The configuration of a finite set on surface (Revised)

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§ 0. Introduction

Let S be a smooth surface in $\mathbf{P}^n$ and m be an integer with $n \geq m \geq 2$. For any m different points on S, if they are linearly dependent we say this set is special. Let M be the collection of all these special sets, then M is a scheme with a natural algebro-geometric structure. We can show that, when $n = 3m - 2$ and S is in general position, M is a finite scheme. We denote the degree of M by $\nu(s)$ which is intuitively the number of the points in M possibly with multiplicities.

S.K. Donaldson posed a conjecture about this case in [2]:

"Conjecture 5. There is a universal formula for expressing $\nu(s)$ in terms of m, the Chern numbers of S, the degree of S in $\mathbf{P}^{3m-2}$, and the intersection number of the canonical class S with the restriction of the hyperplane class."

He pointed out this enumerative problem has something to do with Yang-Mills invariants.

In this paper we give an affirmative answer for the conjecture. But the formula for expressing $\nu(s)$ is complicated for writing down explicitly though there is an algorithm for computing it.

In § 1 we explain the meaning about "general position" in the present case and give the basic construction for computing $\nu(s)$. In § 2 all of the objects considered in § 1 are lifted to some projective vector bundle where it is comparatively easier for computation. In § 3 we prove the main theorem by computing some Segre classes.

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§ 1. Preliminaries

In sequels we assume the ground field is algebraically closed with characteristics $\neq 2$.

Let $m \geq 2$ be an integer and $n = 3m - 2$.

Let $\mathbf{P}^n = \mathbf{P}(V^\vee)$ be the $n-$projective space, where V is a vector space of dimension $n+1$ over the ground field, and we chose a basis $e_0, \ldots, e_n$ for V once for ever. Let $Y = (\mathbf{P}^n)^m$, the $m-$cross product of $\mathbf{P}^n$, and let $X = (S)^m$, where S is a smooth surface in $\mathbf{P}^n$ which is in general position in a sense as follows.

Definition. S is in general position if, execpt for a finite number of the sets consisting of m points on S, every other such set is linearly independent, including the case when k of m points are replaced by a $(k-1)-$plane which tangents to S at a point of S.

We call the exceptional set a special set. For $m = 2$, every smooth surface in $\mathbf{P}^4$ is automatically in general position.

For $m \geq 3$ we have the following proposition:

Proposition 1.1. Let $i : S \rightarrow \mathbf{P}^n$ be a non-degenerate embedding, then there exists a re-embedding $j : S \mapsto \mathbf{P}^n$ by a generic projection from $\mathbf{P}^{n+1}$ to $\mathbf{P}^n$ such that $j(s)$ is in general position.

Proof: Let $i^* O_{\mathbf{P}^n}(1) = O(1)$, then i is determined by a linear system belonging to $O(1)$.

First we shall show that, there exists an integer N_0 such that for every $N \geq N_0$, on the image of the embedding φ determined by $O(N)$ every m points are linearly independent.

In fact, let Z be a subscheme of m points on S with reduced stucture and J_Z be the sheaf of ideal defining Z in S. We have an exact sequence

$$0 \to H^0(S, J_Z(N)) \to H^0(S, O(N)) \xrightarrow{a} H^0(S, O_Z(N))$$
$$\to H^1(S, J_Z(N)) \to H^1(S, O(N)) \to 0 .$$

Since $\dim H^0(S, O_z(N))$ is the number of the points in Z, we see from the sequence that, if $H^1(S, J_Z(N)) = 0$ $\dim H^0(S, J_Z(N))$ is the dimension of the smallest subspace which contains Z. Therefore, if $H^1(S, J_{Z'}(N)) = 0$ for every (reduced) subscheme $Z' \subset Z$, then the points of Z are linearly independent.

By Cartan-Serre Theorem B [4], there exists N_0 such that whenever $N \geq N_0$ we have $H^1(S,_{Z'}(N)) = 0$ for all $Z' \subset Z$. Now we have to prove that N_0 depends only on m rather than on the position of Z. As a standard method we take Z as a subscheme of $\mathbf{P}^n$ and let the ideal defining Z in $\mathbf{P}^n$ be I_Z and the ideal defining S in $\mathbf{P}^n$ be I_S, then we have an exact sequence

$$0 \to I_S \to I_Z \to I_Z \to 0 .$$

In the long exact sequence of the above sequence we see that, the vanishing of $H^1(S, J_Z(N))$ is a consequence of the vanishing of $H^1(\mathbf{P}^n, I_Z(N))$ and $H^2(\mathbf{P}^n, J_S(N))$. But on $\mathbf{P}^n$, $H^1(\mathbf{P}^n, I_Z(N)) = 0$ for N sufficiently large depends only on $\#Z$ (by the homogenety of $\mathbf{P}^n$ or simply by induction on $\#Z$).

We continue to prove our proposition.

