Which 4-manifolds are Toric Varieties?

David Yavin
Stephan Fischli

Stephan Fischli
Mathematisches Institut
Universität Bern
3012 Bern
Switzerland

David Yavin
Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

e-mail addresses:
yavin@eurydice.mpim-bonn.mpg.de
fischli@mai.unibe.ch

MPI / 91- 85

Which 4-manifolds are toric varieties?

Stephan Fischli and David Yavin

August 9, 1991

Abstract

We present a topological definition of (compact) toric varieties due to R. MacPherson and calculate explicitly the intersection form of a smooth 4-dimensional toric variety X. Using the characterisation of 4-manifolds given by Freedman, we conclude that X is homeomorphic either to the complex projective plane $\mathbf{C}P^2$ or to the product of spheres $S^2 \times S^2$ or to the connected sum of $\mathbf{C}P^2$ with a finite number of $-\mathbf{C}P^2$. This result can also be deduced from a theorem of Oda ([7], theorem 8.2) which is based on the algebraic definition of toric varieties and which uses combinatorial properties of regular fans in $\mathbf{R}^2$.

1 Definitions

Let Σ be a complete d-dimensional fan, i.e. a complex of closed convex polyhedral cones in $\mathbf{R}^d$ with apex 0, generated by primitive lattice points $v_1, \dots, v_n \in \mathbf{Z}^d$, such that $\bigcup_{\sigma \in \Sigma} \sigma = \mathbf{R}^d$. Denote the q-skeleton of Σ by $\Sigma^q = \{\sigma \in \Sigma | \dim \sigma = q\}$. The sets obtained by intersecting each cone $\sigma \in \Sigma$, $\sigma \neq \{0\}$, with the unit sphere $S^{d-1} \subset \mathbf{R}^d$ form a spherical complex C. Let $S(C)$ be the barycentric subdivision of C and for $\sigma \in \Sigma$, $\sigma \neq \{0\}$, let $\hat{\sigma}$ be the union of all simplices of $S(C)$ whose vertices are barycenters of elements of C which contain $\sigma \cap S^{d-1}$. For $\sigma = \{0\}$ we set $\hat{\sigma} = B^d$, the unit ball of $\mathbf{R}^d$, and call $\hat{\Sigma} = \{\hat{\sigma} | \sigma \in \Sigma\}$ the dual complex of Σ.

For each cone $\sigma \in \Sigma$ define an equivalence relation $\overset{\sigma}{\sim}$ on the d-dimensional torus $T^d = \mathbf{R}^d / \mathbf{Z}^d$ by

$$t \overset{\sigma}{\sim} t' \iff t - t' \subset \operatorname{span} \sigma + \mathbf{Z}^d.$$

Note that, since the subspace span σ is generated by lattice points of $\mathbf{Z}^d$, the torus T^d is collapsed by the relation $\overset{\sigma}{\sim}$ to a torus of dimension $d - \dim\sigma$. Finally, define an equivalence relation $\sim$ on the product $B^d \times T^d$ by

$$(x,t) \sim (x',t') \iff x = x' \text{ and } t \overset{\sigma}{\sim} t' \text{ where } x \in \operatorname{relint}\hat{\sigma}.$$

Then the quotient space $X_\Sigma = B^d \times T^d/_\sim$ is called the toric variety associated to the fan Σ. We denote the projection of X_Σ onto B^d by p and will identify $p^{-1}(\operatorname{relint}\hat{\sigma})$ with $\operatorname{relint}\hat{\sigma} \times T^d/_{\overset{\sigma}{\sim}}$.

The proofs of the following properties can be found e.g. in [1].

