Conference on
Arithmetical Geometry
Berlin, March 21-26, 1996

Max-Planck-Arbeitsgruppe
Algebraische Geometrie und
Zahlentheorie
Jägerstraße 10-11
D-10117 Berlin

Humboldt-Universität zu
Berlin
Institut für Mathematik
Unter den Linden 6
D-10099 Berlin

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Arithmetical Geometry

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The conference on "Arithmetical Geometry" from March 21 to 26, 1996, has been jointly organized by the Max-Planck-Arbeitsgruppe "Algebraische Geometrie und Zahlentheorie" in Berlin and the Department of Mathematics at Humboldt-University in Berlin.

H. Koch J. Kramer E.-W. Zink

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List of participants of the conference „Arithmetical Geometry"

Name	University
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D. Benois	Universität Essen
J.-M. Bismut	Université Paris Sud, Orsay
J.-B. Bost	IHES, Bures-sur-Yvette
J. Brüning	Humboldt-Universität zu Berlin
N. Dan	Université Paris XIII
B. Edixhoven	Université de Rennes 1
R. Erné	Université de Rennes 1
W. Fischer	Universität Regensburg
G. Frey	Universität Essen
C. Gasbarri	University of Oxford
H. Gillet	University of Illinois at Chicago
S. Günther	Humboldt-Universität zu Berlin
H. Grassmann	Humboldt-Universität zu Berlin
W. Gubler	Eidgenössische Technische Hochschule Zürich
C. Hadan	Humboldt-Universität zu Berlin
G. Harder	MPIM und Universität Bonn
G. Hein	Humboldt-Universität zu Berlin
V. Heiermann	Humboldt-Universität zu Berlin
R.-P. Holzapfel	Humboldt-Universität zu Berlin
R. Hübl	Universität Regensburg
J. Jahnel	Universität Göttingen
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S. Vostokov	University of St. Petersburg
G. Wüstholz	Eidgenössische Technische Hochschule Zürich
Y. Zarhin	Pennsylvania State University
E.-W. Zink	Humboldt-Universität zu Berlin

Program of the Conference „Arithmetical Geometry"

The conference takes place at Humboldt-University,
Unter den Linden 6, room 2097 (look for directions)

Thursday, 21.03.

09.00 - 09.15 h	Opening of the conference
09.15 - 10.10 h	G. Wüstholz: Transcendence and Arakelov Geometry
10.10 - 10.40 h	Break (coffee in front of room 3038, 3rd floor)
10.40 - 11.35 h	F. Oort: Hyperelliptic curves in abelian varieties
11.50 - 12.45 h	J. Nekovář: *p*-adic regulators and filtrations on Chow groups
15.00 - 15.55 h	K. Künnemann: Higher Picard varieties and the height pairing
16.00 - 16.55 h	J.-B. Bost: Arakelov geometry of abelian varieties
16.55 - 17.30 h	Break (coffee in front of room 3038, 3rd floor)
17.30 - 18.25 h	J. Tschinkel: Zeta functions in arithmetic geometry

Friday, 22.03.

09.15 - 10.10 h	A. Abbès: Self-intersection of the dualizing sheaf on modular curves $X_0(N)$
10.10 - 10.40 h	Break (coffee in front of room 3038, 3rd floor)
10.40 - 11.35 h	C. Soulé: Arithmetic Betti numbers
11.50 - 12.45 h	J. Jorgenson: The arithmetic degree of line bundles on abelian varieties
15.00 - 15.55 h	Y. Zarkhin: Reduction and torsion subgroups of abelian varieties
16.00 - 16.55 h	H. Gillet: Motivic weight complexes
16.55 - 17.30 h	Break (coffee in front of room 3038, 3rd floor)
17.30 - 18.25 h	E. Ullmo: Equidistribution of small points

Saturday, 23.03.

09.15 - 10.10 h	J. Schwermer: A decomposition of spaces of automorphic forms and some applications
10.10 - 10.40 h	Break (coffee in front of room 3038, 3rd floor)
10.40 - 11.35 h	G. Frey: Elliptic curves with isomorphic torsion structure
11.50 - 12.45 h	C.-G. Schmidt: p-adic automorphic L-functions
15.00 - 15.55 h	W. Kohnen: Hecke eigenvalues and Fourier coefficients
16.00 - 16.55 h	J. Oesterlé: Torsion of elliptic curves
16.55 - 17.30 h	Break
17.30 - 18.25 h	E. Kani: Applications of diagonal quotient surfaces to a problem of Mazur

Sunday, 24.03.

Excursion: Walk from Schloß Kleinglienecke to Pfaueninsel and back to Wannsee.

Monday, 25.03.

09.15 - 10.10 h	D. Roessler: Lambda structures in K-theory
10.10 - 10.40 h	Break (coffee in front of room 3038, 3rd floor)
10.40 - 11.35 h	J.-M. Bismut: Higher analytic torsion forms in real and complex geometry.
11.50 - 12.45 h	J. Rohlfs: An arithmetic formula for a topological invariant of Siegel modular varieties
15.00 - 15.55 h	W. Gubler: Local heights of subvarities and formal geometry
16.00 - 16.55 h	K. Murty: Non-vanishing of automorphic L-functions
16.55 - 17.30 h	Break (coffee in front of room 3038, 3rd floor)
17.30 - 18.25 h	E. de Shalit: Special values of theta functions with complex multiplication

Tuesday, 26.03.

09.15 - 10.10 h	J. Jahnel: Heights for line bundles on arithmetic varieties
10.10 - 10.40 h	Break (coffee in front of room 3038, 3rd floor)
10.40 - 11.35 h	L. Szpiro: Bounds for the order of the Tate-Shafarevich group
11.50 - 12.45 h	B. Edixhoven: Oort's conjecture for $\mathbf{P}^1 \times \mathbf{P}^1$
14.00 - 14.55 h	B. Venkov: Modular forms and construction of unimodular lattices
15.00 - 15.55 h	G. Harder: Canonical flags on arithmetic Chevalley schemes

Title: Transcendence and arithmetic geometry

Author: G. Wüstholz

Address: ETH Zürich

In the talk we discussed three topics:

1) Projective presentations

2) Differential operators and local geometry

3) Metrics and distances

In the first part of the talk we presented a method basing on work of Lange and Ruppert how to describe projective morphisms by homogeneous polynomials. It depends on the study of the sheaf $\alpha^{*}M^{-1}\otimes L$ where L is a very ample invertible sheaf on the projective variety X, M a very ample invertible sheaf on the projective variety Y and $\alpha : X \to Y$ is a morphism. The method is applied to give addition formulae for group varieties.

The second part of the talk on differential operators presented arithmetical properties of the coefficients of higher order differential operators when evaluated at a point. This is used in the theory of diophantine approximations and in particular in connection with Faltings' product theorem.

In the last part of the talk we discussed the relation between translation invariant hermitian metrics on abelian varieties and and metrics induced from the Fubini-Study metric in projective space.

Title: Hyperelliptic curves in abelian varieties.

Author: Frans OORT (oort@math.ruu.nl) Page: 1

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This is joint work with A.J. de Jong

Theorem B: Let $g \geqslant 3$, and let X be a generic abelian variety of dimension g. Let $C \subset X$ be an (irreducible) algebraic curve. Then the normalization $C^{\sim}$ is not hyperelliptic.

This was proved in characteristic zero by P. Pirola (Duke Math. Journ. 59 (1989), 701–708).

In the proof we use a certain rigidity of hyperelliptic curves in abelian varieties. In characteristic zero, this rigidity holds, in positive characteristic not quite:

- $2 \leqslant g \leqslant 4$: $\exists$ examples of positive dimensional families of hyperelliptic curves in E^g, where E is a supersingular elliptic curve.
- Question: do such examples exist for $g \geqslant 5$?
- Question: what are the dimensions of a component of $j(\mathcal{H}_g \otimes \overline{\mathbb{F}}_p) \cap \mathcal{S}_g \hookrightarrow \mathcal{A}_{g,1} \otimes \overline{\mathbb{F}}_p$? Here: $\mathcal{H}_g$ hyperelliptic locus, $\mathcal{S}_g$: supersingular locus (dim $= [\frac{g^2}{4}]$).

Hyperelliptic curves can only be deformed non-trivially (up to translation) inside a supersingular abelian variety (ss: $\sim E^g$, with E ss ell.c.).

In fact: <u>Theorem A</u>: <u>let X be an abelian variety</u>, $\mathcal{C}/T$ <u>a family of</u> <u>hyperelliptic</u> <u>curves over a curve</u> T, $\mathcal{C} \xleftarrow{w} T$ <u>a Weierstrass section</u>,

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "\psi"] \arrow[r, "\varphi"] \arrow[d, shift left] & X \times T \arrow[r, "p_1"] \arrow[dl] & X \\
T \arrow[u, bend left, "w"] & &
\end{tikzcd}$$

$Y \subset X$
smallest sub ab. var

$$\left.\begin{array}{l} \psi(w(T)) = 0 \\ \dim(\psi(\mathcal{C})) = 2 \\ \psi(\mathcal{C}) \subset Y \subset X \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \text{char} = p > 0 \ \& \\ Y \text{ is } \underline{\text{supersingular.}} \end{array}\right.$$

(the condition $\psi(w(T))$ excludes translations, the condition $\dim(\psi(\mathcal{C})) = 2$ says we have a non-trivial family of hyperelliptic curves in X).

In the proof of Th. A we show: if $S := \overline{\psi(\mathcal{C})}$ is a surface, the image of $H^2(X) \to H^2(S) := H^2_{et}(S \otimes \bar{k}, \mathbb{Q}_\ell)$ consists of Tate classes (for an analogous idea, see C. Schoen, Journ. reine angew. Math. <u>411</u> (1990) pp. 196–204, See lemma 2.5 on page 200).

For the proof that Th. A $\Rightarrow$ Th. B, we choose an ordinary abelian variety B, with $\dim B = g-1$. Let $\Lambda = \bigcup_{\alpha \in A} \Lambda_\alpha = \mathcal{H}^{(p)}_g(B \times E)$ the separable Hecke orbit,

here $\mathcal{E}$ is a universal family of elliptic curves, i.e. a point $t \in \Lambda$ corresponds with a polarized AV, separably isogenous with $B \times E_t$ for some elliptic curve E_t. By a nice result of C.-L. Chai (see Invent. Math. 121 (1995), 439–479) we know that Λ is <u>dense</u> in the moduli space. Suppose $D = C^{\sim} \longrightarrow C \subset X$ is an hyperelliptic curve in a generic abelian variety. Extend D and X to $\mathcal{D} \longrightarrow \mathcal{X}$ over a Zariski open set in a fine moduli space. We see that $\mathcal{D}|_{\Lambda_\alpha}$ is not constant, by theorem A we see that $\mathcal{D}|_{\Lambda_\alpha} \longrightarrow G_\alpha \subset B$ has a constant image, $h = \text{genus}\, D > \text{genus}\, G_\alpha = h_\alpha \geq \dim B = g-1$. For $t \in \Lambda_\alpha$, $\mathcal{D}_t \to I \longrightarrow G_\alpha$ factors in purely insep and separable morphism. We show: $\deg(\mathcal{D}_t \to I)$ is bounded, $\deg(I \longrightarrow G_\alpha) < \frac{2h-2}{2g-4} \leq h-1$, $\therefore \deg(\mathcal{D}_t \longrightarrow G_\alpha)$ is bounded. By the "Mori trick" (existence of Hilbert schemes) we see that ∞ many $G_\alpha \subset B$ extend to some $G \to X$, genus$(G) <$ genus(C), and we obtain a contradiction.
(choose $C \hookrightarrow X$, hyperelliptic, minimal $h = \text{genus}(C)$) $\square$

Frans & Johan de Jong — Hyperelliptic curves in abelian varieties
Preprint #950, Dept. Math. Univ. Utrecht, March 1996.

Talk given on 21-III-1996, the 311th birthday of JSB.
S. D. G.

Title: p-adic regulators and filtrations on Chow groups
Author: JAN NEKOVÁŘ
Page: 1
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§ 1. Motivic motivation

Let $F/\mathbb{Q}$ be a finitely generated extension, $d = \mathrm{tr.deg.}(F/\mathbb{Q})$ and X a smooth projective variety over F. Write $CH^*(X)$ for the Chow groups of X.

<u>Conjecture</u> (Bloch, Beilinson) There is a canonical filtration $CH^n(X)_{\mathbb{Q}} \supset F^1 CH^n(X)_{\mathbb{Q}} \supset \cdots \supset F^{d+2} CH^n(X)_{\mathbb{Q}} = 0$, compatible with correspondences and such that $F^i . F^j \subseteq F^{i+j}$, $F^1 =$ (homologically trivial cycles), $F^2 =$ kernel of the Abel-Jacobi map.

<u>Motivic dream</u> (underlying the conjecture): There is a morphism of "motivic sites" $X_{mot} \xrightarrow{\pi^{mot}} \mathrm{Spec}(F)_{mot}$ such that the corresponding Leray spectral sequence

$$(*) \quad E_2^{i,j} = H^i(\mathrm{Spec}(F)_{mot}, \underbrace{R^j \pi_*^{mot} \mathbb{Q}(n)}_{h^j(X)(n)}) \Longrightarrow H^{i+j}(X_{mot}, \mathbb{Q}(n))$$

degenerates at E_2, and $H^{2n}(X_{mot}, \mathbb{Q}(n))$ is canonically isomorphic to $CH^n(X)_{\mathbb{Q}}$. Hence the filtration $F^{\cdot}$ is induced by $(*)$ and

$$gr_F^i \, CH^n(X)_{\mathbb{Q}} \simeq H^i(\mathrm{Spec}(F)_{mot}, h^{2n-i}(X)(n))$$

It is also expected that $H^i(\mathrm{Spec}(F)_{mot}, -) = 0$ for $i > d+1$.

§ 2. p-adic étale realizations

Fix a prime p. The Leray (= Hochschild-Serre) spectral sequence for $X_{et} \xrightarrow{\pi^{et}} \mathrm{Spec}(F)_{et}$

$$E_2^{i,j} = \underbrace{H^i(\mathrm{Spec}(F)_{et}, R^j\pi^{et}_* \mathbb{Q}_p(n))}_{H^i(G_F,\, H^j(\overline{X}_{et}, \mathbb{Q}_p(n)))} \Longrightarrow H^{i+j}(X_{et}, \mathbb{Q}_p(n))$$

($G_F = G(\overline{F}/F)$, $\overline{X} = X \otimes_F \overline{F}$ and we use continuous cohomology) degenerates at E_2 and defines a filtration $\underline{F^\cdot H^m(X_{et}, \mathbb{Q}_p(n))}$ with

$$gr^i_F\, H^m(X_{et}, \mathbb{Q}_p(n)) = H^i(G_F, H^{m-i}(\overline{X}_{et}, \mathbb{Q}_p(n)))$$

The <u>cycle class</u> map

$$cl_{X,n} : CH^n(X)_{\mathbb{Q}} \longrightarrow H^{2n}(X_{et}, \mathbb{Q}_p(n))$$

is expected to behave as follows:

<u>Conjecture</u>: (1) $\mathrm{Ker}(cl_X) = 0$

(2) $cl_X(F^i CH^n(X)_{\mathbb{Q}}) \subseteq F^i H^{2n}(X_{et}, \mathbb{Q}_p(n))$

(3) $F^i CH^n(X)_{\mathbb{Q}} = cl_X^{-1}(F^i H^{2n}(X_{et}, \mathbb{Q}_p(n)))$

In particular, cl_X should induce <u>injective</u> ("higher Abel-Jacobi") maps

$$gr^i_F\, CH^n(X)_{\mathbb{Q}} \hookrightarrow H^i(G_F, H^{2n-i}(X_{et}, \mathbb{Q}_p(n)))$$

and $$\mathrm{Im}(cl_X) \cap F^{d+2} H^{2n}(X_{et}, \mathbb{Q}_p(n)) \overset{?}{=} 0.$$

<u>Number field case</u>: $[F:\mathbb{Q}] < \infty$, $d = 0$

$$\mathrm{Im}(cl_X) \cap F^2 H^{2n}(X_{et}, \mathbb{Q}_p(n)) \overset{?}{=} 0.$$

§3. Local case, $v \nmid p$

Let $[F:\mathbb{Q}]<\infty$. Fix a prime $v \nmid p\infty$ of F and let X_v be a smooth projective variety over F_v. The cycle class $cl_{X_v,n} : CH^n(X_v)_{\mathbb{Q}} \to H^{2n}((X_v)_{et}, \mathbb{Q}_p(n))$ should satisfy

<u>Conjecture</u>: $\mathrm{Im}(cl_{X_v,n}) \cap F^1 H^{2n}((X_v)_{et}, \mathbb{Q}_p(n)) = 0$.