Let $r+1 = \dim H^0(S, O(N_0))$ and $\psi : S \to \mathbf{P}^r$ be the embedding determined by $O(N_0)$. We show that, for $r \geq n+2 = 3m$ a generic projection from $\mathbf{P}^r$ to $\mathbf{P}^{r-1}$ gives an embedding of S in $\mathbf{P}^{r-1}$ and preserves the independence of any m points on S. Indeed, the subscheme consisting of all the $(m-1)-$planes in $\mathbf{P}^r$ spanned by any m points on S (including the case when k of m points is a $(k-1)-$plane which tangents to S) has dimension $3m-1$. Therefore a projection with a generic point as center meets our need. We proceed like this till we arrive at $\mathbf{P}^{3m-1}$. Then taking a generic point of $\mathbf{P}^{3m-1}$ as center we have a projection which preserves the independence of m points on S except for a finite number of these sets, and in this case anyone of these exceptional sets spans a $(m-2)-$plane.

Hereafter we always assume the surface S is in general position in the above sense.

Our main idea for solving the conjecture is as follows. Let $j : X \to Y$, $j = i \times \ldots \times i$, $P = (p_1, \ldots, p_m) \in X$, and $p_i = (\xi_{i0}, \ldots, \xi_{in})$ be the homogeneous coordinates of p_i in $\mathbf{P}^n$ with respect to the fixed basis of V. Then, $p_1, \ldots p_m$ being in special position means $rk(\xi_{ij}) \leq m-1$. We would like to construct a morphism φ between two locally free sheaves such that φ can be expressed locally by the matrix (ξ_{ij}), then in the $(m-1)-$degeneracy of φ the finite part is what we need.

Let $q_i : X \to S$ be the ith projection and denote $q_i^* O(1)$ by H_i. Then the fiber of H_i^{-1} over P is the 1–subspace of V representing the point p_i. From the following exact sequence

$$0 \to H_i^{-1} \xrightarrow{\varphi_i} V \otimes_{q_i} O_X \to q_i^* \Omega_{P^n}^{\vee}(H_i^{-1}) \to 0$$

we have a morphism

$$\varphi : \oplus H_i^{-1} \xrightarrow{\oplus \varphi_i} \oplus V \otimes_{q_i} O_X \xrightarrow{\Sigma} V \otimes_k O_X .$$

φ is expressed locally by the matrix (ξ_{ij}) with respect to the basis of V. We shall study the degeneracy $D = D_{m-1}(\varphi)$ [5]. By Proposition 1.1, D is divided into two parts, D_0 and D_1, where D_0 is a finite subscheme corresponding to the special sets of S and D_1 is a subscheme of X with positive dimension, supporting on the various diagonals of X. Let ν denote the number of the special sets on S, then $\deg D_0 = m!\nu$.

Formally, at present we may compute ν by Excess Formula [5] § 9.1, but it seems difficult technically. So we lift φ to the desingularization of D [1]. Let $S_{i_1\dots i_k}$ with $i_1 < \dots < i_k$ be the $(i_1, \dots, i_k)-$diagonal of X, i.e. the image of $\Delta_{i_1\dots i_k} \times id : S^{m-k+1} \to S^m$, where $\Delta_{i_1\dots i_k}$ is the diagonal morphism from S to the $i_1th, \dots, i_kth$ factor of X.

Let $Q = P(H_1^{-1} \oplus \dots \oplus H_m^{-1})(= \mathbf{P}(H_1 \oplus \dots \oplus H_m))$, $p : Q \to X$ be the structure projection and $O(-1)$ be the universal subbundle on Q. We have a morphism on Q :

$$O(-1) \to p^*H_1^{-1} \oplus \dots \oplus p^*H_m^{-1} .$$

The composition of the morphism and $p^*\varphi$ gives

$$\psi : O(-1) \to V \otimes O_Q \xrightarrow{\sim} O^{n+1} .$$

Generally we could not assert that $p_*[D_0\psi] = [D_m(\varphi)]$ since φ does not have "correct" dimension [1] Ch. II. But by the assumption of " S being in general position", out of $p^{-1}\left(\bigcup_{k\geq 2} S_{i_1\dots i_k}\right)$ we have $p_*[D_0(\psi)] = [D(\varphi)]$ where the dimension is correct [1]. Therefore, $D(\psi)$ is divided into two parts: V_0 and V_1, where V_0 is a finite scheme with degree $m!\nu$ and V_1 is a scheme supporting on $p^{-1}(\cup S_{i_1\dots i_k})$, defined by the Fitting ideal $F^n(\psi)$ [5]. ψ induces a section $r : Q \to O(1)^{n+1}$ then, if letting r_0 be the zero section, we have the following diagram:

$$\begin{array}{ccc} V_1 \cup V_0 & \to & Q \\ \downarrow & & \downarrow \gamma_0 \\ Q & \to & O(1)^{n+1} . \end{array}$$

By the definition of intersection in [5], § 6.1,

$$\deg V_0 = \deg\left[Q^2\right] - \deg\left(\left(1 + c_1(O(1))\right)^{n+1} \cdot s(V_1, Q)\right)\Big)_0,$$

where $s(V_1, Q)$ is the Segre class of V_1 in Q and c_i is the notation of the ith Chern operator. By an easy computation we have $\deg\left[Q^2\right] = d^m$.