(i) The toric variety X_Σ is compact and simply-connected (since Σ is complete).
(ii) The toric variety X_Σ is a smooth manifold if and only if Σ is a simplicial fan and $|\det(v_{i_1},\ldots,v_{i_d})| = 1$ whenever $v_{i_1},\ldots,v_{i_d} \in \mathbf{Z}^d$ are the spanning vectors of a d-dimensional cone in Σ.
(iii) If Σ and Σ' are two d-dimensional fans, and if there exists a unimodular transformation of $\mathbf{R}^d$ which maps the generating vectors of Σ onto the generating vectors of Σ' and which induces a combinatorial isomorphism between Σ and Σ', then the associated toric varieties X_Σ and $X_{\Sigma'}$ are homeomorphic.

2 CW-cell decomposition and cellular homology

In the following we restrict our attention to a 4-dimensional toric variety $X = X_\Sigma$. Since the underlying fan Σ is 2-dimensional, its dual complex $\hat{\Sigma}$ may be represented as the face complex of a polygon P.

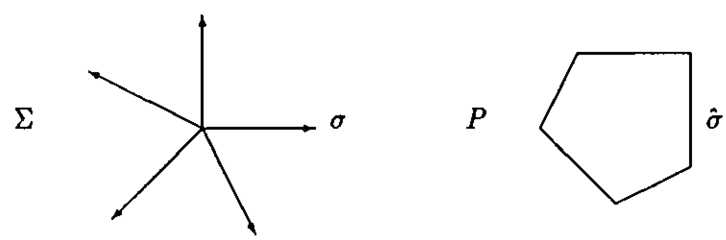

Figure 1: Example of a 2-dimensional fan and its dual complex

We first describe a CW-cell decomposition of X which arises from the face structure of P and the standard CW-cell decomposition of the 2-dimensional torus T^2. The latter is given by the cells τ_0, τ_1, τ_2 and τ_{12} which are the respective images of the subsets $\{0\}$, $]0,1[\times\{0\}$, $\{0\}\times\,]0,1[$ and $]0,1[\times\,]0,1[$ of $\mathbf{R}^2$ under the canonical projection onto T^2. By suitably embedding the tori T^0 and T^1 into T^2, the subcomplexes $\{\tau_0\}$ and $\{\tau_0, \tau_1\}$ can be regarded as cell decompositions of T^0 and T^1, respectively. Note that all these cells represent cycles in the cellular homology of the corresponding tori.

In order to obtain a cell decomposition of the torus $T^2/_{\underset{\sim}{\sigma}}$ which lies over each of the interior points of the face $\hat{\sigma}$ of P, we use an explicit homeomorphism $h_\sigma : T^2/_{\underset{\sim}{\sigma}} \to T^{2-\dim\sigma}$. For a 1-dimensional cone $\sigma \in \Sigma$ generated by the vector $v_\sigma = (v_\sigma^1, v_\sigma^2) \in \mathbf{Z}^2$, such a homeomorphism h_σ is induced by the map $g_\sigma : \mathbf{R}^2 \to \mathbf{R}$ given by $x = (x^1, x^2) \mapsto v_\sigma^1 x^2 - v_\sigma^2 x^1$. (The image $g_\sigma(x)$ is just the length of the orthogonal projection of x onto $(\operatorname{span}\sigma)^\perp$ multiplied by the length of v_σ.) For a 2-dimensional cone $\sigma \in \Sigma$ we set $h_\sigma = 0$ and for $\sigma = \{0\}$, $h_\sigma = \mathrm{id}_{T^2}$.

For each cone $\sigma \in \Sigma$ and each cell τ of the torus $T^{2-\dim\sigma}$ we can now define the cell

$$c_{\sigma,\tau} = \operatorname{relint}\hat{\sigma} \times h_\sigma^{-1}(\tau).$$