<u>Known</u>:
- if X_v has a potentially good reduction
- for 0-cycles
- in a handful of special cases

If X_v has a regular model $\mathfrak{X}_v$, proper over $\mathrm{Spec}(\mathcal{O}_v)$ ($\mathcal{O}_v$ = integers of F_v) with special fibre Y_v, then there is a diagram

$$\begin{array}{ccccc}
CH^n(\mathfrak{X}_v)_{\mathbb{Q}} & \xrightarrow{cl_{\mathfrak{X}_v}} & H^{2n}_{et}(\mathfrak{X}_v, \mathbb{Q}_p(n)) & \xrightarrow[\sim]{\beta} & H^{2n}_{et}(\overline{Y}_v, \mathbb{Q}_p(n))^{G_{k(v)}} \\
\alpha\downarrow & & \downarrow & & \downarrow sp \\
CH^n(X_v)_{\mathbb{Q}} & \xrightarrow{cl_{X_v}} & H^{2n}_{et}(X_v, \mathbb{Q}_p(n)) & \xrightarrow{edge} & H^{2n}_{et}(\overline{X}_v, \mathbb{Q}_p(n))^{G_{F_v}}
\end{array}$$

with α surjective, β an isomorphism (proper base change + Weil II) and $\mathrm{Ker}(edge) = F^1 H^{2n}_{et}(X_v, \mathbb{Q}_p(n))$. One must then analyze the kernel of sp.

§4. Local case at p

Let k be a perfect field of $\mathrm{char}(k) = p > 2$, $\mathcal{O}$ a complete DVR with residue field k and fraction field K of $\mathrm{char}(K) = 0$. Let $\mathfrak{X}$ be proper and smooth over $\mathrm{Spec}(\mathcal{O})$; write $X = \mathfrak{X} \otimes_{\mathcal{O}} K$ (resp. $Y = \mathfrak{X} \otimes_{\mathcal{O}} k$) for its generic (resp. special) fibre.

Syntomic cohomology (Fontaine – Messing):

$$H^*(\mathfrak{X}_{syn}, s_{\mathbb{Q}_p}(n)) \underbrace{\xrightarrow{\lambda_{FM}}}_{\text{"Fontaine – Messing map"}} H^*(X_{et}, \mathbb{Q}_p(n))$$

Cycle classes:

$$\begin{tikzcd}
K_0(\mathfrak{X})^{(n)}_{\mathbb{Q}} \arrow[r, "\sim"] \arrow[d] \arrow[rr, bend left, "\frac{(-1)^{n-1}}{(n-1)!} c_n^{syn}"] & CH^n(\mathfrak{X})_{\mathbb{Q}} \arrow[r, "cl^{syn}_{\mathfrak{X}}"] \arrow[d] & H^{2n}(\mathfrak{X}_{syn}, s_{\mathbb{Q}_p}(n)) \arrow[d, "\lambda_{FM}"] \\
K_0(X)^{(n)}_{\mathbb{Q}} \arrow[r, "\sim"] \arrow[rr, bend right, "\frac{(-1)^{n-1}}{(n-1)!} c_n^{et}"'] & CH^n(X)_{\mathbb{Q}} \arrow[r, "cl^{et}_X"] & H^{2n}(X_{et}, \mathbb{Q}_p(n))
\end{tikzcd}$$

Here ${}^{(n)}$ refers to the weight n eigenspace for Adams operations and c_n^{et} (resp. c_n^{syn}) denotes the n-th Chern class.

Notation: $V^j(n) := H^j(\overline{X}_{et}, \mathbb{Q}_p(n))$

This is a crystalline representation of the Galois group G_K.

For a crystalline representation V of G_K, Bloch and Kato defined groups

$$H^i_f(G_K, V) = \mathrm{Ext}^i_{(\text{crystalline representations of } G_K)}(\mathbb{Q}_p, V)$$

and showed that $H^0_f = H^0$, $H^1_f \subset H^1$, $H^i_f = 0$ $(i>1)$.

Theorem 1. There is a canonical filtration F^i_{syn} on $H^{2n}(\mathfrak{X}_{syn}, s_{\mathbb{Q}_p}(n))$ such that $F^2_{syn} = 0$ and the Fontaine – Messing map λ_{FM} induces

$$gr^0(\lambda_{FM}) : F^0_{syn}/F^1_{syn} \underbrace{\xrightarrow{\sim}}_{\text{isom.}} F^0_{et}/F^1_{et} = H^0(G_K, V^{2n}(n))$$

$$gr^1(\lambda_{FM}) : F^1_{syn} \xrightarrow{\sim} H^1_f(G_K, V^{2n-1}(n)) \hookrightarrow H^1(G_K, V^{2n-1}(n)) = F^1_{et}/F^2_{et}$$

(here $F^i_{et} = F^i H^{2n}(X_{et}, \mathbb{Q}_p(n))$).

Corollary. (1) $\mathrm{Ker}(\lambda_{FM}) = 0$.

(2) $\mathrm{Im}(\lambda_{FM}) \cap F^2_{et} = 0$.

(3) $\mathrm{Im}(cl^{et}_X) \cap F^2_{et} = 0$

(4) $\mathrm{Im}(cl^{et}_X) \cap F^1_{et}$ maps to $H^1_f(G_K, V^{2n-1}(n))$.

Remarks: – (3), (4) hold if X has a potentially good reduction

– there is a version for cohomology with $\mathbb{Z}_p$- coefficients, provided K is absolutely unramified and $p > \max(\dim(X), 2n+1)$.

§5. Global applications

Let $[F:\mathbb{Q}]<\infty$, X smooth projective over F, $V = H^{2n-2}(\overline{X}_{et}, \mathbb{Q}_p(n))$, $S = \{$bad primes for X or primes $v \mid p\}$. The local results imply that $\mathrm{Im}(cl_{X,n}) \cap F^2H^{2n}(X_{et}, \mathbb{Q}_p(n))$ is a subquotient of

$$\mathrm{Ker}\left[H^2(G_{F,S}, V) \xrightarrow{\beta_{S,\Sigma}} \bigoplus_{v\in\Sigma} H^2(G_{F_v}, V)\right],$$

for suitable $\Sigma \subseteq S$. If we are lucky, $\Sigma = S$.

Conjecture: $\mathrm{Ker}(\beta_{S,S}) = 0$. $(**)$

(proved in the function field case by Jannsen and Raskind).

Results of Flach and Rubin give $(**)$ in some special cases. As an application, we get

Theorem 2. Let $E_{/\mathbb{Q}}$ be a modular elliptic curve of conductor N, without complex multiplication.

(1) If $p \nmid 6N$ and $d \geq 1$, then

$$\mathrm{Im}\left[CH^d(E^d)_{\mathbb{Q}} \xrightarrow{cl} H^{2d}((E^d)_{et}, \mathbb{Q}_p(d))\right] \cap F^2 = 0$$

(i.e. $\mathrm{Ker}(\text{Albanese map}) = \mathrm{Ker}(cl)$)

(2) There is an explicit constant $c_E \geq 1$ such that for $d \geq 1$ and $p \nmid c_E (2d)!$

$$\left(\mathrm{Im}\left[CH^d(E^d) \longrightarrow H^{2d}((E^d)_{et}, \mathbb{Z}_p(d))\right]\right)_{tors} = 0.$$

Title: Higher Picard varieties and the height pairing

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Let X_K be a smooth projective variety of dimension d defined over a number field K. Beilinson and Bloch have defined under suitable assumptions height pairings

(1) $\quad \langle\, , \rangle_{X_K} : CH^p(X_K)^0_{\mathbb{Q}} \times CH^{d+1-p}(X_K)^0_{\mathbb{Q}} \longrightarrow \mathbb{R}$

between Chow groups of homologically trivial cycles [1,2]. We denote by $A^p(X_K)$ the subgroup of $CH^p(X_K)^0$ consisting of cycle classes which become algebraically equivalent to zero over an algebraic closure $\overline{K}$ of K. The restricted pairing

(2) $\quad \langle\, , \rangle_{X_K} : A^p(X_K)_{\mathbb{Q}} \times A^{d+1-p}(X_K)_{\mathbb{Q}} \longrightarrow \mathbb{R}$

is well defined under the assumption that X_K has a regular model which is projective and flat over the ring of integers in K. H. Saito has introduced in [4] abelian varieties $\mathrm{Pic}^p(X_{\overline{K}})$ which parametrize cycles algebraically equivalent to zero modulo incidence equivalence on $X_{\overline{K}}$.

The abelian variety $\mathrm{Pic}^p(X_{\bar{K}})$ is the p-th higher Picard variety of $X_{\bar{K}}$. It satisfies a universal property and comes equipped with a natural duality map

(3) $\quad \lambda_{X_{\bar{K}}}^{d+1-p} : \mathrm{Pic}^{d+1-p}(X_{\bar{K}}) \longrightarrow \mathrm{Pic}^p(X_{\bar{K}})^\vee$

and an Abel-Jacobi map

(4) $\quad \Theta^p : A^p(X_{\bar{K}}) \longrightarrow \mathrm{Pic}^p(X_{\bar{K}})(\bar{K}).$

The higher Picard varieties allow the following description of the pairing (2) in terms of the (normalized) Néron-Tate height pairing

(5) $\quad (\ ,\ )_{\mathrm{Pic}^p(X_{\bar{K}})} : \mathrm{Pic}^p(X_{\bar{K}})(\bar{K}) \times \mathrm{Pic}^p(X_{\bar{K}})^\vee(\bar{K}) \to \mathbb{R}.$

<u>Theorem 1</u>: For $x \in A^p(X_K)$, $y \in A^{d+1-p}(X_K)$, we have

$$\frac{1}{[K:\mathbb{Q}]} \langle x, y \rangle_{X_K} = \frac{1}{k_{X_{\bar{K}}}^p} \left(\Theta^p(x), \lambda_X^{d+1-p} \circ \Theta^{d+1-p}(y) \right)_{\mathrm{Pic}^p(X_{\bar{K}})}$$

where k_X^p is a positive integer naturally associated with $X_{\bar{K}}$.

Theorem 1 generalizes a corresponding result of Faltings and Hriljac for curves. In fact, the proof of the theorem works by reduction to this result.

In the case where $X_K = A_K$ is an abelian variety, we have the following application of Theorem 1. Fix a polarization $\lambda: A_K \to A_K^\vee$ and a Poincaré bundle $\mathcal{P}$ on $A_K \times A_K^\vee$. The correspondence

(6) $\mathcal{D}_\lambda^p = (-1)^{g+p}(id \times \lambda)^* c_1(\mathcal{P})^{2d+1-2p} \in CH^\cdot(A_K \times A_K^\vee)$

induces essentially the Hodge $*$-operator on singular cohomology H^{2p-1}. It defines a morphism

(7) $\Lambda_\lambda^p : Pic^p(X_{\bar{K}}) \xrightarrow{Pic(\mathcal{D}_\lambda^p)} Pic^{d+1-p}(X_{\bar{K}}) \xrightarrow{\lambda_{X_{\bar{K}}}^{d+1-p}} Pic^p(X_{\bar{K}})^\vee$

<u>Theorem 2</u>: Λ_λ^p is a polarization.

This follows from a comparison of Saito's higher Picard varieties with intermediate Jacobians over the complex numbers where we can use the positivity of the Hodge $*$-operator (induced by $\mathcal{D}_\lambda^p$) to obtain a polarization.

We define $I^p(X_K)$ to be the group of cycle classes in $A^p(X_K)$ which are incidence equivalent to zero on $X_{\bar{K}}$. Recall that a cycle x in $A^p(X_{\bar{K}})$ is incidence equivalent to zero iff for every smooth projective variety $T_{\bar{K}}$ over $\bar{K}$ and every $\alpha \in CH^{d+1-p}(X_{\bar{K}} \times T_{\bar{K}})$ the image of x in $A^1(T_{\bar{K}})$ under the correspondence α vanishes.

We set $B^p(X_K) = A^p(X_K)/I^p(X_K)$ and obtain:

<u>Theorem 3</u>: Let $\mathcal{L}$ be an ample line bundle on the abelian variety A_K, $2p \leq d+1$, and L the Lefschetz operator associated with $\mathcal{L}$:

i) The operator

$$L^{d+1-2p}: B^p(A_K)_{\mathbb{Q}} \xrightarrow{\sim} B^{d+1-p}(A_K)_{\mathbb{Q}}$$

is an isomorphism.

ii) If $x \in B^p(A_K)_{\mathbb{Q}}$, $x \neq 0$, and $L^{d+2-2p}(x) = 0$ then

$$(-1)^p < x, L^{d+1-2p}x >_{A_K} \; > 0.$$

Beilinson has conjectured that Theorem 3 holds more generally for any variety X_K and the group $CH^p(X_K)^0_{\mathbb{Q}}$ instead of $B^p(A_{\bar{K}})_{\mathbb{Q}}$. Observe that this would imply that the pairing (1) is non-degenerate and the subquotient $B^p(X_K)_{\mathbb{Q}}$ of $CH^p(X_K)^0_{\mathbb{Q}}$ equals the subgroup $A^p(X_K)_{\mathbb{Q}}$.

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Title: Arakelov geometry of abelian varieties

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Page: 1

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Let K be a number field and A an abelian variety of dimension g over K. Let $\pi : \mathcal{A} \longrightarrow S := \operatorname{Spec} \mathcal{O}_K$ be an open subgroup scheme of the Néron-model of A over S; let ε be its zero section.

§1. Arithmetic intersection theory.

a. $\widehat{CH}^*(\mathcal{A}) := \{ (Z, g) \in \hat{Z}^*(\mathcal{A}) \mid$ g is a good current, i.e. $dd^c g + \delta_Z$ translat. invariant on $A_\sigma(\mathbb{C})$, $\forall \sigma : K \hookrightarrow \mathbb{C}$ $\}/\sim$

It has a natural ring structure. $\sim$ = natural equivalence

b. A good hermitian metric $\| \ \|$ on a line bundle L on a complex abelian variety is a C^∞ metric such that the form $c_1(L, \| \ \|)$ is translation invariant. Any such line bundle admits good metrics; these are closely related to classical theta functions.

Lemma: i) $\{ \overline{\mathcal{L}}$ line bundle on $\mathcal{A}$ with good metrics$\}/$isom. $\xrightarrow[\sim]{\hat{c}_1} \widehat{CH}^1(\mathcal{A})$.

ii) If $\mathcal{A}$ has connected fibers (e.g. if A has good reduction), then

$$\widehat{CH}^1(\mathcal{A}) \xrightarrow{\sim} CH^1(A) \oplus \widehat{CH}^1(\operatorname{Spec} \mathcal{O}_K)$$
$$\alpha \longmapsto (\alpha_K, \varepsilon^* \alpha).$$

Heights attached to line bundles $\overline{\mathcal{L}}$ with good metrics s.t. $\varepsilon^* \hat{c}_1(\overline{\mathcal{L}}) = 0$ coincide with the Néron-Tate height and its generalizations for higher dimensional cycles, first introduced by Philippon. The positivity properties of these heights give rise to difficult and important problems.

c. Besides the ring structure provided by intersection theory, the Chow groups $CH^*(A)$ are equipped with a convo-lution or Pontrjagin product $*$: let $pr_1, pr_2, \boxplus : A \times A \to A$ denote the projection maps and the addition map; then for any $\alpha \in CH^p(A)$; $\beta \in CH^q(A)$,

$$\alpha * \beta := \boxplus_* (pr_1^* \alpha \cdot pr_2^* \beta) \in CH^{p+q-g}(A)$$

The same formula defines convolution products of cohomology classes, differential forms and currents on $A_\sigma(\mathbb{C})$, and, when A has good reduction, of elements of $CH^*(\mathcal{A})$.

Applications: i) elementary constructions of Green currents

if $g_{\{0\}}$ is a (good) Green current for 0 in $A_\sigma(\mathbb{C})$, then for any algebraic cycle Z on $A_\sigma(\mathbb{C})$, $g_{\{0\}} * \delta_Z$ is a (good) Green current for Z.

ii) Assume A has good reduction.

