\mathrm{Lie}(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})})$ – it is at this point that we needed the ∂^h_σ-twist; hence

$$\mathrm{Lie}(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})})\xrightarrow{\sim}\mathcal{H}^{can}.\tag{2.7.10}$$

Map (2.7.10) respects all defining relations (1.5.17-18a,b): (1.5.17) is (part of) (2.7.5,6), (1.5.18a) is sect.2.1, Axiom III, and (1.5.18b) is sect.2.1, Axiom II.3 (another point where the ∂^h_σ-twist is necessary). Hence (2.7.10) is a Lie algebra sheaf isomorphism.

2.7.4. Terminology. We have obtained two families of sheaves of vertex Poisson algebras. First, those provided by the combination of Propositions 2.6.1a) and 2.7.1. They can be realized as either $\widehat{\mathcal{O}}_{J^\infty(T^*M_{\Sigma/\Sigma''})}\dot{+}H$, where $H\in H^0(M,\Omega^{3,cl}_M)$ represents a 3-dimensional cohomology class, or $\widehat{\mathcal{O}}_{J^\infty(T^*M_{\Sigma/\Sigma''})}\dot{+}(\{\alpha_{ij}\})$, where $(\{\alpha_{ij}\})$ is a cocycle representing an element of $H^1(M,\Omega^{2,cl}_M)$.

Second, their ∂^h_σ-twisted versions, to be denoted by $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}\dot{+}H$ and $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}\dot{+}(\{\alpha_{ij}\})$. As Proposition 2.7.3 indicates, it is the latter that will be of importance. Note, however, that these choices have arisen only because we have included functions of τ and σ. In fact, both $\widehat{\mathcal{O}}_{J^\infty(T^*M_{\Sigma/\Sigma''})}\dot{+}(\{\alpha_{ij}\})$ and $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}\dot{+}(\{\alpha_{ij}\})$ induce the same vertex Poisson algebra structure on the fiber at any point $(\sigma,\tau)\in\Sigma$. For this reason sheaves such as $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}\dot{+}(\{\alpha_{ij}\})$, where $(\{\alpha_{ij}\})$ will also be referred to as sheaves of SVDOs.

2.8. The Lagrangian interpretation

Let us place ourselves in the situation of 1.6.8 and assume that the Lagrangian $L \in H^0(J^\infty(M_\Sigma), \Omega^{2,0}_{J^\infty(M_\Sigma)})$ is of order 1, globally defined, and convex. A combination of Proposition 2.7.3 and Lemma 1.6.8.1 gives

$$\mathcal{H}^{\bar{\omega}_L}_{Sol/\Sigma''} \xrightarrow{\sim} \mathrm{Lie}(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}),\ \tilde{\mathcal{I}}_L \subset \Gamma(M_\Sigma, \mathrm{Lie}(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})})). \tag{2.8.1}$$

In this sense the universal sheaf of SVDO's $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ governs the theory associated to L.

In order to interpret similarly all the other, twisted, sheaves of SVDO's provided by Proposition 2.6.1a), one needs to consider Lagrangians (1.6.1a,b) that do not glue in a global section of $\Omega^{2,0}_{J^\infty(M_\Sigma)}$. One possibility to construct such a Lagrangian is to add what a physicist might call a *Wess-Zumino term* or an *H-flux*, cf. [GHR, W1].

Fix a global closed 3-form H on M and let $\{U_i\}$ be an open covering of M fine to ensure the existence of a collection of 2-forms

$$\{\beta^{(i)} \in \Omega^2_M(U_i) \text{ s.t. } d\beta^{(i)} = H \text{ on } U_i\}. \tag{2.8.2}$$

define

$$L^H = \{L^{(i)} = L + \beta^{(i)}(\rho(\partial_\tau), \rho(\partial_\sigma))\}. \tag{2.8.3}$$

It follows from (2.8.4) that on double intersection $\beta^{(i)} - \beta^{(j)}$ are closed and, provided $\{U_i\}$ is fine enough, are exact, i.e., there is a collection of 1-forms, $\{\alpha^{(ij)}\}$ such that $\beta^{(j)} - \beta^{(i)} = d\alpha^{(ij)}$. Then a quick computation shows that

$$L^{(j)} - L^{(i)} = d_\rho \left(\left(\iota_{\rho(\partial_\tau)}\alpha^{(ij)}\right) d\tau + \left(\iota_{\rho(\partial_\sigma)}\alpha^{(ij)}\right) d\sigma \right).$$

Therefore, collection (2.8.3) is a new Lagrangian in the sense of (1.6.1a,b).

The L^H is a collection of locally defined Lagrangians, which are still order 1 and convex, hence Sol_{L^H} can still be identified with the universal $J^\infty(T^*M_{\Sigma/\Sigma''})$. One way to define such identification is to use L, as in (2.8.1):

$$d_{TM}L :\ (Sol_{L^H}) \xrightarrow{\sim} J^\infty(T^*M_{\Sigma/\Sigma''}), \tag{2.8.4}$$

but the obvious counterpart of (2.8.1) fails in this case. Instead, (2.8.4) delivers an isomorphism of the twisted sheaf, see (2.4.21),

$$\mathrm{Lie}(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})} \dot{+} H) \xrightarrow{\sim} \mathcal{H}^{\bar{\omega}_{L^H}}_{\Sigma/\Sigma''}. \tag{2.8.5}$$

This attaches the twisted sheaf $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})} \dot{+} H$ to the Lagrangian L^H.

To see how the twist comes about note that the Legendre transform $d_{TM}L$ used in (2.8.4) does not respect the canonical variational 2-form ω_{L^H}, see (1.6.4). This can be straightened out locally. According to (1.6.13), one way to proceed is to choose, over U_i, the mapping to be $d_{TM}L^{(i)}$. Since $L^{(i)} = L + \beta^{(i)}$,

$$d_{TM}L^{(i)}(\xi) = d_{TM}L(\xi) + \frac{1}{2}\iota_\xi \beta^{(i)}, \tag{2.8.6}$$

as follows, e.g., from local formulas (1.6.14). But mappings (2.8.6) are incompatible on double intersections $U_i \cap U_j$, the obstruction being the Čech cocycle

$$d_{\check{C}}\{\beta^{(i)}\} = \{\beta^{(j)} - \beta^{(i)}\} \in Z^1_{\text{Čech}}(M, \Omega^{2,cl}_M). \tag{2.8.7}$$

In order to restore the compatibility, let us introduce the twisted sheaf $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})} \dot{+} \left(d_{\check{C}}\{\beta^{(i)}\}\right)$ obtained by twisting the gluing functions of $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ over $U_i \cap U_j$ by the 2-form $\beta^{(j)} - \beta^{(i)}$, as we did in (2.6.9). Then the collection of mappings

$$\{(d_{TM}L^{(i)})^* : \quad \mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}(U_i) \xrightarrow{\sim} \mathcal{O}_{Sol_{L^H}}(U_i) \tag{2.8.8a}$$

delivers a map of the twisted sheaf

$$\left(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})} \dot{+} \left(d_{\check{C}}\{\beta^{(i)}\}\right)\right) \to \mathcal{O}_{Sol_{L^H}}, \tag{2.8.8b}$$

so that the arising

$$\mathrm{Lie}\left(\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})} \dot{+} \left(d_{\check{C}}\{\beta^{(i)}\}\right)\right) \xrightarrow{\sim} \mathcal{H}^{\bar{\omega}_{L^H}}_{\Sigma/\Sigma''}. \tag{2.8.9}$$

is a Lie algebra sheaf isomorphism. It is explained in some detail in 2.6.2 that this sheaf is the same as $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})} \dot{+} H$, see (2.6.13); hence (2.8.9) is equivalent to (2.8.5).

Incidentally, the classification of automorphisms of SVDO's, Proposition 2.6.1b) is also accurately reflected in the Lagrangian approach. Given a globally defined Lagrangian and a closed 2-form β, *a B-field*, let $L^\beta = L + \beta(\rho(\partial_\tau), \rho(\partial_\sigma))d\tau \wedge d\sigma$, cf. (2.8.3). This does nothing to either the corresponding equations of motion or the corresponding variational 2-form. Hence $Sol_L = Sol_{L^\beta}$, literally, as pre-symplectic manifolds, but there arise two competing Legendre transforms, $d_{TM}L$ and $d_{TM}L^\beta$. A moment's thought shows that the latter is the composition of the former with the *B-field transform*, $\xi \mapsto \xi + \iota_\xi\beta$, and this provides the Lagrangian realization of the automorphism of the SVDO $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$ associated to β in Proposition 2.6.1b.

The subalgebras of integrals of motion $\hat{\mathcal{I}}_L \hookrightarrow \Gamma(Sol_L, \mathcal{H}^{\bar{\omega}_L}_{\Sigma/\Sigma''})$, arising by virtue of Lemma 1.6.9, also tend to come from vertex Poisson subalgebras of $\mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})}$. For example, the three Virasoro algebras, left, right, and "half-twisted", see (1.6.22,23), are the Lie-functor applied to the three subalgebras of $\Gamma(M, \mathcal{O}_{J^\infty(T^*M_{\Sigma/\Sigma''})})$ generated by

$$\begin{aligned} &\frac{1}{4}g^{ij}x_ix_j + \frac{1}{4}g_{ij}\partial_\sigma x^i\partial_\sigma x^j - \frac{1}{2}x_j\partial_\sigma x^j, \\ &-\frac{1}{4}g^{ij}x_ix_j - \frac{1}{4}g_{ij}\partial_\sigma x^i\partial_\sigma x^j - \frac{1}{2}x_j\partial_\sigma x^j, \\ &-x_j\partial_\sigma x^j, \end{aligned} \tag{2.8.10}$$

respectively. The global nature of these local formulas formulas was unraveled in 1.6.9.2.

2.9. An example: WZW model.

Let us see how all of this plays out in the case where the target manifold is a real Lie group G, either compact and simple or $\mathrm{GL}(n, \mathbb{R})$.

2.9.1. *Classification.* Let $\mathfrak{g} = \mathrm{Lie}G$ be the corresponding Lie algebra. Fix an invariant bilinear form $g \in S^2(\mathfrak{g}^*)^\mathfrak{g}$ and an invariant trilinear form

$$H(x, y, z) = g([x, y], z). \tag{2.9.1}$$

The left translates of these generate the invariant metric and 3-form (resp.) on G, which we will take the liberty of denoting by the same letters

$$g \in H^0(G, \mathcal{T}_G^{\otimes 2}),\ H \in H^0(G, \Omega^3_G). \tag{2.9.2}$$

Note that the latter is closed:

$$H \in H^0(G, \Omega_G^{3,cl}). \tag{2.9.3}$$

It is well known that

$$H^3(G, \mathbb{R}) = \mathbb{R} \cdot (\text{class of } H). \tag{2.9.4}$$

Therefore, Proposition 2.6.1a) implies that the set of isomorphism classes of SVDO's on G_Σ form a 1-parameter family:

$$SD_{G,k} \stackrel{\text{def}}{=} \mathcal{O}_{J^\infty(G_{\Sigma/\Sigma''})} \dot{+} \frac{-k}{2} H. \tag{2.9.5}$$

As was explained in sect. 2.6.2, the structure of $SD_{G,k}$ is determined by the following:
there is a fixed splitting

$$(SD_{G,k})_1 = \mathcal{T}_{G_\Sigma/\Sigma} \oplus \Omega_{G_\Sigma/\Sigma}, \tag{2.9.6}$$

and the vertex Poisson algebra structure makes $\left(D^{poiss}_{G,k}\right)_1$ into the Courant algebroid that satisfies

$$(2.4.14, 15) \text{ hold true, and } {}_{(0)_s} = -\frac{k}{2} H, \tag{2.9.7}$$

cf. the definition, (2.4.22), of the canonical Courant algebroid $\mathcal{C}_0$.

Induced by the action on the left and on the right, there are the corresponding Lie algebra $\mathfrak{g} = \mathrm{Lie} G$ embeddings in the space of global vector fields

$$j_l^0 : \mathfrak{g} \hookrightarrow H^0(G_\Sigma, \mathcal{T}_{G_\Sigma/\Sigma}),\ j_r^0 : \mathfrak{g} \hookrightarrow H^0(G_\Sigma, \mathcal{T}_{G_\Sigma/\Sigma}) \text{ s.t.} [j_l^0(\mathfrak{g}), j_r^0(\mathfrak{g})] = 0. \tag{2.9.8}$$

These embeddings respect the SVDO structure on $(SD_{G,0})_1$ in that

$$j_l^0([x,y]) = \left(j_l^0(x)_{(0)} j_l^0(y)\right),\ \left(j_l^0(x)_{(n)} j_l^0(y)\right) \text{ if } n > 0, \tag{2.9.9a}$$

$$j_r^0([x,y]) = \left(j_r^0(x)_{(0)} j_r^0(y)\right),\ \left(j_r^0(x)_{(n)} j_r^0(y)\right) \text{ if } n > 0, \tag{2.9.9b}$$

and

$$\left(j_l^0(x)\right)_{(n)} \left(j_l^0(x)\right) = 0 \text{ if } n \geq 0, \tag{2.9.9c}$$

as follows from either (2.7.6) or (2.4.6).