Note that $\dim c_{\sigma,\tau} = 2 - \dim\sigma + \dim\tau$. As can easily be checked, the cells $c_{\sigma,\tau}$ form a CW-decomposition of X, and if we provide them with appropriate orientations, then the boundaries of the corresponding cellular chains are given by

$$\begin{array}{rrcl}
\sigma \in \Sigma^2: & \partial_0 c_{\sigma,\tau_0} & = & 0 \\
\sigma \in \Sigma^1: & \partial_1 c_{\sigma,\tau_0} & = & c_{\sigma',\tau_0} - c_{\sigma'',\tau_0} \\
 & \partial_2 c_{\sigma,\tau_1} & = & 0 \\
\sigma = \{0\}: & \partial_2 c_{\sigma,\tau_0} & = & \sum c_{\varrho,\tau_0} \\
 & \partial_3 c_{\sigma,\tau_1} & = & \sum(-v_\varrho^2) c_{\varrho,\tau_1} \\
 & \partial_3 c_{\sigma,\tau_2} & = & \sum v_\varrho^1 c_{\varrho,\tau_1} \\
 & \partial_4 c_{\sigma,\tau_{12}} & = & 0
\end{array} \tag{1}$$

where σ', σ'' are the unique 2-dimensional cones which contain $\sigma \in \Sigma^1$ and ϱ ranges over Σ^1 in all the sums. (The multiplicities of the chain c_{ϱ,τ_1} in the boundaries of the two 3-dimensional chains are the images of the vectors $(1,0)$ and $(0,1)$ under the map g_ϱ described above.)

Thus, we finally obtain the following homology groups of X:

$$H_q(X) \cong \begin{cases} \mathbf{Z} & q = 0,4 \\ \mathbf{Z}^n/(\mathbf{Z}v_1^* + \mathbf{Z}v_2^*) \cong \mathbf{Z}^{n-2} \oplus \mathbf{Z}/m\mathbf{Z} & q = 2 \\ 0 & \text{otherwise} \end{cases}$$

where $n = \operatorname{card} \Sigma^1$, $v_1^*, v_2^* \in \mathbf{Z}^n$ are the vectors consisting respectively of the first and the second coordinates of the vectors v_σ, $\sigma \in \Sigma^1$, and m is the greatest common divisor of the determinants $\det(v_\sigma, v_{\sigma'})$, $\sigma, \sigma' \in \Sigma^1$. In particular $H_2(X) \cong \mathbf{Z}^{n-2}$ if X is smooth.

3 The intersection form

In this section we calculate the intersection form of a smooth 4-dimensional toric variety $X = X_\Sigma$.† Let the generating vectors $v_1, \ldots, v_n$ of the underlying 2-dimensional fan Σ be numbered counterclockwise and likewise the corresponding 1-dimensional cones $\sigma_1, \ldots, \sigma_n$. By property (ii) we have

$$\det(v_i, v_{i+1}) = 1 \qquad (i = 1, \ldots, n) \tag{2}$$

(whenever we use indices which exceed n or fall below 1, we consider them to be taken modulo n), and in view of property (iii) we may assume in addition that $v_{n-1} = (1,0)$ and $v_n = (0,1)$. By (1) the group of 2-cycles of X is generated by the chains c_{σ_i, τ_1}, $i = 1, \ldots, n$, and the corresponding homology classes $z_i \in H_2(X)$ satisfy the following relations induced by the boundaries of the two 3-chains:

$$\begin{array}{rcl} v_1^1 z_1 + \ldots + v_{n-2}^1 z_{n-2} + z_{n-1} & = & 0 \\ v_1^2 z_1 + \ldots + v_{n-2}^2 z_{n-2} + z_n & = & 0 \end{array} \tag{3}$$

In order to determine the intersection numbers $z_i \cdot z_j$, we first observe that the spheres $p^{-1}(\hat{\sigma}_i)$ and $p^{-1}(\hat{\sigma}_j)$, which represent the classes z_i and z_j respectively, do not intersect in X if their generating cones σ_i and σ_j are not adjacent in Σ. Therefore

$$z_i \cdot z_j = 0 \qquad (1 < |i - j| < n - 1).$$

†This could be done by using the algebraic description of the cohomology ring of X given by Danilov (see [1], theorem 10.8), however we use the cell decomposition of X to compute the intersection numbers of the generating 2-cycles.