§ 2. Birational transformation

The next step is the computation of $s(V_1, Q)$. Since Segre class is birationally invariant [5], we shall construct a scheme being birationally isomorphic to Q and making the computation easier.

Let $\alpha_m : X_m \to X$ be the blowing-up of X with respect to $S_{1\dots m}$, and $\beta_m : P(\alpha_m^* H_1^{-1} \oplus \dots \oplus \alpha_m^* H_m^{-1}) \to Xm$ be the pull-back of Q by α_m. Q_m is birationally isomorphic to Q.

Let X_{m-1} be the blowing-up of X_m with respect to $\cup S'_{i_1\ldots i_{m-1}}$ where $S'_{i_1\ldots i_{m-1}}$ is the strict transform of $S_{i_1\ldots i_{m-1}}$ with respect to α_m and $\left\{S'_{i_1\ldots i_{m-1}}\right\}$ for $i_1 < \ldots < i_{m-1}$ are disjoint each other. We denote the composition of these two blowing-ups by $\alpha_{m-1} : X_{m-1} \to X$ and the pull-back of Q with respect to α_{m-1} by $\beta_{m-1} : Q_{m-1} = P(\alpha^*_{m-1}H_1^{-1} \oplus \ldots \oplus \alpha^*_{m-1}H_m^{-1}) \to X_{m-1}$. In this case the strict transforms of all of $S_{i_1\ldots i_{m-2}}$ with respect to α_{m-1} are disjoint each other and we do the same thing as above until we arrive at

$$\alpha_2 : \tilde{X} = X_2 \to X$$
$$\beta_2 : \bar{Q} = Q_2 \to X_2 ,$$

where $\bar{Q} = P(\alpha_2^* H_1^{-1} \oplus \ldots \oplus \alpha_2^* H_m^{-1})$.
Let $\gamma : \bar{Q} \to Q$ be the composition of all of these pull-backs, which is a birational morphism. By the elementary property of Fitting ideal, $\gamma^{-1}(D(\psi))$ is defined locally by the 1–minors of the matrix representing $\gamma^*\psi : Q_{\bar{Q}}(-1) \to \alpha_2^* H_1^{-1} \oplus \ldots \oplus \alpha_2^* H_m^{-1} \to V_{\bar{Q}}$. Since γ is an isomorphic out of $\beta_2^{-1}\alpha_2^{-1}\left(\bigcup_{k\geq 2} S_{i_1\ldots i_k}\right)$, $\gamma^{-1}(V_1)$ is the scheme defined by the nth Fitting ideal $F^n(\gamma^*\psi)$ near $\beta_2^{-1}\alpha_2^{-1}(\cup S_{i_1\ldots i_k})$. Now we should know the structure of $\gamma^{-1}(V_1)$ explicitly. For simplicity we denote $\alpha_2^* H_i^{-1}$ by $\mathcal{H}_i^{-1}$, $\alpha_2^{-1}S_{i_1\ldots i_k}$ by $\tilde{S}_{i_1\ldots i_k}$ and $\gamma^*\psi$ by $\tilde{\psi}$. Let k be an integer with $2 \leq k \leq m$. We are going to study the structure of $\gamma^{-1}(V_1)$ near $\beta_2^{-1}\left(\tilde{S}_{1\ldots k}\right)$.
We begin with studying the local structure.
Taking a point $P \in \tilde{S}_{1\ldots k} \backslash \bigcup_I \tilde{S}_{1\ldots kI}$ with $\alpha_2(P) = (p_1, \ldots, p_m)$, and $p_i = (\xi_{i0}, \ldots, \xi_{im})$, where I is an index set and $\{\xi_{ij}\}$, $0 \leq j \leq n$, is the homogeneous coordinate of p_i in $\mathbf{P}^n$ with respect to the basis given in § 1.
Near P (precisely, in the homogeneous local ring of $\bar{Q}$ over P) we have

$$\tilde{\psi}(e) = (a_1\xi_{10} + \ldots + a_m\xi_{m0})e_0 + \ldots + (a_1\xi_{1n} + \ldots + a_m\xi_{mn})e_n ,$$

where e is the base for $O(-1)$ over P , $(a_1, \ldots, a_m)$ is the coordinate of e in $\mathcal{H}_i^{-1} \oplus \ldots \oplus \mathcal{H}_m^{-1}$. Therefore $\gamma^{-1}(V_1)$ is defined by ideal $\mathfrak{a}$ where

$$\mathfrak{a} = (a_1\xi_{10} + \ldots + a_m\xi_{m0}, \ldots, a_1\xi_{1n} + \ldots + a_m\xi_{mn}) \tag{1}$$