Technically, (2.9.9a–c) mean the following. Associated to $\mathfrak{g}$ there is a $\mathbb{Z}_+$-graded vertex Poisson algebra, $V(\mathfrak{g})_k$, see e.g. [FB-Z]. It is the universal vertex Poisson algebra generated by

$$(V(\mathfrak{g})_k)_0 = \mathbb{R},\ (V(\mathfrak{g})_k)_1 = \mathfrak{g}, \tag{2.9.10}$$

subject to relations

$$x_{(n)} y = \begin{cases} kg(x,y) \text{ if } n = 1 \\ [x,y] \text{ if } n = 0 \\ 0 \text{ if } n > 1. \end{cases} \tag{2.9.11}$$

By definition, (2.9.9a–c) imply that maps (2.9.8) can be extended to vertex Poisson algebra maps

$$j_l^0 : V(\mathfrak{g})_0 \hookrightarrow H^0(G_\Sigma, SD_{G,0}),\ j_r^0 : V(\mathfrak{g})_0 \hookrightarrow H^0(G_\Sigma, SD_{G,0}) \tag{2.9.12}$$

such that
$$\left(j_l^0(V(\mathfrak{g})_0)\right)_{(n)}\left(j_r^0(V(\mathfrak{g})_0)\right)=0 \text{ if } n\geq 0. \tag{2.9.13}$$
In order to carry this over to $k\neq 0$, the maps $j^0_{l/r}$ must be deformed. Let
$$j_l^k:\mathfrak{g}\to\left(D_{G,k}^{poiss}\right)_1,\ j_l^k(x)=j_l^0(x)+\frac{k}{2}g\left(j_l^0(x),.\right), \tag{2.9.14}$$
$$j_r^k:\mathfrak{g}\to\left(D_{G,k}^{poiss}\right)_1,\ j_r^k(x)=j_r^0(x)-\frac{k}{2}g\left(j_r^0(x),.\right), \tag{2.9.15}$$

2.9.1.1. Theorem. [FP,F,AG,GMS2] *Maps (2.9.14,15) extend to vertex Poisson algebra embeddings*
$$V(\mathfrak{g})_k\overset{j_l^k}{\hookrightarrow}H^0(G_\Sigma,SD_{G,k})\overset{j_r^k}{\hookleftarrow}V(\mathfrak{g})_{-k} \tag{2.9.16}$$
such that
$$\left(j_l^k(V(\mathfrak{g})_k)\right)_{(n)}\left(j_r^k(V(\mathfrak{g})_{-k})\right)=0 \textit{ if } n\geq 0. \tag{2.9.17}$$

2.9.1.2. Remark. This appealing result has a long and somewhat unhappy history. A version of it first appeared in [FP] (in a more complicated, quantum, situation) but apparently had been known even earlier to E.Frenkel, [F] – all of this before the introduction of sheaves of vertex algebras – and then was thoroughly forgotten. Arkhipov and Gaitsgory [AG] gave a proof in the language of chiral algebras. Our presentation is close to [GMS2].

The algebra $V(\mathfrak{g})_k$ has a well-known family of modules, $V_{\lambda,k}$, induced from V_λ, the simple finite dimensional $\mathfrak{g}$-module with highest weight λ, see e.g. [FBZ]. According to Theorem 2.9.1.1, $H^0(G_\Sigma,SD_{G,k})$ is a $V(\mathfrak{g})_k\otimes V(\mathfrak{g})_{-k}$-module, see sect. 2.3 for the definition of the tensor product of vertex Poisson algebras.

2.9.1.3. Proposition. *If $k\neq 0$, then there is an isomorphism of $V(\mathfrak{g})_k\otimes V(\mathfrak{g})_{-k}$-modules*
$$H^0(G_\Sigma,SD_{G,k})\overset{\sim}{\longrightarrow}C^\infty(\Sigma)\hat{\otimes}\left(\oplus_\lambda V_{\lambda,k}\otimes V_{\lambda^*,-k}\right), \tag{2.9.18}$$
where λ^* stands for the highest weight of the $\mathfrak{g}$-module dual to V_λ.

Sketch of Proof. The validity of decomposition (2.9.18) for the subspace $H^0(G_\Sigma,(SD_{G,k})_0)$ is the content of the Peter-Weyl theorem. It is not hard to deduce from (2.9.14,15) that any $\mathbb{R}$-basis, $\mathcal{B}$, of $j_l^k(\mathfrak{g})\oplus j_r^k(\mathfrak{g})$ is a basis of $H^0(G_\Sigma,(SD_{G,k})_1)$ over functions. Hence, the entire $H^0(G_\Sigma,D_{G,k}^{poiss})$ is the space of differential polynomials in $\mathcal{B}$ over functions. Decomposition (2.9.18) follows at once from the induced nature of modules $V_{\lambda,k}$.

2.9.1.4. Remark. A proof – in the quantum case – of (2.9.18) for a generic k first appeared in [FS]. Our proof goes through in the quantum case for all but one value of k.

Decomposition (2.9.21) is tantalizingly similar to space of states of WZW model to which $SD_{G,k}$ is intimately related.

2.9.2. WZW. Consider the standard σ-model Lagrangian with target G:

$$L_\kappa = \frac{\kappa}{2} g\left(\left(\partial_\tau - \partial_\sigma\right) x, \left(\partial_\tau + \partial_\sigma\right) x\right) d\tau \wedge d\sigma, \tag{2.9.19}$$

cf. (1.6.16), where $g(.,.)$ is the invariant metric (2.9.2) and κ is an arbitrary constant.

Next use the 3-form H of (2.9.2) to obtain $L_\kappa^{-k/2H}$ as explained in (2.8.4-5). The WZW Lagrangian [W1] is

$$L_{WZW} = L_{k/2}^{-\frac{k}{2}H}. \tag{2.9.20}$$

As follows from (2.8.5) and normalization (2.9.5), the sheaf $SD_{G,k}$ governs the theory associated to $L_\kappa^{-k/2H}$ for any κ. It is clear why the H-twist of (2.9.19) is needed – the pleasing decomposition (2.9.18) is valid only if $k \neq 0$.

Let us now explain the choice of κ made in (2.9.20). Recall that Lagrangian (2.9.19) is conformally invartiant, i.e., the corresponding algebra of integrals of motion contains two Virasoro subalgebras $\mathcal{V}ir^\pm$, see (1.6.17). It is easy to see that the twisted version, $L_\kappa^{-k/2H}$, is also, and $\mathcal{V}ir^\pm$ are still the corresponding integrals of motion. By virtue of (2.8.5) the Legendre transform delivers the embeddings

$$\mathcal{V}ir^\pm \hookrightarrow \Gamma(G, \mathrm{Lie}\,(SD_{G,k})). \tag{2.9.21}$$

On the other hand, each $V(\mathfrak{g})_k$ carries its own Virasoro element – well-known fact. By virtue of Theorem 2.9.1.1, there arise then two more Virasoro subalgebras

$$\mathcal{V}ir^l \hookrightarrow \Gamma(G, \mathrm{Lie}\,(SD_{G,k})) \hookleftarrow \mathcal{V}ir^r. \tag{2.9.22}$$

2.9.2.1. Lemma. *Upon taking the images of (2.9.21-22)*

$$\mathcal{V}ir^+ = \mathcal{V}ir^l,\ \mathcal{V}ir^- = \mathcal{V}ir^r \tag{2.9.23}$$

if and only if $\kappa = k/2$.

This allows to compute the left/right moving subalgebra, see Definition 1.6.9.1.

2.9.2.2. Corollary. *The right moving subalgebra of WZW is Lie*$(C^\infty(\Sigma)\otimes V(\mathfrak{g})_k)$ *and the left moving is Lie*$(C^\infty(\Sigma) \otimes V(\mathfrak{g})_{-k})$.

The Lie-functor appearing in 2.9.2.1-2 only obscures the matter, of course. Armed with the notion of a vertex Poisson algebra we can easily refine both Definition 1.6.9.1 and 2.9.2.1-2. The Lie algebra $\mathcal{V}ir$ itself is the Lie-functor applied to a certain vertex Poisson algebra, Vir. Embeddings (2.9.21-22) are engendered by vertex Poisson algebra embeddings of 4 copies of Vir:

$$\mathrm{Vir}^\pm \hookrightarrow \Gamma(G, SD_{G,k}), \tag{2.9.24}$$

$$\mathrm{Vir}^l \hookrightarrow \Gamma(G, SD_{G,k}) \hookleftarrow \mathrm{Vir}^r. \tag{2.9.25}$$

Lemma 2.9.2.1 can be refined as follows: upon taking the images of (2.9.24,25)

$$\mathrm{Vir}^+ = \mathrm{Vir}^l,\ \mathrm{Vir}^- = \mathrm{Vir}^r \text{ iff } \kappa = \frac{k}{2}. \tag{2.9.26}$$

Definition 1.6.9.1 can be similarly refined:

2.9.2.3. Definition. Let the left/right moving subalgebras of $SD_{G,k}$ be

$$SD^+_{G,k} = \{v \in SD_{G,k} \text{ s.t. } v_{(n)}\mathrm{Vir}^- = 0 \text{ if } n \geq 0\} \tag{2.9.27}$$

$$SD^-_{G,k} = \{v \in SD_{G,k} \text{ s.t. } v_{(n)}\mathrm{Vir}^+ = 0 \text{ if } n \geq 0\} \tag{2.9.28}$$

The refined form of Corollary 2.9.2.2 is this:

$$SD^+_{G,k} = C^\infty(\Sigma) \otimes V(\mathfrak{g})_k,\ SD^-_{G,k} = C^\infty(\Sigma) \otimes V(\mathfrak{g})_{-k}. \tag{2.9.29}$$

2.9.2.4. Proofs. We will prove (2.9.26) and (2.9.29) from which Lemma 2.9.2.1 and Corollary 2.9.2.2 follow instantaneously. Proving (2.9.26) amounts to painstakingly translating from sect. 2.9.1 to sect. 2.9.2, the Legendre transform being the main tool.

To facilitate bookkeeping, we will assume that $G = \mathrm{GL}(n, \mathbb{R})$; an extension via a faithful representation to compact Lie groups is immediate. Let then x^{ij} be coordinates, $\partial_{ij} = \partial/\partial x^{ij}$, and $\{E_{ij}\}$ the standard basis of $gl(n, \mathbb{R})$.

The invariant metric is

$$g = x_{t\alpha} dx^{\alpha j} x_{j\beta} dx^{\beta t}, \tag{2.9.30}$$

where $x_{t\alpha}$ are defined so that $x_{t\alpha}x^{\alpha j} = \delta^j_t$, and the summation w.r.t. repeated indices is always assumed.

Embeddings (2.9.8.) take on the form

$$j^0_l(E_{ij}) = x^{\alpha i}\partial_{\alpha j}, \tag{2.9.31}$$

$$j^0_r(E_{ij}) = -x^{j\alpha}\partial_{i\alpha}. \tag{2.9.32}$$

By virtue of (2.9.30), definitions (2.9.14,15) read

$$j^k_l(E_{ij}) = x^{\alpha i}\partial_{\alpha j} + \frac{k}{2} x_{j\gamma}\partial_\sigma x^{\gamma i}, \tag{2.9.33}$$

$$j^k_r(E_{ij}) = -x^{j\alpha}\partial_{i\alpha} + \frac{k}{2} x_{\gamma i}\partial_\sigma x^{j\gamma}. \tag{2.9.34}$$

Finally, the elements that generate the two corresponding Virasoro vertex Poisson algebras inside $SD_{G,k}$, cf. (2.9.25), are

$$\mathrm{Vir}^l =< \frac{1}{k} j^k_l(E_{ij}) j^k_l(E_{ji}) >,\ \mathrm{Vir}^r =< \frac{1}{k} j^k_r(E_{ij}) j^k_r(E_{ji}) > . \tag{2.9.35}$$

To recapitulate all of this in terms intrinsic to the Lagrangian $L^{-k/2H}_\kappa$, one needs to use the twisted version of the Legendre transform, see (2.8.4), i.e., apply (1.6.13-14) not to $L^{-k/2H}_\kappa$ but to L^0_κ. This amounts to letting

$$\partial_{ij} = \frac{\partial L^0_\kappa}{\partial(\partial_\tau x^{ij})};$$

thus

$$\partial_{ij} = \kappa x_{\alpha i}\partial_\tau x^{\beta\alpha} x_{j\beta}. \tag{2.9.36}$$

Plugging this in (2.9.33-34) gives

$$j_l^k(E_{ij}) = \left(\kappa\partial_\tau x^{\alpha i} + \frac{k}{2}\partial_\sigma x^{\alpha i}\right) x_{j\alpha}, \tag{2.9.37}$$

$$j_r^k(E_{ij}) = \left(-\kappa\partial_\tau x^{j\alpha} + \frac{k}{2}\partial_\sigma x^{j\alpha}\right) x_{\alpha i}. \tag{2.9.38}$$

It is pleasing to notice that precisely when $\kappa = k/2$, the latter formulas become the WZW currents, see [W1], (15) or [GW], (2.3),

$$j_l^k(E_{ij}) = k\partial_+ x^{\alpha i} x_{j\alpha}, \tag{2.9.39}$$

$$j_r^k(E_{ij}) = k\partial_- x^{j\alpha} x_{\alpha i}, \tag{2.9.40}$$

where $\partial_\pm = (\partial_\sigma \pm \partial_\tau)/2$.

Now to the Virasoro subalgebras. Plugging (2.9.37,38) in (2.9.35) one finds similarly that precisely when $\kappa = k/2$ the corresponding Virasoro elements are

$$\mathrm{Vir}^l = < kb(\partial_+ x, \partial_+ x) >, \tag{2.9.41}$$

$$\mathrm{Vir}^r = < kb(\partial_- x, \partial_- x) >, \tag{2.9.42}$$

i.e., defined by the familiar, see (1.6.20), formulas for $\mathrm{Vir}^\pm$. This concludes our proof of (2.9.26).

