New examples of obstructed complex manifolds in higher dimension

Masa–Hiko Saito

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
5300 Bonn 3
West Germany

Institute of Mathematics
Faculty of Education
Shiga University
Otsu, 520, Japan

MPI/88 –45

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by

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Institute of Mathematics
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Introduction. In the present paper, we will generalize some results of Burns–Wahl [2] and Kas [10] to varieties of dimension $n \geq 3$ and we will give new examples of obstructed compact complex manifold of dimension ≥ 2.

Let Z be a compact complex manifold and Θ_Z its tangent sheaf. An element $\theta \in H^1(Z,\Theta_Z)$ is obstructed if there are no deformations $\mathfrak{X} \longrightarrow \Delta = \{t \in \mathbb{C} : |t| < \varepsilon\}$ of Z such that the image of Kodaira–Spencer map $\rho\left[\frac{\delta}{\delta t}\right]$ is θ. A complex manifold Z is obstructed if there is an element θ which is obstructed. This is also equivalent to that the Kuranishi space of Z is not smooth.

In the case where Z is a surface (i.e. $\dim Z = 2$), examples of obstructed surfaces are given by (as far as I know), Kas [10], Burns–Wahl [2], Horikawa [8], Pinkham [16] and Catanese [3]. Except for Horikawa's examples, all examples arise from the minimal resolution of surfaces with rational double points.

To be more precise, let V be a surface with only ratinal double points $p = \{p_1, \dots, p_\ell\}$, $r : X \longrightarrow V$ the minimal resolution and $E = r^{-1}(p)$ the exceptional divisor.

Burns–Wahl showed that there exists a natural inclusion $H^1_E(\Theta_X) \longrightarrow H^1(\Theta_X)$ where $H^1_E(\Theta_X)$ is the local cohomology group with support E and they studied the contributions of elements of $H^1_E(\Theta_X)$ to the deformation functor D_X of X . Moreover they showed there is a morphism of the deformation functors $D_X \longrightarrow D_V$ which fits into the commutative diagram: ([2] , [16])

(0.1)
$$\begin{array}{ccc} D_X & \longrightarrow & L_X \\ \downarrow & & \downarrow \\ D_V & \longrightarrow & L_V \end{array}$$

Here L_X and L_V are local deformation functors of small neighborhoods of E and P and the mophism $L_X \longrightarrow L_V$ is obtained by blowdowns. Since $L_X \longrightarrow L_V$ is well understood by a theory of Brieskorn, one can describe the functor D_X or the Kuranishi space of X by D_V and the morphism $D_V \longrightarrow L_V$. From the theory of deformation, we have an exact sequence

(0.2)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(V,\Theta_V) & \longrightarrow & T^1_V & \longrightarrow & H^0(V,\mathcal{T}^1_V) & \xrightarrow{ob} & H^2(V,\Theta_V) \\ & & & & \wr\| & & \wr\| & & \\ & & & & D_V(\mathbb{C}[t]) & & L_V(\mathbb{C}[t]) & & \end{array}$$

where $\mathbb{C}[t] = \mathbb{C}[t]/t^2$ and $D_V(\mathbb{C}[t])$ are the Zariski tangent spaces of functors.

From (0.1) and (0.2), one can show that if <u>ob</u> is non–zero map X is obstructed. (cf. [10], [2] and [16]). Using this result, Burns–Wahl [2] and Kas [10] gave many examples of obstructed surfaces X when the singularities of the surfaces V are only ordinary double $(= A_1)$ points.

Recently, using the result (0.1) and a description of the dual of the map <u>ob</u> in (0.2) (due to Kas [10] and Pinkham [16]), Catanese constructed examples of surfaces of general type whose Kuranishi spaces are isomorphic to the product $T \times S$ of smooth schemes T and nilpotent schemes S. (cf. [3].) These examples contain the former examples of Kas and Miranda.

To generalize these results in [2], [10], [16] and [3] to higher dimensional varieties, we will introduce a kind of n–dimensional singularity which is a generalization of rational double points. A complex space S has equisingular rational double points (RDP) along a subvariety B of codimension 2 in S if for each point $p \in B$, the germ (S,p) is isomorphic to the germ $(B,p) \times$ (rational double points). These types of singularities often appear when one takes a quotient variety or a double covering of a smooth variety.

Let V be a compact complex space of dimension $n \geq 2$ all of whose singularities are equisingular RDP and set $B =$ support of Sing. V. If one wants to generalize the result (0.1) to the case where $n \geq 3$, one should define a suitable local deformation functor L_V of singularity. But since $\dim B \geq 1$, some global structures of B have to make some affects on L_V and I do not know what is the reasonable definition of L_V and how can one generalize the results (0.1) for such singular varieties.

Since these difficulties are not overcomed, (as far as I know), in this paper, we make very strong global assumptions on V. That is, V is a double covering of a smooth proper variety Y whose branched locus is a divisor $D = D_1 + D_2$ where D_1 and D_2 are smooth and intersecting each other transversally. In this case, the support of Sing. V is a smooth subvariety B which is isomorphic to $D_1 \cap D_2$ and V has equisingular A_1 points along B. Moreover, one can obtain a unique resolution $r : X \longrightarrow V$. Though our objects V and X are very simple, these give many examples of obstructed manifolds.

In order to mention the statement of our main theorem (Theorem 6.1), we shall give some notations and results. Let E be the exceptional divisor of $r : X \longrightarrow V$. Then, as in 2 dimensional case, one has an inclusion $H^1_E(\Theta_X) \longrightarrow H^1(\Theta_X)$. Moreover we have an isomorphism $H^1_E(\Theta_X) \cong H^0(B,L_B)$ where L_B is a line bundle on B. On the other hand, we have the exact sequence (0.2) for V and an isomorphism $H^0(V,\mathcal{T}^1_V) \cong H^0(B,L_B^2)$. Considering an element $\overline{\phi} \in H^0(B,L_B)$ as an element of $H^1(\Theta_X)$, we construct a deformation $\eta_1 : \mathcal{X} \longrightarrow S_1 = \mathrm{Spec}\,(\mathbb{C}[t]/t^2)$ of X. Then we have the following

Main Theorem: (Theorem 6.1.) The deformation $\eta_1 : \mathcal{X} \longrightarrow S_1$ can be extended to a deformation over $S_2 = \mathrm{Spec}\,(\mathbb{C}[t]/t^3)$ if and only if $\underline{ob}\,(\overline{\phi}^{\,2}) = 0$ where $\underline{ob}$ is defined as in (0.2).

This theorem shows that the primary obstruction of the element $\overline{\phi} \in H^0(B,L_B)$ is given by $\underline{ob}\,(\overline{\phi}^{\,2})$ up to nonzero constant. (cf. Corollary 6.2.).

Moreover we can construct examples of Y and D_1, D_2 such that for the corresponding V the obstruction map $\underline{ob}$ is nontrivial on the image of the square map $H^0(B,L_B) \longrightarrow H^0(B,L_B^2)$. Thus, by Main Theorem, the corresponding resolution X is obstructed.

We remark that there exist examples of compact complex manifolds of dimension $n \geq 3$ whose Kuranishi spaces are not reduced and which are not products of Catanese's examples and some other complex manifolds. We will discuss such examples elsewhere . (See § 8.)

The organization of this paper is as follows. § 1 is a review from deformation theory of complex spaces. § 2 is definition of double cover V and its resolution X which are main objects in this paper. In § 3, we will generalize some results in Burns–Wahl [2] and Wahl [19] and compute the local cohomology group $H^1_E(\Theta_X)$. In § 4, we will construct the first order deformations of X corresponding to elements of $H^1_E(\Theta_X)$ by using Cech cocycles. In § 5, the first obstruction map ob is introduced and calculated by Cech cocycles. Using the results in § 1 ~ § 5, in § 6, we prove our Main Theorem 6.1. After we study the first obstruction map ob more carefully in § 7, in § 8, we will give two kinds of examples of obstructed manifolds of dimension ≥ 2 .

Acknowledgements. The author would like to thank Professors Miyajima and A. Fujiki for helpful discussions. Professor F. Catanese kindly informed me of his results in [3] and discussions with him at Max–Planck–Institut in Bonn gave some improvements to this paper. Many discussions on the deformation theory with Professor S. Zucker at MPI gave a motivation of this paper. The author would like to thank them for their help. Finally, the author is grateful to the Max–Planck–Institut für Mathematik in Bonn for their hospitality.

§ 1 Tangent cohomology and deformation theory.

In this section, we shall review some facts about tangent cohomology and deformations of compact complex spaces which we will use in the later sections. For details, one may refer to the articles [15], [6] and [5]. (A good summary can be found in [20].)

1.1. Let Z be a compact complex space and let $\mathcal{L}_Z^{\cdot}$ denote the cotangent complex of Z which is defined as an object of derived category. If we denote by Ω_Z^1 the sheaf of Kähler differential of Z, we have a natural morphism $\mathcal{L}_Z^{\cdot} \longrightarrow \Omega_Z^1$. To describe the local deformation of Z, the cohomology groups of the cotangent complex are most important. As in [15], [6] and [20], we define for $i \in \mathbb{N}$

$$(1.1) \qquad T_Z^i = \mathrm{Ext}^i_{\mathcal{O}}(\mathcal{L}_Z^{\cdot}, \mathcal{O}_Z)$$

$$(1.2) \qquad \mathcal{T}_Z^i \equiv \mathcal{E}xt^i_{\mathcal{O}_Z}(\mathcal{L}_Z^{\cdot}, \mathcal{O}_Z) .$$

The objects T_Z^i and $\mathcal{T}_Z^i$ are called the tangent cohomology group and sheaf. The sheaf $\mathcal{T}_Z^i$ is coherent to $\mathcal{O}_Z$ – module for all $i \in \mathbb{N}$. Moreover we have the spectral sequence

$$(1.3) \qquad E_2^{pq} = H^p(Z, \mathcal{T}_Z^q) \Rightarrow T_Z^{p+q} .$$

1.2. A deformation of Z over a germ (S,0) of complex space is a Cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & (S,0) \end{array}$$

with a flat morphism $u : \mathfrak{X} \longrightarrow (S,0)$. Let $\mathcal{A}_0$ (resp. $\mathcal{S}ets$) denote the category of germs of complex spaces (resp. sets). For any base change $(T,0) \longrightarrow (S,0)$, one gets a deformation $\mathfrak{X} \times_S T \longrightarrow (T,0)$. Thus we get the deformation functor

$$(1.4) \qquad D_Z : \mathcal{A}_0 \longrightarrow \mathcal{S}ets.$$

This functor can be extended to the category of formal complex spaces.

Let us set $S\mu = \mathrm{Spec}(\mathbb{C}[t]/t^{\mu+1})$ for $\mu \in \mathbb{N}$ and let $u : \mathfrak{X} \longrightarrow (T,0)$ be a deformation of Z . For any morphism $(S_1,0) \longrightarrow (T,0)$, one gets a deformation $\mathfrak{X} \times_T S_1 \longrightarrow (S_1,0)$. Thus we can define

$$(1.5) \qquad \rho : \Theta_{T,0} = \mathrm{Hom}((S_1,0),(T,0)) \longrightarrow D_Z(S_1) .$$

Here $\Theta_{T,0}$ denotes the Zariski tangent space of $(T,0)$. This map ρ is called the Kodaira–Spencer map.

Definition 1.1. A deformation $\mathfrak{X} \longrightarrow (T,0)$ of Z is called semiuniversal (or simply versal) if

(i) the Kodaira–Spencer map ρ in (1.5) is bijective,

(ii) any deformations of Z are induced by some morphism $(S,0) \longrightarrow (T,0)$.

It follows from the definition that two semiuniversal deformation of Z (after shrinking the parameter spaces) are isomorphic to each other and the parameter space of the semiuniversal deformation is uniquely determined by Z as a germ of a complex space. Hence we denote by Def_Z the germ of this parameter space.

By a work of Kuranishi, the semiuniversal deformation of Z exists if Z is smooth and Def_Z is called the Kuranishi space. Later, Grauert, Forster–Knorr [6] and Palamodov [15] proved the existence of the semiuniversal deformations of all compact complex spaces. Due to Palamodov [15], we have the following theorem.

Theorem 1.2. ([15], Theorem 5)
Let Z be a compact complex space and T^i_Z the tangent cohomology group of Z. Then we have the following:

(1) T^1_Z is the Zariski tangent space of Def_Z, (i.e. $T^1_Z \cong D_Z(S_1)$)

(2) There exists a germ of holomorphic map

$$q : T^1_Z \longrightarrow T^2_Z$$

defined near 0 such that $(\mathrm{Def}_Z, 0)$ is isomorphic to $(q^{-1}(0), 0)$ as a germ.

(3) Let $q = \sum_{k=1}^{\infty} q_k$ be its extension in a series of homogeneous polynomials. Then $q_1 \equiv 0$ and q_2 is the restriction of the Lie bracket $T^1_Z \otimes T^1_Z \longrightarrow T^2_Z$ to the diagonal.