- for any $\alpha \in CH^p(\mathcal{A})$, $\beta \in CH^q(\mathcal{A})$, $p+q = g+1$

$$\widehat{\deg}(\alpha \cdot \beta) = \widehat{\deg}\, \varepsilon^* (\alpha * [-1]^* \beta)$$

- for any $Z \in Z_p(\mathcal{A})$, $T \in Z_q(\mathcal{A})$, $p+q = g+1$

$$\langle Z, T \rangle := \widehat{\deg}\, [Z]_A [T]_A = \widehat{\deg}\, \varepsilon^* [Z \boxminus T]_A.$$

generalized Bloch-Beilinson pairing ($[\ ]_A$ = "Arakelov fundamental class")

These relations reduce some computations of arithmetic intersection numbers on abelian variety to evaluations of $\log|$classical theta function$|$ and their integrals.

The convolution product $*$ on $CH^*(\mathcal{A})$ plays also a key role in the work of Bochnet and Künnemann on arithmetic Fourier transform.

§2. Arithmetic Riemann-Roch for ample line bundles

Notations: . for any non zero hermitian vector bundle $\overline{E}$ on S, $\hat{\mu}(E) := \frac{1}{[K:\mathbb{Q}]} \frac{\widehat{\deg}\,\overline{E}}{\mathrm{rk}\,E}$

. $\overline{\omega}_{\mathcal{A}/S} := (\varepsilon^{*}\Omega^{g}_{\mathcal{A}/S}\,;\, \|\ \|_{L^2})$ where $\|\alpha\|^2_{L^2,\sigma} = \frac{i^{g^2}}{(2\pi)^g}\int_{A_\sigma(\mathbb{C})} \alpha \wedge \overline{\alpha}$.

Faltings height of A : $h(A) := \frac{1}{[K:\mathbb{Q}]}\widehat{\deg}\,\overline{\omega}_{\mathcal{A}/S}$.

. let $\overline{\mathcal{L}}$ be an hermitian line bundle on $\mathcal{A}$; then $\pi_* \mathcal{L}$ is a torsion free coherent sheaf on S; it is equipped with the natural L^2 metric defined by

$$\|s\|^2_{L^2,\sigma} = \int_{A_\sigma(\mathbb{C})} \|s(x)\|^2_{\overline{\mathcal{L}}_\sigma}\, d\mu(x)$$

where $d\mu$ denotes the invariant probability measure on $A_\sigma(\mathbb{C})$.

$\pi_*\overline{\mathcal{L}} := (\pi_*\mathcal{L}, \|\ \|_{L^2})$.

Theorem : Let $\pi: \mathcal{A} \to S$ be an abelian scheme of relative dimension g, and $\overline{\mathcal{L}}$ a line bundle on $\mathcal{A}$ equipped with good metrics. If $\mathcal{L}_K$ is ample then:

$$\hat{\mu}(\pi_*\overline{\mathcal{L}}) = -\frac{1}{2}h(A) + \frac{1}{4}\log\left(\chi(A,\mathcal{L}_K)/(2\pi)^g\right) + \frac{1}{g+1}\frac{\widehat{\deg}\,\hat{c}_1(\overline{\mathcal{L}})^{g+1}}{\deg_K c_1(\mathcal{L}_K)^g}$$

Corollary. If $\mathcal{L}_K$ is symmetric and $\varepsilon^*\hat{c}_1(\overline{\mathcal{L}}) = 0$, then

(KF) $\hat{\mu}(\pi_*\overline{\mathcal{L}}) = -\frac{1}{2}h(A) + \frac{1}{4}\log\frac{\chi(A,\mathcal{L}_K)}{(2\pi)^g}$

(Recall that $\mathrm{rk}\,\pi_*\mathcal{L} = \dim H^0(A_K, \mathcal{L}_K) = \chi(A, \mathcal{L}_K) = \frac{1}{g!}\deg_K c_1(\mathcal{L}_K)^g$)

When $\chi(A, \mathcal{L}_K)=1$, this corollary follows from some works of Moret-Bailly. For general $\overline{\mathcal{L}}$'s, the theorem may be proved by using the arithmetic Riemann-Roch theorem of Gillet and Soulé.

Generalization: The "key formula" (KF) still holds when A has not necessarily good reduction, provided $(\mathcal{A}, \overline{\mathcal{L}})$ is a "Moret-Bailly model" of $(A, \mathcal{L}_K)$, namely when:

- $\mathcal{A}$ is semi-stable (i.e., $\mathcal{A}^\circ$ is semi-abelian)
- $\mathcal{L}_K$ is ample and $\overline{\mathcal{L}}$ is cubist
- the Mumford group $K(\mathcal{L}_K^{\otimes 2})$ extends to a scheme in $\mathcal{A}$ finite over S.

Such models always exist after some finite extension of the ground field K.

This generalization is established by means of the arithmetic Riemann-Roch theorem and of the techniques of Moret-Bailly.

§3. The Faltings height

a The "key formula" (KF), in its generalized form, leads to new simple proofs of the fact that the Faltings height is indeed a height, either by using some basic results of geometric invariant theory, or by deriving from it some explicit relation between the Faltings height and the height defined by means of Thetanullwerte.

b The key formula is also very useful to get explicit bounds on natural invariants of abelian varieties over number fields in terms of their Faltings height.

Definitions and notations:

Let L be an ample line bundle on A.

- for any $\sigma : K \to \mathbb{C}$, let $\omega_{L,\sigma}$ be the invariant Kähler form on $A_\sigma(\mathbb{C})$ defined as $c_1(L_\sigma, \| \|)$ for any good metric on L_σ. These Kählers form define some hermitian structure $\| \|_L$ on $T_{\mathcal{A}/S}$ and on $t_{\mathcal{A}} := \varepsilon^* T_{\mathcal{A}/S}$.
- let Γ_σ the "lattice of periods of A_σ" in $t_{\mathcal{A}_\sigma}$, defined as the kernel of the "exponential" map $t_{\mathcal{A}_\sigma} \to A_\sigma(\mathbb{C})$; we have $A_\sigma := t_{\mathcal{A}_\sigma} / \Gamma_\sigma$, and we let $\rho(A_\sigma, L_\sigma) = \min_{\gamma \in \Gamma_\sigma - \{0\}} \|\gamma\|_{L_\sigma}$
- when $\chi(A, L) = 1$, then for any $\sigma : K \to \mathbb{C}$, there exists a continuous function $\|\vartheta_{A_\sigma, L_\sigma}\| : A_\sigma(\mathbb{C}) \to \mathbb{R}_+$ such that, for any choice of good metric on L_σ, we have:

$$\forall s \in H^0(A_\sigma, L_\sigma),\ \forall x \in A_\sigma(\mathbb{C}),\ \|s(x)\|_{L_\sigma} = \|s\|_{L^2} \|\vartheta_{A_\sigma, L_\sigma}\|(x)$$

Theorem: For any abelian variety A of dimension g over a number field K, equipped with some ample line bundle L s.t. $\chi(A,L)=1$, we have:

$$h(A) \geq -\frac{g}{2}\log(2\pi) - \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma : K \to \mathbb{C}} \int_{A_\sigma(\mathbb{C})} \log \|\vartheta_{A_\sigma, L_\sigma}\|^2(x)\, d\mu(x).$$

Corollary. For any abelian variety A of dimension g over a number field,

$$h(A) \geq -g \, \frac{\log(2\pi)}{2}.$$

Corollary. For any abelian variety A of dimension g over a number field K, equipped with an ample line bundle L,

$$(*) \quad \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma : K \hookrightarrow \mathbb{C}} \rho(A_\sigma, L_\sigma)^{-2} \leq C(g) \max(1, h(A), \log \chi(A,L))$$

(effective form of the "matrix lemma" of D. Masser).

If $\overline{E}$ is a hermitian vector bundle on S, we denote by $\hat{\mu}_{max}(\overline{E})$ its maximal slope, defined as the maximal slope $\hat{\mu}$ of a quotient of $\overline{E}$

Proposition: With the same notation as above

$$(**) \quad \hat{\mu}_{max}(t_A, \| \, \|_L) \leq C(g) \max(1, h(A), \log \chi(A,L))$$

It turns out that the estimates $(*)$ and $(**)$ are the only technical results involving Faltings heights needed to prove the theorem of Masser and Wüstholz on minimal abelian subvarieties, if one systematically uses Arakelov geometry.

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Title: Zeta functions in arithmetic geometry

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Let X be a d-dimensional algebraic variety defined over a number field F. Denote by $\mathcal{L} = (L, \{\|\cdot\|_v\})$ a metrized ample line bundle on X. For any locally closed algebraic subset $Y \subset X$ we denote by $Y(F)$ the set of F-rational points in Y. A metrized line bundle $\mathcal{L}$ defines a height function

$$H_{\mathcal{L}} : X(F) \to \mathbf{R}_{>0}.$$

The asymptotic behavior of the counting function

$$N(Y, \mathcal{L}, B) := \#\{x \in Y(F) \mid H_{\mathcal{L}}(x) \le B\} < \infty$$

as $B \to \infty$ is determined by analytic properties of the *height zeta function* $Z(Y, \mathcal{L}, s)$ defined by the series

$$Z(Y, \mathcal{L}, s) := \sum_{x \in Y(F)} H_{\mathcal{L}}(x)^{-s}$$

which converges for $\mathrm{Re}(s) \gg 0$. One has the following Tauberian statement:

Assume that the series $Z(Y, \mathcal{L}, s)$ is absolutely convergent for $\mathrm{Re}(s) > a > 0$ and that there exists some positive integer b such that

$$Z(Y, \mathcal{L}, s) = \frac{g(s)}{(s-a)^b} + h(s)$$

where $g(s)$ and $h(s)$ are functions holomorphic in the domain $\mathrm{Re}(s) \ge a$ and $g(a) \ne 0$. Then the following asymptotic formula holds:

$$N(Y, \mathcal{L}, B) = \frac{g(a)}{a(b-1)!} B^a (\log B)^{b-1}(1 + o(1)) \ for\ B \to \infty.$$

Let $NS(X)$ be the Neron-Severi group of X, $NS(X)_{\mathbf{R}} = NS(X) \otimes \mathbf{R}$. We denote by $[L]$ the class of L in $NS(X)$. The *cone of effective divisors of* X is the closed cone $\Lambda_{\mathrm{eff}} \subset NS(X)_{\mathbf{R}}$ generated by the classes of effective divisors.

Let $(A, A_{\mathbf{R}}, \Lambda)$ be a triple consisting of a free abelian group A of rank t, a t-dimensional real vector space $A_{\mathbf{R}} = A \otimes \mathbf{R}$ containing A as a sublattice of maximal rank, and a convex t-dimensional finitely generated polyhedral cone $\Lambda \subset A_{\mathbf{R}}$ such that $\Lambda \cap -\Lambda = 0 \in A_{\mathbf{R}}$. Denote by Λ° the interior of Λ. Let $(A^*, A^*_{\mathbf{R}}, \Lambda^*)$ be the triple consisting of the dual abelian group $A^* = \mathrm{Hom}(A, \mathbf{Z})$, the dual real vector space $A^*_{\mathbf{R}} = \mathrm{Hom}(A_{\mathbf{R}}, \mathbf{R})$, and the dual cone $\Lambda^* \subset A^*_{\mathbf{R}}$. We normalize the Haar measure $\mathbf{dy}$ on $A^*_{\mathbf{R}}$ by the condition: $\mathrm{vol}(A^*_{\mathbf{R}}/A^*) = 1$.

The $\mathcal{X}$*-function of* Λ is defined as the integral

$$\mathcal{X}_\Lambda(\mathbf{s}) = \int_{\Lambda^*} e^{-\langle \mathbf{s}, \mathbf{y} \rangle} \mathbf{dy},$$

for $\mathrm{Re}(\mathbf{s}) \in \Lambda^\circ$. One can check that $\mathcal{X}_\Lambda(\mathbf{s}) \neq 0$ for all $\mathbf{s}$ with $\mathrm{Re}(\mathbf{s}) \in \Lambda^\circ$.

Theorem 0.1 *[5] Let X be a smooth projective equivariant compactification of an algebraic torus T defined over a number field F.*

1. *There exists a pairing*

$$H \ : \ X(F) \times NS(X)_{\mathbf{C}} \to \mathbf{C}$$

   such that its restriction to $X(F) \times [L]$ coincides with the height function $H_{\mathcal{L}}$ corresponding to some metrisation on L.

2. *The height zeta function*

$$Z(T, \mathbf{s}) = \sum_{x \in T(F)} H(x, -\mathbf{s})$$

   is holomorphic for $\mathrm{Re}(\mathbf{s}) \in \Lambda^\circ_{\mathrm{eff}} + [-K_X]$.

3. *The function*

$$\frac{Z(T, \mathbf{s})}{\mathcal{X}_{\Lambda_{\mathrm{eff}}}(\mathbf{s} + [K_X])}$$

   is holomorphic for $\mathrm{Re}(\mathbf{s})$ *contained in the closed cone* $\Lambda_{\mathrm{eff}} + [-K_X]$ *and it is not equal to* 0 *for any* $\mathbf{s}$ *with* $\mathrm{Re}(\mathbf{s})$ *contained in the boundary* $\partial(\Lambda_{\mathrm{eff}} + [-K_X])$ *of the shifted cone of effective divisors.*

The following statement, inspired by the Linear Growth conjecture of Manin ([7]), has been expected to be true ([1, 6]):

Conjecture 0.2 *Let X be a smooth Fano variety over a number field E. Then there exist a Zariski open subset $U \subset X$ and a finite extension F_0 of E such that for all number fields F containing F_0 the following asymptotic formula holds*

$$N(U, -\mathcal{K}_X, B) = \frac{c}{(t-1)!} B(\log B)^{t-1}(1 + o(1)), \quad B \to \infty,$$

where t is the rank of the Picard group of X over F.

The conjecture 0.2 was refined by E. Peyre who proposed an adelic interpretation for the constant c introducing Tamagawa numbers of Fano varieties ([8]). The theorem above proves this refined version of the conjecture for smooth projective toric varieties. Moreover, we can compute the asymptotic constant $c = \alpha(X)\beta(X)\tau(\mathcal{K}_X)$, where $\alpha(X) = \mathcal{X}_{\Lambda_{\text{eff}}}([-K_X])$, $\beta(X) = |Br(X)/Br(F)|$ is the order of the non-trivial part of the Brauer group of X and $\tau(\mathcal{K}_X)$ is the Tamagawa number defined by E. Peyre.

However, we were lead to consider the hypersurfaces $X_{n+2} \subset \mathbf{P}^n \times \mathbf{P}^3$ ($n \geq 1$) defined by the equation

$$P(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^{3} l_i(\mathbf{x}) y_i^3 = 0$$

where

$$P(\mathbf{x}, \mathbf{y}) \in \mathbf{Q}[x_0, \ldots, x_n, y_0, \ldots, y_3].$$

and $l_0(\mathbf{x}), \ldots, l_3(\mathbf{x})$ are homogeneous linear forms in $x_0, \ldots, x_n$ ([5]). Put $k = \min(n+1, 4)$. We shall always assume that any k forms among $l_0(\mathbf{x}), \ldots, l_3(\mathbf{x})$ are linearly independent. We checked the following statements:

Proposition 0.3 *1) The hypersurface X_{n+2} is a smooth Fano variety containing a Zariski open subset U_{n+2} which is isomorphic to $\mathbf{A}^{n+2}$.*

2) Let $U_P \subset \mathbf{P}^n$ be the Zariski open subset defined by the condition

$$\prod_{i=0}^{3} l_i(\mathbf{x}) \neq 0.$$

Then the fibers of the natural projection $\pi : X_{n+2} \to \mathbf{P}^n$ over closed points of U_P are smooth diagonal cubic surfaces in $\mathbf{P}^3$.

3) The Picard group of X_{n+2} over an arbitrary field containing $\mathbf{Q}$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$.

Proposition 0.4 *Let $n \geq 3$. Then for any non-empty Zariski open subset $U \subset X_{n+2}$ and for any field F containing $\mathbf{Q}(\sqrt{-3})$, one has*

$$N(U, -\mathcal{K}_{X_{n+2}}, B) \geq cB(\log B)^3$$

for all $B > 0$ and some positive constant c.