Now to (2.9.29). Having it our disposal (2.9.26), we infer from Theorem 2.9.1.1 that

$$C^\infty(\Sigma) \otimes V(\mathfrak{g})_k \subset SD^+_{G,k}, C^\infty(\Sigma) \otimes V(\mathfrak{g})_{-k} \subset SD^-_{G,k}. \tag{2.9.43}$$

To prove the reverse inclusions, let

$$L^l = \frac{1}{k} j_l^k(E_{ij}) j_l^k(E_{ji}), L^r = \frac{1}{k} j_r^k(E_{ij}) j_r^k(E_{ji}).$$

It follows easily from the definition of the modules $V_{\lambda,k}$ that

$$\mathrm{Ker} L^l_{(0)} = V_{0,-k} \stackrel{\text{def}}{=} V(\mathfrak{g})_{-k}, \mathrm{Ker} L^r_{(0)} = V_{0,k} \stackrel{\text{def}}{=} V(\mathfrak{g})_k. \tag{2.9.44}$$

By definition then

$$C^\infty(\Sigma) \otimes V(\mathfrak{g})_k \supset SD^+_{G,k}, C^\infty(\Sigma) \otimes V(\mathfrak{g})_{-k} \supset SD^-_{G,k}, \tag{2.9.45}$$

which concludes the proof of (2.9.29).

3. Supersymmetric Analogues

3.1. Bits of supergeometry.

All of the geometric background of sect. 1 allows for more or less straightforward super-generalization. We will explain this very briefly, and in less generality, because our exposition will be more example-oriented. Such sources as [DM,L,M1] provide an introduction to supermathematics.

3.1.1. Super world-sheet. The world-sheet is now a $2|2$-dimensional real C^∞-manifold either with a fixed coordinate system

$$(u, v, \theta^+, \theta^-) : \hat{\Sigma} \hookrightarrow \mathbb{R}^{2|2} \tag{3.1.1a}$$

or a fixed étale coordinate system

$$(u, v, \theta^+, \theta^-) : \mathbb{R}^{2|2} \to \hat{\Sigma}, \tag{3.1.1b}$$

where (u, v) are even and $\theta^\pm$ are odd.

We have the underlying even manifold

$$\Sigma = \{\theta^+ = \theta^- = 0\} \hookrightarrow \hat{\Sigma} \tag{3.1.2}$$

and the bundle

$$\hat{\Sigma} \stackrel{(u,v)}{\to} \Sigma. \tag{3.1.3}$$

The time-fibration will be defined to be the composition

$$\hat{\Sigma} \stackrel{(u,v)}{\to} \Sigma \stackrel{\tau}{\to} \Sigma'' \subset \mathbb{R} \tag{3.1.4}$$

for some fibration τ, where Σ is an even manifold underlying $\hat{\Sigma}$.

The Lie algebra of vector fields on $\hat{\Sigma}$ contains two remarkable elements

$$D_+ = \frac{\partial}{\partial \theta^+} - \theta^+ \frac{\partial}{\partial u},\ D_- = \frac{\partial}{\partial \theta^-} - \theta^- \frac{\partial}{\partial v}. \tag{3.1.5}$$

The following relations hold true

$$\begin{aligned} &[D_+, D_+] = -2\frac{\partial}{\partial u}, [D_-, D_-] = -2\frac{\partial}{\partial v}, [D_+, D_-] = 0 \\ &[\frac{\partial}{\partial v}, D_\pm] = [\frac{\partial}{\partial u}, D_\pm] = 0. \end{aligned} \tag{3.1.6}$$

3.1.2. Super-jets. Let M be a C^∞-supermanifold with underlying even manifold M^{even}. Define

$$M_{\hat{\Sigma}} = M \times \hat{\Sigma}. \tag{3.1.7}$$

It is fibered over $\hat{\Sigma}$:

$$M_{\hat{\Sigma}} \to \hat{\Sigma}. \tag{3.1.8}$$

The manifold of ∞-jets of sections of this bundle, $J^\infty(M_{\hat{\Sigma}})$, is defined in a straightforward manner as follows (cf. [BD p.80]).

3.1.2.1. Definition. *$J^\infty(M_{\hat{\Sigma}})$ is a supermanifold with underlying even manifold $J^\infty(M^{even}_\Sigma)$ and the structure sheaf $\mathcal{O}_{J^\infty(M_{\hat{\Sigma}})}$ defined to be the symmetric algebra on*

$$D_{\hat{\Sigma}} \otimes_{\mathcal{O}_{\hat{\Sigma}}} \mathcal{O}_{M_\Sigma}$$

modulo the relations

$$\begin{aligned} &1 \otimes f \cdot 1 \otimes g = 1 \otimes fg,\ 1 \otimes 1 = 1, \\ &\xi \otimes fg = (\xi \otimes f) \cdot (1 \otimes g) + (-1)^{\tilde{\xi}\tilde{f}} (1 \otimes f) \cdot (\xi \otimes g). \end{aligned} \tag{3.1.9}$$

for any $\xi \in \mathcal{T}_{\hat{\Sigma}}$, $f, g \in \mathcal{O}_{M_\Sigma}$, where $\tilde{}$ stands for the parity.

There arises a fiber bundle

$$J^\infty(M_{\hat{\Sigma}}) \to \hat{\Sigma} \tag{3.1.10}$$

with connection

$$\rho : \mathcal{T}_{\hat{\Sigma}} \to \mathcal{T}_{J^\infty(M_{\hat{\Sigma}})} \text{ s.t. } \rho(\eta)(\xi \otimes f) = (\eta\xi) \otimes f \tag{3.1.11}$$

in complete analogy with (1.1.5).

The relative versions, such as $J^\infty(M_{\hat{\Sigma}/\Sigma''})$, are immediate.

Note that connection (3.1.11) is constant in the direction of (θ^+, θ^-), i.e., if we let $J^\infty(M_{\hat{\Sigma}})^o = \{\theta^+ = \theta^- = 0\} \hookrightarrow J^\infty(M_{\hat{\Sigma}})$, then there is a diffeomorphism

$$(J^\infty(M_{\hat{\Sigma}}), \rho(\partial_{\theta^\pm})) \to \left(J^\infty(M_{\hat{\Sigma}})^o \times \mathbb{R}^{0|2}, \rho^o(\partial_{\theta^\pm}) = \partial_{\theta^\pm}\right) \tag{3.1.12}$$

of $\mathbb{R}^{0|2}$-manifolds with connection.

Indeed, given a local coordinate system X^i on M, the collection

$$\{X^i_{(m),(\epsilon)}, u, v, \theta^+, \theta^-; (m) \in \mathbb{Z}^2_+, (\epsilon) \in \mathbb{Z}^2_2\} \tag{3.1.13a}$$

constitutes a local coordinate system on $J^\infty(M_{\hat{\Sigma}})$, where

$$X^i_{(m_1,m_2),(\epsilon_1,\epsilon_2)} = (\partial_u^{m_1}\partial_v^{m_2}\partial_{\theta^+}^{\epsilon_1}\partial_{\theta^-}^{\epsilon_2}) \otimes X. \tag{3.1.13b}$$

Letting

$$\begin{aligned}
\tilde{F}^i &= (\partial_{\theta^-}\partial_{\theta^+}) \otimes X^i, \\
\psi^i_+ &= (\partial_{\theta^+}) \otimes X^i - \theta^-\tilde{F}^i, \\
\psi^i_- &= (\partial_{\theta^-}) \otimes X^i + \theta^+\tilde{F}^i, \\
x^i &= X^i - \theta^+\psi^i_+ - \theta^-\psi^i_- - \theta^+\theta^-\tilde{F}^i,
\end{aligned} \tag{3.1.14}$$

we obtain another local coordinate system

$$\{x^i_{(m)}, \psi^i_{\pm,(m)}, \tilde{F}^i_{(m)}; u, v, \theta^+, \theta^-; (m) \in \mathbb{Z}^2_+, (\epsilon) \in \mathbb{Z}^2_2\} \tag{3.1.15}$$

such that

$$\partial_{\theta^\pm} x^i_{(m)} = \partial_{\theta^\pm}\psi^i_{\pm,(m)} = \partial_{\theta^\pm}\tilde{F}^i_{(m)} = 0, \tag{3.1.16}$$

and (3.1.12) follows.

Note that change of variables (3.1.14) is nothing but the formal Taylor series expansion at $J^\infty(M_{\hat{\Sigma}})^o$:

$$X^i = x^i + \theta^+\psi^i_+ + \theta^-\psi^i_- + \theta^+\theta^-\tilde{F}^i. \tag{3.1.17}$$

Along M, $\{x^i\}$ are coordinates and

$$\psi^i_\pm \text{ transform as (even or odd) } dx^i. \tag{3.1.18}$$

3.1.3. Differential equations. The definition and discussion of a submanifold $Sol \subset J^\infty(M_{\hat{\Sigma}})$ as the zero locus of a differential ideal $\mathcal{J}$ is quite parallel to sect. 1.4. Since our exposition is strongly focused on one particular example, that of the (2,2)-supersymmetric σ-model, we will restrict ourselves to the case where $\mathcal{J}$ is locally generated by 4n functions, E^i_α, $1 \le i \le n$, $1 \le \alpha \le 4$, such that (cf. (1.4.1))

$$\begin{aligned}
E^i_1 &= \tilde{F}^i + \cdots \\
E^i_2 &= \partial_\tau\psi^i_- + \cdots \\
E^i_3 &= \partial_\tau\psi^i_+ + \cdots \\
E^i_4 &= \partial^2_\tau x^i + \cdots ,
\end{aligned} \tag{3.1.19}$$

where the omitted terms are independent of $\tilde{F}^{\cdot}$, of non-zero order jets of $\psi^{\cdot}_{\pm}$ in the direction of τ, and of order > 1 jets of $x^{\cdot}$ also in the direction of τ. (τ is time-function (3.1.4) tacitly assumed to have been included in a coordinate system.)

Letting

$$Sol^o = \{\theta^+ = \theta^- = 0\} \hookrightarrow Sol, \tag{3.1.20}$$

one obtains a diffeomorphism of $\mathbb{R}^{0|2}$-manifolds with connection

$$(Sol, \rho(\partial_{\theta^\pm})) \xrightarrow{\sim} (Sol^o \times \mathbb{R}^{0|2}, \rho^o(\partial_{\theta^\pm}) = \partial_{\theta^\pm}), \tag{3.1.21}$$

by restricting (3.1.12).

Note that Sol^o is a D_Σ-, hence a $D_{\Sigma/\Sigma''}$-, supermanifold; to emphasize this we will often write Sol^o_Σ or $Sol^o_{\Sigma/\Sigma''}$.

If (3.1.19) holds, then (3.1.18) implies a diffeomorphism of $D_{\Sigma/\Sigma''}$-manifolds

$$Sol^o_{\Sigma/\Sigma''} \xrightarrow{\sim} J^\infty\left(T\left(\Pi TM\right)_{\Sigma/\Sigma''}\right), \tag{3.1.22}$$

where Π is the familiar parity change functor. Similarly,

$$Sol_{\hat{\Sigma}/\Sigma''} \xrightarrow{\sim} J^\infty\left(T\left(\Pi TM\right)_{\Sigma/\Sigma''}\right) \times \mathbb{R}^{0|2} \tag{3.1.23}$$

as $D_{\hat{\Sigma}/\Sigma''}$-manifolds. Both (3.1.22,23) are analogous to (1.4.2).

3.2. Functional pre-symplectic structure.

The right framework for super-generalization of 1.5 is provided by integral, rather than differential, forms on $\hat{\Sigma}$ [L,M1,DM].

3.2.1. Recall that the sheaf of integral forms is defined to be

$$I^*_{\hat{\Sigma}} = \bigoplus_{i=-\infty}^{4} I^i_{\hat{\Sigma}} \text{ s.t. } I^{4-i}_{\hat{\Sigma}} = \Lambda^i \mathcal{T}_{\hat{\Sigma}} \otimes_{\mathcal{O}_{\hat{\Sigma}}} \mathrm{Ber}(\Omega_{\hat{\Sigma}}), \tag{3.2.1}$$

where $\mathrm{Ber}(\Omega_{\hat{\Sigma}})$ is the Berezinian of $\Omega_{\hat{\Sigma}}$.

By definition, $I^*_{\hat{\Sigma}}$ is a locally free $\Lambda^*\mathcal{T}_{\hat{\Sigma}}$-module defined by

$$\mathcal{T}_{\hat{\Sigma}} \to End_{\mathcal{O}_{\hat{\Sigma}}}(I^*_{\hat{\Sigma}}),\ \xi \mapsto \iota_\xi \text{ where } \iota_\xi\beta \stackrel{\text{def}}{=} \xi \wedge \beta. \tag{3.2.2}$$

Next, $I^*_{\hat{\Sigma}}$ carries a unique structure of a module over the Clifford algebra, $Cl(\mathcal{T}_{\hat{\Sigma}} \oplus \Omega_{\hat{\Sigma}})$, such that

$$\Omega^i_{\hat{\Sigma}} \otimes_{\mathcal{O}_{\hat{\Sigma}}} I^j_{\hat{\Sigma}} \to I^{i+j}_{\hat{\Sigma}},\ \alpha \otimes \beta \mapsto \alpha \wedge \beta,\ [\iota_\xi, \alpha\wedge] = \alpha(\xi). \tag{3.2.3}$$

The Berezinian, $\mathrm{Ber}(\Omega_{\hat{\Sigma}})$, carries the Lie derivative operation

$$\mathcal{T}_{\hat{\Sigma}} \otimes_{\mathbb{R}} \mathrm{Ber}(\Omega_{\hat{\Sigma}}) \to \mathrm{Ber}(\Omega_{\hat{\Sigma}}),\ \xi \otimes \beta \mapsto \mathrm{Lie}_\xi \beta, \tag{3.2.4}$$

which is naturally extended to

$$\mathcal{T}_{\hat{\Sigma}} \otimes_{\mathbb{R}} I^i_{\hat{\Sigma}} \to I^i_{\hat{\Sigma}},\ \xi \otimes \beta \mapsto \mathrm{Lie}_\xi \beta. \tag{3.2.5}$$

The sheaf of integral forms is a complex with differential

$$d : I^i_{\hat{\Sigma}} \to I^{i+1}_{\hat{\Sigma}} \tag{3.2.6a}$$

determined by

$$[d, \iota_\xi] = \mathrm{Lie}_\xi,\ \xi \in \mathcal{T}_{\hat{\Sigma}}. \tag{3.2.6b}$$

Many other differential-geometric identities, such as

$$[\mathrm{Lie}_\xi, \iota_\eta] = \iota_{[\xi,\eta]},\ [\mathrm{Lie}_\xi, \beta\wedge] = (\mathrm{Lie}_\xi\beta)\wedge,\ \xi,\eta\in\mathcal{T}_{\hat\Sigma}, \beta\in\Omega_{\hat\Sigma}, \tag{3.2.7}$$

keep on holding true.