From the assumption of " S being in general position" we see that $a_{k+1} = \ldots = a_m = 0$. In fact, since $p_k, \ldots, p_m$ are different points on S and $k \geq 2$, the vectors representing these points in V are linearly independent; otherwise we would have an infinite number of special sets. Hence the degeneracy $D\left(\tilde{\psi}\right)$ over P is defined by

$$(a_1\xi_{10} + \ldots + a_k\xi_{k0}, \ldots, a_1\xi_{1n} + \ldots + a_k\xi_{kn},\ a_{k+1}, \ldots, a_m) .$$

Since $p_1 = \ldots = p_k$, near P we may assume $\xi_{10} = \ldots = \xi_{k0} = 1$ without loss of generality, and then we may write $a_1\xi_{1i} + \ldots + a_k\xi_{ki}$ as $(a_1 + \ldots + a_k)\xi_{1i} + a_2(\xi_{2i} + \xi_{1i}) + \ldots + a_k(\xi_{ki} - \xi_{1i})$. Therefore $\mathfrak{a}$ is reduced to

$$\begin{gathered}(a_1 + \ldots + a_k,\ a_{k+1}, \ldots, a_m,\ a_2(\xi_{21} - \xi_{11}) + \ldots + a_k(\xi_{k1} - \xi_{11}), \ldots, \\ a_2(\xi_{2n} - \xi_{1n}) + \ldots + a_k(\xi_{kn} - \xi_{1n}))\end{gathered} \tag{2}$$

or by emphasizing the symmetry we write $(a_1 + \ldots + a_k,\ a_{k+1}, \ldots, a_m)$ as

$$(a_1 + \ldots + a_k, a_1 + \ldots + a_k + a_{k+1}, \ldots, a_1 + \ldots + a_m)$$

time to time. Denoting the zero locus of an ideal I by $\mathsf{V}(I)$, then

$$\mathsf{V}(\mathfrak{a}) = \mathsf{V}(a_1 + \ldots + a_k,\ a_{k+1}, \ldots, a_m) \cap \mathsf{V}(\mathfrak{a}'),$$

where $\mathfrak{a}'$ is generated by those last n elements in the above expressing for $\mathfrak{a}$. We write the above argument as a proposition but in its global form.

Proposition 2.1. *Over $\tilde{S}_{i_1 \ldots i_k}$ with $2 \le k \le m$, $D\left(\tilde{\psi}\right)$ has a component $W_{i_1 \ldots i_k}$ which is a projective bundle $P(E_{i_1 \ldots i_k})$ over $\tilde{S}_{i_1 \ldots i_k}$, where $E_{i_1 \ldots i_k}$ is the kernel of the surjective morphism: $\mathcal{H}_{i_1}^{-1} \oplus \ldots \oplus \mathcal{H}_{i_k}^{-1} \to H_{i_1}^{-1}$ with $\mathcal{H}_{i_k} = \ldots = \mathcal{H}_{i_k} = \mathcal{H}$.*

Proof: We still work with $\tilde{S}_{1 \ldots k}$ without loss of generality. By the above local argument $\tilde{\psi}$ is splitted into two parts:

$$O(-1) \to \oplus \mathcal{H}_i^{-1} \begin{array}{ccc} \nearrow & \mathcal{H}_1^{-1} \oplus \ldots \oplus \mathcal{H}_k^{-1} & \searrow \\ & \oplus & \\ \searrow & \mathcal{H}_{k+1}^{-1} \oplus \ldots \oplus \mathcal{H}_m^{-1} & \nearrow \end{array} V_{\bar{Q}},$$

and $D\left(\tilde{\psi}\right) = D\left(\tilde{\psi}_1\right)$, where $\tilde{\psi}_1$ is the top arrow in the diagram.

The local assumption of " $\xi_{i0} \ne 0$ for $1 \le i \le k$ " globally means that, in $V^\vee = H^0(\mathbf{P}^n, O(1))$ we have chosen the sections which has non-zero coordinate ξ_0 at p_i. Therefore all of these sections generate $O(1)$, i.e.

$$H^0(\mathbf{P}^n, O(1)) \otimes O_{\mathbf{P}^n} \to O(1) \to 0.$$

Pulling back the morphism to $\overline{Q}$ by anyone of these projection, restricting it over $\tilde{S}_{1 \ldots k}$ and taking the duality we have an exact sequence

$$0 \to \mathcal{H}^{-1} \to V_{\bar{Q}} \to T_{\mathbf{P}^n}(\mathcal{H}^{-1})_{\bar{Q}} \to 0$$