Proposition 0.5 *Let $n = 2$. Then there exists a number field F_0 depending only on X_{n+2} such that for any non-empty Zariski open subset $U \subset X_{n+2}$ for any field F containing F_0 one has*

$$N(U, -\mathcal{K}_{X_{n+2}}, B) \geq cB(\log B)^3$$

for all $B > 0$ and some positive constant c.

Proposition 0.6 *Let $n = 1$. Then for any non-empty Zariski open subset $U \subset X_{n+2}$, there exists a number field F_0 (which depends on U) such that for any field F containing F_0, one has*

$$N(U, -\mathcal{K}_{X_{n+2}}, B) \geq cB(\log B)^3$$

for all $B > 0$ and some positive constant c.

The statements in the propositions 0.4, 0.5, 0.6 and 0.3 show that Conjecture 0.2 is not true for Fano cubic bundles X_{n+2} ($n \geq 1$). Let us remark that the lower bounds in the above propositions follow from 0.1.

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Title: Self-intersection of the dualizing sheaf of modular curves $X_0(N)$

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Let N be a square free integer and let $X_0(N)$ be the compactification of the coarse moduli of elliptic curves with a cyclic subgroup of order N. It is a projective smooth curve over $\mathbb{Q}$.

Let $X_0(N)_{\mathbf{Z}}$ be the minimal regular model of $X_0(N)$ over $\mathbf{Z}$. The Arakelov theory associates an arithmetic invariant to such arithmetic surface : *the self-intersection of the relative dualizing sheaf.* Recall for this that by a theorem of Deligne and Rapoport $X_0(N)_{\mathbf{Z}}$ is semi-stable (N is square free) so we can consider its relative dualizing sheaf ω over $\mathbf{Z}$.

In the geometric case, an upper bound of the self-intersection of the dualizing sheaf enabled Arakelov, Parshin and Szpiro to prove an effective Mordell conjecture for function fields in all characteristics. Some arithmetic conjectures specially Parshin's conjecture give analogous bounds in the arithmetic case. In this talk, I will compute this invariant for modular curves $X_0(N)$.

Notations

The Riemann surface $X_0(N)_{\mathbf{C}}$ is canonically isomorphic to the quotient $\Gamma_0(N)\backslash(\mathcal{H}\cup\mathbb{P}^1(\mathbb{Q}))$, where

$$\mathcal{H}=\{z\in\mathbf{C}\ ;\ \Im(z)>0\}\qquad \Gamma_0(N)=\{\begin{pmatrix} a & b\\ Nc & d\end{pmatrix}\}.$$

Over this Riemann surface we have two metrics :

1) The Poinacaré metric : $d\mu_0=\dfrac{dxdy}{y^2}$.

2) The Arakelov metric : $\nu=F(z)d\mu_0(z)$ where $F(z)$ is the $\Gamma_0(N)$–invariant function over $\mathcal{H}$ given by

$$F(z)=\frac{y^2}{g}\sum_{i=1}^{g}|f_i(z)|^2$$

and $(f_i)_{1\leq i\leq g}$ is an orthonormal basis of the space $\mathcal{S}(2,\Gamma_0(N))$ of weight–2 parabolic modular forms for $\Gamma_0(N)$ equiped with the Peterson product.

Arakelov defined intersection product for line bundles equiped with admissible metrics (which means that the curvature is proportional to ν).

The dualizing sheaf is canonically equiped with such metric :

i) For any point $P\in X(\mathbb{C})$, the line bundle $\mathcal{O}_X(P)$ is canonically equiped with an admissible metric; let s_P be the canonical section of $\mathcal{O}_X(P)$, this metric is given by

$$\|s_P\|(Q)=\exp(g_{Ar}(P,Q))$$

where g_{Ar} is the Arakelov–Green function. It is the unique C^∞ symmetrical real function defined over $X(\mathbb{C})\times X(\mathbb{C})-\Delta$ (where Δ is the diagonal) and satisfying :

[1] Joined work with E. Ullmo (Orsay, France).

- $\partial_z\partial_{\bar z}g_{Ar}(z,w) = i\pi(\nu(z) - \delta_w)$,
- $\int_{X(\mathbb{C})} g_{Ar}(z,w)\nu(z) = 0$ for any w.

ii) The canonical admissible metric on Ω^1 is given by imposing that the residue isomorphism

$$\Omega^1(P)|_P \xrightarrow{\sim} \mathbb{C}$$

is an isometry if $\mathcal{O}_X(P)$ is equiped with the metric i) and $\mathbb{C}$ with its canonical metric.

These are some well-known results on the self-intersection ω^2 :

1. Faltings proved that $\omega^2 \geq 0$ and $\omega^2 = 0$ for elliptic curves. It is conjectured that $\omega^2 > 0$ for genus $g \geq 2$.
2. This conjecture is proved by Zhang for curves with at least a fiber of bad reduction.
3. Zhang and Burnol proved that $\omega^2 > 0$ for many smooth cases.
4. Szpiro gave the arithmetic meaning of this invariant : he proved that *the non-vanishing of ω^2 in the smooth case is equivalent to the discreetness of the arithmetic points of the curve in its jacobian for the Néron-Tate topology i.e.*

$$\begin{array}{ccc} X(\overline{K}) & \to & J(\overline{K}) \\ P & \mapsto & x_P = \mathrm{cl}(\omega - (2g-2)P) \end{array}$$

   there exists $\varepsilon > 0$ such that $\{P \in X(\overline{K}) \mid h_{NT}(x_P) < \varepsilon\}$ is finite. It is a generalization of Raynaud's theorem on the finiteness of points of $X(\overline{K})$ which are torsion points when the curve is imbedded in its jacobian.
5. Bost, Mestre and Moret-Bailly computed ω^2 for $g = 2$.
6. Why ω^2 for $X_0(N)$? The choice of this family of curves was motivated by their relation with elliptic curves. We know by Wiles' theorem that every elliptic curve E semi-stable over $\mathbb{Q}$ has a modular parametrization $\varphi : X_0(N) \to E$ where N is the conductor of E. We try to use this parametrization to get informations on the elliptic curve from informations on the modular curve. This is an example; Szpiro conjectured that the minimal discriminant of E depends polynomially on its conductor. We hope to attack this question by first relating the discriminant of E to some arithmetic invariants of $X_0(N)$ and second establishing good estimates on these invariants of $X_0(N)$ as functions of N.

Let $E_\infty(z,s)$ be the Eisenstein series at the cusp ∞ for $\Gamma_0(N)$ and let *vol* be the volume of $X_0(N)$ for the Poincaré metric.

Theorem 1 *For any square free integer N prime to 6, we have*

$$\begin{aligned}\omega^2 &\leq 16\pi g(g-1)\lim_{s\to1}\left(\int_{X_0(N)} E_\infty(z,s)\nu(z) - \frac{1}{vol(s-1)}\right)\\ &+\frac{g}{2}\log(N) + g\sum_{p/N}\frac{p+1}{p-1}\log(p) + o(g\log(N))\end{aligned}$$

Theorem 2 *For any square free integer N prime to 6, we have*

$$\begin{aligned}\omega^2 &\leq -8\pi\frac{g-1}{vol}\lim_{s\to1}\left(\frac{Z'(s)}{Z(s)} - \frac{1}{s-1}\right)\\ &+\frac{g}{2}\log(N) + g\sum_{p/N}\frac{p+1}{p-1}\log(p) + o(g\log(N))\end{aligned}$$

where $Z(s)$ is the Selberg zeta function for $\Gamma_0(N)$.

Selberg zeta function

Let $\gamma \in \Gamma_0(N)$ be a hyperbolic matrix ($|\mathrm{tr}\gamma| > 2$). Let $N(\gamma) = v^2$ where v is the eigenvalue of γ with a square bigger than 1. Let γ_0 be the matrix with positive trace such that

$$\mathrm{Commutant}_{\Gamma_0(N)}(\gamma) = \{\pm\gamma_0^n \;.\; n \in \mathbf{Z}\}$$

γ is said to be primitive if $\gamma = \gamma_0$.

$$Z(s) = \prod_{\substack{\gamma_0 \text{ primitive}\\ \text{non conj.}}} \prod_{k=0}^{\infty}\left(1 - e^{-(s+k)\log(N(\gamma_0))}\right)$$

This function has many spectral propreties. Let $L^2(\Gamma_0(N)\backslash\mathcal{H}, d\mu_0)$ be the space of $\Gamma_0(N)$–invariant functions on $\mathcal{H}$ with an integrable square relatively to the Poincaré metric. The Laplacian $D_0 = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ acts on this space and gives the decomposition :

$$L^2(\Gamma_0(N)\backslash\mathcal{H}, d\mu_0) = \oplus_{n\geq0}\mathbb{C}[\varphi_n] \oplus \mathcal{E}$$

where $\mathcal{E}$ is the continuous spectrum and φ_n is an orthonormal family of eigenfunctions of D_0 with eigenvalues $-\lambda_n$ and $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$. Fix $\lambda_n = s_n(1-s_n) = \frac{1}{4} + r_n^2$ and $s_n = \frac{1}{2} + ir_n$. Then, the r_n are real except a finite number.

1. $Z(s)$ is holomorphic on a neighbourhood of 1 with a simple zero at $s = 1$.
2. The non trivial zeroes of Z are the $s_k = \frac{1}{2} + ir_k$ and $1 - s_k$

3. $\lambda_n \sim \dfrac{4\pi n}{vol}$ and we have the "vague" relation :

$$\frac{1}{2s-1}\frac{Z'(s)}{Z(s)} = \sum_0^\infty \left(\frac{1}{s(s-1)-\lambda_n} - \frac{vol}{4\pi(s+n)}\right)$$

which can be obtained from the Selberg trace formula.

Hence, theorem 2 establishes a relationship between an arithmetic and a spectral invariants. As I will explain, theorem 1 gives a relationship between an arithmetic and a modular invariants.

Our first object was to give an estimate for ω^2 depending on the level N. This object cannot be reached because the asymptotic behaviour of $\lim_{s\to 1}\left(\frac{Z'(s)}{Z(s)} - \frac{1}{s-1}\right)$ on N is unknown for us. But two remarks can be made :

- theorem 2 gives an upper bound for this term depending only on N ($\omega^2 > 0$);
- I think that this term is negligeable compared to $g\log N$. Such an estimate will imply that ω^2/g is logarithmic in N. Indeed, we have the following :

Proposition 3 *For any square free integer N,*

$$\omega^2 \geq \frac{g}{3}\sum_{p/N}\frac{p-1}{p+1}\log(p) - O(\log(N))$$

This proposition follows from a theorem of Zhang which gives a lower bound of ω^2 based on the dual graph of the bad fibers of the arithmetic surface.

Proof (of theorem 1). Consider the cusps 0 and ∞ of $X_0(N)$ and the sections E_0 and E_∞ of $X_0(N)_{\mathbf{Z}}(\mathbf{Z})$ associated to these cusps. Fix two vertical divisors with rational coefficients Φ_0 and Φ_∞ such that

$$(\omega - (2g-2)E_i + \Phi_i, F) = 0 \text{ for any vertical divisor } F$$

where $i = 0, \infty$. The following inequality relates ω^2 to the geometry of the cusps :

$$\omega^2 \leq -4g(g-1)(E_0, E_\infty) + \frac{1}{g-1}\left(g(\Phi_0, \Phi_\infty) - \frac{\Phi_0^2 + \Phi_\infty^2}{2}\right)$$

It follows from :

1. the Manin–Drinfeld theorem : *the class of $(0-\infty)$ is torsion in the jacobian of $X_0(N)$*,
2. the Faltings–Hriljac formula relating Arakelov intersection on the curve and Néron–Tate height on the jacobian.

This inequality reduces theorem 1 to

a) the computation of Φ_0 and Φ_∞. This is done using the Deligne–Rapoport description of $X_0(N)_{\mathbf{Z}}$.

b) the computation of (E_0, E_∞). These sections don't intersect over finite places. Hence, $(E_0, E_\infty) = -g_{Ar}(0,\infty)$. Theorem 1 follows by comparing the Arakelov–Green function to more classical Green functions on $\mathcal{H}$ which are associated to the Poincaré metric.

Theorem 1 establishes a relationship between ω^2 and the constant term at 1 of the Rankin–Selberg transform of the Arakelov metric :

$$R_F(s) = \int_{X_0(N)} F(z) E_\infty(z,s) d\mu_0(z)$$

This transform is absolutely convergent for $\Re(s) > 1$ and admits a meromorphic continuation to the complex plane with a simple pole at $s = 1$ whose residue is $(vol)^{-1}$.

We classically compute this transform for modular forms $f \in \mathcal{S}(2, \Gamma_0(N))$, we then consider

$$\int_{X_0(N)} y^2 |f(z)|^2 E_\infty(z,s) d\mu_0(z).$$

The residue at $s = 1$ of this transform is $\langle f, f\rangle / vol$ where $\langle f, f\rangle$ is the Peterson product. If the q-expansion of f is $f(z) = \sum_{n\geq 1} a_n \exp(2i\pi n z)$, then the Rankin–Selberg method gives

$$\int_{X_0(N)} y^2 |f(z)|^2 E_\infty(z;s) d\mu_0(z) = \frac{\Gamma(s+1)}{(4\pi)^{s+1}} \sum_{n\geq 1} \frac{|a_n|^2}{n^{s+1}}$$

When f is a newform (a normalized new Hecke eigenform), this transform coincides, up to some zeta factors, with the L_2 series of f and then admits an Eulerian product. *We think of R_F as a mean of these Rankin–Selberg transforms for an orthonormal basis relatively to the Peterson product.*

Theorem 2 is obtained from theorem 1 by adapting a startegy suggested by Zagier which enabled him to refind the Selberg trace formula from the Rankin–Selberg method.

Zagier's strategy in weight 0

Let k be C^4-function of $\mathbb{R}_+$ such that there exist two constants $\alpha > 1$ and $C > 0$

$$|k(t)| \leq \frac{C}{t^\alpha} \qquad t >> 0 \tag{1}$$

Define the $\Gamma_0(N)$-invariant function on $\mathcal{H} \times \mathcal{H}$

$$K_0(z,w) = \frac{1}{2} \sum_{\gamma\in\Gamma_0(N)} k(u(z,\gamma w))$$

where u is the hyperbolic distance

$$u(z,w) = \frac{|z-w|^2}{\Im(z)\Im(w)}$$

The spectral decomposition of $K_0(z,w)$ is

$$K_0(z,w) = h(\frac{i}{2})\frac{1}{vol} + \sum_{i>0} h(r_i)\varphi_i(z)\overline{\varphi}_i(w) + \text{continuous term}$$

I will forget the continuous terms in the rest of this talk.

The function h is the Selberg transform of k. It has the following propreties :

i) $h(r) = h(-r)$ for any $r \in \mathbb{R}$,

ii) h admits an analytic continuation to the strip $\Im(r) < A/2$ for a real $A > 1$ (in our case $A = 2\alpha - 1$),

iii) h decreases quickly on this strip.

Conversely, if we start with a function h satisfying i)-ii)-iii) then the Selberg inverse transform k of h satisfies (1).

$$k \underset{\text{Selberg t.}}{\longrightarrow} h \underset{\text{Selberg inverse t.}}{\longrightarrow} k$$

These two transforms do not depend on the group.

Consider the expression :

$$K_0(z,z) = \sum_{i>0} h(r_i)|\varphi_i(z)|^2 + (CT + h(\frac{i}{2})\frac{1}{vol}) \tag{2}$$

We will take the integral of the product of this equality by the Eisenstein series. First, we group in (2) all terms giving a divergent contribution (they need a special analysis). On the left hand side, this term is the parabolic one.

$$K_0(z,z) = \frac{1}{2}\sum_{l=-\infty}^{+\infty} \sum_{\substack{\gamma \in \Gamma_0(N) \\ \mathrm{tr}\gamma = l}} k(u(z,\gamma z))$$

Every sum for a fixed trace is $\Gamma_0(N)$–invariant, so let

$$\widetilde{K}_0(z,z) = \frac{1}{2}\sum_{|l|\neq 2} \sum_{\substack{\gamma \in \Gamma_0(N) \\ \mathrm{tr}\gamma = l}} k(u(z,\gamma z))$$

By integrating, we get

$$\int_{X_0(N)} \widetilde{K}_0(z,z)E_\infty(z,s)d\mu_0(z) = \sum_{i>0} h(r_i)R_{|\varphi_i|^2}(s) + \dots$$

The residue at $s=1$ of the right hand side, up to the forgotten terms, is $(vol)^{-1}\sum_{i>0} h(r_i)$. The residue at $s=1$ of left hand side follows by writing

$$\int_{X_0(N)} \widetilde{K}_0(z,z)E_\infty(z,s)d\mu_0(z) = \sum_{|l|\neq 2} \zeta_N(s,l)I(s,k,l)$$

- $\zeta_N(s,l)$ is a zeta function which depends on N and l but not on the function k. Under some conditions

$$\zeta_N(s,l) = \prod_{p/N} \frac{1}{1+p^s} \prod_{p/N} \left(1 + \left(\frac{l^2-4}{p}\right)\right) \zeta_1(s,l)$$

  $\zeta_1(s,l)$ is the product of a finite Dirichlet series by the Dedekind zeta function of $\mathbb{Q}(\sqrt{l^2-4})$.