Since our particular $\hat\Sigma$ carries a fixed (étale) coordinate system $(u, v, \theta^+, \theta^-)$, there is an integral form $[d\theta^+ d\theta^-]$ such that $du\wedge dv\wedge[d\theta^+ d\theta^-]$ trivilaizes the Berezinian $\mathrm{Ber}(\Omega_{\hat\Sigma})$. Letting

$$[d\theta^\pm] = \iota_{\partial_{\theta^\mp}}[d\theta^+ d\theta^-] = [d\theta^\pm]$$

one discovers a part of $I^*_{\hat\Sigma}$ pleasingly – and deceptively – similar to the de Rham complex; e.g.,

$$\mathrm{Lie}_{\partial_{\theta^\pm}}[d\theta^+ d\theta^-] = 0,\ d([d\theta^+ d\theta^-]) = d([d\theta^\pm]) = 0, d(\theta^\pm[d\theta^\mp] = [d\theta^+ d\theta^-]. \tag{3.2.8}$$

Once a projection

$$\hat\Sigma\to\Sigma$$

is given, integration over fibers delivers a morphism

$$I^4_{\hat\Sigma}\to\Omega^2_\Sigma,\ \alpha\mapsto\alpha^o \tag{3.2.9}$$

which, in the case where the projection is (3.1.3), means that

$$f(u, v, \theta^+, \theta^-)du\wedge dv\wedge[d\theta^+ d\theta^-]\mapsto\partial_{\theta^-}\partial_{\theta^+} f(u, v, \theta^+, \theta^-)du\wedge dv, \tag{3.2.10}$$

cf. (3.2.8). This is often referred to as integrating out θ^+ and θ^-.

3.2.2. Back to super-presymplectic forms. Let $\mathcal{M}$ be either *Sol* or any version of an ∞-jet space considered in 3.1.2 that is fibered over $\hat\Sigma$. Let

$$\tilde\Omega^{*,*}_{\mathcal{M}} = \Omega^*_{\mathcal{M}/\hat\Sigma}\otimes_{\mathcal{O}_{\hat\Sigma}} I^*_{\hat\Sigma}. \tag{3.2.11}$$

If we wish to work in a relative situation determined by τ, see (3.1.4), then we write

$$\tilde\Omega^{*,*}_{\mathcal{M}/\Sigma''} = \Omega^*_{\mathcal{M}/\hat\Sigma}\otimes_{\mathcal{O}_{\hat\Sigma}} I^*_{\hat\Sigma/\Sigma''}. \tag{3.2.12}$$

In any case, we get a bi-complex with an obvious vertical differential

$$\delta : \tilde\Omega^{*,i}_{\mathcal{M}/S}\to\tilde\Omega^{*,i+1}_{\mathcal{M}/S} \tag{3.2.13}$$

and a horizontal differential

$$d_{\rho/S} : \tilde\Omega^{i,*}_{\mathcal{M}/S}\to\tilde\Omega^{i+1,*}_{\mathcal{M}/S}, \tag{3.2.14}$$

which owes its existence to connection (3.1.11) and is defined in exactly the same way as its counterpart in sect. 3.1; here and elsewhere S is either Σ'' or a point.

With $\tilde\Omega^{*,*}_{\mathcal{M}/\Sigma''}$ taken as a replacement of $\Omega^{*,*}_{\mathcal{M}/\Sigma''}$, the discussion of sect.1.5.1-3 carries over to the super-case practically word for word. For example, cf. (1.5.2), a functional pre-symplectic form is $\omega\in H^0(\mathcal{M}, \tilde\Omega^{3,2}_{\mathcal{M}/S}/d_\rho\tilde\Omega^{2,2}_{\mathcal{M}/S})$ such that,

$$\delta\omega\in H^0\left(\mathcal{M}, d_{\rho/S}\left(\tilde\Omega^{2,3}_{\mathcal{M}/S}\right)\right). \tag{3.2.15}$$

The outcome is the Lie superalgebra sheaf over $\mathcal{M}$

$$\mathcal{H}^\omega_{\mathcal{M}/S}. \tag{3.2.16}$$

Here is an operation that does not have an adequate purely even analogue. In all our examples,

$$\mathcal{M}\stackrel{\sim}{\longrightarrow}\mathcal{M}^o\times\mathbb{R}^{0|2},$$

in a way respecting the connection, cf. (3.1.12, 21, 23). Given $\omega\in H^0(\mathcal{M},\tilde{\Omega}^{3,2}_{\mathcal{M}/S}/d_\rho\tilde{\Omega}^{2,2}_{\mathcal{M}/S})$, operation (3.2.9) produces $\omega^o\in H^0(\mathcal{M}^o,\tilde{\Omega}^{3,2}_{\mathcal{M}^o/S}/d_{\rho^o}\tilde{\Omega}^{2,2}_{\mathcal{M}^o/S})$. Integrating over fibers one obtains a Lie algebra sheaf morphism, an isomorphism in fact,

$$\mathcal{H}^\omega_{\mathcal{M}/S}\stackrel{\sim}{\longrightarrow}\mathcal{H}^\omega_{\mathcal{M}^o/S}. \tag{3.2.17}$$

As a practical matter, (3.2.17) amounts to carrying out a Taylor expansion as in (3.1.17) and then extracting the coefficient of $\theta^+\theta^-[d\theta^+d\theta^-]$ as in (3.2.10).

Similar in spirit is a morphism

$$\mathcal{H}^\omega_{\mathcal{M}}\to\mathcal{H}^\omega_{\mathcal{M}/\Sigma''} \tag{3.2.18}$$

that relates the relative and absolute versions and amounts to letting $d\tau=0$, cf. (1.5.11).

3.2.3. *Example: canonical commutation relations.*

Let M be an n-dimensional purely even C^∞-manifold. The $2n|2n$-dimensional supermanifold $T^*(\Pi TM)$ carries a well-known closed 2-form ω^o. If we let $\{x^i\}$ be coordinates on M, then $\{x^i,x_i=\partial_{x^i}\}$ along with their superpartners $\{\phi^i,\phi_i\}$ form a system of local coordinates on $T^*(\Pi TM)$, and

$$\omega^o=\delta x_i\wedge\delta x^i+\delta\phi_i\wedge\delta\phi^i. \tag{3.2.19}$$

Now use the projection

$$\pi: J^\infty(T^*(\Pi TM)_{\Sigma/\Sigma''})\to T^*(\Pi TM)$$

to introduce $\pi^*\omega^o$, a closed 2-form on $J^\infty(T^*(\Pi TM)_{\Sigma/\Sigma''})$.

A suitable analogue of $\mathcal{H}^{can}$, see 1.5.4, is provided by fixing a suitable σ so that σ, τ is a coordinate system on Σ, see (3.1.4), letting

$$\omega=\pi^*\omega^o\wedge d\sigma, \tag{3.2.20}$$

and defining

$$\tilde{\mathcal{H}}^{can}=\mathcal{H}^\omega_{J^\infty(T^*(\Pi TM)_{\Sigma/\Sigma''})}. \tag{3.2.21}$$

The rest of the discussion in 1.5.4 carries over to the present situation practically word for word; we will not dwell upon this any longer.

Note that in this example integral forms do not appear. The reader interested in an example of a full-fledged Lie superalgebra sheaf $\mathcal{H}^\omega_{\mathcal{M}/S}$ will have to wait for the discussion of the calculus of variations in sect.3.3.

3.2.4. *Legendre transform?* In practice, the manifold $J^\infty(T(\Pi TM)_{\Sigma/\Sigma''})$ may be more important than $J^\infty(T^*(\Pi TM)_{\Sigma/\Sigma''})$ because of (3.1.22). The possibility to apply $\tilde{\mathcal{H}}^{can}$ then rests on the existence of the diffeomorphism, cf. (1.5.19),

$$g: J^\infty(T(\Pi TM)_{\Sigma/\Sigma''})\to J^\infty(T^*(\Pi TM)_{\Sigma/\Sigma''}), \tag{3.2.22}$$

because given (3.2.21) there arises at once a Lie algebra sheaf ismorphism, cf. Lemma 1.5.5.1,

$$g^\#:\mathcal{H}^{g^*\omega}\stackrel{\sim}{\longrightarrow}g^{-1}\tilde{\mathcal{H}}^{can}. \tag{3.2.23}$$

Isomorphism (3.2.22), however, is a more subtle matter in the present situation than the usual Legendre transform. While the purely even manifolds underlying both the manifolds in (3.2.22) are the familiar $J^\infty(TM_{\Sigma/\Sigma''})$ and $J^\infty(T^*M_{\Sigma/\Sigma''})$, and they are easy to identify via a metric, the structure sheaves are more substantially different. The essence of this difference is that while

$$\Omega_M \hookrightarrow \mathcal{O}_{T(\Pi TM)} \tag{3.2.24}$$

as a direct summand, its $T^*(\Pi TM)$-counterpart, $\mathcal{T}_M$, appears via the extension

$$0 \to \mathrm{End}\Omega_M \to \mathcal{A}_{\Omega_M} \to \mathcal{T}_M \to 0, \tag{3.2.25}$$

where $\mathcal{A}_{\Omega_M}$ is the Atiyah algebra, i.e., the algebra of order 1 differential operators acting on the sections of Ω_M. One way to construct (3.2.22) seems to be this: split (3.2.25) by means of a connection

$$0 \to \mathrm{End}\Omega_M \to \mathcal{A}_{\Omega_M} \overset{\nabla}{\leftrightarrows} \mathcal{T}_M \to 0, \tag{3.2.26}$$

and then identify

$$\Omega_M \xrightarrow{\sim} \mathcal{T}_M \tag{3.2.27}$$

by means of a metric. This is exactly what the Lagrangian of a (1,1)-supersymmetric σ-model allows to do.

3.3. Calculus of variations.

3.3.1. The discussion of sect. 1.6 carries over in a straightforward manner. Here are a few highlights.

An action is

$$A \in \Gamma\left(J^\infty\left(M_{\hat{\Sigma}}\right), \tilde{\Omega}^{4,0}_{J^\infty\left(M_{\hat{\Sigma}}\right)}/d_\rho\tilde{\Omega}^{3,0}_{J^\infty\left(M_{\hat{\Sigma}}\right)}\right). \tag{3.3.1}$$

It is represented by a collection of Lagrangians

$$\{L^{(j)} \in \tilde{\Omega}^{4,0}_{J^\infty\left(M_{\hat{\Sigma}}\right)}(U_j)\} \tag{3.3.2}$$

determined up to d_ρ-exact terms and equal to each other on intersections $U_i \cap U_j$ up to d_ρ-exact terms, cf. (1.6.0-1ab).

An analogue of (1.6.2) is immediate, the outcome is a $D_{\hat{\Sigma}}$-supermanifold Sol_L with variational 1-form γ_L and 2-form $\omega_L = \delta\gamma_L$.

The definition of a symmetry of L is also an obvious modification of (1.6.5). Nöther's Theorem 1.6.3 establishes a bijection between symmetries and integrals of motion as follows

$$\xi \leftrightarrow \alpha_\xi + (-1)^{\tilde{\xi}+1}\iota_\xi\gamma_L; \tag{3.3.3}$$

the change of sign occurs when swapping ι_ξ and d_ρ as in (1.6.8).

Thus there arise the Lie algebra sheaf $\mathcal{H}^{\omega_L}_{Sol_L}$, containing the algebra of integrals of motion $\tilde{I}_L$, its relative version, $\mathcal{H}^{\omega_L}_{Sol_L/\Sigma''}$, and morphisms

$$\mathcal{H}^{\omega_L}_{Sol_L} \to \mathcal{H}^{\omega_L}_{Sol_L/\Sigma''}, \ \tilde{I}_L \hookrightarrow \Gamma(Sol_L, \mathcal{H}^{\omega_L}_{Sol_L}) \to \Gamma(Sol_L, \mathcal{H}^{\omega_L}_{Sol_L/\Sigma''}) \tag{3.3.4}$$

whose composition is an injection provided (3.1.19) holds.