where $T_{\mathbf{P}^n} = \Omega_{\mathbf{P}^n}^\vee$. In this case $D_0\left(\tilde{\psi}_1\right) = D_0(O(-1) \to \mathcal{H}^{-1}) \cap D_0\left(O(-1) \to T_{\mathbf{P}^n}(\mathcal{H}^{-1})_{\overline{Q}}\right)$.
On the other hand, the ideal $(\xi_{21} - \xi_{11}, \ldots, \xi_{2n} - \xi_{1n}, \ldots, \xi_{k1} - \xi_{11}, \ldots, \xi_{kn} - \xi_{1n})$ defines the diagonal $S_{1 \ldots k}$ in X. Then on $\tilde{X}$ we have a principal factor for defining $\tilde{S}_{1 \ldots k}$ from the ideal. Denoting the factor by $s_{1 \ldots k}$, then at a generic point of $\tilde{S}_{1 \ldots k}$ we have

$$\mathfrak{a}' = s_{1 \ldots k} \cdot \mathfrak{a}''.$$

This implies that

$$\begin{aligned} \mathsf{V}(\mathfrak{a}) &= \mathsf{V}(a_1 + \ldots + a_k, a_{k+1}, \ldots, a_m, s_{1 \ldots k}) \\ &\cup \mathsf{V}(a_1 + \ldots + a_k, a_{k+1}, \ldots, a_m) \cap \mathsf{V}(\mathfrak{a}''). \end{aligned}$$

Since $\tilde{S}_{1 \ldots k}$ is irreducible then by taking closure we see that the first component in the expression is what we expect.

For the structure of $\mathfrak{a}''$ we have the following proposition.

Proposition 2.2 *Over $\tilde{S}_{12\ldots k}$ with $k \geq 3$, $\mathfrak{a}''$ is the first Fitting ideal of $O(-1) \to (\beta_2\alpha_2)^* T_s \otimes \mathcal{H}^{-1} \otimes \tilde{S}^{-1}_{12\ldots k}$, and it defines a subscheme of $W_{12\ldots k}$ consisting of $\bigcup_I \tilde{S}_{1\ldots kI}$ and a component of codimension 2.*

Proof: Let $P^1_s(1)$ denote the sheaf of the first principal part of $O_s(1)$, then we have a diagram as follows [4] [7].

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \Omega_s(1) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & N^\vee & \to & V_s & \xrightarrow{a_1} & P^1_s(1) & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & \Omega^{(1)}_{\mathbf{P}^n} \mid_S & \to & V_s & \xrightarrow{a_0} & O_s(1) & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & \Omega_s(1) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

The two rows in the diagram is exact because $O_s(1)$ is very ample, and the column on the left is exact by using Five Lemma. Obviously $N^\vee$ is the sheaf of conormal of S in $\mathbf{P}^n$. From the definition of $P^1(1)$ we see that the local sections of $\Omega_{\mathbf{P}^n}(1)$ has its Taylor expansion with order ≥ 1 at any point on S . We claim that the local sections of $\Omega_{\mathbf{P}^n}(1)$ has its power expansion at any point of S exactly with order 1, i.e. $T_s(\mathcal{H}^{-1}) = s_{1\ldots k} T_s(\mathcal{H}^{-1} \otimes S^{-1}_{1\ldots k})$, $N(\mathcal{H}^{-1}) = S^2_{1\ldots k} N(\mathcal{H}^{-1} \otimes S^{-2}_{1\ldots k})$, where $T_s(\mathcal{H}^{-1} \otimes S^{-1}_{1\ldots k})$ and $N(\mathcal{H}^{-1} \otimes S^{-1}_{1\ldots k})$ is regular.

In fact, taking any smooth curve c on S (for example a section of $O(1)$) then by [8] we see that the first two gap numbers in the gap sequence of c with respect to V_c is 0, 1 since the ground field has characteristics $\neq 2$. This means the power series of one section of V_c has the form $at + \ldots$ with $a \neq 0$ and where t is the local parameter of C and hence the claim is true.

In the proof of Proposition 2.1 we saw that over $\tilde{S}_{1\ldots k}$ $\mathfrak{a}''$ is the $(n-1)th$ Fitting ideal of $O(-1) \to T_{\mathbf{P}^n} \mid {}_s(\mathcal{H}^{-1})$, now we have a splitting for $T_{\mathbf{P}^n}(\mathcal{H}^{-1})$ (shown in the above diagram). Therefore the Fitting ideal is the product of the first Fitting ideal of $O(-1) \to T_s(\mathcal{H}^{-1})$ and the $(n-3)th$ Fitting ideal of $O(-1) \to N_s(\mathcal{H}^{-1})$. Since $N\left(\mathcal{H}^{-1} \otimes \tilde{S}^{-2}_{1\ldots k}\right)$ is generated locally by some sections of $sym^i T_s\left(\mathcal{H}^{-1} \otimes \tilde{S}^{-1}_{1\ldots k}\right)$ with $i \geq 2$, [7], then over $\tilde{S}_{1\ldots k}$ the $(n-1)th$ Fitting ideal of $O(-1) \to T_{\mathbf{P}^n}(\mathcal{H}^{-1})$ is the same as $F^1\left(O(-1) \to \tilde{S}_{1\ldots k} T_s\left(\mathcal{H}^{-1} \otimes \tilde{S}^{-1}_{1\ldots k}\right)\right)$.