1.3. We will restrict to the following situation. Let Z be a compact complex space which is embedded as a hypersurface in a smooth variety W. In this case, if I_Z denotes the ideal sheaf of Z, we have the following exact sequence

$$(1.6) \qquad 0 \longrightarrow I_Z/I_Z^2 \longrightarrow \Omega^1_{W|Z} \longrightarrow \Omega^1_Z \longrightarrow 0\,.$$

The cotangent complex $\mathcal{L}_Z^{\bullet}$ is isomorphic to $I_Z/I_Z^2 \longrightarrow \Omega^1_{W|Z}$, thus we have

$$(1.7) \qquad T_Z^i = \mathrm{Ext}^i(\Omega^1_Z, \mathcal{O}_Z)$$

$$(1.8) \qquad \mathcal{T}_Z^i = \mathcal{E}xt^i_{\mathcal{O}_Z}(\Omega^1_Z, \mathcal{O}_Z)\,.$$

Dualizing (1.6) yields an exact sequence

$$(1.9) \qquad \begin{array}{l} 0 \longrightarrow \Theta_Z \longrightarrow \Theta_{W|Z} \xrightarrow{\ \alpha\ } (I_Z/I_Z^2)^* \\ \qquad\quad \| \\ \qquad\quad \mathcal{T}_Z^0 \end{array}$$

where we set $\Theta_Z = \mathcal{T}_Z^0 = \mathcal{H}om(\Omega^1_Z, \mathcal{O}_Z)$.

<u>Lemma 1.3</u>. Let $Z \subset W$ be as above.

(1) $\mathcal{T}_Z^1 = \mathrm{coker}\,(\Theta_{W|Z} \longrightarrow (I_Z/I_Z^2)^*)$

(2) $\mathcal{T}_Z^i = 0 \quad$ if $i \geq 2$.

(3) There exists an exact sequence

$$0 \longrightarrow H^1(Z,\Theta_Z) \longrightarrow T_Z^1 \longrightarrow H^0(Z,\mathcal{T}_Z^1) \longrightarrow H^2(Z,\Theta_Z)$$
$$\longrightarrow T_Z^2 \longrightarrow H^1(Z,\mathcal{T}_Z^1)$$

<u>Proof</u>. The assertions (1) and (2) follow directly from (1.6) and the locally freeness of $\Omega^1_{W\,|\,Z}$ and I_Z/I_Z^2. The assertion (3) follows from the spectral sequence (1.3) and (2).

§ 2. Singular double covering V and its resolution X.

In this section, we will introduce a special variety V which has singularities along a subvariety of codimension 2. We will also introduce a "minimal" resolution X of V and we will calculate the tangent cohomology groups and sheaves of V and X.

2.1. From this section to the end of this paper, we will consider the following quadruplet (Y,D_1,D_2,L) which consists of:

(2.1)

(i) Y : a smooth projective variety over $\mathbb{C}$ of dimension $n \geq 2$,

(ii) D_1, D_2 : smooth divisors on Y intersecting transversaly each other,

(iii) L : a line bundle on Y satisfying that $L^{\otimes 2} = L_1 \otimes L_2$ where L_1 and L_2 are line bundles corresponding to D_1 and D_2 respectively.

For each quadruplet (Y,D_1,D_2,L) of (2.1), we can define the double covering $\varphi : V \longrightarrow Y$ as follows.

Let $\pi : \mathbb{P} = \mathbb{P}(\mathcal{O}_Y \oplus L) \longrightarrow Y$ be the projective bundle associated to $\mathcal{O}_Y \oplus L$ over Y, $\mathcal{O}_{\mathbb{P}}(1)$ the tautological line bundle of $\mathbb{P}$, and $y \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ and $w \in H^0(\mathbb{P}, \pi^*(L^{-1}) \otimes \mathcal{O}_{\mathbb{P}}(1))$ sections corresponding to the natural inclusions $\mathcal{O}_Y \longrightarrow \mathcal{O}_Y \oplus L$ and $L \longrightarrow \mathcal{O}_Y \oplus L$ respectively. Moreover, let $f \in H^0(Y,L_1)$ and $g \in H^0(Y,L_2)$ denote the sections defining the divisors D_1 and D_2 respectively. Considering $f \cdot g$ as a section in $H^0(\mathbb{P}, \pi^*(L^2))$, we can define section of $\mathcal{O}_{\mathbb{P}}(2)$

(2.2) $$H = y^2 - f \cdot g \cdot w^2 \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2)) .$$

Then we define the hypersurface

(2.3) $$V = \{H = 0\} \longrightarrow \mathbb{P} .$$

The natural projection $\pi : Z \longrightarrow Y$ induces the morphism $\varphi : V \longrightarrow Y$ of degree 2.

It follows from the definition of V that:

(i) V is a double cover of Y branched along the divisor $D = D_1 + D_2$,

(ii) $\varphi_* \mathcal{O}_V \cong \mathcal{O}_Y \oplus L^{-1}$,

(iii) V is a normal projective variety whose singularities are analytically isomorphic to $(y^2 - xz = 0) \times$ (smooth $(n-2)$–dim. variety),

(iv) the singular locus of V coincides with the subvariety $B = \{y = f = g = 0\} \subset \mathbb{P}$.

The subvariety B in (iv) is isomorphic to $D_1 \cap D_2 \subset Y$. Thus we will identify B in $\mathbb{P}$ with $D_1 \cap D_2$ in Y.

2.2. Tangent cohomology groups of V.

Let V be as in 2.1. We will calculate the tangent cohomology groups T^i_V and sheaves $\mathcal{T}^i_V$.

Proposition 2.1. Let V be as in 2.1 and $B = \mathrm{Sing}\, V$. Then we have the following.

(1) $\mathcal{T}_V^{\,1} = L^2 \otimes_{\mathcal{O}_Y} \mathcal{O}_B$ as a sheaf of B.

(2) $\mathcal{T}_V^{\,i} = 0$ for $i \geq 2$.

Proof. From lemma 1.3, we have an isomorphism

$$\mathcal{T}_V^{\,1} \cong \mathrm{coker}\,(\Theta_{\mathbb{P}\,|\,V} \xrightarrow{\alpha} N_V)$$

where $N_V = (I_V/I_V^2)^*$. Since V is a hypersurface and $\mathcal{O}_{\mathbb{P}}(V) \cong \mathcal{O}_{\mathbb{P}}(2)$, N_V is the line bundle isomorphic to $\mathcal{O}_{\mathbb{P}}(2)_{\,|\,V} \cong \mathcal{O}_V(2)$. Let $I_B = \langle y,f,g \rangle$ denote the ideal of B in $\mathbb{P}$. Then by a local calculation, one can easily see that the image of α coincides with $I_B N_V$. Thus we have an isomorphism

$$\mathcal{T}_V^{\,1} \cong N_V \otimes_{\mathcal{O}_V} \mathcal{O}_B \cong \mathcal{O}_{\mathbb{P}}(2) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_B \,. \tag{2.4}$$

Identifying Y with a section $\{y = 0\}$ in $\mathbb{P}$, we also get $\mathcal{O}_{\mathbb{P}}(2) \otimes \mathcal{O}_Y \cong L^2$. Combining this with (2.4), we have

$$\mathcal{T}_V^{\,1} \cong \mathcal{O}_{\mathbb{P}}(2) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_B \cong L^2 \otimes_{\mathcal{O}_Y} \mathcal{O}_B \,.$$

The assertion (2) follows directly from Lemma 1.3, (2). q.e.d.

2.3. Let V be as in 2.1. By assumption on the quadruplet (Y,D_1,D_2,L) in (2.1), if we once blow up V along $B = \mathrm{Sing}\, V$, we obtain a smooth projective variety X. Let us denote by

$$r : X \longrightarrow V \tag{2.5}$$

this resolution and let E be the exceptional divisor of r.

Since X is smooth, the tangent cohomology groups T^i_X are isomorphic to $H^i(X,\Theta_X)$.

Next we will see that X can be embedded into a projective bundle over Y. Let $\mathcal{E}' = \mathcal{O}_Y \oplus L \otimes L_1^{-1}$ be the rank 2 vector bundle over Y and $\tau : \mathbb{P}' = \mathbb{P}(\mathcal{E}') \longrightarrow Y$ the associated projective bundle. Let $x \in H^0(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(1))$ and $z \in H^0(\mathbb{P}', \tau^*(L^{-1} \otimes L_1) \otimes \mathcal{O}_{\mathbb{P}'}(1))$ denote the section corresponding to the natural inclusions $\mathcal{O}_Y \longrightarrow \mathcal{E}'$ and $L \otimes L_1^{-1} \longrightarrow \mathcal{E}'$ respectively. Then we have the section

$$G = fx^2 - gz^2 \in H^0(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(2) \otimes \tau^*(L_1)). \tag{2.6}$$

It is easy to see that the hypersurface $\{G = 0\}$ in $\mathbb{P}'$ is isomorphic to X and the exceptional divisor E of r in (2.5) is given by $X \cap \{f = g = 0\} = X \cap \tau^{-1}(B)$. From this fact, we have the following proposition.

<u>Proposition 2.2</u>. Let $E \subset X$ be as above.

(1) The $\mathbb{P}^1$–bundle $\tau_{|E} : E \longrightarrow B$ is isomorphic to $\mathbb{P}(\mathcal{O}_B \oplus L \otimes L_1^{-1} \otimes \mathcal{O}_B) \longrightarrow B$.

(2) Let N_E denote the normal bundle of E in X. Then we have the isomorphism

$$N_E \cong \mathcal{O}_E(-2) \otimes \tau^*(L_2) .$$

Here $\mathcal{O}_E(1)$ denotes the tautological line bundle of $\tau : E \longrightarrow B$.

The proof is easy and left for the reader.

Remark 2.3. There exists an elementary transformation $e : \mathbb{P}' \longrightarrow \mathbb{P}$. It is easy to check that the resolution $r : X \longrightarrow V$ is induced by this birational map e.

§ 3 Local cohomology group $H^1_E(X,\Theta_X)$ and its contribution.

Let X and E be as in 2.2. In this section we will compute the local cohomology group $H^1_E(X,\Theta_X)$ and its contribution to global deformations of X. Moreover we will consider the relation between deformations of X and those of V.

We note that if $\dim V = 2$, V has only isolated (A_1–) singularities. In this case, Burns–Wahl [2] and Wahl [19] have dealt with these problems in the context of the deformation theory of normal two–dimensional singularity. Our work in this section is based on their works and may be viewed as a generalization to special non–isolated singularities.

3.1. Let V be as defined in 2.1 and $r : X \longrightarrow V$ its resolution defined in 2.2. Recall that we set $B = \mathrm{Sing}\, V$ and $E = r^{-1}(B)$ denote the exceptional divisor of r. In this section, we will consider V and X as complex spaces and all cohomology groups are computed by the analytic topology.

Lemma 3.1.

(1) $j_*(\Theta_{V-B}) = \Theta_V$ where $j : V - B \longrightarrow V$ is the inclusion.

(2) $r_*\Theta_X \cong \Theta_V$.

Proof.: (1): Since V has only quotient singularities, this follows from a general argument (cf. [7]). (2): Since $r : X \longrightarrow V$ is the blowing–up of the maximal ideal I_B of B, we have this assertion.

From this lemma, we can prove the following

Proposition 3.2. (cf. [2], 1.1 and [19] Prop. 1.8.)

Let X , E , V be as above.

(1) There is a natural inclusion $H^1_E(X,\Theta_X) \longrightarrow H^1(X,\Theta_X)$

(2) We have the following commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H^1_E(X,\Theta_X) & \longrightarrow & H^1(X,\Theta_X) & \longrightarrow & H^1(X-E,\Theta_X) . \\
 & & & & \downarrow \beta & & || \\
 & & 0 & \longrightarrow & T^1_V & \longrightarrow & H^1(V-B,\Theta_V) .
\end{array} \tag{3.1}
$$

Proof: (1) From the long exact sequence for local cohomology, we set

$$H^0(X,\Theta_X) \longrightarrow H^0(X-E,\Theta_X) \longrightarrow H^1_E(X,\Theta_X) \longrightarrow H^1(X,\Theta_X) .$$

Set $U = X - E = V - B$. By lemma, one obtains an isomorphism $H^0(X,\Theta_X) \cong H^0(V,r_*\Theta_X) \cong H^0(V,\Theta_X)$. Thus it suffices to show that $H^0(V,\Theta_V) \longrightarrow H^0(U,\Theta_V)$ is surjective. The local cohomology sequence on V gives

$$H^0(V,\Theta_V) \longrightarrow H^0(U,\Theta_V) \longrightarrow H^1_B(V,\Theta_V) .$$

Let $\mathscr{H}^i_B(\Theta_V)$ denote the local cohomology sheaves of Θ_V . Since $j_*(\Theta_{V-B}) = \Theta_V$, one obtains $\mathscr{H}^0_V(\Theta_V) = \mathscr{H}^1_V(\Theta_V) = 0$ (see [9], § 1) . From the spectral sequence, we have $H^1_B(V,\Theta_V) = 0$. Thus we get the assertion (1).