A familiar novelty is that in all of this $\theta^{\pm}$ can be integrated out. The result is this: the action

$$A \in \Gamma\left(\left(J^{\infty}\left(M_{\hat{\Sigma}}\right)\right)^{o}, \Omega^{2,0}_{\left(J^{\infty}\left(M_{\hat{\Sigma}}\right)\right)^{o}} / d_{\rho^{o}} \Omega^{1,0}_{\left(J^{\infty}\left(M_{\hat{\Sigma}}\right)\right)^{o}}\right), \tag{3.3.5}$$

the Lagrangian

$$\{L^{(j)} \in \Omega^{2,0}_{\left(J^{\infty}\left(M_{\hat{\Sigma}}\right)\right)^{o}}(U_j)\} \qquad (3.3.6),$$

and, since nothing is gained or lost, the integrated version of (3.3.4) as follows

$$\mathcal{H}^{\omega_L^o}_{Sol_L^o} \to \mathcal{H}^{\omega_L^o}_{Sol_L^o/\Sigma''}, \; \tilde{I}_L \hookrightarrow \Gamma(Sol_L^o, \mathcal{H}^{\omega_L^o}_{Sol_L^o}) \to \Gamma(Sol_L^o, \mathcal{H}^{\omega_L^o}_{Sol_L^o/\Sigma''}), \tag{3.3.7}$$

where Sol_L^o is defined in (3.1.20)

In view of (3.1.22), this means $J^{\infty}(T(\Pi TM)_{\Sigma/\Sigma''})$ equipped with ω_L and an embedding

$$\tilde{I}_L \hookrightarrow \Gamma(M, \mathcal{H}^{\omega_L}_{J^{\infty}(T(\Pi TM)_{\Sigma/\Sigma''})}). \tag{3.3.8}$$

We will now exhibit an example where

$$\mathcal{H}^{\omega_L}_{J^{\infty}(T(\Pi TM)_{\Sigma/\Sigma''})} \xrightarrow{\sim} \tilde{\mathcal{H}}^{can}. \tag{3.3.9}$$

3.4. An example: (1,1)-supersymmetric σ-model.

3.4.1. Let M be an n-dimensional purely even Riemannian manifold with metric $(.,.)$. Analogously to sect. 1.6.9, we observe that a point in $J^1(M_{\hat{\Sigma}})$ is a triple $(\hat{t}, X, \partial X)$, a point in $\hat{\Sigma}$, a point in M, and a map

$$\partial X : T_{\hat{t}}\hat{\Sigma} \to T_X M, \; \xi \mapsto \partial_{\xi} X, \tag{3.4.1}$$

cf. (3.1.13a,b). Hence for fixed vector fields ξ, η, $(\partial_{\xi}X, \partial_{\eta}X)$ is a global section of $\mathcal{O}_{J^1(M_{\hat{\Sigma}})}$. (Of course, to be precise, we should have used B-points.) We will unburden the notation by letting ξX stand for $\partial_{\xi}X$. Here is a coordinate expression for this function

$$g_{ij}(X)\xi X^i \eta X^j. \tag{3.4.1}$$

The (1,1)-supersymmetric σ-model Lagrangian is defined to be

$$L = (D_+X, D_-X) du \wedge dv \wedge [d\theta^+ d\theta^-], \tag{3.4.2}$$

where the vector fields $D_{\pm}$ are from (3.1.5), cf. (1.6.13). Integrating out θ^+ and θ^- gives (an exercise in differential geometry, see e.g. [QFS, p. 666])

$$\begin{aligned} L_{11} =& (-(\partial_u x, \partial_v x) + (\nabla_{\partial_v x}\psi_+, \psi_+) + (\nabla_{\partial_u x}\psi_-, \psi_-) + \\ & (R(\psi_+, \psi_-)\psi_+, \psi_-) - (F, F)) du \wedge dv, \end{aligned} \tag{3.4.3}$$

In this formula

$$\partial_u x = \partial_u X|_{\theta^+=\theta^-=0}, \partial_v x = \partial_v X|_{\theta^+=\theta^-=0}, \psi_{\pm} = \partial_{\theta^{\pm}} X|_{\theta^+=\theta^-=0} \tag{3.4.4a}$$

and coincide with their namesakes from (3.1.14), ∇ is the Levi-Civita connection associated to the metric $(.,.)$, R is the curvature tensor, and

$$F = \nabla_{D_+X} D_-X|_{\theta^+=\theta^-=0}, \tag{3.4.4b}$$

which is somewhat different from its counterpart $\tilde{F}$ of (3.1.14).

In fact, with a little extra effort the entire Taylor series expansion of L in θ^+, θ^-, cf. (3.1.17), can be computed to the effect that

$$\begin{aligned} L = ((\psi_+, \psi_-)-& \\ &\theta^+((\partial_u x, \psi_-) + (\psi_+, F)) + \theta_-((\partial_v x, \psi_+) - (\psi_-, F))+ \\ &\theta^+\theta^- L_{11}) du \wedge dv \wedge [d\theta^+ d\theta^-]. \end{aligned} \tag{3.4.5}$$

To see better what all of this means, let us write down the first three terms of (3.4.3) in local coordinates (3.1.14,17); the result is

$$\begin{aligned} L_{11} =&(-g_{ij}(x)\partial_u x^i \partial_v x^j + \partial_v x^\alpha \psi_+^i \psi_+^j \Gamma^s_{\alpha i} g_{sj}(x) + \partial_u x^\alpha \psi_-^i \psi_-^j \Gamma^s_{\alpha i} g_{sj}(x) + \\ &g_{ij}(x)\partial_v \psi_+^i \psi_+^j + g_{ij}(x)\partial_u \psi_-^i \psi_-^j \cdots) du \wedge dv. \end{aligned} \tag{3.4.6}$$

Computation of δL_{11}, cf. (1.6.2), yields the Euler-Lagrange equations and a variational 1-form. The former are as follows, see also [QFS, p.666],

$$\begin{aligned} F &= 0 \\ \nabla_{\partial_u x}\psi_- &= -R(\psi_+, \psi_-)\psi_+ \\ \nabla_{\partial_v x}\psi_+ &= -R(\psi_+, \psi_-)\psi_- \\ \nabla_{\partial_u x}\partial_v x &= \frac{1}{2}\left(R(\psi_-, \psi_-)\partial_v x + R(\psi_+, \psi_+)\partial_u x\right) - (\nabla_{\psi_+} R)(\psi_-, \psi_-)\psi_+ \end{aligned} \tag{3.4.7}$$

The latter is

$$\begin{aligned} \gamma_L = &\left(-g_{ij}(x)\partial_v x^j \delta x^i + \psi_-^i \psi_-^j \Gamma^s_{\alpha i} g_{sj}(x)\delta x^\alpha\right) dv + \\ &\left(g_{ij}(x)\partial_u x^i \delta x^j - \psi_+^i \psi_+^j \Gamma^s_{\alpha i} g_{sj}(x)\delta x^\alpha\right) du + \\ &- g_{ij}(x)\psi_+^j \delta\psi_+^i du + g_{ij}(x)\psi_-^j \delta\psi_-^i dv. \end{aligned} \tag{3.4.8}$$

This, unlike more challenging (3.4.7), is a straightforward consequence of (3.4.6). Note that we have computed after projection (3.2.9), i.e., with $\theta^\pm$ integrated out, (3.2.10); nothing is gained or lost because γ_L matters only modulo d_ρ-exact terms.

3.4.2. *(1,1)-supersymmetry.*
In addition to $D_\pm$, (3.1.5), $\hat{\Sigma}$ carries another pair of distinguished vector fields,

$$\Xi_+ = \frac{\partial}{\partial\theta^+} + \theta^+\frac{\partial}{\partial u},\ \Xi_- = \frac{\partial}{\partial\theta^-} + \theta^-\frac{\partial}{\partial v}. \tag{3.4.9}$$

They enjoy similar properties

$$\begin{aligned} &[\Xi_+, \Xi_+] = 2\frac{\partial}{\partial u}, [\Xi_-, \Xi_-] = 2\frac{\partial}{\partial v}, [\Xi_+, \Xi_-] = 0 \\ &[\frac{\partial}{\partial v}, \Xi_\pm] = [\frac{\partial}{\partial u}, \xi_\pm] = 0, \end{aligned} \tag{3.4.10}$$

and

$$[\Xi_\bullet, D_\bullet] = 0. \tag{3.4.11}$$

Relations (3.4.10) imply that

$$\begin{aligned} \mathcal{N}1^+ &\stackrel{\text{def}}{=} \text{span}\{f(u)\Xi_+\} \subset \mathcal{T}_{\hat{\Sigma}}(\hat{\Sigma}) \\ \mathcal{N}1^- &\stackrel{\text{def}}{=} \text{span}\{f(v)\Xi_-\} \subset \mathcal{T}_{\hat{\Sigma}}(\hat{\Sigma}) \end{aligned} \tag{3.4.12}$$

are two commuting copies of the N=1- supersymmetric superalgebra Lie realized in vector fields on $\hat{\Sigma}$; note that each contains a copy, $\mathcal{V}ir^{\pm}$, of the algebra of vector fields on Σ.

In fact, both are subalgebras of the algebra of symmetries of L:

$$\mathcal{N}1^{\pm} \hookrightarrow \tilde{I}_L. \tag{3.4.13}$$

Indeed, using (3.4.5), one computes easily that

$$\mathrm{Lie}_{\rho f(u)\Xi_+} L_{11} = d_\rho(f(u)L_{01})dv,\ \mathrm{Lie}_{\rho f(v)\Xi_-} L_{11} = d_\rho(f(v)L_{10})du. \tag{3.4.14}$$

It is then rather straightforward, and pleasing, to use (3.3.3, 3.4.8) in order to compute the corresponding integrals of motion

$$\begin{aligned} Q_f^+ &\stackrel{\text{def}}{=} Q_{\rho f(u)\Xi_+} = 2f(u)g_{ij}(x)\psi_+^i\partial_u x^j du - g_{ij}(x)F^i\psi_-^j dv \\ Q_f^- &\stackrel{\text{def}}{=} Q_{\rho f(v)\Xi_-} = -2f(v)g_{ij}(x)\psi_-^i\partial_v x^j dv + g_{ij}(x)F^i\psi_+^j, \end{aligned} \tag{3.4.15}$$

which, upon the imposition of the Euler-Lagrange equation $F=0$, becomes

$$Q_f^+ = 2f(u)g_{ij}(x)\psi_+^i\partial_u x^j du,\ Q_f^- = -2f(v)g_{ij}(x)\psi_-^i\partial_v x^j dv. \tag{3.4.16}$$

This furnishes the embeddings

$$\mathcal{V}ir^{\pm} \hookrightarrow \mathcal{N}1^{\pm} \hookrightarrow \Gamma(Sol_L^o, \mathcal{H}^{\omega_L}_{Sol_L^o}), \tag{3.4.17a}$$

and the definitions (cf. 1.6.11.1) of right/left moving subalgebras

$$\mathcal{H}^{\omega_L,\pm}_{Sol_L^o} = \{F \in \mathcal{H}^{\omega_L}_{Sol_L^0} :\ [F, \mathcal{V}ir^{\mp}] = 0\}. \tag{3.4.17b}$$

Next, we will see that all of this unfolds within the canonical Lie algebra sheaf $\tilde{\mathcal{H}}^{can}$ of sect. 3.2.3.

3.4.3. Proposition. *There is a diffeomorphism*

$$\hat{g}:\ Sol_L^o \stackrel{\sim}{\longrightarrow} J^\infty(T^*(\Pi TM)_{\Sigma/\Sigma''})$$

of $D_{\Sigma/\Sigma''}$-manifolds, which delivers the Lie algebra sheaf isomorphism

$$g^{\#}: \mathcal{H}^{\omega_L,\pm}_{Sol_L^o/\Sigma''} \stackrel{\sim}{\longrightarrow} \hat{g}^{-1}\tilde{\mathcal{H}}^{can},$$

cf. Lemmas 1.5.5.1 and 1.6.8.1.

3.4.4. *Proof. Super-Legendre transform.*

In order to proceed, we need to make sure that Sol_L as defined by (3.4.7) satisfies Cauchy-Kovalevskaya condition (3.1.19). Apparently neither u nor v can play the role of time, but the change variables as follows

$$u = \sigma + \tau,\ v = \sigma - \tau \tag{3.4.18}$$

so that

$$\partial_u = \frac{1}{2}(\partial_\sigma + \partial_\tau),\ \partial_v = \frac{1}{2}(\partial_\sigma - \partial_\tau) \tag{3.4.19}$$

does the job. Therefore, cf. (3.1.22),

$$Sol_L^o \stackrel{\sim}{\longrightarrow} J^\infty(T(\Pi TM)_{\Sigma/\Sigma''}), \tag{3.4.20}$$

and our task is to find

$$\hat{g}: J^{\infty}(T(\Pi TM)_{\Sigma/\Sigma''}) \xrightarrow{\sim} J^{\infty}(T^{*}(\Pi TM)_{\Sigma/\Sigma''}) \tag{3.4.21}$$

that identifies ω_L on the L.H.S. with the pull-back of the canonical ω^o, (3.2.19), on the R.H.S

$$(\hat{g})^{*}\omega|_{d\tau=0} = \omega_L|_{d\tau=0} + d_{\rho/\Sigma''}(\cdots) \tag{3.4.22}$$

modulo $d_{\rho/\Sigma''}$-exact terms. Variational 1-form γ_L, computed in (3.4.8), is not well suited for this purpose. In addition to (3.4.18), let us introduce variables

$$\rho^j = \frac{1}{2}(\psi_-^j + \psi_+^j), \phi^j = \frac{1}{2}(\psi_-^j - \psi_+^j). \tag{3.4.23}$$

(Since $\psi_{\pm}^j$ are sections of 2 copies of the bundle of 1-forms, see (3.4.4a) and (3.1.17,18), this change of variables makes sense globally.) Plugging these variables in Lagrangian (3.4.3) gives

$$L_{11} = (\frac{1}{2}(\partial_\tau x, \partial_\tau x) + 2(\nabla_{\partial_\tau x}\rho, \phi) + 2(\nabla_{\partial_\tau x}\phi, \rho) + \cdots d\tau\wedge)d\sigma$$

where $\cdots$ stand for the terms not containing ∂_τ. Since

$$(\nabla_{\partial_\tau x}\phi, \rho) = -(\phi, \nabla_{\partial_\tau x}\rho) + \partial_\tau(\phi, \rho) = (\nabla_{\partial_\tau x}\rho, \phi,) + \partial_\tau(\phi, \rho), \tag{3.4.24}$$

we obtain

$$\tilde{L}_{11} = (\frac{1}{2}(\partial_\tau x, \partial_\tau x) + 4(\nabla_{\partial_\tau x}\rho, \phi) + \cdots)d\tau \wedge d\sigma = L_{11}\ mod\ d_\rho(...) \tag{3.4.25}$$

It is immediate to derive from (3.4.25) that the corresponding variational 1-form

$$\gamma'_L = (g_{ij}\partial_\tau x^j + 4\Gamma^s_{i\alpha}\rho^\alpha\phi^j g_{sj}(x))\delta x^i \wedge d\sigma + 4g_{ij}(x)\delta\rho^i\phi^j \wedge d\sigma, \tag{3.4.26}$$

which is equal to $\gamma_L|_{d\tau=0}$ modulo d_ρ-exact terms.