- $I(s,l,k)$ doesn't depend on N. It is holomorphic at $s=1$ with a value depending on the Fourier transform of h.

The equality of the residues of the left and the right hand sides is the Selberg trace formula.

Weight 2

Our goal is to compute the constant term at 1 of R_F. We should then work with differential forms and consider $F(z)d\mu_0(z)$ as the contribuation of the Laplacian eigenvalue 0. We prefer to work over $\mathcal{H}$ and to consider automorphic forms of weight 2.

From a function k satisfying (1) we construct a Selberg kernel $K_2(z,w)$ automorphic of weight 2 in each variable and take its spectral decomposition relatively to the eigenfunctions of the Laplacian $\Delta_2 = D_0 - 2iy\frac{\partial}{\partial x}$. We get a Selberg transform h_2 satisfying also i)-ii)-iii). Finally we integrate the product of the spectral decomposition (taken for $z=w$) by the Eisenstein series. These are the supplementary difficulties in comparison with the weight 0 :

1. We must isolate the contribuation of the eigenvalue 0 ($r=\frac{i}{2}$). We would like to take $h=\delta_{i/2}$. So, we consider $h(t,r)=\exp(-t(\frac{1}{4}+r^2))$, we work with a fixed t and let t goes to infinity at the end.

2. In addition to the residue, we need the constant term.

This is the result : Let g be the Fourier transform of $h(t,r)$

$$g(t,u) = \frac{1}{\sqrt{4\pi t}}\exp(-\frac{t}{4}-\frac{u^2}{4t})$$

Let

$$\Theta(t) = \sum_{l>2} \sum_{\substack{[\gamma] \\ \mathrm{tr}\gamma = l}} \frac{\log(N(\gamma_0))}{\sqrt{l^2-4}} g(t, \log(N(\gamma)))$$

The contant term at 1 of $R_F(s)$ is

$$C_F = -\frac{1}{2g\ vol}\int_0^{+\infty}(\Theta(t)-1)dt + \frac{a}{4\pi g} + O_\varepsilon(\frac{1}{N^{2-\varepsilon}})$$

where $a < 0$.

I haven't succeeded in getting the asymptotic behaviour of C_F on N from this expression. The final result follows from the identity :

$$\int_0^{+\infty}(\Theta(t)-1)dt = \lim_{s\to 1}\left(\frac{Z'(s)}{Z(s)} - \frac{1}{s-1}\right) - 1.$$

which, using the Selberg trace formula, also gives the vague relation I wrote in the begining.

Title: Arithmetic Betti numbers

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Page: (1)

Joint work with H. GILLET.

Let X be a projective variety over $\mathbb{Z}$ and $\bar{E} = (E, h)$ an hermitian vector bundle over X. Our goal is to define real numbers $h^q(X, \bar{E})$ satisfying properties similar to Betti numbers of coherent sheaves on algebraic varieties over a field.

1. The case of dimension one:

Let M be a finitely generated $\mathbb{Z}$-module of rank n and h a euclidean scalar product on the real vector space $M \underset{\mathbb{Z}}{\otimes} \mathbb{R}$. The arithmetic degree of (M, h) is the real number

$$\widehat{\deg}(M, h) := \log \#(M_{tors}) - \log \operatorname{vol}\left(\frac{\mathbb{R}^n}{M}\right),$$

where M_{tors} is the (finite) torsion subgroup of M and $M \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ is identified with $\mathbb{R}^n$ by means of an orthonormal basis for h.

We let

$$h^0(M, h) := \log \# \{ m \in M \ / \ \|m\| \leq 1 \},$$

and, if $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ is equipped with the dual metric h^* to h,

$$h^1(M, h) := h^0(M^*, h^*).$$

Clearly $h^0 \geq 0$ and $h^1 \geq 0$. Furthermore we have the following analogy of the Riemann-Roch ~~theorem~~ on curves:

Theorem 1:

$$\left| h^0(M,h) - h^1(M,h) - \widehat{\deg}(M,h) \right| \leq C(n)$$

where

$$C(n) := n \log(6) - \log\left(2\pi^{n/2} / \Gamma\left(\tfrac{n}{2}+1\right)\right).$$

2. Higher dimensions:

Let X be a regular projective flat scheme over $\mathbb{Z}$, $d = \dim(X/\mathbb{Z})$, E an algebraic vector bundle on X, and h a C^∞ metric on the corresponding holomorphic vector bundle $E_{\mathbb{C}}$ on $X(\mathbb{C})$ (invariant under complex conjugation). If a Kähler metric h_X on $X(\mathbb{C})$ is chosen, the finitely generated $\mathbb{Z}$-modules $H^q(X, E)$ are equipped with the L^2-scalar product, obtained by identifying

$$H^q(X, E) \otimes_{\mathbb{Z}} \mathbb{C} = H^q(X(\mathbb{C}), E_{\mathbb{C}})$$

with harmonic forms of type $(0,q)$ with coefficients in $E_{\mathbb{C}}$, and letting

$$\langle \alpha, \beta \rangle_{L^2} = \int_{X(\mathbb{C})} \langle \alpha(x), \beta(x) \rangle \frac{\mu^d}{d!}$$

if α, β are such forms. Here μ is the (normalized) Kähler ~~forms~~ form attached to h_X.

Let $\Delta_q = \bar\partial \bar\partial^* + \bar\partial^* \bar\partial$ on $A^{0q}(X, E)$ be the Laplace operator and $\zeta_q(s) = \operatorname{tr}(\Delta_q^{-s} \mid \ker(\Delta_q)^\perp)$ its zeta function. The Euler characteristic of $\bar E$ ~~[illegible]~~ is defined by

$$\chi_Q(\bar E) := \sum_{q \geq 0} (-1)^q \widehat{\deg}(H^q(X, E), h_{L^2}) + \sum_{q \geq 0} (-1)^q q \, \zeta_q'(0).$$

It is computed by the arithmetic Riemann-Roch theorem. To express it as ~~[illegible]~~ an alternated sum of Betti numbers we let $B^q = \operatorname{Im}(\bar\partial) \subset A^{0q}(X, E)$ and $\zeta_{B^q}(s) = \operatorname{tr}(\Delta_q^{-s} \mid B^q)$. Then we define

$$h^q(X, (E, h)) := h^0(H^q(X, E), h_{L^2}) + h^1(H^{q-1}(X, E), h_{L^2}) + \frac{1}{2} \zeta_{B^q}'(0)$$

(we use §1 for each cohomology group).

Theorem 1 implies

If $r_q = \dim_{\mathbb{C}} H^q(X(\mathbb{C}), E_{\mathbb{C}})$,

$$\left| \chi_Q(\bar E) - \sum_{q \geq 0} (-1)^q h^q(X, \bar E) \right| \leq \sum_{q=0}^{d} C(r_q).$$

Furthermore, if ω_X is the relative dualizing sheaf of X over $\mathbb{Z}$ (recall that X is regular), we have

$$h^q(X, \overline{E}) = h^{d+1-q}(\overline{X}, \overline{\omega_X} \otimes \overline{E}^*),$$

where $\overline{E}^*$ is the dual of $\overline{E} = (E, h)$ and ω_X is equipped with the metric induced by h_X.

3. Positivity:

One would like $h^q(X, \overline{E})$ to be bounded below, at least by a constant independent of the metric h on E. Investigating this question led ~~to~~ us to the following

Theorem 2: Let X be a Riemann surface and L an holomorphic line bundle on X. Assume that

$\dim_{\mathbb{C}} H^0(X, L) + \dim_{\mathbb{C}} H^1(X, L) \leq 2$.

Then there exists a constant C such that, for any choice of a metric on L, the Laplace operator Δ on C^∞ sections of L over X is such that

$$\det{}'(\Delta) \leq C,$$

where $\det{}'(\Delta) = \exp(-\zeta'_\Delta(0))$ is the regularized determinant of Δ.

The proof of Theorem 2 uses an inequality of Trudinger, Moser and Fontana.

Teil 2

Title: The arithmetic degree of line bundles on abelian varieties

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This talk was a report on joint work with Jürg Kramer.

Let $\mathcal{L}_{\tau_0}$ be an ample line bundle of type $D = \mathrm{diag}(d_1, \dots, d_n)$ on the abelian variety $A_{\tau_0} = \mathbb{C}^n / (\tau_0 \mathbb{Z}^n \oplus D\mathbb{Z}^n)$. Assume $\mathcal{L}_{\tau_0}$ has $n+1$ sections $s_{1,\tau_0}, \dots, s_{n+1,\tau_0}$ such that their divisors $\Theta_{1,\tau_0}, \dots, \Theta_{n+1,\tau_0}$ intersect properly on A_{τ_0}. Let $g_{j,\tau_0} = -\log \|s_{j,\tau_0}\|^2$ and consider the integral

$$(1) \quad \int_{A_{\tau_0}} g_{1,\tau_0} * \cdots * g_{n+1,\tau_0} = \int_{A_{\tau_0}} g_{1,\tau_0} \wedge \Omega_n + \int_{A_{\tau_0}} g_{2,\tau_0} \wedge \delta_{\Theta_{1,\tau_0}} \wedge \Omega_{n-1} + \cdots + \int_{A_{\tau_0}} g_{n+1,\tau_0} \wedge \delta_{\Theta_{1,\tau_0}} \wedge \cdots \wedge \delta_{\Theta_{n,\tau_0}}.$$

We used the notation $d_z d_z^c\, g_{k,\tau_0} + \delta_{\Theta_{k,\tau_0}} = \omega$, where ω is the $(1,1)$ form $\omega = \frac{i}{2}\, dz^t \cdot (\mathrm{Im}\,\tau)^{-1} \wedge d\bar{z}$ and $\Omega_k = \wedge^k \omega$. The integral (1) can be considered in its own right, although its importance appears as the contribution from analysis to the arithmetic degree of $\mathcal{L}_{\tau_0}$.

Author: Jay Jorgenson Page: 2

In the case $n=1$, one has the following formula. Let $d_1=1$, and then we are left to consider

$$(2)\quad \underset{\sim}{I}_1 = \int_0^1\int_0^1 \log\left|\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\tau,0)\right|^2 d\alpha\, d\beta$$

where $\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\tau,0) = \sum_{n=-\infty}^{\infty} \exp(\pi i(n+\alpha)^2\tau + 2\pi i(n+\alpha)\beta)$. It can be shown in a number of ways that one has the identity

$$I_1 = \frac{1}{12}\log|\Delta(\tau)| \quad \text{where } \Delta(\tau) = e^{2\pi i\tau}\prod_{n=1}^{\infty}(1-e^{2\pi i\tau})^{24}$$

In general, our work yields similar evaluations in all situations. The first result we obtain is the following.

<u>Theorem</u> - With assumptions as above, there is a Hausdorff nbhd $\mathcal{U}_0$ of τ_0 in $\mathfrak{H}_n$ (Siegel upper half space of dim n) such that the above situation extends to all $\tau\in\mathcal{U}_0$ and one has the variational formula

$$(3)\quad d_\tau d_\tau^c\left[\int_{A_\tau} g_{1,\tau}(z) * \cdots * g_{n+1,\tau}(z) + \frac{d_1\cdots d_n}{2}(n+1)!\,\log\det[\operatorname{Im}\tau)\right] = 0.$$

Having established the harmonicity of

$$(4)\quad \int_{A_\tau} g_{1,\tau} * \cdots * g_{n+1,\tau} + \frac{d_1\cdots d_n}{2}(n+1)!\,\log\det(\operatorname{Im}\tau)$$

On $\mathcal{U}_0$, we immediately conclude there exists a non-vanishing harmonic function $F_V(\tau)$ for every open, connected, simply connected $V \subseteq U_0$ such that

$$(A) = -\log|F_V(\tau)|.$$

In the remainder of our work, we extend $F_V(\tau)$ to a globally defined holomorphic function on all of $\mathfrak{H}_n$, and we show that the extended function $F(\tau)$ is a Siegel modular form with respect to a paramodular group $\Gamma_D(N)$ (where N is some additional level structure on A_τ). In addition, we are able to explicitly evaluate $F(\tau)$. Our main result is the following.

Theorem - Let A_{τ_0} be an abelian variety of dim $n \geq 2$, Let $\mathcal{L}_{\tau_0}$ an ample line bundle of type $D = \mathrm{diag}(d_1 \cdots d_n)$. Let $A_{n,D,N}$ be the universal a.v. with this data, and $\mathcal{L}$ the universal bundle on $A_{n,D,N}$, defined over the moduli space $\mathcal{A}_{n,D,N}$. Let $\mathcal{K} = e^*(\det \Omega^1_{A_{n,D,N}/\mathcal{A}_{n,D,N}})$ be the (locally trivial) bundle obtained as the pullback of the determinant of the relative cotangent bundle $\Omega^1_{A_{n,D,N}/\mathcal{A}_{n,D,N}}$ via the zero section e.

Let $\mathcal{M}(m_1, m_2) = \pi^* \mathcal{K}^{m_1} \otimes \mathcal{L}^{m_2}$ where $\pi: A_{n,D,N} \to A_{n,D,N}$.
For $m_1 >> m_2 >> 0$, let

$$s_{1,\tau}, \ldots, s_{n+1,\tau} \in \Gamma(A_{n,D,N}, \mathcal{M}(m_1, m_2))$$

be a "generic set" of $(n+1)$ sections, which are shown to intersect properly for all $\tau \in A_{n,D,N} \setminus S$, codim $S = 1$, and ~~some~~ subset of cardinality n intersects properly for $\tau \in A_{n,D,N} \setminus T$, codim $T = 2$. Then, for $\tau \notin S$, we have

$$\int_{A_\tau} g_{1,\tau} * \cdots * g_{n+1,\tau} = -\log |F(\tau)| - C_{n,D,m_1,m_2} \log \det(\operatorname{Im} \tau)$$

where $C_{n,D,m_1,m_2} = (m_1 + \frac{m_2}{2}) m_2^n \cdot d_1 \cdots d_n \cdot (n+1)!$,

$$F(\tau) = J \prod_{1 \le i_1 < \cdots < i_n \le n+1} \prod_{P \in \Theta_{i_1,\tau} \cap \cdots \cap \Theta_{i_n,\tau}} s_{i_{n+1},\tau}(P) e^{2\pi i m_2 d_P^t \tau d_P}$$

with $P = -\tau\alpha + \beta$ and $J \in \mathbb{C}$ is independent of the choice of sections.

As an example of such evaluations, we consider the case $n = 2$, $\mathcal{L}$ be the principal polarization. Take S to be those abelian surfaces which are not Jacobians of non-singular genus two curves. Let

P_1 be the 2-torsion point associated to the classical vector of Riemann constants and set $S_1 = \Theta(\tau, z+P_1)$. There are 20 choices of pairs of 2-torsion points P_2 and P_3 such that the sections $S_1 = \Theta(\tau, z+P_1)$, $S_2 = \Theta(\tau, z+P_1-P_2)$ and $S_3 = \Theta(\tau, z+P_1-P_3)$ of $\mathcal{L}^{\otimes 2}$ intersect properly for $\tau \notin S$. Our techniques apply to yield the following formula:

$$\sum_{P_2, P_3} \int g_{1,\tau} * g_{2,\tau} * g_{3,\tau} = -30 \log(\det \operatorname{Im} \tau) - 6 \log |\chi_{10}(\tau)| - c$$

where c is independent of τ and $\chi_{10}(\tau)$ is the weight 10 cusp form on $\mathfrak{h}_2$ with respect to $Sp_4(\mathbb{Z})$, as defined by Igusa, and is shown to equal
$$\chi_{10}(\tau) = \prod_{\substack{\text{even} \\ 2\text{-torsion}}} \Theta(\tau, z+Q)^2 \exp(2\pi i \, {}^t\!\alpha_Q \tau \alpha_Q).$$

Further examples, as well as extension of our techniques, are under consideration.