Second, the spheres $p^{-1}(\hat{\sigma}_i)$ and $p^{-1}(\hat{\sigma}_{i+1})$ intersect in the unique point which lies over the vertex $\hat{\sigma}_i \cap \hat{\sigma}_{i+1}$ of P. Since $\det(v_i, v_{i+1}) = 1$, it can be seen that this intersection is transversal and hence

$$z_i \cdot z_{i+1} = \pm 1 \qquad (i = 1, \dots, n)$$

where the signs are all equal and only depend on the orientation of X. In the following we fix them to be $+1$.

Third, by multiplying both relations (3) with z_i and taking a suitable linear combination of the resulting equations, we obtain the self intersection numbers

$$z_i \cdot z_i = -\det(v_{i-1}, v_{i+1}) \qquad (i = 1, \dots, n)$$

where we have also used the smoothness condition (2).

4 Characterisation of the intersection form

Having calculated the intersection form of X, we now characterise it up to equivalence, i.e. up to a change of basis of $H_2(X)$. (For a treatment of symmetric bilinear forms in general see [6].) By (3) the classes $z_1, \dots, z_{n-2}$ form a basis of $H_2(X)$, and since by Poincaré duality the intersection form is always non-singular, its **rank** equals $n-2$.

In order to determine the **signature**, we first calculate the principal minors $D_k = \det((z_i \cdot z_j)_{1 \le i,j \le k})$ of the intersection matrix. (Henceforth we will not distinguish between the intersection form and its matrix.) By the results of the previous section we have $D_1 = \det(v_2, v_n)$ and

$$D_k = -\det(v_{k-1}, v_{k+1})\, D_{k-1} - D_{k-2} \qquad (k = 2, \dots, n-2),$$

where we have set $D_0 = 1$. By induction one can easily prove, e.g. by using the Grassmann-Plücker relation in $\mathbf{R}^2$, that

$$D_k = (-1)^{k+1} \det(v_{k+1}, v_n) \qquad (k = 1, \dots, n-2).$$

From this equation we now see that if none of the vectors v_k is equal to $-v_n$, then all the principal minors are non-zero and have alternating signs, except for the unique pair (D_{k-1}, D_k) for which the vectors v_k and v_{k+1} lie on different sides of the y-axis. Hence by Jacobi's theorem the signature of the intersection form equals $4-n$. By a rule of Gundenfinger this still holds even if there exists a vector $v_k = -v_n$ in which case $D_{k-1} = 0$ (see [4], note 1 on page 304).

If $n = 3$ then there is only one possible vector $v_1 = (-1, -1)$ and the intersection form of X given by the matrix (1) is positive definite. If $n > 3$ then the absolute value of the signature of the intersection form of X is less than its rank, hence the form is indefinite. Thus it can be characterised by finally determining its **type**.

The even type is only possible if $n = 4$.[‡] Indeed, if the intersection form of X is even, then all the determinants $\det(v_{i-1}, v_{i+1})$, $i = 1, \ldots, n-2$, have to be even. Since the vectors v_i are primitive, it follows that they all must have one even and one odd coordinate, thus they are contained in the lattice $\Gamma = (1,0) + \mathbf{Z}(1,1) + \mathbf{Z}(-1,1)$. Hence by Pick's formula (see e.g. [5]) the area $A(S)$ of the star-shaped polygon $S = \bigcup_{i=1}^{n} \operatorname{conv}\{0, v_i, v_{i+1}\}$ is given by

$$\frac{A(S)}{\det \Gamma} = \operatorname{card}(\Gamma \cap \operatorname{relint} S) + \frac{1}{2} \operatorname{card}(\Gamma \cap \operatorname{relbd} S) - 1$$

where $\det \Gamma$ denotes the determinant of a basis of Γ. But by condition (2) the area $A(S)$ equals $\frac{n}{2}$ and S does not contain any points of Γ other than its vertices. Therefore the equality can hold only if $n = 4$.