If we let

$$\rho_i = \frac{1}{4}g_{ij}(x)\phi^j, \tag{3.4.27}$$

then

$$\gamma'_L = (g_{ij}\partial_\tau x^j + \Gamma^s_{i\alpha}\rho^\alpha\rho_s)\delta x^i \wedge d\sigma + \delta\rho^i\rho_i \wedge d\sigma, \tag{3.4.28}$$

The substitution

$$x_i \mapsto g_{ij}\partial_\tau x^j + \Gamma^s_{i\alpha}\rho^\alpha\rho_s, \phi^i \mapsto \rho^i, \phi_i \mapsto \rho_i \tag{3.4.29}$$

makes sense as a globally defined map

$$d_{TM}\tilde{L}_{11}: T(\Pi TM) \to T^{*}(\Pi TM). \tag{3.4.30}$$

It is a super-analogue of the Legendre transform, (1.6.13-14), which was envisaged in sect.3.2.4; indeed, if $x_i = \partial_{x^i}$, then the first of assignments (3.4.29) is exactly splitting (3.2.26). The $D_{\Sigma/\Sigma''}$-manifold property allows to extend this map unambiguously to the jet-spaces, and it is clear that such map identifies $\delta\gamma'_L$ with ω from (3.2.19).

3.4.5. Therefore, $\tilde{\mathcal{H}}^{can}$ is to the (1,1)-supersymmetric σ-model what $\mathcal{H}^{can}$ is to the ordinary σ-model. In particular, writing integrals of motion (3.4.15) in terms of the new variables introduced in sect. 3.4.4 provides a free field realization of $\mathcal{N}1^{\pm}$. The result, which we will discuss in the context of the Kähler geometry, see the next section, is presumably the quasiclassical limit of the formulas obtained in [B-ZHS].

3.4.6. *The Kähler case: (2,2)-supersymmetry and the Witten Lie algebra.*

It is an exciting discovery going back to [Zu, A-GF] that in the Kähler case the supersymmetry algebra becomes twice as large.

3.4.6.1. Let then M be a complex manifold and $(.,.)$ a Kähler metric on it. To handle this case, we will change the notation somewhat: the natural vector bundles, such as TM, will be assumed to be complexified, and decompositions, such as $TM = T^{10}M \oplus T^{01}M$, will arise. What has been treated as a vector field, e.g. $\partial_\tau x$, $\partial_\tau \psi_+$, will become a section of $T^{10}M$, and $\partial_\tau \bar{x}$, $\partial_\tau \bar{\psi}_+$ will stand for the complex conjugate sections. We will also let, sloppily but customarily,

$$\partial_\tau \overline{x^j} = \partial_\tau x^{\bar{j}}, \overline{\psi^j_\pm} = \psi^{\bar{j}}_\pm. \tag{3.4.31}$$

The defining property of the Kähler metric

$$\nabla(T^{10}) \subset T^{10},\ \nabla(T^{01}) \subset T^{01} \tag{3.4.32}$$

is crucial for what follows.

Computing as in 3.4.4 (and using (3.4.32)) one obtains

$$L_{11} = (-(\partial_u x, \partial_u \bar{x}) - (\partial_v x, \partial_u \bar{x}) + (\nabla_{\partial_v}\psi_+, \bar{\psi}_+) + (\nabla_{\partial_u}\psi_-, \bar{\psi}_-) + \cdots)du \wedge dv \ mod\ d_\rho, \tag{3.4.33}$$

where the terms not containing ∂_u, ∂_v are omitted.

Property (3.4.32) implies (and (3.4.33) supports) that w.r.t. the grading on $\mathcal{O}_{J^\infty(M_{\hat{\Sigma}})^o}$ defined by

$$\psi_\pm \mapsto 1,\ \bar{\psi}_\pm \mapsto -1 \tag{3.4.34}$$

L_{11} is homogeneous of degree 0. Therefore, any homogeneous component of a symmetry of L_{11} is also a symmetry. Integrals of motion (3.4.15) afford decomposition

$$Q^+_f = Q^{++}_f + Q^{+-}_f, Q^-_f = Q^{-+}_f + Q^{--}_f \tag{3.4.35}$$

into the sum of degree ± 1 components, which implies that the entire quadruple

$$\{Q^{++}_f, Q^{+-}_f, Q^{-+}_f, Q^{--}_f\} \subset \tilde{\mathcal{I}}_L, \tag{3.4.36}$$

and this extends (3.4.17a) to an embedding of a pair of N=2-superconformal Lie algebras

$$\mathcal{V}ir^\pm \hookrightarrow \mathcal{N}1^\pm \hookrightarrow \mathcal{N}2^\pm \hookrightarrow \tilde{\mathcal{I}}_L \hookrightarrow \tilde{\mathcal{H}}^{can}. \tag{3.4.37}$$

In particular, (and it follows from the consideration of the degree)

$$[Q^{++}_f, Q^{++}_g] = [Q^{--}_f, Q^{--}_g] = 0. \tag{3.4.38}$$

Witten has used these relations [W2,W3] to define what in the present context becomes *Witten Lie algebra sheaves*:

$$\mathcal{W}^\pm \stackrel{\text{def}}{=} \frac{\{X \in \tilde{\mathcal{H}}^{can} :\ [Q^{\mp,\mp}_1, X] = 0\}}{\{[Q^{\mp,\mp}_1, X] \text{ all } X \in \tilde{\mathcal{H}}^{can}\}}. \tag{3.4.39}$$

(There are, of course, two more versions of these sheaves.)

3.4.6.2. *Some formulas.* For the purpose of writing embeddings such as (3.4.37) explicitly, rewrite (3.4.33) using σ and τ which were defined in (3.4.18)

$$\begin{aligned} L_{11} =&(g_{i\bar{j}}\partial_\tau x^i\partial_\tau x^j - 2\partial_\tau x^i\Gamma^s_{ij}g_{s\bar{t}}\psi^j_+\psi^{\bar{t}}_+ + 2\partial_\tau x^{\bar{i}}\Gamma^{\bar{s}}_{\bar{i}\bar{j}}g_{\bar{s}\bar{t}}\psi^{\bar{j}}_-\psi^t_-)d\tau\wedge d\sigma \\ &+ (2g_{i\bar{j}}\psi^{\bar{j}}_+\partial_\tau\psi^i_+ - 2g_{\bar{i}j}\psi^j_-\partial_\tau\psi^{\bar{i}}_-)d\tau\wedge d\sigma\cdots \end{aligned} \tag{3.4.40}$$

where the terms not containing ∂_τ are omitted. It follows that, cf. (3.4.26),

$$\begin{aligned} \gamma'_L =&(g_{i\bar{j}}\partial_\tau x^{\bar{j}} - 2\Gamma^s_{ij}g_{s\bar{t}}\psi^j_+\psi^{\bar{t}}_+)\delta x^i\wedge d\sigma + (g_{i\bar{j}}\partial_\tau x^i + 2\Gamma^{\bar{s}}_{\bar{j}\bar{i}}g_{\bar{s}t}\psi^{\bar{i}}_-\psi^t_-)\delta x^{\bar{j}}\wedge d\sigma \\ &(-2g_{i\bar{j}}\psi^{\bar{j}}_+\delta\psi^i_+ + 2g_{\bar{i}j}\psi^j_-\delta\psi^{\bar{i}}_-)\wedge d\sigma. \end{aligned} \tag{3.4.41}$$

If we let

$$\psi = \psi_+, \bar{\psi} = \bar{\psi}_-, \tag{3.4.42a}$$

and

$$\psi_i = -2g_{i\bar{j}}\psi^{\bar{j}}_+, \psi_{\bar{i}} = 2g_{\bar{i}j}\psi^j_-, \tag{3.4.42b}$$

then

$$\begin{aligned} \gamma'_L =&(g_{i\bar{j}}\partial_\tau x^{\bar{j}} + \Gamma^s_{ij}\psi^j\psi_s)\delta x^i\wedge d\sigma + (g_{i\bar{j}}\partial_\tau x^i + \Gamma^{\bar{s}}_{\bar{j}\bar{i}}\psi^{\bar{i}}\psi_{\bar{s}})\delta x^{\bar{j}}\wedge d\sigma \\ &(\psi_i\delta\psi^i + \psi_{\bar{i}}\delta\psi^{\bar{i}})\wedge d\sigma. \end{aligned} \tag{3.4.43}$$

Therefore, the coordinate form of the super-Legendre transform (3.4.30) is

$$\begin{aligned} &x_i \mapsto g_{i\bar{j}}\partial_\tau x^{\bar{j}} + \Gamma^s_{ij}\psi^j\psi_s \\ &x^\bullet \mapsto x^\bullet \\ &x_{\bar{j}} \mapsto g_{i\bar{j}}\partial_\tau x^i + \Gamma^{\bar{s}}_{\bar{j}\bar{i}}\psi^{\bar{i}}\psi_{\bar{s}} \\ &\phi^\bullet \mapsto \psi^\bullet,\ \phi_\bullet \mapsto \psi_\bullet \end{aligned} \tag{3.4.44}$$

Plugging these in (3.4.17) and extracting homogeneous components as in (3.4.35) one obtains, upon letting $d\tau = 0$,

$$\begin{aligned} Q^{--}_f &= f(\sigma-\tau)(-x_{\bar{j}}\phi^{\bar{j}} + g_{i\bar{j}}\partial_\sigma x^i\phi^{\bar{j}})d\sigma, \\ Q^{-+}_f &= 2f(\sigma-\tau)(\partial_\sigma x^{\bar{j}}\phi_{\bar{j}} - g^{\bar{j}i}x_i\phi_{\bar{j}} + g^{\bar{j}i}\Gamma^s_{i\alpha}\phi^\alpha\phi_s\phi_{\bar{j}})d\sigma. \end{aligned} \tag{3.4.45$^-$}$$

$$\begin{aligned} Q^{++}_f &= f(\sigma+\tau)(x_j\phi^j + g_{\bar{j}i}\partial_\sigma x^{\bar{j}}\phi^i)d\sigma, \\ Q^{+-}_f &= -2f(\sigma+\tau)(\partial_\sigma x^i\phi_i + g^{i\bar{j}}x_{\bar{j}}\phi_i - g^{i\bar{j}}\Gamma^{\bar{s}}_{\bar{j}\bar{\alpha}}\phi^{\bar{\alpha}}\phi_{\bar{s}}\phi_i)d\sigma. \end{aligned} \tag{3.4.45$^+$}$$

One may wish at this point to use these formulas to compute Witten's Lie algebra sheaf (3.4.39). Two things transpire immediately: first, the role played by f in all of this is rather superficial and, second, if one removes from the first of (3.4.45$^-$) the annoying $g_{i\bar{j}}\partial_\sigma x^i\phi^j$ (and $g_{i\bar{j}}\partial_\sigma x^i\phi^j$ from the first of (3.4.45$^+$) resp.), then it becomes exactly the $\bar{\partial}$- (∂- resp.) differential; and so, perhaps, $\mathcal{W}^\pm$ should be of completely holomorphic (antiholomorphic resp.) nature. This is all true, but the language suited to analysis of such issues is that of vertex Poisson algebras.