Title: REDUCTION of ABELIAN VARIETIES
(joint work with ALICE SILVERBERG).
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Let F be a field, $\bar{F}$ its separable algebraic closure, $G_F = Gal(\bar{F}/F)$ its Galois group. Let n be a positive integer, which is not divisible by char(F).

Let X be an abelian variety over F. We write X_n for the kernel of multiplication by n in $X(\bar{F})$.

1. LOCAL CASE. Let F be a field with a discrete valuation $v: F^* \to \mathbb{Z}$. In this section we assume that n is not divisible by the residual characteristic. We write

$e_n: X_n \times X_n^t \to \mu_n$

for the Weil pairing. It is non-degenerate and G_F-equivariant. Here X_1^t is the dual of X and $\mu_n \subset F^*$ is the group of roots of power n in $\bar{F}^*$.

Let us choose an extension (valuation map) $\bar{v}: \bar{F}^* \to \mathbb{Q}$ of v to $\bar{F}^*$ and let $I = I(\bar{v}) \subset G_F$ be the corresponding inertia group. The group I acts on all X_n. The action of I on μ_n is trivial.

According to Serre-Tate, the celebrated Néron-Ogg-Shafarevich criterion may be reformulated as follows.

X has good reduction at v if and only if $X_n = X_n^I$ for infinitely many n.

Theorem. Let $n \geq 5$. Then X has semistable reduction at v if and only if there exists a subgroup $A \subset X_n$, enjoying the following properties:

1) $A = A^I$;

2) Let $B = A^{\perp} \subset X_n^t$ be the orthogonal complement of A in X_n^t with respect to e_n. Then $B = B^I$.

Corollary (Raynaud criterion). Let $m \geq 3$ be a positive integer not divisible by the residual characteristic. If $X_m = X_m^I$ then X has semistable reduction at v.

Proof of the Corollary. Put $n = m^2 \geq 9 > 5$, $A = X_m$, $B = X_m^t$.

Corollary. Let $\lambda : X \to X^t$ be a polarization. Let us consider the pairing

$$e_{\lambda,n} : X_n \times X_n \to \mu_n, \quad e_{\lambda,n}(x,y) = e_n(x, \lambda y).$$

Assume that $(n, \deg(\lambda)) = 1$ and $n \geq 5$. Then X has semistable reduction at v if and only if there exists a subgroup $A \subset X_n$, enjoying the following properties:

1) A is a maximal isotropic subgroup in X_n with respect to $e_{\lambda,n}$;

2) $A = A^I$.

Hint! $B = \lambda(A) \subset X_n^t$.

2. GLOBAL CASE. Now, let F be a global field, ℓ a prime number different from char(K),

$$T_\ell(X) = \varprojlim_i X_{\ell^i}, \quad V_\ell(X) = T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

the corresponding $\mathbb{Z}_\ell$-Tate module and $\mathbb{Q}_\ell$-Tate module of X,

$$\rho_{\ell,X} : G_F \to \operatorname{Aut} V_\ell(X)$$

the corresponding ℓ-adic representation attached to X. We write $G_\ell(F,X) \subset GL(V_\ell(X))$ for the Zariski closure of $\rho_{\ell,X}(G_F)$. Let $G_\ell(F,X)^0$ be the connected identity component of $G_\ell(F,X)$. Serre proved that the group of connected components $G_\ell(F,X)/G_\ell(F,X)^0$.

Let $\operatorname{End}^0 X = \operatorname{End} X \otimes \mathbb{Q}$ be the algebra of $\bar{F}$-endomorphisms of X, Z the center of $\operatorname{End}^0 X$, $\mu_Z \subset Z^*$ the (finite) group of all elements of finite multiplicative order in Z^*. The algebra $\operatorname{End}^0 X$ acts naturally on $V_\ell(X)$; this action is faithful and commutes with the action of $G_\ell(F,X)^0(\mathbb{Q}_\ell)$. In particular, the intersection

$$\operatorname{End}^0 X \cap G_\ell(F,X)^0(\mathbb{Q}_\ell) \subset Z^*.$$

Theorem The intersection

$$\mu_{\ell,X} = \mu_Z \cap G_\ell(F,X)^0(\mathbb{Q}_\ell)$$

does not depend on the choice of ℓ.

Remark. The group $\mu_{\ell,X}$ always contains -1.

Remark. Assume that F is a number field and fix an embedding $\bar{F} \subset \mathbb{C}$. Let MT_X be the Mumford-Tate group of the complex abelian variety $X_{\mathbb{C}}$. If the Mumford-Tate conjecture is true for X then one may easily observe that $\mu_{\ell,X}$ coincides with the torsion subgroup of the center of $MT(\mathbb{Q})$. In other words, our Theorem is a corollary of Mumford-Tate conjecture in the number field case.

The proofs are contained in the following preprints.

A. Silverberg, Yu. G. Zarhin „Reduction of abelian varieties", alg-geom/9602014.

A. Silverberg, Yu. G. Zarhin „Images of ℓ-adic representations and automorphisms of abelian varieties", alg-geom/9603001.

Title: Motivic Weight Complexes (Joint work with C. Soulé)

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Page: 1

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The following question, asked by Serre, has its origin in the early 1960's. First some notation:

- Let k be an algebraically closed field of characteristic 0.
- Write Mot_k for the category of Grothendieck Chow motives over k. That is, the pseudo-abelian completion of the category of correspondences modulo rational equivalence between smooth projective varieties $/k$. N.B. We do not tensor with $\mathbb{Q}$.
- There is a contravariant functor h: Smooth Proj Vtys$/k \to \mathrm{Mot}_k$.

Question Is there a function

$$\chi_c : \text{Arbitrary varieties}/k \longrightarrow K_0(\mathrm{Mot}_k) \quad \begin{cases} = \text{The Grothendieck} \\ \text{group of } \mathrm{Mot}_k. \end{cases}$$

such that if X is smooth and projective,

$$\chi_c(X) = [h(X)]$$

and if $Y \subset X$ is Zariski closed,

$$\chi_c(X) = \chi_c(Y) + \chi_c(X-Y). \quad ?$$

I.e. χ_c behaves like the Euler Characteristic of cohomology with compact supports. Note It is easy to see from resolution of singularities that there is at most one such function.

Our first result is:

Theorem 1 The answer to the above question is YES.

In fact we can prove more. First we note that Mot_k is an additive category in which "contractible" short exact sequences (i.e. $0 \to A \xrightarrow{i} B \xrightarrow{\varepsilon} C \to 0$ s.t. $\exists\, g : C \to B$, $\varepsilon \cdot g = 1_C$) split. Following Verdier (unpublished work, IHES preprint) there is a well defined triangulated category of homotopy classes of maps between bounded below chain complexes in Mot_k, $\mathrm{Ho}^b(\mathrm{Mot}_k)$. In this category a map has a homotopy inverse $\Leftrightarrow$ its mapping cone is contractible.

<u>Theorem 2</u> There is a contravariant functor,

$$W : \text{Varieties}/k \longrightarrow \text{Hot}(\text{Mot}_k)$$

with the following properties:

1) For any variety X, $W(X)$ is homotopy equivalent to a bounded complex.
2) If $Y \subset X$, there is a triangle in $\text{Hot}(\text{Mot}_k)$

$$\begin{tikzcd} W(X-Y) \arrow[r] & W(X) \arrow[dl] \\ W(Y) \arrow[u, "\partial"] & \end{tikzcd}$$

degree 1

[This may be represented by an exact sequence of chain complexes]

3) If X is smooth and projective,

$$W(X) = h(X)$$

i.e. the motive of X, viewed as a complex concentrated in degree 0.

The relevance of Theorem 2 to Theorem 1 is given by:

$$\chi_c(X) = [W(X)] \in K_0(\text{Mot}_k).$$

Note that the class of $[W(X)]$ in K_0 is defined by virtue of 1) above.

<u>Remark</u> This complex is not quite the "mixed motive" of X, but it carries information about the weight filtration on cohomology, and can be used to give a weight filtration on cohomology with <u>integral</u> coefficients. The complex $W(X)$ is the E_1-term of the weight spectral sequence converging to the mixed motive of X.

Theorem 3 If $k = \mathbb{C}$, there is a spectral sequence, for any variety $X/\mathbb{C}$:

$$E_r^{p,q} \Rightarrow H_c^{p+q}(X, \mathbb{Z})$$

$$E_1^{p,q} = H^q(W^p(X))$$

This spectral sequence is independant, from E_2 onward, of the choice of complex of motives representing X. If we tensor this spectral sequence with $\mathbb{Q}$, we obtain the weight spectral sequence defined by Deligne in "Théorie de Hodge, III".

Remarks 1) This spectral sequence provides a purely algebraic-geometric construction of the weight filtration on cohomology, without the use of Hodge theory.

2) By look at the filtration on torsion, as well as the torsion in E_2, we ~~pa~~ obtain new ~~~~ invariants of singular non compact varieties

3) This spectral does not, as far as we know degenerate at E_2.

— · —

Methods of Proof

Definition A map $f: X \to Y$ of varieties is an envelope if

1) f is proper

2) $\forall$ field F, f induces a surjective map $X(F) \to Y(F)$

(Equivalently, each integral subscheme of Y is the birational image of a subscheme of X).

Author: Henri Gillet Page: 4

<u>Properties</u> 1) If $f: X \to Y$ and $g: Y \to Z$ are envelopes, then $g \cdot f$ is an envelope. (<u>Closed under Composition</u>)

2) If $f: X \to Y$ is an envelope, and $g: Y' \to Y$ is arbitrary, the induced map $f': X \times_Y Y' \to Y'$ is an envelope (<u>stability under base change</u>)

3) In characteristic zero, if X is an arbitrary variety, there exists an envelope $p: \tilde{X} \to X$ with $\tilde{X}$ non-singular.

<u>Remarks</u> • Even if X is irreducible, $\tilde{X}$ in 3) above need not be connected

• An étale map is <u>not</u> in general an envelope, even if it is proper.

• We can define a Grothendieck topology on the category of varieties by taking covering maps to be envelopes.

<u>Definition</u> A <u>hyperenvelope</u> of X $\pi: \tilde{X}_\cdot \to X$ is a simplicial scheme, augmented toward X, which is a hypercovering for the topology above. Equivalently, for all fields F, $\pi: \tilde{X}_\cdot(F) \to X(F)$ is a Kan fibration

<u>Proposition</u> $\exists$ nonsingular hyperenvelope $\pi: \tilde{X}_\cdot \to X$ for any variety X. If $\pi_i: \tilde{X}_{\cdot,i} \to X$ are two hyperenvelopes, they are dominated by a nonsingular hyperenvelope. Any map $f: Y \to X$ of varieties can be dominated by a map of nonsingular hyperenvelopes:

$$\begin{array}{ccc} \tilde{Y}_\cdot & \to & \tilde{X}_\cdot \\ \downarrow & & \downarrow \\ Y & \to & X \end{array}$$

Provisional Definition

1) If X is a projective, possibly singular, variety we choose a non-singular hyperenvelope $\pi : \hat{X}_\cdot \to X$, and set:

$$W(X) := h(\tilde{X}_0) \to h(\tilde{X}_1) \to \dots \to h(\tilde{X}_n) \xrightarrow{\delta_n} h(\tilde{X}_{n+1}) \to \dots$$

where $\delta_n = \sum_{i=0}^{n+1} (-1)^i h(d_i)$, $d_i : \tilde{X}_{n+1} \to X_n$ being the "face" maps of $\tilde{X}_\cdot$.

2) If X is quasi-projective, Let $\bar{X}$ = a compactification of X, $Y = \bar{X} - X$. Choose a map of non-singular envelopes dominating $Y \hookrightarrow \bar{X}$:

$$\begin{array}{ccc} \tilde{Y}_\cdot & \longrightarrow & \tilde{X}_\cdot \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \bar{X} \end{array}$$

and set

$$W(X) := \mathrm{Cone}\left(h(\tilde{X}_\cdot) \to h(\tilde{Y}_\cdot)\right)[1]. \qquad //$$

To make this definition effective, we must show that in $\mathrm{Hot}(\mathrm{Mot}_k)$, $W(X)$ is independant of the choice of hyperenvelope. We can reduce to the following, for X projective: Let $f : \tilde{X}'_\cdot \to \tilde{X}''_\cdot$ be a map of nonsingular hyperenvelopes of X. Then $h(\tilde{X}''_\cdot) \to h(\tilde{X}'_\cdot)$ is a homotopy equivalence.

The main ingredient here is the notion of Gersten Complex: For all $q \geq 0$ and variety X, define:

$$R_{*,q}(X) := \quad \cdots \to \bigoplus_{x \in X_{(q+2)}} K_2 k(x) \to \bigoplus_{x \in X_{(q+1)}} k(x)^* \xrightarrow{\text{div}} \bigoplus_{x \in X_{(q)}} \mathbb{Z}$$

Properties: • $H_0(R_{*,q}(X)) = CH_q(X)$

• $R_{*,q}(X)$ is covariant w.r.t. proper maps $X \to Y$.

• $H_*(R_{*,q}(X))$ is motivic i.e. the functors $X \mapsto H_*(R_{*,q}(X))$ factor through h.

• If $p: \tilde{X}_\cdot \to X$ is a hyperenvelope, $R_{*,q}(\tilde{X}_\cdot) \to R_{*,q}(X)$ is a quasi-isomorphism for all q.

We now use a generalized Manin principal:

Let $X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow$ be a chain complex in the free additive category on the category of ~~[illegible]~~ smooth projective varieties. Then:

Thm $h(X_\cdot)$ is split acyclic (i.e. contractible) $\Leftrightarrow \forall T, \forall q \geq 0$ $R_{*,q}(X_\cdot \times T)$ is acyclic

– We omit the proof.

Now if

$$\begin{tikzcd} \tilde{X}'_\cdot \arrow[rr, "f"] \arrow[dr, "p'"'] & & X''_\cdot \arrow[dl, "p''"] \\ & X & \end{tikzcd}$$

is a map of non-singular hyper-envelopes of X, we get

$$\begin{tikzcd} R^*_{*,q}(\tilde{X}'_\cdot \times T) \arrow[rr] \arrow[dr, "\simeq"'] & & R_{*,q}(\tilde{X}''_\cdot \times T) \arrow[dl, "\simeq"] \\ & R_{*,q}(X \times T) & \end{tikzcd}$$

in which the vertical maps are quasi-isomorphisms since they are ~~env~~ induced by hyperenvelopes. Hence the horizontal map is a quasi-isomorphism for all q & T. Therefore

$$h(\tilde{X}''_\cdot) \to h(\tilde{X}'_\cdot)$$

is a homotopy equivalence. //

Theorem 3 follows from cohomological descent since envelopes are proper.

Title: Equidistribution of small points

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We present a joint work with L. Szpiro and S. Zhang.

Thm 1 Let K a number field, A an abelian variety over K and $\sigma : K \hookrightarrow \mathbb{C}$ a morphism from K to $\mathbb{C}$. For each "strict" sequence $(P_n)_n$ of torsion points, (stric = no subsequences in a translate, by a torsion point of an abelian subvariety) we have

$$\frac{1}{\# \mathcal{O}(P_n)} \sum_{y \in \sigma(\mathcal{O}(P_n))} \delta_y$$

is weakly convergent to the normalized Haar measure $d\mu_\sigma$ of $A_\sigma = A \otimes_\sigma \mathbb{C}$.

$$\begin{cases} \delta_x = \text{dirac measure at } x \\ \mathcal{O}(P_n) = \text{Galois orbit of } P_n. \end{cases}$$

Thm 2 Let A/K an abelian variety over K.
Let x_n a strict sequence of "small" points (small := the Néron-Tate's height $h_{NT}(x_n) \longmapsto 0$).
The 3 following statements are equivalent.

(i) (Bogomolov conjecture)
$\{\sigma(\mathcal{O}(x_n)), n \in \mathbb{N}\}$ is dense for the Zariski topology of $A_\sigma(\mathbb{C})$

(ii) $\{\sigma(\mathcal{O}(x_n)), n \in \mathbb{N}\}$ is dense for the complex topology of $A_\sigma(\mathbb{C})$

(iii) $\{\sigma(\mathcal{O}(x_n)), n \in \mathbb{N}\}$ are equidistributed for the normalized Haar measure $d\mu_\sigma$:

$$\frac{1}{\# \mathcal{O}(x_n)} \sum_{y \in \sigma(\mathcal{O}(x_n))} \delta_y \xrightarrow{c.w} d\mu_\sigma .$$

Remarks ① The theorem of Raynaud, on torsion points of subvarieties of an abelian variety and theorem 2 implies theorem 1.