(2) Let $\mathscr{L}^{\cdot}_{X/V}$ denote the cotangent complex of the morphism $r : X \longrightarrow V$. We get a short exact sequence in the derived category

$$0 \longrightarrow \mathbb{L}\, r^{*} \mathscr{L}_V^{\cdot} \longrightarrow \mathscr{L}_X^{\cdot} \longrightarrow \mathscr{L}_{XV}^{\cdot} \longrightarrow 0 .$$

(cf. [20]). This induces the exact sequence of cohomology

$$\begin{array}{ccccccc} T^1_{X/V} & \longrightarrow & T^1_X & \longrightarrow & \mathrm{Ext}^1(\mathbb{L}\, r^{*} \mathscr{L}_V^{\cdot}, \mathcal{O}_X) & \longrightarrow & T^2_{X/V} \\ & & \| & & & & \\ & & H^1(X,\Theta_X) & & & & \end{array}$$

Since V has only rational singularities, we can see $\mathbb{R}\, r_* \mathcal{O}_X \cong \mathcal{O}_V$ (in the derived category). By this, we get (by projection formula)

$$\mathrm{Ext}^1(\mathbb{L}\, r^{*} \mathscr{L}_V^{\cdot}, \mathcal{O}_X) \cong \mathrm{Ext}^1(\mathscr{L}_V^{\cdot}, \mathbb{R}\, r_* \mathcal{O}_X) \cong \mathrm{Ext}^1(\mathscr{L}_V^{\cdot}, \mathcal{O}_V) \cong T^1_V .$$

This defines $\beta : T^1_X \longrightarrow T^1_V$.

Next we will show that there exists a natural inclusion

$T^1_V \hookrightarrow H^0(V-B, \Theta_V)$. On $U = V - B$, we have the exact sequence

$$0 \longrightarrow \Theta_{V\,|\,U} \longrightarrow \Theta_{\mathbb{P}\,|\,U} \longrightarrow N_{V\,|\,U} \longrightarrow 0$$

(cf. 2.2). This gives the exact sequence of cohomology

$$(3.2) \qquad 0 \longrightarrow H^0(U,\Theta_V) \longrightarrow H^0(U,\Theta_{\mathbb{P}\,|\,V}) \longrightarrow H^0(U,N_V) \xrightarrow{\alpha} H^1(U,\Theta_V)$$

Since $\Theta_{\mathbb{P}|V}$ and N_V are free sheaves on V and V is normal, we have $j_*(\Theta_{\mathbb{P}|U}) = \Theta_{\mathbb{P}|V}$ and $j_*(N_{V|U}) = N_V$. Together with $j_*(\Theta_U) = \Theta_V$, from (3.2), we get the exact sequence

$$0 \longrightarrow H^0(V,\Theta_V) \longrightarrow H^0(V,\Theta_{\mathbb{P}|V}) \longrightarrow H^0(V,N_V) \xrightarrow{\alpha} H^1(V,\Theta_V)\,.$$

Since the dual of cotangent complex $\mathcal{L}_V^{\boldsymbol{\cdot}}$ is represented by the complex $\Theta_{\mathbb{P}|V} \longrightarrow N_V$, the image of α is nothing but T_V^1. Thus we obtain the assertion. q.e.d.

<u>3.2</u>. We will compute the local cohomology group $H_E^1(X,\Theta_X)$. The result is as follows.

<u>Proposition 3.3.</u> Let X,E be as above. Then we have the following isomorphism

$$H_E^1(X,\Theta_X) \cong H^1(E,N_E) \cong H^0(B,L_B)\,. \tag{3.3}$$

Here N_E denotes the normal bundle of E in X and L_B denotes the line bundle $L \otimes_{\mathcal{O}_Y} \mathcal{O}_B$ on $B \subset Y$.

Proof: Let

$$H^1_{[E]}(X,\Theta_X) \cong \varinjlim_{m} \operatorname{Ext}^1(\mathcal{O}_{mE},\Theta_X) \tag{3.4}$$

denote the algebraic local cohomology group where $\mathcal{O}_{mE} \cong \mathcal{O}_X/\mathcal{O}_X(-mE)$. Since E can be contracted to $B = \operatorname{Sing} V$ and $\operatorname{codim}(B \subset V) = 2$, we can check that the local cohomology group $H^1_E(X,\Theta_X)$ is isomorphic to the algebraic one. (cf. Proposition 1.6. in [9]). Thus we will compute the algebraic one in (3.4). It is clear that:

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{mE},\Theta_X) = 0 \quad \text{and}$$

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_{mE},\Theta_X) = \Theta_X \otimes N_{mE}$$

where $N_{mE} := \mathcal{O}_X(mE)/\mathcal{O}_X$. Therefore by a spectral sequence we get

$$H^1_E(X,\Theta_X) = \varinjlim_m H^0(mE,\Theta_X \otimes N_{mE}) . \tag{3.5}$$

To compute the right hand side of (3.5), we consider the exact sequence

$$0 \longrightarrow N_{(m-1)E} \longrightarrow N_{mE} \longrightarrow \mathcal{O}_E(mE) \longrightarrow 0 \quad (m \geq 1) . \tag{3.6}$$

First we claim that for $m \geq 2$

$$H^0(E,\Theta_X \otimes \mathcal{O}_E(mE)) = 0 . \tag{3.7}$$

Consider the following two exact sequences:

$$0 \longrightarrow \Theta_E \otimes \mathcal{O}_E(mE) \longrightarrow \Theta_X \otimes \mathcal{O}_E(mE) \longrightarrow \mathcal{O}_E((m+1)E) \longrightarrow 0 \tag{3.8}$$

$$0 \longrightarrow \Theta_{E/B} \otimes \mathcal{O}_E(mE) \longrightarrow \Theta_E \otimes \mathcal{O}_E(mE) \longrightarrow \tau^*(\Theta_B) \otimes \mathcal{O}_E(mE) \longrightarrow 0 \tag{3.9}$$

where $\tau : E \longrightarrow B$ is the natural map and $\Theta_{E/B}$ is the relative tangent sheaf of τ.

Since by (2) of Proposition 2.2, we have

$$\mathcal{O}_E((m+1)E) = N_E^{m+1} \cong \mathcal{O}_E(-(2m+2)) \otimes \tau^*(L_2^{m+1})$$

$$\Theta_{E/B} \otimes \mathcal{O}_E(mE) = \mathcal{O}_E(2) \otimes \tau^*(L^{-1} \otimes L_1) \otimes \mathcal{O}_E(-2m) \otimes \tau^*(L_2^m)$$

$$= \mathcal{O}_E(-2(m-1)) \otimes \tau^*(L^{-1} \otimes L_1 \otimes L_2^m) .$$

Therefore we have

$$\tau_* \mathcal{O}_E((m+1)E) = 0 \quad \text{if} \quad m \geq 0 ,$$

$$\tau_*(\tau^*(\Theta_B) \otimes \mathcal{O}_E(mE)) = 0 \quad \text{if} \quad m \geq 1 ,$$

$$\tau_*(\Theta_{E/B} \otimes \mathcal{O}_E(mE)) = 0 \quad \text{if} \quad m \geq 2 .$$

From these and (3.8), (3.9), we can show that the assertion (3.7) is true. By means of (3.6) and (3.7), we have the isomorphisms

$$H^0(mE, \Theta_X \otimes N_{mE}) \cong H^0(E, \Theta_X \otimes N_E) \quad \text{for} \quad m \geq 1 .$$

Moreover putting $m = 1$ in (3.8) and (3.9), one gets

$$H^0(E, \Theta_X \otimes N_E) \cong H^0(E, \Theta_E \otimes N_E) \cong H^0(E, \Theta_{E/B} \otimes N_E) .$$

Therefore we obtain the isomorphism

$$H^1_E(X,\Theta_X) \cong H^0(\Theta_{E/B} \otimes N_E) .$$

Moreover from the relative Euler sequence for $\tau : E \longrightarrow B$, one has the exact sequence

$$0 \longrightarrow N_E \longrightarrow \tau^*(\mathcal{E}) \otimes \mathcal{O}_E(-1) \otimes \tau^*(L_2) \longrightarrow \Theta_{E/B} \otimes N_E \longrightarrow 0 .$$

Since $\tau_* \mathcal{O}_E(-1) = R^1\tau_* \mathcal{O}_E(-1) = 0$, one can easily see that

$$H^0(E,\Theta_{E/B} \otimes N_E) \cong H^1(E,N_E) .$$

Thus we have the isomorphism

$$H^1_E(X,\Theta_X) \cong H^1(E,N_E) .$$

By Proposition 2.2, one can also check that $\tau_* N_E = 0$ and $R^1\tau_* N_E \cong R^1\tau_* \mathcal{O}_E(-2) \otimes L_2 \otimes \mathcal{O}_B$. Since $R^1\tau_* \mathcal{O}_E(-2) \cong L^{-1} \otimes L_1 \otimes \mathcal{O}_B$, we get $R^1\tau_* N_E \cong L^{-1} \otimes L_1 \otimes L_2 \otimes \mathcal{O}_B \cong L \otimes \mathcal{O}_B$. Thus we have the isomorphism

$$H^1(E,N_E) \cong H^0(B,L_B)$$

from the Leray spectral sequence for $\tau : E \longrightarrow B$. q.e.d.

3.3. Let $E \subset X$ be as in 3.2. Let $\Theta_X(-\log E)$ denote the sheaf of holomorphic vector fields which preserve the ideal of E. This is a locally free sheaf on X and there exists an exact sequence

$$(3.10) \qquad 0 \longrightarrow \Theta_X(-\log E) \longrightarrow \Theta_X \longrightarrow N_E \longrightarrow 0 .$$

The sheaf $\Theta_X(-\log E)$ plays the same role in the deformation theory of the pair (X,E) as Θ_X plays in that of X. (cf. [11], [19], see also [17], § 4. I). It is known that the semiuniversal family of the deformation of the pair exists ([11]). Moreover Wahl defined the functor of equisingular deformation of the resolution ES which is convenient to our context ([19], § 2). It is easy to see that the cohomology group $H^1(X,\Theta_X(-\log E))$ is the Zariski tangent space of the semiuniversal deformation of (X,E) and $H^2(X,\Theta_X(-\log E))$ is the set of the obstructions to the deformations.

In our context, the following proposition is important.

Proposition 3.4. Let $E \subset X$ be as in 3.2. Then we have the following:

(1) there exists an exact sequence

$$(3.11) \qquad 0 \longrightarrow H^1(X,\Theta_X(-\log E)) \longrightarrow H^1(X,\Theta_X) \longrightarrow H^1(E,N_E) \longrightarrow 0 ,$$

(2) $H^1_E(\Theta_X(-\log E)) = 0$,

(3) the subspace $H^1_E(\Theta_X)$ in $H^1(\Theta_X)$ is isomorphic to $H^1(E,N_E)$ and to the set of the first order deformations of X to which the divisor E does not lift.

Proof. The assertion (3) follows from Proposition 3.3 and the assertion (2) can be proved by the same argument as in the proof of Proposition 3.3. Hence we left the proofs for the reader. To prove (1), we first note that $H^0(E,N_E) = 0$ because E is the exceptional divisor of $r : X \longrightarrow V$. In view of (3.10), therefore, it suffices to show that the map

$$H^1(X,\Theta_X) \longrightarrow H^1(E,N_E)$$

is surjective. Consider the following diagram

$$\begin{array}{ccccccc} & & H^1_E(\Theta_X(-\log E)) & \longrightarrow & H^1_E(\Theta_X) & \xrightarrow{\gamma'} & H^1_E(N_E) \\ & & \downarrow & & \delta\downarrow & & \| \\ 0 & \longrightarrow & H^1(\Theta_X(-\log E)) & \longrightarrow & H^1(X,\Theta_X) & \xrightarrow{\gamma} & H^1(E,N_E) \end{array}$$

Since δ is injective by (1) of Proposition 3.2 and $\dim H^1_E(\Theta_X) = \dim H^1(E,N_E)$, it suffices to show that $\gamma\cdot\delta$ is injective. By the commutativity of the diagram, this is equivalent to the injectivity of γ' which follows from the assertion (2). q.e.d.

<u>Remark 3.5</u>. The exact sequence (3.11) has the splitting $H^1(E,N_E) \longrightarrow H^1(X,\Theta_X)$ if we identify $H^1(E,N_E)$ with $H^1_E(\Theta_X)$. Thus we can write as

$$H^1(X,\Theta_X) = H^1(\Theta_X(-\log E)) \oplus H^1_E(\Theta_X)$$

$$= H^1(\Theta_X(-\log E)) \oplus H^1(E,N_E) .$$

<u>3.4</u>. Let $r : X \longrightarrow V$ be as in (2.5). Let $D_{(r)} : \mathscr{A}_0 \longrightarrow \mathscr{S}ets$ denote the deformation functor of the morphism $r : X \longrightarrow V$. We have the natural morphisms of functors

(3.12) $$\Phi : D_{(r)} \longrightarrow D_X ,$$

(3.13) $$\Psi : D_{(r)} \longrightarrow D_V .$$

Let $\mathscr{A}_0^f$ denote the category of germs of complex spaces of dimension 0 and $D_{(r)}^f$, D_X^f and D_V^f the functors restricted to $\mathscr{A}_0^f$.