3.5. Vertex Poisson algebra interpretation. Witten's models

The sheaf $\tilde{\mathcal{H}}^{can}$ is the tip of an iceberg. It is, just as its purely even counterpart $\mathcal{H}^{can}$ was, sect. 1.5.4, a Lie algebra sheaf attached to a certain sheaf of vertex Poisson superalgebras

3.5.1. The notion of a super-SVDO is quite analogous to the one we discussed in sect. 2. It is a $\mathbb{Z}_+$-graded vertex Poisson superalgebra $V = V_0 \oplus V_1 \oplus \cdots$ such that

$$V_0 = C^\infty(\Pi T U_\Sigma),\ U \subset \mathbb{R}^n, \tag{3.5.1}$$

and, non-canonically,

$$V_1 = \mathcal{T}_{U_\Sigma/\Sigma}(U_\Sigma) + \Omega_{U_\Sigma/\Sigma}(U_\Sigma) + \Pi\left(\mathcal{T}_{U_\Sigma/\Sigma}(U_\Sigma) + \Omega_{U_\Sigma/\Sigma}(U_\Sigma)\right). \tag{3.5.2}$$

Classification of such algebras [GMS3], under some obvious non-degeneracy assumptions, is obtained in a way similar to sect. 2.4.3, 2.5. They form an $\Omega^{3,cl}(U)$-torsor, i.e., given a super-SVDO V and a closed 3-form $H \in \Omega^{3,cl}(U)$, an operation

$$(V, H) \mapsto V \dot{+} H \tag{3.5.3}$$

is defined, where $V \dot{+} H$ is a super-SVDO different from V only in that the operation

$$_{(0)} : \mathcal{T}_{U_\Sigma/\Sigma} \otimes \mathcal{T}_{U_\Sigma/\Sigma} \to V_1$$

is replaced with

$$_{(0)_H} =_{(0)} +H, \tag{3.5.4}$$

cf. (2.4.21). (This involves only even components of V_1.) One has, cf. (2.4.24),

$$Mor(V, V \dot{+} H) = \{\alpha \in \Omega^2(U) \text{ s.t. } d\alpha = H\}. \tag{3.5.5}$$

In particular, cf. (2.4.25),

$$\Omega^{2,cl}(U) \xrightarrow{\sim} \mathrm{Aut}(V), \tag{3.5.6a}$$

where the automorphism corresponding to α is the one determined by the shear, cf. (2.4.19),

$$\mathcal{T}_{U_\Sigma/\Sigma}(U_\Sigma) \ni \xi \mapsto \xi + \iota_\xi\alpha. \tag{3.5.6b}$$

All of this can be spread over manifolds. There is a distinguished such sheaf of super-SVDOs, the vertex Poisson de Rham complex [MVS], Ω_M^{poiss}. As an $\mathcal{O}_M$-module,

$$\Omega_M^{poiss} = \pi_*\mathcal{O}_{J^\infty(T^*(\Pi TM)_{\Sigma/\Sigma''})}, \tag{3.5.7}$$

where π is the projection $J^\infty(T^*(\Pi TM)_{\Sigma/\Sigma''}) \to M_\Sigma$, The operations are determined by the requirement that they all be of classical origin – as in Proposition 2.7.1. Here are some examples written down in local coordinates:

$$\begin{aligned}&(x_i)_{(0)}f(x) = \partial_{x^i}f(x), (x_{\bar{i}})_{(0)}f(x) = \partial_{x^{\bar{i}}}f(x)\ (\phi_i)_{(0)}\phi^j = \delta_i^j, (\phi_{\bar{i}})_{(0)}\phi^{\bar{j}} = \delta_{\bar{i}}^{\bar{j}}\\ &\xi_{(0)}\eta = [\xi, \eta],\ \xi_{(o)}\alpha = \mathrm{Lie}_\xi\alpha,\end{aligned} \tag{3.5.8a}$$

where $\xi = f^i(x)x_i + f^{\bar{i}}(x)x_{\bar{i}}$, $\eta = g^i(x)x_i + g^{\bar{i}}(x)x_{\bar{i}}$, $\alpha = h_i(x)\rho(\partial_\sigma)x^i + h_{\bar{i}}(x)\rho(\partial_\sigma)x^{\bar{i}}$, the vertex algebra derivation being

$$T = \rho(\partial_\sigma). \tag{3.5.8b}$$

(The twist that takes care of functions of σ and imposed in the even case in Definition 2.7.2 has been tacitly assumed throughout.)

One has, analogously to Proposition 2.6.1,

3.5.1.1. Proposition. *a) The set of isomorphism classes of sheaves of super-SVDOs on M is identified with $H^3(M, \mathbb{R})$.*

b)If $\mathcal{V}$ is a sheaf of super-SVDOs, then

$$Aut\mathcal{V} \xrightarrow{\sim} H^0(M, \Omega_M^{2,cl}).$$

Let

$$\mathrm{Lie}(\mathcal{V}) = \mathcal{V}/T(\mathcal{V}). \tag{3.5.9}$$

Operation $_{(0)}$ makes Lie$\mathcal{V}$ into a sheaf of Lie superalgebras. One has, cf. Proposition 2.7.3,

3.5.2. Proposition. *The algebra sheaves $\tilde{\mathcal{H}}^{can}$ and $Lie(\Omega_M^{poiss})$ are isomorphic.*

3.5.3. Some of the constructions above are simplified when performed in the framework of of vertex Poisson superalgebras because some of the Lie algebras considered are the value of the Lie-functor. For example, there are N=1,2 supersymmetric vertex Poisson algebras [K], $N1$ and $N2$, such that the N=1,2 supersymmetric Lie superalgebras, which appeared in (3.4.12), are

$$\mathcal{N}1 = \mathrm{Lie}(C^\infty(\Sigma)\otimes N1),\ \mathcal{N}2 = \mathrm{Lie}(C^\infty(\Sigma)\otimes N2). \tag{3.5.10}$$

The elements, see (3.4.45),

$$\begin{aligned} Q^{--} &= -x_{\bar{j}}\phi^{\bar{j}} + g_{i\bar{j}}\partial_\sigma x^i\phi^{\bar{j}}, \\ Q^{-+} &= 2(\partial_\sigma x^{\bar{j}}\phi_{\bar{j}} - g^{\bar{j}i}x_i\phi_{\bar{j}} + g^{\bar{j}i}\Gamma^s_{i\alpha}\phi^\alpha\phi_s\phi_{\bar{j}}) \end{aligned} \tag{3.5.11$^-$}$$

$$\begin{aligned} Q^{++} &= x_j\phi^j + g_{\bar{j}i}\partial_\sigma x^{\bar{j}}\phi^i \\ Q^{+-} &= -2(\partial_\sigma x^i\phi_i + g^{i\bar{j}}x_{\bar{j}}\phi_i - g^{i\bar{j}}\Gamma^{\bar{s}}_{\bar{j}\bar{\alpha}}\phi^{\bar{\alpha}}\phi_{\bar{s}}\phi_i). \end{aligned} \tag{3.4.45$^+$}$$

define global sections of Ω_M^{poiss}. By definition, the following analogue of (2.8.12) and (2.9.25) holds true.

3.5.3.1. Lemma. *The two pairs of global sections (Q^{--}, Q^{-+}) and (Q^{++}, Q^{+-}) generate, inside $H^0(M, \Omega_M^{poiss})$, two pairwise Poisson-commuting copies of the vertex Poisson N=2 superalgebra:*

$$N2^+ \hookrightarrow H^0(M, \Omega_M^{poiss}) \hookleftarrow N2^-,\ (N2^+_{(n)}(N2^-) = 0 \textit{ if } n \geq 0. \tag{3.5.12}$$

A streamlined version of Witten's Lie algebra sheaf (3.4.39) is Witten's vertex Poisson algebra sheaf defined as follows. Relations (3.4.38) in the vertex algebra context imply that each element of the quadruple $\{Q^{\bullet,\bullet}_{(0)}, \bullet = \pm\})$ and various linear combinations thereof are differentials of the sheaf Ω_M^{poiss}. Letting $Q_{(0)}$ be one such differential, we obtain a cohomology sheaf

$$H_Q(\Omega_M^{poiss}) \stackrel{\text{def}}{=} \frac{\mathrm{Ker}Q_{(0)}}{\mathrm{Im}Q_{(0)}}. \tag{3.5.13}$$

It is a vertex Poisson algebra sheaf – a well-known fact and an immediate consequence of $_{(0)}$ being a derivation of all $_{(n)}$-products (super-analogue of Jacobi identity, sect. 2.1, II.2).

Of sheaves (3.5.13) the following 3 will be of interest to us:

3.5.3.2. Definition. (cf. [W2])

$$\text{A-model sheaf: } W_A = H_{Q^{--}+Q^{++}}(\Omega_M^{poiss}), \tag{3.5.14a}$$

$$\text{B-model sheaf: } W_B = H_{Q^{--}+Q^{+-}}(\Omega_M^{poiss}), \tag{3.5.14b}$$

$$\text{half-twisted model sheaf: } W_{1/2} = H_{Q^{--}}(\Omega_M^{poiss}), \tag{3.5.14c}$$

The relation of (3.5.13-14) to (3.4.39) is that

$$\mathcal{W}^{-} = \text{Lie}(W_{1/2}), \tag{3.5.15}$$

to give but one example.

The cohomology, $H^*(M,\mathcal{V})$, of a sheaf of vertex Poisson algebras $\mathcal{V}$ is a vertex Poisson algebra, of course. We are led then, following [W2], to

3.5.3.3. Definition.

$$\text{A-model vertex Poisson algebra: } H^*(M, W_A), \tag{3.5.16a}$$

$$\text{B-model vertex Poisson algebra: } H^*(M, W_B), \tag{3.5.16b}$$

$$\text{half-twisted model vertex Poisson algebra: } H^*(M, W_{1/2}), \tag{3.5.16c}$$

3.5.4. Theorem. *Let M be Kählerian. Then*
1) the following isomorphisms are valid:

$$H^*(M, W_A) \xrightarrow{\sim} H^*(M, \mathbb{C}), \tag{3.5.17a}$$

$$H^*(M, W_B) \xrightarrow{\sim} H^*(M, \Lambda^*\mathcal{T}_M), \tag{3.5.17b}$$

$$H^*(M, W_{1/2}) \xrightarrow{\sim} H^*(M, \Omega_M^{poiss,an}), \tag{3.5.17c}$$

where $\Omega_M^{poiss,an}$ is a purely holomorphic version of the sheaf Ω_M^{poiss} [MSV];
2) embedding $N2^+ \hookrightarrow \Omega_M^{poiss}$, (3.5.12), descends to an embedding $N2^+ \hookrightarrow \Omega_M^{poiss,an}$ whose image coincides with N=2 superconformal structure introduced on $\Omega_M^{poiss,an}$.

3.5.4.1. *Remark.* 1) Of these, the first two are finite dimensional supercommutative algebras and as such are trivial examples of a vertex Poisson algebra with zero derivation T as noted in sect.2.3. Contrary to this, the last one is a full-fledged infinite dimensional vertex Poisson algebra. Being infinite dimensional it is characterized by its character (q-dimension), which is closely related to the elliptic genus of M. The algebra can be quantized, and the character of the quantum version has provided some insights into the elliptic genus [BL,MS,GM1, GM2].

2) This theorem, especially (3.5.17c) is a refined version of [Kap]. In fact, Kapustin deals with the quantum version of this result; we will discuss quantization in the next section.

3.5.4.2. *Sketch of proof.*

Apply to Ω_M^{poiss} automorphism (3.5.6a-b) determined by the Kähler 2-form $g_{i\bar{j}}dx^i \wedge dx^{\bar{j}}$. As a result, $(Q^{--})_{(0)}$ will be replaced with a vertex analogue of the $\bar{\partial}$-differential:

$$\bar{\partial}_{vert} = (x_{\bar{j}}\phi^{\bar{j}})_{(0)} \tag{3.5.18}$$

Essentially by definition,

$$(\Omega_M^{poiss,an}, 0) \hookrightarrow (\Omega_M^{poiss}, \bar{\partial}_{vert}),$$

is a quasiisomorphism [MSV]. Indeed, a glance at (3.5.8a) convinces that $x^{\bar{j}}$ $\phi^{\bar{j}}$ are not $\bar{\partial}_{vert}$-cocycles, and $x_{\bar{j}}$ $\phi_{\bar{j}}$ are $\bar{\partial}_{vert}$-cohomologous to 0. Therefore $\bar{\partial}_{vert}$ effectively kills all antiholomorphic variables, leaving holomorphic ones intact. This defines a purely holomorphic analogue of Ω_M^{poiss}, that is, $\Omega_M^{poiss,an}$.

Hence a quasiisomorphism

$$(\Omega_M^{poiss,an}, 0) \hookrightarrow (\Omega_M^{poiss}, (Q^{--})_{(0)}),$$

which proves (3.5.17c).

In (3.5.17a-b) one more differential is turned on. Definition (3.5.11) implies that upon the same shear by the Kähler form,

$$Q^{++} = x_j\phi^j. \tag{3.5.19}$$

Therefore, $(Q^{--})_{(0)} + (Q^{++})_{(0)}$ is a vertex analogue of total de Rham differential, and (3.5.17b) becomes essentially [MSV], Theorem 2.4.

Similarly, in the $(Q^{--})_{(0)}$-cohomology,

$$Q^{+-} = -4\partial_\sigma x^i\phi_i, \tag{3.5.20}$$

and a simple analysis along the lines of [MSV], sect. 2.3-2.4, shows that

$$H_{\partial_\sigma x^j\phi_j}(\Omega_M^{poiss,an}) \xrightarrow{\sim} H^*(M, \Lambda^*\mathcal{T}_M),$$

as desired.

Item 2) is a result of checking (3.5.19,20) against [MSV], (2.3b).

Next, we establish concrete complexes which compute vertex Poisson algebras of the A-, B-, and half-twisted models.

3.5.5. Corollary.

$$H^*(M, W_A) \xrightarrow{\sim} H_{Q^{--}+Q^{++}}(\Gamma(M, \Omega_M^{poiss})), \tag{3.5.21a}$$

$$H^*(M, W_B) \xrightarrow{\sim} H_{Q^{--}+Q^{+-}}(\Gamma(M, \Omega_M^{poiss})), \tag{3.5.21b}$$

$$H^*(M, W_{1/2}) \xrightarrow{\sim} H_{Q^{--}}(\Gamma(M, \Omega_M^{poiss})), \tag{3.5.21c}$$

Proof. The sheaf Ω_M^{poiss} is a complex w.r.t. the 3 differentials used above. Hence there arise 3 different hypercohomology groups, $\mathbb{H}_A(\Omega_M^{poiss})$, $\mathbb{H}_B(\Omega_M^{poiss})$, $\mathbb{H}_{1/2}(\Omega_M^{poiss})$. Each can be computed by any of the two spectral sequences. The computation using one of them is the content of Theorem 3.5.4. It says that the result is the vertex Poisson algebra of A-, B-, and half-twisted models resp. The computation using another will then prove the corollary, because the sheaf Ω_M^{poiss} being flabby, $H^j(M, \Omega_M^{poiss}) = 0$ if $j > 0$.