In fact, if $n = 4$ and $v_1 = (-1, 0)$, $v_2 = (0, -1)$ the resulting form is even. On the other hand, every odd indefinite intersection form of rank $n \geq 4$ can also be realised, e.g. by setting $v_i = (i-2, -1)$, $i = 1, \ldots, n-2$. Thus we have completely characterised the possible intersection forms of X and we summarize the results in the following

THEOREM. *A non-singular integral symmetric bilinear form B can be realised as the intersection form of an oriented smooth 4-dimensional toric variety X if and only if either*
(i) $\operatorname{rank}(B) = 1$ *and* $B = (\pm 1)$, *or*
(ii) $\operatorname{rank}(B) = 2$ *and B is indefinite, or*
(iii) $\operatorname{rank}(B) > 2$, $|\operatorname{signature}(B)| = \operatorname{rank}(B) - 2$ *and B is of odd type.*

[‡]This also follows from a theorem of Donaldson (see [2], theorem B) which says that if the intersection form of a smooth simply-connected 4-manifold is indefinite and even and if the absolute value of its signature is 2 less than its rank, then its rank equals 2.

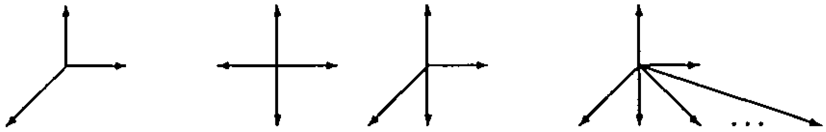

Figure 2: Representative fans with 3, 4 and n generators

5 Classification

In 1982 Freedman characterised topological 4-manifolds by showing that every non-singular integral symmetric bilinear form can be realised as the intersection form of an oriented closed simply-connected 4-manifold, and that any two such manifolds realising the same form are homeomorphic if the form is even, whereas if the form is odd there are two homeomorphism classes, one with trivial and the other with non-trivial Kirby-Siebenmann obstruction (see [3], theorem 1.5).

In our case it is easy to give representatives of 4-manifolds which realise the intersection forms described in the theorem of the previous section. Namely, let us consider the oriented complex projective plane $\mathbf{C}P^2$ which has intersection form $(+1)$. Then $\mathbf{C}P^2$ with the opposite orientation, which we denote by $-\mathbf{C}P^2$, has intersection form (-1). Furthermore, if we take the connected sum of $\mathbf{C}P^2$ with a finite number of copies of $-\mathbf{C}P^2$, we obtain a 4-manifold whose intersection form is the orthogonal sum of $(+1)$ with a finite number of (-1) and hence satisfies condition (iii) of the theorem. Finally, the even indefinite form of rank 2 is the intersection form of the product of spheres $S^2 \times S^2$. All these manifolds are smooth and hence have trivial Kirby-Siebenmann obstruction, and since the same is true for the toric varieties in question, we can state the following

COROLLARY. *A smooth 4-dimensional toric variety is homeomorphic either to the complex projective plane* $\mathbf{C}P^2$ *or to the product of spheres* $S^2 \times S^2$ *or to the connected sum of* $\mathbf{C}P^2$ *with a finite number of copies of* $-\mathbf{C}P^2$.

References

[1] V.I. Danilov, *The Geometry of Toric Varieties*, Russian Math. Surveys **33** (1978), 97–154

[2] S.K. Donaldson, *Connections, Cohomology and the Intersection Forms of 4-Manifolds*, J. Diff. Geom. **24** (1986), 275–341

[3] M.H. Freedman, *The Topology of 4-Dimensional Manifolds*, J. Diff. Geom. **17** (1982), 357–453

[4] F.R. Gantmacher, *The Theory of Matrices*, Volume One, Chelsea Publishing Company, 1959

[5] I.G. Macdonald, *The Volume of a Lattice Polyhedron*, Proc. Camb. Phil. Soc. **59** (1963), 719–726

[6] J. Milnor, D. Husemoller, *Symmetric Bilinear Forms*, Springer 1973

[7] T. Oda, *Lectures on Torus Embeddings and Applications*, Tata Inst. of Fund. Research, Springer 1978