Proposition 3.6. Let $r : X \longrightarrow V$ be as above. Then we have a blow down morphism $\beta : D_X^f \longrightarrow D_V^f$ and $\tilde{\beta} : D_X^f \longrightarrow D_{(r)}^f$ which are compatible with (3.12) and (3.13).

Proof: The most important fact is $\mathbb{R}\, r_* \mathcal{O}_X \cong \mathcal{O}_V$. Let $(S,0)$ be an element of $\mathscr{A}^f$ and $\mathscr{X} \longrightarrow (S,0)$ an element of $D_X(S)$. Considering $\mathcal{O}_{\mathscr{X}}$ as the sheaf algebras on X, the sheaf $r_*(\mathcal{O}_{\mathscr{X}})$ on V defines a deformation of V over $(S,0)$. This is verified by the same argument as in Proposition (2.3) in [2] because of the isomorphism $\mathbb{R}\, r_* \mathcal{O}_X \cong \mathcal{O}_V$. Thus we have $\beta : D_X^f \longrightarrow D_V^f$ and $\tilde{\beta} : D_X^f \longrightarrow D_{(r)}^f$ as desired. q.e.d.

For the functor $D_{(r)}$, D_X and D_V, and we can prove the following

Proposition 3.7. Let $r : X \longrightarrow V$ be as in (2.5). Let $(S,0) \in \mathscr{A}_0$ be any germ of a complex space. Then the natural map

$$D_{(r)}(S) \longrightarrow D_X(S) \tag{3.14}$$

is surjective.

Proof: Let us denote by

$$\beta^i : T_V^i \longrightarrow \mathrm{Ext}^i(\mathbb{L}\, r^* \mathscr{L}_V^{\cdot}, \mathcal{O}_X) \quad i \geq 0 \tag{3.15}$$

the natural induced map. By virture of Proposition 1.10 in [20], if β^1 is surjective and β^2 is injective, the assertion is true. On the other hand, we have

$$\begin{aligned}\mathrm{Ext}^i(\mathbb{L}\, r^* \mathscr{L}^{\cdot}_V, \mathcal{O}_X) &\cong \mathrm{Ext}^i(\mathscr{L}^{\cdot}_V, \mathbb{R}\, r_* \mathcal{O}_X) \\ &\cong \mathrm{Ext}^i(\mathscr{L}^{\cdot}_V, \mathcal{O}_V) \\ &\cong T^i_V .\end{aligned}$$

Thus β^i is isomorphism for each $i \geq 0$. q.e.d.

Corollary 3.8. Let $r : X \longrightarrow V$ be as in (2.5). Let $\mathscr{X} \longrightarrow \mathrm{Def}_X$ and $\mathscr{V} \longrightarrow \mathrm{Def}_V$ be the Kuranishi families of X and V . Then we have a commutative diagram

(3.16)
$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\tilde{\eta}} & \mathscr{V} \\ \downarrow & & \downarrow \\ \mathrm{Def}_X & \xrightarrow{\eta} & \mathrm{Def}_V \end{array}$$

such that $\tilde{\eta}_0 = r$.

Proof. By proposition 3.7, we have a commutative diagram

$$\begin{tikzcd} \mathscr{X} \arrow[r] \arrow[dr] & \mathscr{V}' \arrow[d] \\ & \mathrm{Def}_X \end{tikzcd}$$

with a flat morphism $\mathscr{V}' \longrightarrow \mathrm{Def}_X$. By the semiuniversality of $\mathscr{V} \longrightarrow \mathrm{Def}_V$, we get the morphism $\eta : \mathrm{Def}_X \longrightarrow \mathrm{Def}_V$ and $\eta' : \mathscr{V}' \longrightarrow \mathscr{V}$. Thus we obtain the assertion.

Remark 3.8. In the above case, β^i are all isomorphisms. This implies that

$$T^1{}_{[r]} = D_{[r]}(\mathrm{Spec}\, \mathbb{C}[t]/t^2) \xrightarrow{\sim} T^1_X$$

and

$$T^2_{[r]} \cong T^2_X .$$

Moreover if S is Artinian (i.e. $S \in \mathscr{A}^f_0$) or S is a formal analytic space, we have a canonical section of Φ_S in (3.14)

$$\tilde{\beta} : D_X(S) \longrightarrow D_{(r)}(S)$$

such that $(\Phi \circ \tilde{\beta})_S = \mathrm{id}$.

By using the existence of the relative Doudady space and a Artin's theorem in [1], we can prove that for any $(S,0) \in \mathscr{A}_0$ the section $\tilde{\beta}_S : D_X(S) \longrightarrow D_{(r)}(S)$ exists and hence so does $\beta_S : D_X(S) \longrightarrow D_V(S)$.

§ 4 First order deformations of X via Čech cocycles.

4.1. By a first order deformation of a compact complex space Z, we mean a deformation of Z over $S_1 = \mathrm{Spec}(\mathbb{C}[t]/t^2)$. The set of first order deformations of Z is isomorphic to T^1_Z.

Let X be as in (2.2). By Proposition 3.4, (1), we have an isomorphism

$$T^1_X = H^1(X,\Theta_X) = H^1(X,\Theta_X(-\log E)) \oplus H^1(E,N_E) . \tag{4.1}$$

We will construct first order deformations of X corresponding to elements of $H^1(E,N_E)$ by using Čech cocycles. To proceed to this, we shall introduce the followings notations. Let us define the followings:

(i) $\mathcal{U} = \{U_i\}$: a Stein covering of Y such that $\mathcal{U}_B = \{U_i \cap B\}$ is also a Stein covering of B,

(ii) $\{h_{ij}\}$, $\{f_{ij}\}$ and $\{g_{ij}\} \in H^1(\mathcal{U}\mathcal{O}^*_Y)$: sets of transition functions of the line bundles L, L_1 and L_2 repectively with respect to $\mathcal{U}$. We also assume that $h^2_{ij} = f_{ij}g_{ij}$ on $U_{ij} = U_i \cap U_j$,

(iii) $\{f_i \in \Gamma(U_i,\mathcal{O}_Y)\}$, $\{g_i \in T(v_i,\mathcal{O}_Y)\}$: sets of defining equations D_1 and D_2 respectively satisfying that

$$f_i = f_{ij}f_j \text{ and } g_i = g_{ij}g_j \text{ on } U_{ij}, \tag{4.2}$$

(iv) $(t^\alpha_i) = (t^1_i, \dots, t^n_i)$: a local coordinate system on U_i with transision functions $\{F^\beta_{ij}\}$ satisfying that

(4.3) $$t^{\alpha}_{ij} = F^{\alpha}_{ij}(t^{\beta}_{j}) \text{ on } U_{ij},$$

(v) $\{y_i\}$, $\{x_i\}$ and $\{z_i\}$: fiber coordinates of L, $\mathcal{O}_Y$ and $L^{-1} \otimes L_1$ satisfying that

(4.4) $$y_i = h_{ij} y_j, \quad x_i = x_j, \quad z_i = h^{-1}_{ij} f_{ij} z_j .$$

The $\mathbb{P}^1$–bundle $\tau : \mathbb{P}' = \mathbb{P}(\mathcal{O}_Y \oplus L \otimes L_1^{-1}) \longrightarrow Y$ has a trivialization $\tau^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{P}^1$ with a local coordinate $((t^{\alpha}_{i}), (x_i ; z_i))$. The transition matrix of this $\mathbb{P}^1$–bundle is given by

(4.5) $$A_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & h^{-1}_{ij} f_{ij} \end{bmatrix} \quad \text{on} \quad U_{ij} .$$

A local equation of $X \cap U_i \times \mathbb{P}^1$ is given by

(4.6) $$G_i = f_i x_i^2 - g_i z_i^2 .$$

(cf. (2.6).) Note that on $U_{ij} \times \mathbb{P}^1$, we have the equality

(4.7) $$G_i = f_{ij} G_j .$$

We finally set $X_i = \{G_i = 0\} \subset U_i \times \mathbb{P}^1$.

4.2. Let B denote the submanifold of Y defined by the ideal $I_B = \{f = g = 0\}$ (Recall that B is isomorphic to Sing V.) Set $L_B = L \otimes \mathcal{O}_B$. By Proposition 3.3, we have an isomorphism

$$H^1_E(X,\Theta_X) = H^1(E,N_E) = H^0(B,L_B)\,. \tag{4.8}$$

Let $\overline{\phi} \in H^0(B,L_B)$ be a section of L_B and $\{\overline{\phi}_i \in \Gamma(U_i \cap B, \mathcal{O}_B)\}$ a Čech cocycle representing $\overline{\phi}$. On $U_{ij} \cap B$, we have

$$\overline{\phi}_i = h_{ij}\overline{\phi}_j\,. \tag{4.9}$$

Let us construct the first order deformation of X corresponding to this $\overline{\phi} \in H^0(B,L_B)$.

Since U_i is a Stein open set, we can take an extension $\overline{\phi}_i \in \Gamma(U_i,\mathcal{O}_Y)$ of $\overline{\phi}_i$. Take an extension ϕ_i of $\overline{\phi}_i$ for each i, and set

$$\phi_i - h_{ij}\phi_j = h_{ij}\,\gamma_{ij} \quad \text{on } U_{ij}\,. \tag{4.10}$$

By (4.9), γ_{ij} vanishes on $U_{ij} \cap B$, therefore we can set (not uniquely) as follows:

$$\gamma_{ij} = f_j a_{ij} + g_j b_{ij}\,. \tag{4.11}$$

By definition (4.10), $\{\gamma_{ij}\}$ satisfies the 1–cocycle conditions

$$h_{jk}^{-1}\gamma_{ij} + \gamma_{jk} = \gamma_{ik} \quad \text{on } U_{ijk} = U_i \cap U_j \cap U_k\,. \tag{4.12}$$

Substituting (4.11) to (4.12), we have the identity

$$f_k(h_{jk}^{-1}f_{jk}a_{ij} + a_{jk} - a_{ik}) = -\,g_k(h_{jk}^{-1}g_{jk}b_{ij} + b_{jk} - b_{ik})\,. \tag{4.13}$$

Since f_k and g_k are coprime, we get $\gamma_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}_Y)$ such that

$$h_{jk}^{-1} f_{jk} a_{ij} + a_{jk} - a_{ik} = g_k \gamma_{ijk} \tag{4.14}$$

$$h_{jk}^{-1} g_{jk} b_{ij} + b_{jk} - b_{ik} = - f_k \gamma_{ijk} \,. \tag{4.15}$$

Remark 4.1. Since B is a complete intersection of D_1 and D_2, we have the resolution of I_B :

$$0 \longrightarrow L^{-2} \longrightarrow L_1^{-1} \oplus L_2^{-1} \longrightarrow I_B \longrightarrow 0 \,. \tag{4.16}$$

Tensoring L to this sequence, we have

$$0 \longrightarrow L^{-1} \longrightarrow L \otimes L_1^{-1} \oplus L \otimes L_2^{-1} \longrightarrow I_B L \longrightarrow 0 \,. \tag{4.17}$$

From this sequence, we can see that if $H^2(Y, L^{-1}) = 0$ we can choose an extension $\{\phi_i\}$ such that $\gamma_{ijk} = 0$ for all (i,j,k).

4.3. Now let us define a deformation of $X_i = \{G_i = 0\}$ by a hypersurface

$$\mathcal{X}_i : = \{\bar{G}_i = f_i x_i^2 - 2t\phi_i x_i z_i - g_i z_i^2 = 0\} \tag{4.18}$$

in $U_i \times \mathbb{P}^1 \times S_1$. We have the commutative diagram

$$\mathscr{X}_i \longrightarrow U_i \times \mathbb{P}^1 \times S_1 , \tag{4.19}$$

$$S_1 = \mathrm{Spec}(\mathbb{C}[t]/t^2) .$$

Let $u_i = x_i/z_i$ be an inhomogeneous coordinate of $U_i \times \mathbb{P}^1$. Define the following automorphism

$$\begin{array}{ccc} U_j \times \mathbb{P}^1 & & U_i \times \mathbb{P}^1 \\ \uparrow & & \uparrow \\ \eta_{ij} : U_{ij} \times \mathbb{P}^1 & \longrightarrow & U_{ij} \times \mathbb{P}^1 \end{array}$$

by

$$\tilde{\eta}_{ij}(t_j^{\beta}, u_j) = (t_i^{\alpha} = F_{ij}^{\alpha}(t_j^{\beta}),\ u_i = \eta_{ij}(u_j)) , \tag{4.20}$$

where η_{ij} is given by the projective automorphism

$$u_i = \eta_{ij}(u_j) = \frac{u_j + t a_{ij}}{h_{ij}^{-1} f_{ij}(-\, t\, b_{ij} u_j + 1)} \tag{4.21}$$

Since $t^2 = 0$, we can express (4.21) as

$$u_i = \eta_{ij}(u_j) = h_{ij} f_{ij}^{-1}(u_j + t(a_{ij} + b_{ij} u_j^2))). \tag{4.22}$$

By an easy calculation using (4.14) and (4.15), we have the following

Lemma 4.2. On U_{ijk}, we have

$$\eta_{ij} \circ \eta_{jk}(u_k) - \eta_{ik}(u_k) = -h_{ik} f_{ik}^{-1} \cdot \gamma_{ijk} \cdot G_k \cdot t \,. \tag{4.23}$$

Moreover, set $\tilde{G}_i' = \tilde{G}_i / z_i^2 = f_i u_i^2 - 2t\phi_i u_i - g_i$. Then we have the following lemma.