Remark. In view of Theorem 3.5.4, isomorphisms (3.5.21a,b) are vertex Poisson algebra versions of the de Rham complex and $\bar{\partial}$-resolution of the algebra of polyvector fields resp., while (3.5.21c) is the $\bar{\partial}$-resolution of the vertex Poisson de Rham complex.

3.5.6. *H-flux.*

Let us now give, along the lines of sect.2.8, the Lagrangian interpretation of twisted sheaves of super-SVDOs which arise via (3.5.3) and are parametrized by $H^3(M,\mathbb{R})$, see Proposition 3.5.1.1.

Fix $H \in \Gamma(M, \Omega_M^{3,cl})$, a closed 3-form; a cover $\{U_i\}$ of M; and a collection of 2-forms $\beta^{(i)} \in \Gamma(U_i, \Omega_M^2)$ s.t.

$$d\beta^{(i)} = H|_{U_i}.$$

Denote by $X^*\beta^{(i)} \in \Omega^{2,0}_{J^\infty(M_{\hat{\Sigma}})}$ the 2-form determined by the condition:

$$X^*\beta^{(i)}(\xi,\eta) = \beta^{(i)}(\rho(\xi),\rho(\eta)),\ \xi,\eta \in \mathcal{T}_{\hat{\Sigma}}. \tag{3.5.22}$$

Informally speaking, the meaning of this definition is this: if $X : \hat{\Sigma} \to M$ is a smooth map, then $X^*\beta^{(i)} \in \Omega^2_{\hat{\Sigma}}$ is the standard pull-back of $\beta^{(i)}$ via X. It depends only on the 1-jet of X and, therefore, comes from a universal 2-form on $J^\infty(M_{\hat{\Sigma}})$; the latter is defined in (3.5.22).

Analogously to (2.8.3), consider the H-twist of Lagrangian (3.4.2):

$$L^H = \{L + X^*\beta^j \wedge [d\theta^+ d\theta^-]\}. \tag{3.5.23}$$

The argument completely parallel to that leading to (2.8.5) proves the following.

3.5.6.1. Lemma.

$$\tilde{\mathcal{H}}^{\omega^o_{L^H}}_{Sol^o_{L^H}} \xrightarrow{\sim} \mathrm{Lie}(\Omega_M^{poiss} \dot{+} H).$$

where $\Omega_M^{poiss} \dot{+} H$ is defined as in (3.5.3).

Therefore, all the constructions, originating in [GHR] and further explored in papers such as [BLPZ,KL], translate into different vertex Poisson subalgebras of $\Omega_M^{poiss} \dot{+} H$ depending on the choice of a generalized Kähler structure.

3.6. Quantization. B-model moduli.

This section is an announcement. It will be assumed throughout that the automorphism by the Kähler form has been performed so that

$$Q^{--} = \phi^{\bar{j}} x_{\bar{j}},\ Q^{++} = \phi^i x_i, \tag{3.6.0}$$

cf. sect.3.5.4.2.

3.6.1. The differential graded sheaves of vertex Poisson algebras, $(\Gamma(M, \Omega_M^{poiss}), Q_{(0)})$, where Q is any of the differentials appearing in (3.5.21a,b,c), can be quantized. What we mean by this is that, first, there is a sheaf of vertex algebras Ω_M^{vert} [MSV] whose quasiclassical limit is Ω_M^{poiss} and, second, this sheaf carries quantum analogues of each of the 3 differentials. In fact, quantum versions of $(Q^{--}_{(0)})$ and $(Q^{++}_{(0)})$ are in [MSV], and $(Q^{+-})_{(0)}$ has been recently proposed in [B-ZHS]; in what follows the use of the latter is easy to avoid. Thus there arise 3 vertex algebra versions of A-, B-, and half-twisted models resp.:

$$H^*(M, W_A^{quant}) \xrightarrow{\sim} H_{Q^{--}+Q^{++}}(\Gamma(M, \Omega_M^{vert})), \tag{3.6.1a}$$

$$H^*(M, W_B^{quant}) \xrightarrow{\sim} H_{Q^{--}+Q^{+-}}(\Gamma(M, \Omega_M^{vert})), \tag{3.6.1b}$$

$$H^*(M, W_{1/2}^{quant}) \xrightarrow{\sim} H_{Q^{--}}(\Gamma(M, \Omega_M^{vert})), \tag{3.6.1c}$$

The first two, $H^*(M, W_A^{quant})$ and $H^*(M, W_B^{quant})$, coincide with their quasiclassical limits (3.5.21a,b). The 3rd is quite different from its quasiclassical limit and equals the cohomology of the chiral de Rham complex, $H^*(M, \Omega_M^{ch})$ [MSV].

Relation of this naive quantization to the genuine quantum string theory is expressed by saying, in physics language, that the latter equals the former "perturbatively", [Kap]. But let us show that both (3.6.1b,c) can be further deformed along the Barannikov-Kontsevich moduli space [BK]. We will focus on the half-twisted model (3.6.1c).

3.6.2. Recall that associated to any differential Lie superalgebra ($\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1, d$) there is a deformation functor, $\mathrm{Def}_{\mathfrak{g}}$, with domain the category of Artin algebras and range the category of sets [Kon,BK]. In order to define it, introduce the space of solutions to the Maurer-Cartan equation with values in an Artin algebra A:

$$MC_{\mathfrak{g}}(A) = \{\gamma : d\gamma + \frac{1}{2}[\gamma, \gamma] = 0,\ \gamma \in (\mathfrak{g} \otimes A)^1\}. \tag{3.6.2}$$

The operation

$$(\mathfrak{g} \otimes A)^1 \ni \gamma \mapsto d\beta + [\gamma, \beta] \text{ if } \beta \in (\mathfrak{g} \otimes A)^0 \tag{3.6.3}$$

does not preserve the set $MC_{\mathfrak{g}}(A)$, but it does so infinitesimally, see a lucid explanation in [M2], Ch.2, sect.9. Exponentiating (3.6.3) gives a group action

$$G(A)^0 \times MC_{\mathfrak{g}}(A) \to MC_{\mathfrak{g}}(A). \tag{3.6.4}$$

Define

$$\mathrm{Def}_{\mathfrak{g}}(A) = MC_{\mathfrak{g}}(A)/G(A)^0. \tag{3.6.5}$$

The motivation behind this ([M2], Ch.2, sect.9) is that

(i) if γ is a solution of the Maurer-Cartan equation, then $d + [\gamma, .]$ is also a differential, and

(ii) the adjoint action of $\mathfrak{g}^0$ results in the action on solutions of the Maurer-Cartan equation defined in (3.6.3).

Barannikov and Kontsevich apply this functor in the case where

$$\begin{gathered}\mathfrak{g}_{BK} = \Gamma(M, \Omega_M^{0,*} \otimes T_M^{*,0}),\ d = \bar{\partial} \\ [.,.] \text{ is the Schouten-Nijenhuis bracket.}\end{gathered} \tag{3.6.6}$$

Our task is similar but somewhat different. We need, see (3.6.1c), to deform $(Q^{--})_{(0)}$ within the class of differentials on the vertex algebra $\Gamma(M, \Omega_M^{vert})$. Even though the latter is not a Lie algebra, this deformation problem is governed by the differential Lie superalgebra

$$(\hat{\mathfrak{g}}, d, [.,.]) \overset{\text{def}}{=} \left(\Gamma(M, \mathrm{Lie}(\Omega_M^{vert})), (Q^{--})_{(0)}, {}_{(0)}\right), \tag{3.6.7a}$$

where
$$\Gamma(M, \mathrm{Lie}(\Omega_M^{vert})) = \Gamma(M, \Omega_M^{vert}/T(\Omega_M^{vert})). \tag{3.6.7b}$$
Indeed,
$$_{(0)} : \hat{\mathfrak{g}} \otimes \Gamma(M, \Omega_M^{vert}) \to \Gamma(M, \Omega_M^{vert}) \tag{3.6.8}$$
makes $\Gamma(M, \Omega_M^{vert})$ a $\hat{\mathfrak{g}}$-module, on which $\hat{\mathfrak{g}}$ operates by derivations. Furthermore,
$$((Q^{--})_{(0)} + \gamma_{(0)})^2 = (Q^{--}_{(0)}\gamma)_{(0)} + \frac{1}{2}(\gamma_{(0)}\gamma)_{(0)}. \tag{3.6.9}$$
Hence, if γ satisfies the Maurer-Cartan equation, then $(Q^{--})_{(0)} + \gamma_{(0)}$ is a differential. Let us define then
$$\mathrm{Def}_{\Gamma(M,\Omega_M^{vert})} = \mathrm{Def}_{\hat{\mathfrak{g}}}. \tag{3.6.10}$$
By definition,
$$\mathfrak{g}_{BK} \subset \Gamma(M, \Omega_M^{vert}), \tag{3.6.11}$$
which, by virtue of (3.6.7b), gives a map, an injection, in fact,
$$\iota : \mathfrak{g}_{BK} \hookrightarrow \hat{\mathfrak{g}}. \tag{3.6.12}$$
It is not a differential Lie algebra homomorphism, but its twisted version
$$\iota_{Q^{++}} : \mathfrak{g}_{BK} \to \hat{\mathfrak{g}},\ a \mapsto Q^{++}_{(0)}\iota(a) \tag{3.6.13}$$
is; here Q^{++} is a vertex analogue of the ∂-differential; it has appeared in (3.6.1) and is defined by the same formula as its quasiclassical limit (3.6.0). Indeed, it is a pleasing exercise to check that the Schouten-Nijenhuis bracket can be expressed in purely vertex algebra terms, cf. Proposition 1.1 in [Get],
$$\iota\left([a, b]\right) = \iota(a)_{(0)}(Q^{++}_{(0)}\iota(b)). \tag{3.6.14}$$
Therefore
$$\iota_{Q^{++}}\left([a, b]\right) = Q^{++}_{(0)}\left(\iota(a)_{(0)}\left(Q^{++}_{(0)}\iota(b)\right)\right) = \left(Q^{++}_{(0)}\iota(a)\right)_{(0)}\left(Q^{++}_{(0)}\iota(b)\right).$$
Note that morphism (3.6.13) changes the parity, as it should, because $\mathfrak{g}_{BK}$ is an odd Lie superalgebra.

This proves

3.6.2.1. Lemma. *Map (3.6.13) defines a morphism of* functors
$$\mathrm{Def}_{\mathfrak{g}_{BK}} \to \mathrm{Def}_{\Gamma(M,\Omega_M^{vert})}. \tag{3.6.15}$$

If M is a Calabi-Yau manifold, then $\mathrm{Def}_{\mathfrak{g}_{BK}}$ is represented by a formal scheme that is the formal neighborhood of 0 of the superspace $H^*(M, \Lambda^*\mathcal{T}_M)$ [BK]. In particular, there exists a generic formal solution of the Maurer-Cartan equation in variables chosen to be any basis of the dual space $(H^*(M, \Lambda^*\mathcal{T}_M))^*$. Therefore,

3.6.2.2. Corollary. *If M is a Calabi-Yau manifold, then there is a formal family of vertex algebras*
$$H^*(M, W^{quant}_{1/2})_t \xrightarrow{\sim} H_{Q_t^{--}}(\Gamma(M, \Omega_M^{vert})), \tag{3.6.16}$$
with base the formal neighborhood of 0 of the superspace $H^*(M, \Lambda^*\mathcal{T}_M)$.

Some of these deformations are not so formal; for example, $(Q^{--})_{(0)}$ itself depends quite explicitly on the choice of a complex structure, see (3.6.0); this can be extended by including generalized complex structures [G]; and considerable work has been done in order to interpret other points of the Barannikov-Kontsevich moduli space.

3.6.3. *Vertex Frobenius manifolds?*

It appears that there is more than just that to this story. The events unfolding in the conformal weight zero component of $H_{Q_t^{--}}(\Gamma(M,\Omega_M^{vert}))$ is precisely the Barannikov-Kontsevich construction of the Frobenius manifold structure on $\mathrm{Def}_{\mathfrak{g}_{BK}}$. Furthermore, it is plausible that each line of [BK] has a vertex algebra analogue valid up to homotopy. For example, operation $_{(-1)}$ makes each vertex algebra into a homotopy associative commutative algebra [LZ]. Furthermore, the order 2 differential operator Δ defined on $\mathfrak{g}$, which is essential for [BK], has a vertex analogue; this analogue is $(Q^{++})_{(1)}$, which is well defined precisely when M is a Calabi-Yau manifold [MSV]. It is also an order 2 differential operator of sorts in that

$$[(Q^{++})_{(1)}, a_{(-1)}] - (Q^{++}_{(1)}a)_{(-1)} = (Q^{++}_{(0)}a)_{(0)}, \tag{3.6.17}$$

which is a derivation of all $_{(n)}$-multiplications – a remark of Lian and Zuckerman, [LZ], Lemma 2.1.

What all of this seems to indicate is that there is a reasonable definition of a a vertex Frobenius manifold of which $H_{Q_t^{--}}(\Gamma(M,\Omega_M^{vert}))$ is an important example.

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