As for the last assertion in this proposition we note that, when we write down $F^{n-1}(O(-1) \to T_{\mathbf{P}^n}(\mathcal{H}^{-1}))$ explicitly we have by (2)

$$s_{1\ldots k} \prod_{\#I \geq 1} s_{1\ldots kI}(a_2 x_{21} + \ldots + a_k x_{k1}, \ldots, a_2 x_{2n} + \ldots + a_k x_{kn})$$

where $x_{ij} = (\xi_{ij} - \xi_{ij}) \cdot \left(s_{1\ldots k} \prod_{\#I \geq 1} s_{1\ldots kI} \right)^{-1}$ and $(x_{21}, \ldots, x_{2n}, \ldots, \widehat{x_{i1}, \ldots x_{in}}, \ldots, x_{k1}, \ldots x_{kn})$ is the ideal for defining the strict transform of $S_{1\ldots\hat{i}\ldots k}$ with respect to α_{k+1} . $\left\{ S'_{i\ldots\hat{i}\ldots k} \right\}$ are disjoint each other for $i = 1, \ldots, k$ and $k \geq 3$, therefore the rank of (x_{ij}) does not zero on an open set of $W_{1\ldots k}$, and then the Fitting ideal defines $\sum_{\#I \geq I} \tilde{S}_{1\ldots kI}$ and a subscheme of $W_{1\ldots k}$ with codimension 2 which is the zero locus of 2–bundle $T_s \left(\mathcal{H}^{-1} \otimes \tilde{S}_{1\ldots k}^{-1} \otimes \prod_{\#I \geq 1} \tilde{S}_{1\ldots kI}^{-1} \otimes O(1) \right)$ on $W_{1\ldots k}$.

§ 3. Main Theorem

From Proposition 2.1 and 2.2 we see that $\gamma^{-1}(V_1)$ is defined by an ideal with the form $(s_{1\ldots m}\mathfrak{a}, a_1 + \ldots + a_m)$ over an open set containing $\beta_2^{-1}\left(\tilde{S}_{1\ldots m}\right)$ and hence on $\overline{Q}$ by taking closure. By the definition of the Segre class $s\left(\gamma^{-1}(V_1), \overline{Q}\right)$ we should blow up $\overline{Q}$ with respect to the ideal $(s_{1\ldots m}\mathfrak{a}, a_1 + \ldots + a_m)$. In order to make the following statement simplier we fix some terms.

Definition. *Let Y, $\tilde{Y}$ be two schemes, J be an ideal sheaf on Y . The morphism $\pi : \tilde{Y} \to Y$ is said to be an effective resolution of Y with respect to J if π is a composition of $\pi_1, \ldots, \pi_l$, where $\pi_i : Y_i \to Y_{i-1}$ is a birational morphism with $Y_0 = Y$, $Y_l = \tilde{Y}$ such that*

> *(1) $\pi^{-1}(J)$ is an invertible sheaf,*
> *(2) each π_i is a blowing up of Y_{i-1} with respect to a locally free sheaf.*

What we shall show in this section is that, $\overline{Q}$ has an effective resolution with respect to ideal $(s_{1\ldots m}\mathfrak{a}, a_1 + \ldots + a_m)$ and, for each π_i the Chern class of the locally free sheaf involved is expressed by the Chern classes of S, $\mathcal{H}$ and $\tilde{S}_{1\ldots k}$ with $k \geq 2$.

Theorem. *$\nu(s)$ is expressed by a polynomial of the Chern numbers of S , the degree of S in $\mathbf{P}^{3m-2}$ and the intersection number of the canonical class of S with the restriction of the hyperplane section; the coefficients and the degree of the polynomial depend only on m .*

Proof: First we show that there exists an affective resolution of $\overline{Q}$ with respect to the nth Fitting ideal of $\tilde{\psi}$ over $\bigcup_{1 \leq i < j \leq m} \tilde{S}_{ij}$ and the Chern classes of all of the normal bundles involved are expressed by the Chern classes of Ω_s, $c_1(O(1))$, H and $\tilde{S}_{ij}$, $\tilde{S}_{ijI}$ with $\#I \geq 1$.