Lemma 4.3. On U_{ij}, we have

$$\tilde{G}_i' \circ \eta_{ij} = g_{ij}(1 + 2tb_{ij}u_j)\tilde{G}_j' \,. \tag{4.24}$$

The proof of Lemma 4.2 and 4.3 is straightforward and left for the readers. By these lemmas, we have the following

Proposition 4.4. The collection of hypersurfaces $\{\mathscr{X}_i\}$ in (4.18) with automorphism $\{\eta_{ij}\}$ in (4.20) defines a deformation $\mathscr{X} \longrightarrow S_1$ which corresponds to $\overline{\phi} \in H^0(B, L_B)$.

§ 5 The first obstruction map for V .

5.1. Let V be as in defined in (2.3). From (3) of Lemma 1.3 and Proposition 2.1, one has the exact sequence

$$(5.1)\qquad 0 \longrightarrow H^1(V,\Theta_V) \longrightarrow T_V^1 \longrightarrow H^0(B,L_B^2) \xrightarrow{\ ob\ } H^2(V,\Theta_V)\ .$$

We call the map $\underline{ob} : H^0(B,L_B^2) \longrightarrow H^2(V,\Theta_V)$ the first obstruction map for V . In this section, we shall describe the map $\underline{ob}$ by means of Čech cocycles.

5.2. First we recall that the tangent complex of $V \subset \mathbb{P} = \mathbb{P}(\mathcal{O}_Y \oplus L)$ is given by $\alpha : \Theta_{\mathbb{P}\,|\,V} \longrightarrow N_V$ which gives the two exact sequences (cf. Proposition 2.1):

$$(5.2)\qquad 0 \longrightarrow \Theta_V \longrightarrow \Theta_{\mathbb{P}\,|\,V} \longrightarrow I_B N_V \longrightarrow 0$$

$$(5.3)\qquad 0 \longrightarrow I_B N_V \longrightarrow N_V \longrightarrow L_B^2 \longrightarrow 0\ .$$

By definition of the spectral sequence, the map $\underline{ob}$ is the composition map of two connecting homomorphisms:

$$(5.4)\qquad \begin{tikzcd} H^0(B,L_B^2) \arrow[r, "\delta_1"] \arrow[dr, "\underline{ob}"'] & H^1(I_B N_V) \arrow[d, "\delta_2"] \\ & H^2(V,\Theta_V) \end{tikzcd}$$

Moreover we have the following

Lemma 5.1. Let $\varphi = \pi_{|V} : V \longrightarrow Y$ be the natural projection. Then we have the following:

(1) $\varphi_* \mathcal{O}_V \cong \mathcal{O}_Y \oplus L^{-1}$,

(2) $\varphi_* N_V \cong L \oplus L^2$,

(3) $\varphi_* I_B N_V \cong L \oplus I_B L^2$,

(4) $\varphi_* \Theta_V \cong \Theta_Y(-\log D) \oplus \Theta_Y \otimes L^{-1}$ (where $D = D_1 + D_2$),

(5) there exists an exact sequence

$$(5.5) \qquad 0 \longrightarrow \mathcal{O} \oplus L \longrightarrow \varphi_*(\Theta_{\mathbb{P}|V}) \longrightarrow \Theta_Y \oplus \Theta_Y \otimes L^{-1} \longrightarrow 0 .$$

Proof. The assertion (1) is a standard fact of the double covering. Since $N_V \cong \mathcal{O}_{\mathbb{P}}(2) \otimes \mathcal{O}_V$ and $\varphi_* \mathcal{O}_{\mathbb{P}}(2) \cong \mathcal{O} \oplus L \oplus L^2$, we have $\varphi_* N_V \cong L \oplus L^2$. From the exact sequence (5.3), we obtain

$$0 \longrightarrow \varphi_* I_B N_V \longrightarrow L \oplus L^2 \longrightarrow L_B^2 \longrightarrow 0$$

From a local computation and this sequence, the assertion (3) follows. The assertion (4) follows from Proposition 2.1 in [12]. Let $\Theta_{\mathbb{P}/Y}$ denote the relative tangent sheaf of $\pi : \mathbb{P} \longrightarrow Y$. Then $\Theta_{\mathbb{P}/Y}$ is isomorphic to $\mathcal{O}_{\mathbb{P}}(2) \otimes \pi^*(L^{-1})$. Moreover we have the exact sequence

$$0 \longrightarrow \Theta_{\mathbb{P}/Y|V} \longrightarrow \Theta_{\mathbb{P}|V} \longrightarrow \varphi^*(\Theta_Y) \longrightarrow 0 .$$

Taking a direct images and using $\varphi_*\Theta_{\mathbb{P}/Y|V} \cong \varphi_*\mathcal{O}_V(2) \otimes L^{-1} \cong \mathcal{O} \oplus L$, we get the exact sequence in (5). q.e.d.

Let $\iota : V \longrightarrow V$ be the natural involution corresponding to the double covering $\varphi : V \longrightarrow Y$. All sheaves in (5.2) and (5.3) have natural actions of this involution ι , hence we can consider the ι–invariant direct image φ_*^+ for these sheaves.

Lemma 5.2. Let $\varphi : V \longrightarrow Y$ be as in Lemma 5.1. Then we have the following isomorphisms:

(1) $\varphi_*^+ \mathcal{O}_V = \mathcal{O}_Y$

(2) $\varphi_*^+ N_V = L^2$

(3) $\varphi_*^+ I_B N_V = I_B L^2$

(4) $\varphi_*^+ \Theta_V = \Theta_Y(-\log D)$

(5) $\varphi_*^+(\Theta_{\mathbb{P}|V}) \cong \Sigma_L$ where Σ_L is the sheaf of germs of differential operator of L of degree ≤ 1 . Equivalently Σ_L is defined by the following extension

(5.6) $$0 \longrightarrow \mathcal{O}_Y \longrightarrow \Sigma_L \longrightarrow \Theta_Y \longrightarrow 0$$

whose extension class is $C_1(L) \in H^1(Y, \Omega_Y^1)$.

Proof. The assertions (1) – (4) are clear. By using the exact sequence (5.5) and local computation, we get the assertion (5). q.e.d.

Taking the ι–invariant direct image of (5.2) and (5.3), we get the exact sequence

$$0 \longrightarrow \Theta_Y(-\log D) \longrightarrow \Sigma_L \longrightarrow I_B L^2 \longrightarrow 0\,, \tag{5.7}$$

$$0 \longrightarrow I_B L^2 \longrightarrow L^2 \longrightarrow L_B^2 \longrightarrow 0\,. \tag{5.8}$$

Since $H^0(B,L_B^2)$ is clearly ι–invariant, the diagram (5.4) becomes

$$\begin{tikzcd} H^0(B,L_B^2) \arrow[r, "\delta_1^+"] \arrow[dr] & H^1(Y,I_B L^2) \arrow[r] \arrow[d, "\underline{ob}\ \ \delta_2^+"] & H^1(I_B N_V^2) \arrow[d] \\ & H^2(Y,\Theta_Y(-\log D)) \arrow[r, hook] & H^2(V,\Theta_V) \end{tikzcd} \tag{5.9}$$

Proposition 5.3. The map $\underline{ob} : H^0(B,L_B^2) \longrightarrow H^2(V,\Theta_V)$ coincides with the composite map $\delta_2^+ \circ \delta_1^+$ in (5.9).

5.3. Next we will calculate the map $\underline{ob}$ by means of Čech cocycles. We keep the notations in 4.1. The $\mathbb{P}^1$–bundle $\pi : \mathbb{P} = \mathbb{P}(\mathcal{O}_Y \oplus L) \longrightarrow Y$ has a trivialization $\pi^{-1}(U_i) \cong U_i \times \mathbb{P}^1$ with a local coordinate system (t_i^α, y_i) where y_i denotes an inhomogeneous coordinate of $\mathbb{P}^1$. On $U_{ij} \times \mathbb{P}^1$, we have an identity $t_i^\alpha = F_{ij}^\alpha(t_j^B)$ and $y_i = h_{ij} y_j$. The hypersurface V in $\mathbb{P}$ defined in (2.3) is locally defined by

$$V_i = \{H_i = y_i^2 - f_i g_i = 0\} \subset U_i \times \mathbb{P}^1\,. \tag{5.10}$$

(5.10) $$V_i = \{H_i = y_i^2 - f_i g_i = 0\} \subset U_i \times \mathbb{P}^1 .$$

Note that on $U_{ij} \times \mathbb{P}^1$, we have

(5.11) $$H_i = h_{ij}^2 H_j .$$

Let $\overline{K} = \{\overline{K}_i\}$ be an element of $H^0(B, L_B^2)$ which is represented by cocycles $\overline{K}_i \in \Gamma(U_i \cap B, \mathcal{O}_B)$. Taking an extension $K_i \in \Gamma(U_i, \mathcal{O}_Y)$ of each $\overline{K}_i$, we set

(5.12) $$\tilde{K}_{ij} = h_{ij}^2 K_{ij} = K_i - h_{ij}^2 K_j$$

Then $\{\tilde{K}_{ij}\}$ defines an element of $H^1(Y, I_B L^2)$. In fact, we have the cocycle conditions

$$\tilde{K}_{ij} + h_{jk}^2 \tilde{K}_{jk} = \tilde{K}_{ik} \quad \text{on} \quad U_{ijk}$$

from (5.12) and $\tilde{K}_{ij|B} = 0$ by definition. Therefore we have

Lemma 5.4. Let δ_1^+ be as in (5.9). Then we have

$$\delta_1^+(\overline{K}) = \{\tilde{K}_{ij}\} \text{ in } H^1(Y, I_B L^2) .$$

Next we consider the map δ_2^+ in (5.9). By definition, δ_2^+ is the map which fits into the exact sequence

(5.13) $$\longrightarrow H^1(Y, \Sigma_L) \xrightarrow{\mu} H^1(Y, I_B L^2) \xrightarrow{\delta_2^+} H^2(Y, \Theta_Y(-\log D)) .$$

Let us analyse the image of μ in (5.13).

Let $\tilde{\theta} = \{\tilde{\theta}_{ij}\}$ be a 1–cocycle of Σ_L . Then it can be written as

$$\tilde{\theta}_{ij} = \theta_{ij} + \beta_{ij}y_i \frac{\partial}{\partial y_i} \tag{5.14}$$

where $\theta_{ij} = \sum_{\alpha=1}^{n} \theta_{ij}^{\alpha} \frac{\partial}{\partial t_i^{\alpha}} \in \Gamma(U_{ij},\Theta_Y)$ and $\beta_{ij} \in \Gamma(U_{ij},\mathcal{O}_Y)$.

Moreover the 1–cocycle condition of $\tilde{\theta}$ is equivalent to the following:

$$\theta_{jk} - \theta_{ik} + \theta_{ij} = 0 \quad \text{on} \quad U_{ijk} , \tag{5.15}$$

i.e. $\{\theta_{ij}\} = \theta$ defines an element in $H^1(Y,\Theta_Y)$, and

$$\theta_{jk} \cdot \log h_{ij} = -(\beta_{jk} - \beta_{ik} + \beta_{ij}) . \tag{5.16}$$

By definition, $\mu(\tilde{\theta})$ is represented by the 1–cocycle $\{\tilde{\theta}_{ij} \cdot H_i\}$. This is given by

$$\tilde{\theta}_{ij} \cdot H_i = -\theta_{ij} \cdot (f_i g_i) + 2\beta_{ij} f_i g_i . \tag{5.17}$$

From these considerations, we have the following proposition .

From these considerations, we have the following proposition.