② Bogomolov conjecture implies equidistribution for all sequences of small points

③ This is the case if A is an elliptic curve

④ Let $\mathbb{F}$ the field of totaly real algebraic numbers. K a number field with $K \subset \mathbb{F}$. For any abelian variety A over K we have:

a) $\#\left(A_{Tors}(\mathbb{F})\right) < \infty$

b) $\displaystyle \inf_{\substack{x \in A(\mathbb{F}) \\ x \notin A_{Tors}(\mathbb{F})}} h_{NT}(x) > 0 .$

<u>Question</u>: Let x_n a sequence of algebraic numbers such that $h(x_n) \to 0$, ($h(\cdot)$ = canonical height). Do we have equidistribution for the Galois orbits of (x_n) respectively to the circle measure?

<u>Thm 3</u> Let $X/\mathcal{O}_K$ a projective arithmetic variety of dimension d. Let $\overline{L} = (L, \| \ \|)$ an hermitian line bundle on X such that
- L_K is ample
- $c_1(\overline{L}_\sigma)$ = (curvature form of $\overline{L}_\sigma$) is positive.

Let (x_n) a sequence of points of $X(\overline{K})$ converging to the generic point (= the subsequences (x_{n_p}), $p \in \mathbb{N}$ are Zariski dense)

① <u>Zhang</u>

$$\liminf_{x \in X(\overline{K})} h_{\overline{L}}(x_i) \geq \frac{\widehat{c}_1(\overline{L})^d}{d \cdot c_1(L_K)^{d-1}}$$ where

. $h_{\overline{L}}$ = height associated to $(L, \| \ \|)$.
. $\widehat{c}_1(\overline{L})$ = Gillet–Soulé's first arithmetic Chern classes.

② If $\lim h_{\overline{L}}(x_i) = \dfrac{\widehat{c}_1(\overline{L})^d}{d \cdot c_1(L_K)^{d-1}}$,

for each $\sigma : K \to \mathbb{C}$:

$$\frac{1}{\# \mathcal{O}(x_n)} \sum_{x_n^? \in \sigma(\mathcal{O}(x_n))} \delta_{x_n^?} \xrightarrow{c.w} d\mu_\sigma = \frac{c_1(\overline{L}_\sigma)^{d-1}}{c_1(L_K)^{d-1}}.$$

Title: A decomposition of spaces of automorphic forms, and some applications

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Page: 1

Let G be a connected reductive algebraic group defined over $\mathbb{Q}$, A_G the maximal $\mathbb{Q}$-split torus in the center of G, K_∞ a maximal compact subgroup of $G(\mathbb{R})$. Given an open compact subgroup $K_f \subset G(\mathbb{A}_f)$ the space

$$X_{K_f} := G(\mathbb{Q})\, A_G(\mathbb{R})^\circ \backslash G(\mathbb{A}) / K_\infty K_f$$

has finitely many connected components each of which has the form $\Gamma \backslash G(\mathbb{R})^\circ / K_\infty$ for an appropriate arithmetic subgroup Γ of G. Let (ν, E) be a finite-dimensional algebraic representation of $G(\mathbb{C})$; suppose that A_G acts on E by a central character χ_E. Varying K_f the cohomology groups $H^*(X_{K_f}, E)$ form a directed system, and the inductive limit

$$H^*(G, E) = \varinjlim_{K_f} H^*(X_{K_f}, E)$$

carries a natural $G(\mathbb{A}_f)$-module structure. Let $C^\infty(G(\mathbb{Q})\, A_G(\mathbb{R})^\circ \backslash G(\mathbb{A}))$ be the space of all complex valued C^∞-functions on

$G(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash G(\mathbb{A})$, and let $m_G \subset Lie(G(\mathbb{R}))$ be the intersection of $\ker(d\chi)$ for all rational characters χ of G. Then there is an isomorphism of $G(\mathbb{A}_f)$-modules

$$H^*(G,E) = H^*(m_G, K_\infty; C^\infty(G(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash G(\mathbb{A}))\otimes E)_{\lambda_E}$$

where the right hand side denotes the relative Lie algebra cohomology 'χ_E' means a twist of the natural $G(\mathbb{A}_f)$-action.

Let V_G be the subspace of functions of uniform moderate growth in the space $C^\infty(G(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash G(\mathbb{A}))$. Given a class $\{P\}$ of associate parabolic $\mathbb{Q}$-subgroups of G we denote by $V_G(\{P\})$ the space of elements which are negligible along Q for every parabolic $\mathbb{Q}$-subgroup $Q \notin \{P\}$, i.e. the constant term f_Q with respect to the $Q = M_Q A_Q N_Q$ is orthogonal to the space of cusp forms on M_Q. It was proved by Langlands that one has a direct sum decomposition

$$V_G = \bigoplus_{\{P\}\in\mathcal{C}} V_G(\{P\})$$

ranging over the set $\mathcal{C}$ of classes of associate parabolic $\mathbb{Q}$-subgroups of G (cf. [L], [BLS])

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra of $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))$, and let $J \subset Z(\mathfrak{g})$ be the annihilator of the dual representation $E^\vee$ in $Z(\mathfrak{g})$. Let

$$A_E \subset V_G = C^\infty_{umg}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \backslash G(\mathbb{A}))$$

be the space of functions which are annihilated by a power of J, and we put

$$A_{E,\{P\}} := A_E \cap V_G(\{P\})$$

for $\{P\} \in \mathcal{C}$. One obtains

$$A_E = \bigoplus_{\{P\} \in \mathcal{C}} A_{E,\{P\}}$$

as a direct sum. The inclusion $A_E \longrightarrow C^\infty(G(\mathbb{Q})A_G(\mathbb{R})^\circ \backslash G(\mathbb{A}))$ induces an isomorphism (of $G(\mathbb{A}_f)$-modules) in cohomology ([F], [W]), i.e.

$$H^*(G, E) = H^*(\mathfrak{m}_G, K_\infty; A_E \otimes E)_{\chi_E}$$

$$= \bigoplus_{\{P\} \in \mathcal{C}} H^*(\mathfrak{m}_G, K_\infty; A_{E,\{P\}} \otimes E)_{\chi_E}$$

In order to relate this general approach (and the cohomological consequences) to the use of Eisenstein series or residues of such to construct cohomology classes

in $H^*(G, E)$ at infinity (cf. [H], [S])
one can define a refinement

$$A_E = \bigoplus_{\{P\} \in \mathcal{C}} \bigoplus_{\varphi} A_{E,\{P\},\varphi}$$

of the decomposition above where the second term ranges over the set $\Phi_{E,\{P\}}$ of classes $\varphi = \{\varphi_P\}_{P \in \{P\}}$ of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$. For the exact definition of this set we refer to [FS]. One can give two alternate definitions for the spaces $A_{E,\{P\},\varphi}$) one by use of the constant term,

$$A_{E,\{P\},\varphi} = \{ f \in A_{E,\{P\}} \mid \text{for each } P \in \{P\} \text{ the constant term } f_P \text{ belongs to } \bigoplus_{\pi \in \varphi_P} L^2_{cusp,\pi}(L_P(\mathbb{Q}) \backslash L_P(\mathbb{A}))_{\chi_\pi} \otimes S(\check{\mathfrak{a}}_P) \}$$

the other one in terms of Eisenstein series, namely, we define

$\tilde{A}_{E,\{P\},\varphi}$ = space spanned by all possible residues and derivatives with respect to the parameter $\lambda \in (\check{\mathfrak{a}}_P)_{\mathbb{C}}$ of Eisenstein series

$E(\varphi, \lambda)$ starting from cusp forms φ of type φ at values λ in the positive Weyl chamber defined by P for which the infinitesimal character of $E^\vee$ is matched.

Theorem ([FS], 1.4)

(1) $A_{E,\{P\},\varphi} = \widetilde{A}_{E,\{P\},\varphi}$

(2) One has the direct sum decomposition ($\{P\} \in \mathcal{C}$)

$$A_{E,\{P\}} = \bigoplus_{\varphi \in \Phi_{E,\{P\}}} A_{E,\{P\},\varphi}$$

*

In turn, this gives rise to a direct sum decomposition of the cohomology

$$H^*(G,E) = \bigoplus_{\{P\}\in\mathcal{C}} \bigoplus_{\varphi\in\Phi_{E,\{P\}}} H^*(m_G, K_\infty; A_{E,\{P\},\varphi} \otimes E)_{\chi_\pi}$$

This decomposition confirms in a precise form certain ideas of Harder. However, it is still necessary to analyze the cohomological contribution of the spaces $A_{E,\{P\},\varphi}$. Note that the summand indexed by $\{G\}$ is the cuspidal cohomology. For non-vanishing results for the cuspidal cohomology we refer to [BLS]. In the cases $\{P\} \neq \{G\}$, this analysis amounts to a systematic treatment of possible poles and (partial) residues of Eisenstein series. We refer to [H, 1987], [S2-3] for some specific cases.

In view of arithmetic applications the relation of the spaces $A_{E,\{P\},\varphi}$ to Hecke eigenvalues is of interest. Given a finite place v of $\mathbb{Q}$ let $\mathcal{H}_v(G)$ be the Hecke algebra of K_v-biinvariant functions on $G(\mathbb{Q}_v)$, where K_v denotes the local component of a fixed good maximal compact subgroup of $G(\mathbb{A})$. Then the space $A_E^{K_v}$ of K_v-invariant vectors in A_E has a decomposition into associated Hecke eigenspaces $A_{E,\mathrm{I}}^{K_v}$ where the sum ranges over all maximal ideals I in $\mathcal{H}_v(G)$ and where $A_{E,\mathrm{I}}^{K_v}$ denotes the space of vectors in $A_E^{K_v}$ which are annihilated by a power of I. Then it is shown ([FS], 3.3) that the $A_{E,\{P\},\varphi}^{K_v}$ of K_v-invariant vectors is contained in such a generalized Hecke eigenspace $A_{E,\mathrm{I}_{\varphi,v}}^{K_v}$ for a suitable Hecke ideal $\mathrm{I}_{\varphi,v}$ depending only on φ.

In the case of the $\mathbb{Q}$-group $\mathrm{Res}_{F/\mathbb{Q}}(GL_{n}\times F)$ obtained by restriction of scalars from the general linear group defined over an algebraic number field F certain rationality results for the cohomological subspaces $H^{*}(m_{G}, K_{\infty}; A_{E,\{P\},\varphi} \otimes E)$ can be derived from this study.

As an example for the necessary analysis of the possible poles of Eisenstein series in determining the precise cohomological contribution of the spaces $A_{E,\{P\},\varphi}$ the case of the group GSp_{2} of symplectic similitudes defined over a totally real algebraic number field was discussed. The structure depends in a very subtle way on arithmetic data attached to $\varphi = \{\varphi_{P}\}$ resp. $\pi \in \varphi_{P}$, in particular special values of automorphic L-functions attached to automorphic representations of GL_{2}. For the case $GSp_{2}/\mathbb{Q}$ see [S3]

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Title: Elliptic curves with isomorphic Galois representations

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Page: 1

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Reference: Elliptic curves ~~with~~ and ternary Diophantine equations; Preprint IEM, Essen, 1996

In the following K is either a number field or a function field over a constant field K_0. For simplicity we shall assume that K_0 is algebraically closed and that $\mathrm{char}(K) = 0$.

1.) Some philosophy

Let A/K be an abelian variety with multiplicity 1, i.e. we assume that the simple factors of A are pairwise non-isogenous.

Definition: Let $H < A$ be a finite K-rational group scheme which is not contained in a proper abelian subvariety and which does not contain a non-trivial kernel of an endomorphism of K. Then H is exceptional.

The existence of exceptional subgroup H of large order should be related to interesting diophantine properties of A and K. One would expect finiteness results of the following type: let $h_K(A)$ (resp. $h_{geom}(A)$) be the height of A (resp. its geometric height). If $H < A$ is large enough then $h_K(A)$ (resp $h_{geom}(A)$) should be bounded by a number M depending on invariants of A and K and on dimension of the simple subvarieties of A, the genus of K or the irrationality degree of K.

We recall that if K is a number field then results of Faltings and, in an effective version, of Masser-Wüstholz give estimates for $|H|$ in terms of $\dim(A)$, $g(K)$ and the degree of the conductor of A.

2.) <u>An example</u>

Assume that $E/\mathbb{Q}$ is modular with minimal parametrization $\varphi_E : X_0(N_E) \longrightarrow E$.
Then $h_{\mathbb{Q}}(E) = \frac{1}{2}\log(\deg\varphi) + O(\log(N\log N))$, and so the height conjecture for elliptic curves (which is a sharpening of Szpiro's conjecture and which predicts that $h_K(j) \leq c\deg(N_E) + d(g(K))$ with $d(g(K))$ a linear function) is in this case

equivalent with an analogous estimate of $\deg \varphi_E$. Since the size of $\deg \varphi_E$ is (essentially) equal to $|\varphi^*(E) \cap \operatorname{Ker} \varphi_* \cap J_0^{new}(N_E)|$ it becomes obvious how the question about exceptional subgroups applied to $J_0(N)$ is related to the height conjecture. We get even more: Due to Wiles' result that all elliptic curves $/\mathbb{Q}$ with $\mathbb{Q}$-rational points of order 2 are modular we can relate $\deg \varphi_E$ to the A-B-C-conjecture of Masser-Oesterlé which predicts that (for any global field K) for all $x \in K \setminus \{0,1\}$ we have: $h_K(x) \leq \deg(\operatorname{supp}(x(x-1)))$.

3. Elliptic curves

We restrict us to the case that A is either an elliptic curve or a product of two (non-isogenous) elliptic curves.

If $\dim A = 1$ then our question boils down to

Conjecture 1: For $N \geq N_0\left(\substack{K \\ g(K) \\ d(K)}\right)^{*}$ there is a number $M = M\left(\substack{K \\ g(K) \\ d(K)}\right)$ such that all elliptic curves E

---

* make your choice!

with cyclic K-rational isogeny of degree N we have: $h_{geom}(E) \leq M$.

If K is a number field this implies that for N large enough E has to have complex multiplication.

If K is a function field we get at once that $h_{geom}(E)=0$ for $N \geq 12 g(K)$ and so Conjecture 1 is true with N depending on $g(K)$.

For number fields K the irrationality degree of $X_0(N)$ can be estimated from below by $c(d(K)) \cdot N$ with $c(d(K)) > 0$. Hence we get a very weak version of Conjecture 1: For $d \leq c(d(K)) \cdot \frac{N}{2}$ there is a number $M(d,N)$ such that for fields L with $[L:K] \leq d$ and elliptic curves E/L with L-rational cyclic isogeny of degree N we get: $h_{geom}(E) \leq M(d,N)$.

Now assume that $A = E_1 \times E_2$, $E_1 \not\sim E_2$. We can assume that exceptional subgroups H in $E_1 \times E_2$ don't contain cyclic subgroups and hence

$$H = \{(x, \alpha x);\ x \in E_{1,n} \text{ and } \alpha : E_{1,n} \xrightarrow[G_K]{\sim} E_{2,n}\}.$$

Conjecture 2: There are numbers $N_0 = N_0\begin{pmatrix} K \\ g(K) \\ d(K) \end{pmatrix}$ and $M = M\begin{pmatrix} K \\ g(K) \\ d(K) \end{pmatrix}$ such that for $|H| > N_0$ we get: $h_{geom}(E_i) < M$.

For K a number field this conjecture implies that the set $\{(E_1, E_2)$; E_1 not isogenous to E_2 and $H < E_1 \times E_2$ with $|H| > N_0\}$ mod twists is finite.

In this conjecture both N_0 and M depend on K.

Conjecture 2': There is a number N_0 such that for $N > N_0$ there are numbers $M = M\left(\frac{g(K)}{d(K)}, N\right)$ such that for exceptional $H < E_1 \times E_2$ with $|H| > N_0$ it follows that $h_{\text{geom}}(E_i) < M$.