In fact, in this case the nth Fitting ideal of $\tilde{\psi}$ is the product of $(s_{ij}\mathfrak{a}'', a_i + a_j, \ldots, a_1 + \ldots + a_m)$ since $\left\{ \tilde{S}_{ij} \right\}$ are disjoint by Proposition 2.1. Locally $\mathfrak{a}_{ij}$ is written as

$\prod_{\#I \geq 1} s_{ijI}\left(a_j(\xi_{j1} - \xi_{i1})', \ldots, a_j(\xi_{jn} - \xi_{in})'\right)$, but $(\xi_{j1} - \xi_{i1}, \ldots, \xi_{jn} - \xi_{in})$ is the ideal for defining S_{ij} in X and hence, after the blowing-ups in § 1, $\left((\xi_{j1} - \xi_{i1})', \ldots, (\xi_{jn} - \xi_{in})'\right) = 1$. Indeed, this shows that near $\tilde{S}_{12}$, $F^n\left(\tilde{\psi}\right)$ is simply W_{12} described in Proposition 2.1. The normal bundle of W_{12} in $\overline{Q}$ is

$$\left\{ \tilde{S}_{12} \oplus O(-1)\mathcal{H}_2^{-1} \otimes \oplus \ldots \oplus O(-1) \otimes O(-1) \otimes \mathcal{H}_m^{-1} \right\} \otimes \prod_{\#I \geq 1} \tilde{S}_{12I}^{-1} .$$

Inductively assuming there exists an effective resolution of $\overline{Q}$ with respect to $F^n\left(\tilde{\psi}\right)$ near $\tilde{S}_{1\ldots k-1}$ and the normal bundles of each blowing-up is expressed by $\tilde{S}_{1\ldots kI}$ with $\#I \geq 0$, Ω_s, $\mathcal{H}_i$ and $O(1)$, we shall construct an effective resolution of $\overline{Q}$ with respect to $F^n\left(\tilde{\psi}\right)$ near $\tilde{S}_{1\ldots k}$. We saw in the beginning of § 2 $F^n\left(\tilde{\psi}\right) = (s_{1\ldots k}\mathfrak{a}'', a_1+\ldots+a_k, \ldots, a_1+\ldots+a_m)$ near $\tilde{S}_{1\ldots k}$. By induction we have had an effective resolution with respect to $F^n\left(\tilde{\psi}\right)$ near $\cup\tilde{S}_{i_1\ldots i_{k-1}}$, denoted by $\tilde{W}$, then

$$\begin{aligned}&\left(s_{12\ldots k}\mathfrak{a}'', a_1+\ldots+a_k, \ldots, a_1+\ldots+a_m\right)\\ &= w\left(s_{12\ldots k}\mathfrak{a}''', (a_1+\ldots+a_k)', \ldots, (a_1+\ldots a_m)'\right)\end{aligned}$$

and $s_{12\ldots k}$, $(a_1+\ldots+a_k)', \ldots, (a_1+\ldots+a_m)'$ have no any common factor. The reason of this assertion is from the local expression (1), (2) in § 2. In fact, the intersection $(w, s_{1\ldots k}, \mathfrak{a}'', a_1+\ldots+a_k, \ldots, a_1+\ldots+a_m)$ is the same as $(w, s_{1\ldots k}, a_1+\ldots+a_k, \ldots, a_1+\ldots+a_m)$. On the other hand, what we just did means $\left(\mathfrak{a}''', (a_1+\ldots+a_k)', \ldots, (a_1+\ldots+a_m)'\right)$ is the residue ideal of $(\mathfrak{a}'', a_1+\ldots+a_k, a_1+\ldots+a_m)$ with respect to W, and then $\left(\mathfrak{a}''', (a_1+\ldots+a_k)', \ldots, (a_1+\ldots+a_m)'\right)$ defines a subscheme which is supported over $\tilde{S}_{12\ldots k}$. Therefore, if we blow up $\overline{Q}'$ (the effective resolution of $\overline{Q}$ with respect to W) along $\tilde{W}\cap\tilde{S}_{12\ldots k}$ and denote the exceptional divisor by $\tilde{Z}_{12\ldots k}$ we see from Proposition 2.2,

$$\begin{aligned}&\left(\mathfrak{a}''', (a_1+\ldots+a_k)', \ldots, (a_1+\ldots+a_m)'\right)\\ &= Z_{12\ldots k}\left(\prod_{\#I\geq 1} s_{1\ldots kI}\mathfrak{a}''', (a_1+\ldots+a_k)'', \ldots, (a_1+\ldots+a_m)''\right)\end{aligned}$$

where $\left(\mathfrak{a}''', (a_1+\ldots+a_k)'', \ldots, (a_1+\ldots+a_m)''\right)$ is the locally free sheaf $T_s\left(\mathcal{H}^{-1}\otimes O(1)\otimes \prod_{\#I\geq 0}\tilde{S}^{-1}_{1\ldots kI}\right)$ over $\tilde{S}_{12\ldots k}$. Blowing up $\overline{Q}''$ (the blowing-up of $\overline{Q}'$ with respect to $\tilde{W}\cap\tilde{S}_{12\ldots k}$) with respect to the locally free sheaf and denoting the exceptional divisor by $\tilde{G}_{1\ldots k}$ we have

$$\begin{aligned}&\left(s\mathfrak{a}'', a_1+\ldots+a_k, \ldots, a_1+\ldots+a_m\right)\\ &= wzg\left(\prod_{\#I\geq 0} s_{1\ldots kI}, (a_1+\ldots+a_k)'', \ldots, (a_1+\ldots+a_m)''\right).\end{aligned}$$

Finally we blow up $\overline{Q}'''$ with respect to $\left(\prod_{\#I\geq 0} s_{1\ldots kI}, (a_1+\ldots+a_k)'', (a_1+\ldots+a_m)''\right)$ and denote the exceptional divisor by $\tilde{T}_{1\ldots k}$. As for the normal bundles involved are as follows.