Proposition 5.5. Let $\overline{K} = \{\overline{K}_i\}$ be an element of $H^0(B, L_B^2)$. Then $\underline{ob}(\overline{K})$ is zero in $H^2(V, \Theta_V)$ if and only if there exists an extension K_i of $\overline{K}_i$ and $(\theta, \beta) = (\{\theta_{ij}\}, \{\beta_{ij}\}) \in H^1(Y, \Theta_Y) \times C^1(\mathcal{U}, \mathcal{O}_Y)$ satisfying the conditions (5.15) and (5.16) such that the following equality holds:

$$-\theta_{ij}(f_i g_i) + 2\beta_{ij} f_i g_i = K_i - h_{ij}^2 K_j . \tag{5.18}$$

Remark 5.6. The condition (5.16) is equivalent to $\theta \cdot C_1(L) = 0$ in $H^2(Y, \mathcal{O}_Y)$ and this implies that under the first order deformation of Y corresponding to $\theta = \{\theta_{ij}\}$, the line bundle L can be lifted.

5.4. Let $\overline{K} = \{\overline{K}_{ij}\}$ be an element of $H^0(B, L_B^2)$. If $\underline{ob}\,(\overline{K}) = 0$, the exact sequence (5.1) implies that $\overline{K}$ comes from an element of T_V^1, that is, the local deformation near the singularities of V defined by $\overline{K}$ can be globalized to a first order deformation of V. By using Čech cocycles, we will give a first order deformation of V corresponding to $\overline{K}$ such that $\underline{ob}\,(\overline{K}) = 0$. Let us choose $K = \{K_i\}$ and (θ, β) as in Proposition 5.5. Set

$$h_{ij}^2 K_{ij} = K_i - h_{ij}^2 K_j . \tag{5.19}$$

Let us consider a deformation of V_i in (5.10) for each i defined by

$$\tilde{\mathcal{V}}_i = \{\tilde{H}_i = y_i^2 - f_i g_i - tK_i = 0\} \subset U_i \times \mathbb{P}^1 \times S_1 \tag{5.20}$$

$$t_i^{\alpha} = F_{ij}^{\alpha}(t_j^{\beta}) + t \cdot \theta_{ij} \tag{5.21}$$

$$y_i = h_{ij}(1 + t\beta_{ij})y_j \, . \tag{5.22}$$

Setting $\tilde{h}_{ij} = h_{ij}(1 + t\beta_{ij})$, we can verify the following equality by using (5.16) and (5.18).

$$\tilde{h}_{ij}(\varphi_{ij})\tilde{h}_{jk} = \tilde{h}_{ik} \text{ on } U_{ijk} \times S_1 \tag{5.23}$$

$$\tilde{H}_i(\varphi_{ij}) = \tilde{h}_{ij}^2 \cdot \tilde{H}_j \text{ on } U_{ijk} \times \mathbb{P}^1 \times S_1 \, . \tag{5.24}$$

Therefore, we can define a deformation $\tilde{\mathscr{V}} \longrightarrow S_1$ of V by patching $\tilde{\mathscr{V}}_i$ by the automorphisms φ_{ij} .

§ 6. Proof of Main Theorem.

6.1. In this section, we shall prove the following theorem which we mentioned in the Introduction.

Theorem 6.1. Let $\eta_1 : \mathscr{X} \longrightarrow S_1$ be the first order deformation of X corresponding to an element $\bar{\Phi} \in H^0(B,L_B)$ (see Proposition 4.4). Then this deformation $\eta_1 : \mathscr{X} \longrightarrow S_1$ can be extended to $\eta_2 : \tilde{\mathscr{X}} \longrightarrow S_2 = \mathrm{Spec}(\mathbb{C}[t]/t^3)$ if and only if

$$\underline{\mathrm{ob}}\,(\bar{\Phi}^2) = 0 \tag{6.1}$$

where the map $\underline{\mathrm{ob}}$ is defined as in (5.1).

Corollary 6.2. Let $\theta_{\bar{\Phi}} \in H^1(X,\Theta_X)$ be an element corresponding to $\bar{\Phi} \in H^0(B,L_B)$ (cf. Proposition 3.3). Then the primary obstruction $[\theta_{\bar{\Phi}},\theta_{\bar{\Phi}}]$ defined in $H^2(X,\Theta_X)$ lies in $H^2(X,\Theta_X)^+ = H^2(Y,\Theta_Y(-\log D))$ and we have an equality

$$[\theta_{\bar{\Phi}},\theta_{\bar{\Phi}}] = c \cdot \underline{\mathrm{ob}}\,(\bar{\Phi}^2)$$

where c is a non–zero constant.

For the definition of the primary obstruction, see the book [13].

6.2. We first prove the "if" part of Theorem 6.1.

Let $\bar{\Phi} = \{\bar{\phi}_i\} \in H^0(B,L_B)$, $\phi_i \in \Gamma(U_i,\mathcal{O}_Y)$, $\xi_{ij} = h^{-1}_{ij}(\phi_i - h_{ij}\phi_j) = f_j a_{ij} + g_j b_{ij}$ be as in 4.2. Moreover we define A_{ij} by

(6.2) $$\phi_i + h_{ij}\phi_j = h_{ij}A_{ij}\,.$$

Then we have

(6.3) $$\phi_i^2 - h_{ij}^2\phi_j^2 = h_{ij}^2\,\delta_{ij}A_{ij}$$

$$= h_{ij}^2(f_ja_{ij}A_{ij} + g_jb_{ij}A_{ij})\,.$$

Lemma 6.3. The following conditions are equivalent to each other.

(i) $\underline{\mathrm{ob}}\,(\overline{\phi}^{\,2}) = 0$.

(ii) There exists $(\theta,\beta) = (\{\theta_{ij}\},\{\beta_{ij}\}) \in H^1(Y.\Theta_Y) \times C^1(\mathcal{U},\mathcal{O}_Y)$ satisfying (5.16) and $\{a_i\}$, $\{b_i\} \in C^0(\mathcal{U},\mathcal{O}_Y)$ such that

(6.4) $$-\theta_{ij}(f_ig_i) + 2\beta_{ij}f_ig_i = (\phi_i^{\,2} + f_ia_i + g_ib_i) - h_{ij}^2(\phi_j^2 + f_ja_j + g_jb_j)\,.$$

Proof: This follows from Proposition 5.5.

Now assume that $\underline{\mathrm{ob}}\,(\overline{\phi}^{\,2}) = 0$ and choose $(\theta,\beta) \in H^1(Y,\Theta_Y) \times C^1(\mathcal{U},\mathcal{O}_Y)$ and $\{a_i\}$, $\{b_i\}$ as in Lemma 6.3.

From (6.3), we can see that (6.4) is equivalent to

(6.5) $$f_i\{\theta_{ij}(g_i) + g_{ij}a_{ij}A_{ij} + (a_i - g_{ij}a_j) - \beta_{ij}g_{ij}\}$$

$$+ g_i\{\theta_{ij}(f_i) + f_{ij}b_{ij}A_{ij} + (b_i - f_{ij}b_j) - \beta_{ij}f_i\} = 0\,.$$

Since f_i and g_i are coprime, we can get $c_{ij} \in \Gamma(U_{ij}, \Theta_Y)$ satisfying that

$$\begin{cases} (6.6) \quad \theta_{ij}(g_i) + g_{ij}a_{ij}A_{ij} + (a_{ij} - g_{ij}a_j) - \beta_{ij}g_i = g_i c_{ij} \\ \\ (6.7) \quad \theta_{ij}(f_i) + f_{ij}b_{ij}A_{ij} + (b_i - f_{ij}b_j) - \beta_{ij}f_i = -f_i c_{ij} . \end{cases}$$

As we see in 5.4, we can construct a first order deformation $\mathcal{V} \longrightarrow S_1 = \mathrm{Spec}\,(\mathbb{C}[s]/s^2)$ of V corresponding to $\overline{\Phi}^2$satisfying that $\mathrm{ob}(\overline{\Phi}^2) = 0$. Set

$$(6.8) \qquad \tilde{H}_i = y_i^2 - f_i g_i - s(\phi_i^2 + f_i a_i + g_i b_i)$$

$$= y_i^2 - (f_i + sb_i)(g_i + sa_i) - s\phi_i^2 ,$$

and define the hypersurface

$$(6.9) \qquad \tilde{\mathcal{V}}_i = \{\tilde{H}_i = 0\} \subset U_i \times \mathbb{P}^1 \times S_1 .$$

Moreover let $\varphi_{ij} : U_{ij} \times \mathbb{P}^1 \times S^1 \longrightarrow U_{ji} \times \mathbb{P}^1 \times S_1$ denote the automorphism defined in (5.21) and (5.22). Then $\{ \tilde{\mathcal{V}}_i \}$ are patched together by automorphisms φ_{ij} . We denote the corresponding deformation by

$$(6.10) \qquad \tilde{\mathcal{V}} \longrightarrow S_1 = \mathrm{Spec}\,\mathbb{C}[s]/s^2 .$$

The following lemma implies the "if" part of Theorem 6.1.

Lemma 6.4. Let $\overline{\phi} \in H^0(B,L_B)$ be an element satisfying that $\mathrm{ob}(\overline{\phi}^{\,2}) = 0$ and $\tilde{\mathscr{V}} \longrightarrow S_1 = \mathrm{Spec}\,\mathbb{C}[s]/s^2$ the first order deformation defined in (6.10). Let $\tilde{\mathscr{V}}' \longrightarrow S_2 = \mathrm{Spec}\,\mathbb{C}[t]/t^3$ be the deformation induced from (6.10) by the base extension $\mathbb{C}[s]/s^2 \longrightarrow \mathbb{C}[t]/t^3$, $s \longrightarrow t^2$. Then $\tilde{\mathscr{V}}' \longrightarrow S_2$ can be simultaneously resolved, that is, we obtain a deformation $\tilde{\mathscr{X}} \longrightarrow S_2$ of X and a morphism $\tilde{\mathscr{X}} \longrightarrow \tilde{\mathscr{V}}'$. This deformation $\tilde{\mathscr{X}} \longrightarrow S_2$ is an extension of $\mathscr{X} \longrightarrow S_1$ defined in Proposition 4.4.

Proof: Setting $s = t^2$ in (6.8) and (6.9), we obtain

$$\tilde{H}'_i = y_i^2 - (f_i + t^2 b_i)(g_i + t^2 a_i) - t^2 \phi_i^2 \tag{6.11}$$

$$\tilde{\mathscr{V}}'_i = \{\tilde{H}'_i = 0\} \subset U_i \times \mathbb{P}^1 \times S_2 \tag{6.12}$$

and $\varphi_{ij} : U_{ij} \times \mathbb{P}^1 \times S_2 \longrightarrow U_{ji} \times \mathbb{P}^1 \times S_2$.

We also define the equation by

$$G_i^{(2)} = (f_i + t^2 b_i)x_i^2 - 2t\phi_i x_i z_i - (g_i + t^2 a_i) z_i^2 . \tag{6.13}$$

Moreover, setting $u_i = x_i/z_i = \frac{1}{v_i}$, we can write (6.13) as

$$G_{i,1} = (f_i + t^2 b_i)u_i^2 - 2t\phi_i u_i - (g_i + t^2 a_i)\,, \quad (z_i \neq 0) \tag{6.14}$$

(6.15) $$G_{i,2} = (f_i + t^2 b_i) - 2t\phi_i v_i - (g_i + t^2 a_i)v_i^2, \quad (x_i \neq 0).$$

Moreover we define

(6.16) $$\tilde{\mathscr{X}}_i = \{G_i^{(2)} = 0\} \subset U_i \times \mathbb{P}^1 \times S_2 ,$$

(6.17) $$\tilde{\mathscr{X}}_i^1 = \{G_{i,1} = 0\} \subset U_i \times \mathbb{C}(u_i) \times S_2 ,$$

and

(6.18) $$\tilde{\mathscr{X}}_i^2 = \{G_{i,2} = 0\} \subset U_i \times \mathbb{C}(v_i) \times S_2 .$$

Note that $\tilde{\mathscr{X}}_i = \tilde{\mathscr{X}}_i^1 \cup \tilde{\mathscr{X}}_i^2$ and $\tilde{\mathscr{X}}_i$ is smooth. For each i, we can define the morphism $\eta_i : \tilde{\mathscr{X}}_i \longrightarrow \tilde{\mathscr{V}}_i'$ by

(6.19) $$y_i = \begin{cases} (f_i + t^2 b_i)u_i - t\phi_i & \text{on } \tilde{\mathscr{X}}_i^1 \\ \\ (g_i + t^2 a_i)v_i + t\phi_i & \text{on } \tilde{\mathscr{X}}_i^2 . \end{cases}$$

This gives the local resolution of $\tilde{\mathscr{V}}_i'$.

Next we show that the isomorphism φ_{ij} lifts to ϕ_{ij} satisfying that

$$\begin{tikzcd} \tilde{\mathscr{X}}_{ij} \arrow[r, "\phi_{ij}"] \arrow[d] & \tilde{\mathscr{X}}_{ji} \arrow[d] \\ \tilde{\mathscr{V}}_{ij}' \arrow[r, "\varphi_{ij}"] & \tilde{\mathscr{V}}_{ji}' \end{tikzcd} .$$

By using (6.19), we have for each i

(6.20) $$u_i = \frac{y_i + t\phi_i}{(f_i + t^2 b_i)} .$$

This equality shows that the automorphism φ_{ij} induces a birational map $\phi_{ij} : \tilde{\mathscr{X}}_{ij} \dashrightarrow \tilde{\mathscr{X}}_{ji}$ and by using (5.21), (5.22), (6.7), (6.14), after a long but straightforward calculation, we can show that ϕ_{ij} can be written as

(6.21) $$u_i = f_{ij}^{-1}\, h_{ij}(1 + t^2\beta_{ij})\{(1 + t^2 R_{ij})u_j + t(a_{ij} + b_{ij}u_j^2) + t^2 b_{ij}^2 u_j^3\}$$

where $R_{ij} = c_{ij} - \beta_{ij} + a_{ij}b_{ij}$. Thus ϕ_{ij} gives the isomorphism $: \tilde{\mathscr{X}}_{ij} \xrightarrow{\sim} \tilde{\mathscr{X}}_{ji}$. (For the coordinate v_j, the argument is similar.)

Now let us consider the following automorphism on $\tilde{\mathscr{X}}_{ijk}$;

$$\gamma_{ijk} = (\phi_{ij} \cdot \phi_{jk} \circ \phi_{ik}^{-1}) .$$

Since $\tilde{\mathscr{X}}_i$ and ϕ_{ij} are extensions of $\mathscr{X}_i$ and $\tilde{\varphi}_{ij}$ in 4.3 to S_2, we can write as $\gamma_{ijk} = \mathrm{id} + t^2\xi_{ijk}$ where $\xi_{ijk} \in \Gamma(X_{ijk}, \Theta_X)$. Since $(\eta_i)_*(\gamma_{ijk}) = (\varphi_{ij} \circ \varphi_{jk} \circ \varphi^{-1}_{ik}) = \mathrm{id}$, we have $r_*\xi_{ijk} = 0$ in $\Gamma(V_{ijk}, r_*\Theta_X)$. By the equality $r_*\Theta_X = \Theta_V$ (Lemma 3.1), this implies that $\xi_{ijk} = 0$, and thus $\gamma_{ijk} = \mathrm{id}$. Therefore $\{\tilde{\mathscr{X}}_i\}$ together with isomorphism $\{\phi_{ij}\}$ gives the deformation $\tilde{\mathscr{X}} \longrightarrow S_2$ of X which is an extension of $\mathscr{X} \longrightarrow S_1$ in Proposition 4.4. q.e.d.

<u>6.3</u>. Next we prove the "only if" part of Theorem 6.1. Let $\mathscr{X} \longrightarrow S_1$ be the first order deformation corresponding to $\overline{\Phi} \in H^0(B,L_B)$ as we defined in 4.3. Assume that there exists an extension $\tilde{\mathscr{X}} \longrightarrow S_2$ of $\mathscr{X} \longrightarrow S_1$. By Proposition 3.6, we have a deformation $\tilde{\eta} : \tilde{\mathscr{V}} \longrightarrow S_2$ of V and a morphism $\tilde{u} : \tilde{\mathscr{X}} \longrightarrow \tilde{\mathscr{V}}$ which make the following diagram commutative:

(6.22)
$$\begin{tikzcd} \tilde{\mathscr{X}} \arrow[r, "\tilde{u}"] \arrow[dr] & \tilde{\mathscr{V}} \arrow[d, "\tilde{\eta}"] \\ & S_2 \end{tikzcd}$$

This gives an extension of the following diagram:

(6.23)
$$\begin{tikzcd} \mathscr{X} \arrow[r, "u"] \arrow[dr] & \tilde{\mathscr{V}} \times_{S_2} S_1 = \mathscr{V} \arrow[d, "\eta"] \\ & S_1 \end{tikzcd}$$

Let $\mathscr{X}_i \longrightarrow \mathscr{V}_i$ denote the blow down morphism $u_{|\mathscr{X}_i}$. This is written by the local coordinate in 4.3 as

(6.24)
$$y_i = u(u_i) = f_i u_i - t\phi_i \; .$$

The image $\mathcal{V}_i$, which is a deformation of V_i over S_1, is the hypersurface $U_i \times \mathbb{P}^1 \times S_1$ defined by

$$(6.25) \qquad y_i^2 - f_i g_i - t^2\phi_i^2 = y_i^2 - f_i g_i = 0$$

Thus $\mathcal{V}_i \longrightarrow S_1$ is isomorphic to $V_i \times S_1 \longrightarrow S_1$. Since $\tilde{u}_{|\mathscr{X}_i}$ is an extension of $u_{|\mathscr{X}_i}$, $\tilde{\mathcal{V}}_i$ is isomorphic to a hypersurface in $U_i \times \mathbb{P}^1 \times S_2$ defined by

$$(6.26) \qquad y_i^2 - f_i g_i - t^2\phi_i^2 = 0 \qquad (\mathrm{mod}\ f_{i,} g_i)\,.$$

Moreover let $\eta_{ij} : \tilde{\mathcal{V}}_{ij} \longrightarrow \tilde{\mathcal{V}}_{ji}$ denote the patching isomorphism of $\tilde{\mathcal{V}} \longrightarrow S_2$. By the commutative diagram (3.1) in Proposition 3.2, we can see that $\mathscr{X} \longrightarrow S_1$ can be blown down to the trivial deformation $V \times S_1 \longrightarrow S_1$ since $\bar{\phi} \in H^0(B,L_B)$ corresponds to an element $H_E^1(X,\Theta_X)$. Thus the deformation $\eta : \tilde{\mathcal{V}} \times_{S_2} S_1 \longrightarrow S_1$ is isomorphic to the trivial deformation and this implies that we can write η_{ij} by

$$(6.27) \qquad \eta_{ij} = \eta_{ij}^{\circ} + t^2\rho_{ij}$$

where $\eta_{ij}^{\circ} : V_{ij} \longrightarrow V_{ji}$ denote the patching isomorphism for V and $\rho_{ij} \in \Gamma(V_{ij},\Theta_V)$. If we put $t^2 = s$ in (6.26) and (6.27), we obtain

$$(6.28) \qquad \tilde{\mathcal{V}}'_i = \{y_i^2 - f_i g_i - s\phi_i^2 = 0\} \longrightarrow S_1 = \mathrm{Spec}\ \mathbb{C}[s]/s^2$$

$$(6.29) \qquad \eta'_{ij} = \eta_{ij}^0 + s\rho_{ij} : \tilde{\mathcal{V}}'_{ij} \longrightarrow \tilde{\mathcal{V}}'_{ji}\,.$$

After we modify $\tilde{\mathscr{V}}_i'$ and η_{ij}', we get the first order deformation

$$\mathscr{V}' \longrightarrow S_1 = \mathrm{Spec}\, \mathbb{C}[s]/s^2$$

which corresponds to $\overline{\phi}^{\,2} \in H^0(B, L_B^2)$ and this implies that $\underline{\mathrm{ob}}(\overline{\phi}^{\,2}) = 0$. q.e.d.

§ 7. More analysis for the map $\underline{ob}$.

$\underline{7.1}$. For the general compact complex space Z , the first obstruction map $\underline{ob} : H^0(Z, \mathcal{T}^1_Z) \longrightarrow H^2(Z,\Theta_Z)$ is not easily computed. For a surface with rational double points, the dual of the map $\underline{ob}$ is easily computed by the natural exact sequence

$$0 \longrightarrow \Omega^1{}_Z(K_Z) \longrightarrow \Omega^1{}_{\tilde{Z}}(K_{\tilde{Z}}) \longrightarrow (\mathcal{T}^1_Z)^* \tag{7.1}$$

where $\tilde{Z}$ is the minimal resolution of Z (cf. [10], [2], [16] and [3]). Since our examples V in this paper have the good global structure (cf.. 2.1), the obstruction map is easily computed by some connected homomorphisms of cohomology groups.

Lemma 7.1. Let Y , L , and D_1 , D_2 be as in (2.1). Then we have the following commutative diagrams:

(7.2)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_Y \arrow[r, phantom, "\sim"] \arrow[d, "xf \cdot g"] & \mathcal{O}_Y \arrow[d, "xf \cdot g"] & & \\
0 \arrow[r] & I_B L^2 \arrow[r] \arrow[d] & L^2 \arrow[r] \arrow[d] & L^2_B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N_{D_1} \oplus N_{D_2} \arrow[r] \arrow[d] & N_{D_1+D_2} \arrow[r] \arrow[d] & L^2_B \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

.

(7.3)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_Y \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[d] & \\
0 \arrow[r] & \Theta_Y(-\log D) \arrow[r] \arrow[d, equal] & \Sigma_L \arrow[r] \arrow[d] & I_B L^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Theta_Y(-\log D) \arrow[r] & \Theta_Y \arrow[r] \arrow[d] & N_{D_1} \oplus N_{D_2} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

<u>Proof</u>. Except for the map $I_B L^2 \longrightarrow N_{D_1} \oplus N_{D_2}$, the definitions of the morphisms in (7.2) are obvious. The map $I_B L^2 \longrightarrow N_{D_1} \oplus N_{D_2}$ is locally given by $fa + gb \longrightarrow (b|_{D_1}, a|_{D_2})$. Then it is easy to see the commutativity of (7.2) and exactness of each row and column. The commutative diagram (7.3) follows from (5.6), (5.7), (7.2) and the standard exact sequence $0 \longrightarrow \Theta_Y(-\log D) \longrightarrow \Theta_Y \longrightarrow N_{D_1} \oplus N_{D_2} \longrightarrow 0$.

q.e.d.

From (7.2), we have the commutative diagram

(7.4)
$$\begin{tikzcd}
H^0(B, L_B^2) \arrow[r, "\delta_1^+"] \arrow[d, "\wr"] & H^1(Y, I_B L^2) \arrow[d, "\gamma_1"] \\
H^0(B, L_B^2) \arrow[r, "\eta_1^+"] & H^1(D_1, N_{D_1}) \oplus H^1(D_2, N_{D_2}) .
\end{tikzcd}$$

Moreover from (7.3), we also obtain the commutative diagram

$$(7.5)\qquad \begin{tikzcd} H^1(\Sigma_L) \arrow[r, "\mu"] \arrow[d] & H^1(Y, I_B L^2) \arrow[r, "\delta_2^+"] \arrow[d] & H^2(\Theta_Y(-\log D)) \arrow[d, equal] \\ H^1(\Theta_Y) \arrow[r, "\mu^1"] & H^1(N_{D_1}) \oplus H^1(N_{D_2}) \arrow[r, "\eta_2^+"] & H^2(\Theta_Y(-\log D)) \end{tikzcd}$$

(see, 5.2 and 5.3).

Then the following proposition follows from Proposition 5.3, (7.4) and (7.5).

<u>Proposition 7.2</u>. The obstruction map $\underline{ob} : H^0(B, L_B^2) \longrightarrow H^2(V, \Theta_V)$ is given by the composite map

$$\begin{tikzcd} H^0(B, L_B^2) \arrow[r, "\eta_1^+"] \arrow[ddr, "\underline{ob}"'] & H^1(D_1 N_{D_1}) \oplus H^1(D_2, N_{D_2}) \arrow[d] \\ & H^2(\Theta_Y(-\log D)) \arrow[d] \\ & H^2(V, \Theta_V) . \end{tikzcd}$$

We will next consider the map η_2^+ . The map μ^1:
$H^1(Y, \Theta_Y) \longrightarrow H^1(D_1, N_{D_1}) \oplus H^1(D_2, N_{D_2})$ is given by $\mu^1(\theta) = ((\theta \cdot f)_{|D_1},$ $(\theta \cdot g)_{|D_2})$. It is known that $(\theta \cdot f)_{|D_1}$ and $(\theta \cdot g)_{|D_2}$ are obstructions to the lifting of divisors D_1 and D_2 to the first order deformation of Y corresponding to $\theta \in H^1(Y, \Theta_Y)$. Moreover we can see that

$$\mathrm{Im}\, \eta_1^+ \cap \mathrm{Im}\, \mu^1 = \mathrm{Im}\, \eta_1^+ \cap \mathrm{Im}\, \mu^1(H^1(Y, \Theta_Y)_{C_1(L)})$$

where $H^1(Y,\Theta_Y)_{C_1(L)} = \{\theta \in H^1(Y,\Theta_Y) \mid \theta \cdot C_1(L) = 0\}$. This consideration with Proposition 7.2 yields the following

Proposition 7.3.

Assume that $\mu^1(H^1(Y,\Theta_Y)_{C_1(L)}) = 0$, that is, all elements $\theta \in H^1(Y,\Theta_Y)_{C_1(L)}$ preserve the divisors D_1 and D_2 . Then we have the following

(i) The map ob is non–zero map if and only if η_1^+ is non zero map.

(ii) The map ob is injective if and only if η_1^+ is injective .

§ 8 Examples of obstructed manifolds.