$N_W\overline{Q}$ is given by induction hypothesis.
$N_Z\overline{Q}'$ is $O\left(\tilde{W}\right)\oplus O\left(\tilde{S}_{1\ldots k}\right)$
$N_G\overline{Q}''$ is $T_s\left(\mathcal{H}^{-1}\otimes O(1)\otimes\prod_{\#I\geq 0}\tilde{S}^{-1}_{1\ldots kI}\right)\oplus\tilde{S}_{12\ldots k}\oplus\bigoplus_{i\geq 0}\mathcal{H}^{-1}_{k+i}\otimes O(1)\otimes\tilde{z}^{-1}$

$$N_T\overline{Q}''' \text{ is } \left(\sum_{\#I\geq 0} \tilde{S}_{1\ldots kI}\right) \oplus \bigoplus_{i\geq 0} \mathcal{H}_{k+1}^{-1} \otimes O(1) \otimes \tilde{z}^{-1} \otimes \tilde{G}^{-1} .$$

Therefore there exists an effective resolution of $\overline{Q}$ with respect to the ideal $F_0\left(\tilde{\psi}\right)$ near $\cup\tilde{S}_{i_1\ldots i_k}$, and the total exceptional divisor is

$$\tilde{W} + \Sigma\tilde{Z}_{i_1\ldots i_k} + \Sigma\tilde{G}_{i_1\ldots i_k} + \Sigma\tilde{T}_{i_1\ldots i_k} .$$

The last step we need to do is to show $\left(\beta_{2*}\left(1 + c_1(O(1))^{n+1} \cdot s(V_1, \overline{Q})\right)\right)_0$ can be expressed by $c_2(\Omega_s)$, K^2, KH, d, where $K = c_1(\Omega_s^\vee)$, H is the hyperplane section of $\mathbf{P}^n$, $d = H_s^2$. But on $\overline{Q}$, $\beta_{2*}\left(c_1(O(1))^{(m-1)+i} \cdot \beta_2^* a\right) = \frac{1}{(1-h_1)\ldots(1-h_m)_i} \cdot a$, where a is a cycle on $\tilde{X}$. By the above argument, $\left(\beta_{2*}\left(1 + c_1(O(1))^{n+1} \cdot s(V_1, \overline{Q})\right)\right)$ is a polynomial in terms of $\tilde{S}_{i_1\ldots i_k}$, H_i, $c_2(\Omega_s)$, $c_1(\Omega_s)$. We push down $\left(\beta_{2*}\left(1 + c_1(O(1))^{n+1} \cdot s(V_1, \overline{Q})\right)\right)_0$ by α_{2*} , the terms involved with $\tilde{S}_{i_1\ldots i_k}$ give us the terms involved with $c_2(\Omega_s)$, $c_1(\Omega_s)$. The theorem is proved.

<u>**Example 1**</u> $m = 2$. In this case S is a surface in $\mathbf{P}^4$ and $\nu = 0$. Theorem tells us

$$2\nu = d^2 - 10d - 5HK + c_2(S) - K^2 = 0$$

This is the well-known condition for a smooth surface embedded in $\mathbf{P}^4$ [4].

<u>**Example 2**</u> [6] $m = 3$. In this case, S is a surface in $\mathbf{P}^7$ and ν is the number of trisecants of S . The total exceptional divisor is

$$\sum_{1\leq i<j\leq 3} \tilde{W}_{ij} + \sum_{1\leq i<j\leq 3} \tilde{Z}_{ij} + \tilde{G}_{123} + \tilde{T}_{123} ,$$

where W_{12} is the projective bundle $\mathbf{P}(E_{12})$ over $\tilde{S}_{12}$ with $c_1(E_{12}) = c_1\mathcal{H}$, $Z_{12} = \left[\tilde{W}_{12}\right] \cdot \left[\tilde{S}_{123}\right]$, $G_{123} =$ the Zero locus of a generic section of $T_s \otimes \left(\mathcal{H}^{-1} \otimes \tilde{S}_{123}^{-1}\right)$ $T_{123} = \left[\tilde{S}_{123}\right] \cdot \left[O(1) \otimes \mathcal{H}^{-1} \otimes \left(\Sigma\tilde{Z}_{ij}^{-1}\right) \otimes \tilde{G}_{123}^{-1}\right]$. We have the following formula

$$6\nu = d^3 - 3d(10d + 5KH + K^2 - c_2) + 224d + 192KH + 56K^2 - 40c_2$$

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