8.1. Let Z be a compact complex manifold and θ an element of $H^1(Z,\Theta_Z)$. Then θ is obstructed if there are no deformations $\mathfrak{X} \longrightarrow \Delta = \{t \in \mathbb{C}; |t| < \varepsilon\}$ of Z such that $\rho\left[\frac{\partial}{\partial t}\right] = \theta$ where ρ is the Kodaira–Spencer map. We say that a complex manifold Z is obstructed if it has an obstructed element $\theta \in H^1(Z,\Theta_Z)$. Moreover the followings are equivalent:

$$Z \text{ is obstructed} \longleftrightarrow \dim \mathrm{Def}_Z < \dim H^1(Z,\Theta_Z) .$$

An element $\theta \in H^1(Z,\Theta_Z)$ is obstructed if the primary obstruction $[\theta,\theta] \in H^2(Z,\Theta_Z)$ is not zero. (Kodaira [13]). In this section, we will show that by using Theorem 6.1 many examples of obstructed manifolds of dimension ≥ 2 can be constructed.As far as I know, examples of obstructed surfaces are given by Kas [10], Burns–Wahl [2], Catanese [3], Pinkham [16] and Horikawa [8]. Moreover Douady [4] and Kodaira–Spencer [14] showed that the products of complex torus and $\mathbb{P}^1$ are obstructed.

8.2. First examples. Our examples are compact complex manifolds X which are resolutions of V constructed from the quadruplet (Y,D_1,D_2,L) in 2.1. Let Y be a smooth projective variety of dimension $n \geq 2$ and L an ample line bundle, D_1, $D_2 \in |L|$ satisfying the conditions (2.1). Let X be a compact complex manifold defined in (2.5), that is, a resolution of the double cover V of Y branched along the normal crossing divisor $D_1 + D_2$.

We assume that:

(8.1) $\qquad H^2(Y,\mathcal{O}_Y) \neq 0$, (8.2) the cup product map $H^1(Y,\mathcal{O}_Y) \otimes H^1(Y,\mathcal{O}_Y) \longrightarrow H^2(Y,\mathcal{O}_Y)$ is non–trivial,

(8.3) $L \otimes K_Y^{-1}$ is ample.

Proposition 8.1. Under the conditions (8.1) – (8.3), the manifold X above is obstructed. In fact, there exists an element $\theta \in H^1(X,\Theta_X)$ whose primary obstruction $[\theta,\theta] \neq 0$.

Proof: First we assume that $n = \dim Y \geq 3$. Set $B = D_1 \cap D_2$. By Main Theorem 6.1, it suffices to show that there exists an element $\bar{\phi} \in H^0(B,L_B)$ such that $\underline{\mathrm{ob}}(\bar{\phi}^2) \neq 0$. From the exact sequences (cf. (4.16))

$$0 \longrightarrow L^{-1} \longrightarrow \mathcal{O}_Y \oplus \mathcal{O}_Y \longrightarrow I_B L \longrightarrow 0 \tag{8.4}$$

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow L \oplus L \longrightarrow I_B L^2 \longrightarrow 0 \tag{8.5}$$

and Kodaira vanishing theorem (L^{-1} and $K_Y \otimes L^{-1}$ negative) , we have

$$H^1(I_B L) \cong H^1(\mathcal{O}_Y) \oplus H^1(\mathcal{O}_Y) , \tag{8.6}$$

$$H^1(I_B L^2) \cong H^2(Y,\mathcal{O}_Y) . \tag{8.7}$$

Moreover, by a standard exact sequence and Kodaira vanishing theorem, we obtain the exact sequences

$$H^0(Y,L) \longrightarrow H^0(B,L_B) \longrightarrow H^1(I_B L) \longrightarrow 0 \tag{8.8}$$

$$H^0(Y,L^2) \longrightarrow H^0(B,L_B^2) \xrightarrow{\delta_1^+} H^1(I_BL^2) \longrightarrow 0 . \tag{8.9}$$

Take an element $\bar{\phi} \in H^0(B,L_B)$ and set $\gamma(\bar{\phi}) = (a,b) \in H^1(I_BL) \cong H^1(Y,\mathcal{O}_Y) \oplus H^1(Y,\mathcal{O}_Y)$. Then by an easy calculation, we can see that

$$\delta_1^+(\bar{\phi}^2) = 2(a \cup b) \in H^2(Y,\mathcal{O}_Y) \cong H^1(I_BL^2) \tag{8.10}$$

where $\cup$ denote the cup product $\cup : H^1(Y,\mathcal{O}_Y) \otimes H^1(Y,\mathcal{O}_Y) \longrightarrow H^2(Y,\mathcal{O}_Y)$. Thus from (8.6) and (8.7) with assumption (8.2), we obtain an element $\bar{\phi} \in H^0(B,L_B)$ such that $\delta_1^+(\bar{\phi}^2) \neq 0$. Again from Kodaira vanishing, theorem, we have $H^1(N_{D_i}) \cong H^2(Y,\mathcal{O}_Y)$ and thus from (7.4), we obtain the following commutative diagram.

(8.11)

$$\begin{array}{ccc}
H^0(B,L_B^2) & \xrightarrow{\delta_1^+} & H^2(Y,\mathcal{O}_Y) \cong H^1(I_BL^2) \\
\downarrow \wr & & \downarrow \gamma_1 \\
H^0(B,L_B^2) & \xrightarrow{\eta_1^+} & H^1(N_{D_1}) \oplus H^1(N_{D_2}) \\
& & \| \wr \\
& & H^2(\mathcal{O}_Y) \oplus H^2(\mathcal{O}_Y)
\end{array}$$

It is easy to see that γ_1 is injective. Therefore if $\delta_1^+(\bar{\phi}^2) \neq 0$, then $\eta_1^+(\bar{\phi}^2) \neq 0$. Moreover the natural map $\mu^1 : H^1(Y,\Theta_Y)_{C_1(L)} \longrightarrow \bigoplus_{i=1}^{2} H^1(D_i,N_{D_i}) = H^1(Y,\mathcal{O}_Y)^{\oplus 2}$ defined in (7.5) is zero map by definition. Hence $\underline{ob}(\bar{\phi}^2) \neq 0$ if $\eta_1^+(\bar{\phi}^2) \neq 0$ by Proposition 7.3 and this completes the proof for the case of $\dim Y = n \geq 3$. Even if $\dim Y = 2$, (8.7), (8.9) and (8.11) remain to be true. Since B is a set of finite of points,

$S^2H^0(B,L_B) \longrightarrow H^0(B,L_B^2)$ clearly surjective. Moreover it follows from (8.11) that the map η_1^+ is non–trivial. Therefore the assertion again follows from Proposition 7.3. q.e.d.

Remark 8.2. A typical example of Y and L satisfying (8.1) ~ (8.3) is an abelian variety and its ample line bundle. In this case, our example X are closely related to the example of Douady and Kodaira–Spencer (cf. [4], [14]). In fact, X can be embedded as a hypersurface into the product $\mathbb{P}^1 \times Y$.

8.3. Second examples. Next we will give examples for which the obstruction map ob is injective. (cf. § 5, § 7).

Let W be a smooth projective variety of dimension $n-1 \geq 1$ and C a curve of genus g. Let D_1' be a smooth ample effective divisor on W divisible in Pic(W) by 2 and D_2' an efficitive divisor on C with degree $2d$ without multiple points.

We set $Y = W \times C$ and $D_1 = P_1^*(D_1')$, $D_2 = P_2^*(D_2')$ where P_i denote the projection to the i–th factor. We take a line bundle L such that $L^2 \cong \mathcal{O}_Y(D_1 + D_2)$.

Now we assume that:

$$H^0(W,\Theta_W) = H^0(C,\Theta_C) = 0, \text{ (in particular } g(C) \geq 2), \tag{8.12}$$

$$H^1(W,D_1') = H^2(W,\mathcal{O}_Y) = 0 \text{ and } \dim H^0(W,D_1') \geq 2. \tag{8.13}$$

By assumption (8.12), we have the isomorphism

$$H^1(Y,\Theta_Y) \cong H^1(W,\Theta_W) \oplus H^1(C,\Theta_C). \tag{8.14}$$

Lemma 8.2. The natural map $\mu^1 : H^1(Y.\Theta_Y)_{C_1(L)} \longrightarrow \overset{2}{\underset{i=1}{\oplus}} H^1(N_{D_i})$ is zero map.

Proof. Since $D_1 = D_1' \times C$ and $D_2 = W \times D_2'$, we have the following isomorphisms

(8.14) $$H^1(D_1,N_{D_1}) \cong H^1(D_1',N_{D_1'}) \oplus H^0(D_1',N_{D_1'}) \otimes H^1(C,\mathcal{O}_C),$$

(8.15) $$H^1(D_2,N_{D_2}) \cong H^1(D_2',N_{D_2'}) \oplus H^2(D_2',N_{D_2'}) \otimes H^1(W,\mathcal{O}_W).$$

Then μ^1 is decomposed into the following maps

(8.16) $$H^1(W,\Theta_W) \longrightarrow H^1(N_{D_1'})$$

(8.17) $$H^1(C,\Theta_C) \longrightarrow H^1(N_{D_2'}).$$

Since C is a curve, $H^1(N_{D_2'}) = 0$ and by (8.13) we have $H^1(N_{D_1'}) = 0$. These imply that $\mu^1 = 0$. q.e.d.

From this lemma, the obstruction map ob is injective (resp. non–trivial) if the map η_1^+ in (7.4) is injective (resp. non–trivial). Moreover we can prove the following

Lemma 8.3. Under the above assumptions, we have the followings:

(i) The map η_1^+ is always non–trivial.

(ii) The map η_1^+ is injective if $2d = \deg D_2' \leq g$ and D_2' is general or more precisely if $\dim H^0(C,D_2') = 1$.

Proof. From the commutative diagram (7.2), we get

(8.18)

$$\begin{tikzcd}
 & 0 \arrow[r] & H^0(I_B L^2) \arrow[r] \arrow[d] & H^0(L^2) \arrow[r, "\tau"] \arrow[d] & H^0(B, L_B^2) \arrow[r, "\delta_1^+"] \arrow[d, equal] & {} \\
0 \arrow[r] & H^0(N_{D_1}) \oplus H^0(N_{D_2}) \arrow[rr] \arrow[d] & & H^0(N_{D_1+D_2}) \arrow[r, "\tau'"] \arrow[d] & H^0(B, L_B^2) \arrow[r, "\eta_1^+"] & {} . \\
 & H^1(Y, \mathcal{O}_Y) \arrow[rr, equal] & & H^1(Y, \mathcal{O}_Y) & &
\end{tikzcd}$$

It follows from this diagram that $\operatorname{Im} \tau = \operatorname{im} \tau'$.

Since B is isomorphic to $2d$ copies of $D_1' \subset W$ and $L^2 \otimes D_1' \cong N_{D_1'}$. Thus

(8.19) $$\dim H^0(B,L_B^2) = 2d \times \dim H^0(D_1',N_{D_1'}) \geq 2d \times (\dim H^0(W,D_1') - 1).$$

On the other hand, we can easily see that

(8.20) $$\dim H^0(Y,L^2) = \dim H^0(W,D_1') \times \dim H^0(C,D_2').$$

Moreover an exact sequence

(8.21) $$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(D_1) \oplus \mathcal{O}(D_2) \longrightarrow I_B L^2 \longrightarrow 0$$

implies that

$$\dim H^0(I_B L^2) \geq \dim H^0(W, D_1') + \dim H^0(C, D_2') - 1 . \tag{8.22}$$

From (8.20) and (8.22), it follows that

$$\dim(\mathrm{Im}\ \tau) \leq (\dim H^0(W, D_1') - 1)(\dim H^0(C, D_2') - 1) . \tag{8.23}$$

From this inequality, the assertion (ii) follows. If D_2' is not a special divisor on C, by Riemann–Roch, $\dim H^0(C, D_2') - 1 = 2d - g < 2d$. If D_2' is a special divisor, by Clifford's theorem, $\dim H^0(C, D_2') - 1 \leq d < 2d$. Thus by (8.19) and (8.23) with assumption that $\dim H^0(W, D_1') \geq 2$, we have

$$\dim(\mathrm{Im}\ \tau) < \dim H^0(B, L_B^2)$$

which implies the assertion (i). q.e.d.

Let (Y, D_1, D_2, L) be a quadruplet as above. From Lemma 8.2 and Lemma 8.3, we have the following theorem.

<u>Theorem 8.4</u>. Let X be the manifold defined in § 2 from (Y, D_1, D_2, L). Under the assumptions (8.12) and (8.13), we have the followings:

(i) X is always obstructed.

(ii) If degree of $D_2' = 2d \leq g(C)$ and D_2' is general, or more precisely if $\dim H^0(C, D_2') = 1$, all the elements in $H^1_E(\Theta_X) \cong H^1(N_E)$ are obstructed.

<u>Remark 8.5</u>. In case (ii) of Theorem 8.4, under some suitable conditions on W, we can prove that the Kuraniski space X is non–reduced. We will discuss this topic in the future.

<u>Remark 8.6</u>. If W is also a curve, the above examples are given by Kas [10]. Moreover Catanese generalizes the example to surfaces which have A_n singularities in [3].

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