For number fields these conjectures are essentially due to Darmon. They can be translated into properties of curves of small genus on diagonal surfaces $Z_{N,\varepsilon}$. Since for $N \geq 13$ these surfaces are of general type one can combine a conjecture of E. Kani about curves of genus ≤ 1 on $Z_{N,\varepsilon}$ with Lang's conjecture to get conjecture 2' for number fields. (cf. lecture of E. Kani).

Now fix one of the curves, let's say E_1, and look at the twisted modular curve $X_{\rho_{E_1,N},\varepsilon}(N)$ on $Z_{N,\varepsilon}$ parametrizing elliptic curves E with $\rho_{E_1,N} \cong \rho_{E,N}$

and the isomorphism has determinant ε.

Conjecture 3: If $E|K$ corresponds to a K-point on $X_{\rho_{E_1,N}}$ for some $N \geq N_0$ then $h_K(E) \leq M\left(\frac{g(K)}{d(K)}, E_0\right)$.

4.) Relations with ternary equations

There are strong arithmetical conditions imposed on E by E_1 if $(E_1, E) \hat{=} P \in X_{\rho_{E_1}}(N)(K)$. For instance E has to be semistable outside of a set S_0 determined by E_1 and K, and the discriminant divisor of E is (up to a small summand) divisible by N, so the height conjecture for elliptic curves implies conjecture 3 (with dependence on $g(K)$) and since this conjecture is true over function fields conjecture 3 is true if K is a function field.

The diophantine background of conjecture 3 becomes even more obvious if we assume that $E_2 \subset E(K)$. Let (x_1, x_2, x_3) be the x-coordinates of the points of order 2. Then there are elements $a, b, c \in K$ with $\mathrm{supp}(abc) \subset S_0$

and that (x_1, x_2, x_3) is a solution of $aX^p - bY^p = cZ^p$.
Hence conjecture 3 for E with $E_2 \subset E(K)$ follows from the

<u>Asymptotic Fermat Conjecture:</u> Let S_0 be a finite set of places of K. There exists a number $M = M(g_{(K)}^K, S_0)$ such that for all a, b, c with $\mathrm{supp}(abc) \subset S_0$, and for all $N \geq 5$ and $(x_1, x_2, x_3) \in K^3$ with
$a x_1^N - b y_1^N = c z_1^N$ we get: $h_K(x_1, x_2, x_3) \leq M$.

5.) $K = \mathbb{Q}$

For $K = \mathbb{Q}$ we can apply the results of Wiles, Ribet and Mazur about modularity of elliptic curves, congruence primes and bounds for cyclic isogenies to get:
Conjecture 3 restricted to elliptic curves with rational points of order 2 is equivalent with the Asymptotic Fermat conjecture.
The relation between elliptic curves with isomorphic Galois representations for some level N and solutions of equations of Fermat type can be used to

set finiteness results for solution of these equations.

We end by citing a nice example of Mazur:

If q is a prime such that $2^4 x^N \pm y^N = q z^N$ has solutions for infinitely many numbers N then $q = 17$.

Title: P-adic automorphic L-functions

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The original motivation for constructing p-adic L-functions was the demand for analogues of the Kummer congruences satisfied by Bernoulli numbers viewed as special values of the Riemann zeta function. A standard technique is to try to find a p-adic measure $\mu = \mu_p$ on $\mathbb{Z}_p^\times$ whose p-adic Mellin transform becomes an interpolating function for a given infinite set of algebraic special L-values. Hence we concentrate on measures and more general on distributions μ.

All successfully treated cases are L-functions attached to automorphic representations $\pi = \bigotimes_{\ell} \pi_\ell$. But we have the unpleasant situation that the distribution μ_p is by construction a *global* invariant of π.

Question: Can we somehow associate μ_p to the local component π_p? Or, to what extend is μ_p a *local* invariant?

In this talk I consider representations of $GL(n)$ and want to explain a purely algebraic local construction of a certain p-adic distribution $\tilde{\mu}$ attached to π_p, which is the key object for p-adic interpolation.

The first ingredient of the construction is

(1) Extension of the standard Hecke algebra

Let

$G := GL_n(\mathbb{Q}_p) > B$ standard Borel $> U$ unipotent subgr.

$K := GL_n(\mathbb{Z}_p) > K_B := K \cap B$.

and $\mathcal{H} := \mathcal{H}(K, G)$ the Hecke algebra of $\mathbb{C}$-valued, compactly supported, K-biinvariant functions on G. The Satake Isomorphism

$$\mathcal{H} \xrightarrow{\sim} \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{\mathrm{Sym}}$$

sends the generators

$$T_\nu := K \begin{pmatrix} I_{n-\nu} & \\ & pI_\nu \end{pmatrix} K \longmapsto p^{\langle \nu \rangle} \cdot \sigma_\nu(X_1, \dots, X_n),$$

where $\langle \nu \rangle := \frac{1}{2}\nu(\nu+1)$.

<u>Gritsenko</u> (1992): The Hecke algebra $\mathcal{H}_B := \mathcal{H}(K_B, B)$ is a ring extension of $\mathcal{H}$ s.that the Hecke polynomial

$$H(X) := \sum_{\nu=0}^{n} (-1)^\nu p^{\langle \nu-1 \rangle} T_\nu X^{n-\nu} \in \mathcal{H}[X]$$

decomposes over $\mathcal{H}_B$ as $H(X) = \prod_i (X - U_i)$ where

$$U_i := K_B \begin{pmatrix} I_{i} & \\ & p I_{n-i} \end{pmatrix} K_B .$$

In this non-commutative ring $\mathcal{H}_B$ there are interesting elements which do commute. Let $V_\nu := p^{-\langle \nu-1 \rangle} U_1 \cdots U_\nu$ and $t := \mathrm{diag}(p^{n-1}, p^{n-2}, \dots, 1)$.

<u>Prop. 1</u>: The V_ν's commute and

$$\prod_{\nu=1}^{n-1} V_\nu = \sum_u u t K_B$$

where u runs over a representative system for $U(\mathbb{Z}_p) / t\, U(\mathbb{Z}_p) t^{-1}$.

This operator is usually used to characterize distributions.

(2.) Hecke operators and distributions

Let

$\mathcal{M}$:= space of $\mathbb{C}$-valued, K_B-right invariant functions on G

$\mathcal{H}_B$ acts naturally on $\mathcal{M}$. For any $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{C}^{n-1}$ define the Hecke operator in $\mathcal{H}_B$

$$\mathcal{S}_{\underline{\lambda}} := \prod_{i=1}^{n-1} \prod_{\substack{j=1 \\ j \neq i}}^{n} (\lambda_i p^{1-j} V_{j-1} - V_j) .$$

Prop. 2: Suppose $\varphi \in \mathcal{M}$ and $\underline{\lambda}$ are such that $H(\lambda_\nu)\varphi = 0$ for $\nu = 1, \dots, n-1$. Then $\varphi_{\underline{\lambda}} := \mathcal{S}_{\underline{\lambda}} \varphi$ is a simultaneous eigenfunction of $V_1, \dots, V_{n-1}$ and with $\eta_\nu := p^{-\langle \nu - 1 \rangle} \cdot \prod_{i=1}^{\nu} \lambda_i$ we have $V_\nu \varphi_{\underline{\lambda}} = \eta_\nu \cdot \varphi_{\underline{\lambda}}$.

$\mathcal{U}^{(0)} := \mathcal{U}(\mathbb{Z}_p)$ has a filtration $\mathcal{U}^{(0)} \supset \mathcal{U}^{(1)} \supset \mathcal{U}^{(2)} \supset \dots$ by $\mathcal{U}^{(k)} := t\, \mathcal{U}^{(k-1)} t^{-1}$. A distribution on $\mathcal{U}^{(0)}$ with values in any addition group A is by definition a function

$$\tilde{\mu} : \{ u\,\mathcal{U}^{(k)} ;\ u \in \mathcal{U}^{(0)},\ k \geq 0 \} \longrightarrow A$$

such that

$$\tilde{\mu}(u\,\mathcal{U}^{(k)}) = \sum_{v} \tilde{\mu}(uv\,\mathcal{U}^{(k+1)}) \quad (v \text{ runs in } \mathcal{U}^{(k)} \bmod \mathcal{U}^{(k+1)})$$

Prop. 3: If $\varkappa_{\underline{\lambda}} := \prod_{\nu=1}^{n-1} \eta_\nu \neq 0$, then

$$\mu_{\underline{\lambda}}(u\,\mathcal{U}^{(k)}) := \varkappa_{\underline{\lambda}}^{-k} \cdot \varphi_{\underline{\lambda}}(g\,u\,t^k)$$

defines a function valued distribution.

(3) Application to unramified generic representations

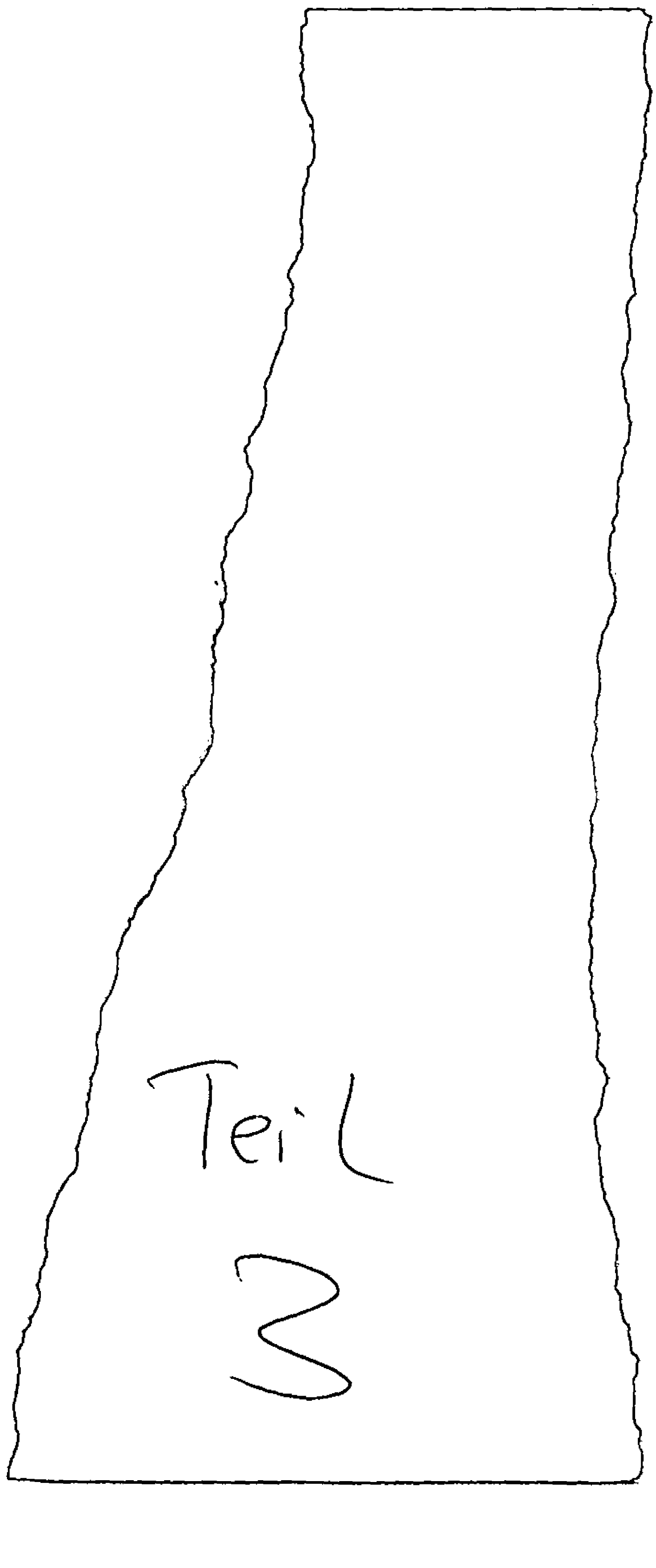
Suppose π_p unramified (i.e. $\pi_p^K \cong \mathbb{C}$) and generic (i.e. π_p has Whittaker model). In particular $\exists$ Whittaker fctn w^0 K-invariant such that $w^0(e) = 1$. By the scalar action of $\mathcal{H}$ on π_p^K we have an algebra homom. $\chi: \mathcal{H} \to \mathbb{C}$ and hence get a $\underline{\lambda} \in \mathbb{C}^n$ by the zeros of image of the Hecke polynomial

$$\chi(H(X)) = \prod_{i=1}^{n} (X - \lambda_i) \quad \text{in } \mathbb{C}[X].$$

Now apply Prop. 2/3 to $w^0 \ (\in \mathcal{M})$!

<u>**Theorem:**</u> We have $w^0_{\underline{\lambda}}(e) \neq 0$ and hence we get a non-trivial distribution $\mu_{\underline{\lambda}}$ on $\mathcal{U}$ with values in the Whittaker space.

If π_p is <u>ordinary</u>, i.e. all λ_i are algebraic and can be arranged s.th. $|\lambda_i|_p = p^{1-i}$ (in all $|\,\cdot\,|_{\mathfrak{P}} = 1$), we renormalize $w^0_{\underline{\lambda}}$ by demanding $w^0_{\underline{\lambda}}(e) \overset{!}{=} 1$. This "<u>distribution new vector</u>" is the key to p-adic interpolation. A global zeta integral usually leads to a $\mathbb{C}$-valued distribution attached to special L-values. The really hard problem then is to prove algebraicity and the p-adically boundedness of the distribution, which is far from being settled in general. For instance for ~~a~~ cuspidal (cohomological) automorphic representations of GL_3 the algebraicity ~~it~~ was only recently proven by Mahnkopf 96.


Teil
3

Title: Hecke eigenvalues and Fourier coefficients

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We will discuss various results on the estimation of eigenvalues and Fourier coefficients of Siegel cusp forms of arbitrary genus $g \geq 1$. Precise references for (most of) the material discussed here can be found in [1].

Let $\mathfrak{H}_g = \{ Z \in \mathbb{C}^{(g,g)} \mid Z' = Z, \operatorname{Im}(Z) > 0 \}$ be the Siegel upper half-space of genus g and $\Gamma_g = Sp_g(\mathbb{Z})$. For $k \in \mathbb{N}$ we denote by $S_k(\Gamma_g)$ the space of Siegel cusp forms of weight k w.r.t. Γ_g, i.e. the space of holomorphic functions $F: \mathfrak{H}_g \to \mathbb{C}$ which satisfy $F((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k F(Z)$ $\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ and which have a Fourier expansion $F(Z) = \sum_{T>0} a(T) e^{\pi i \, tr(TZ)}$, where T runs over all positive definite even integral matrices of size g.

For a prime p, let T_p be the usual Hecke operator on $S_k(\Gamma_g)$ (one may consider more general Hecke operators, too).

Problem A. Given a Hecke eigenform $F \in S_k(\Gamma_g)$ with eigenvalues λ_p, how does λ_p $(p \to \infty)$ grow? (More generally, how do the Satake p-parameters $\alpha_{1,p}, \ldots, \alpha_{g,p}$ $(p \to \infty)$ grow?)

Problem B. Given $F \in S_k(\Gamma_g)$, how do the Fourier coefficients $a(T)$ grow? More precisely, one wants estimates of the form $a(T) \ll (\det T)^c$, where $c > 0$ is "small" and depends only on g and k.

Of course, if $g=1$ then A and B coincide, and one has Deligne's theorem (previously the Ramanujan-Petersson conjecture) stating that $\lambda_p \ll_\varepsilon p^{\frac{k-1}{2}+\varepsilon}$ $(\varepsilon>0)$. However, for $g \geq 2$, A and B in general do not coincide; for example, estimates for the Fourier coefficients in general cannot be deduced from those for the eigenvalues. Since the Hecke operators act on Fourier coefficients, one can nevertheless ask

<u>Problem C.</u> What bounds for the Fourier coefficients imply what bounds for the eigenvalues?

Concerning A, one has the

<u>Generalized Ramanujan-Petersson conjecture</u> (Kurokawa, Satake, 1963). One should have

$$\lambda_p \ll_\varepsilon p^{\frac{gk}{2}-\frac{g(g+1)}{4}+\varepsilon} \qquad (\varepsilon>0).$$

(More generally, all the p-Satake parameters should have absolute value 1).

If $g \geq 3$, the conjecture so far is not known in a single case. For $g=2$, it is known to be wrong if F is a Saito-Kurokawa lift of a form $f \in S_{2k-2}(\Gamma_1)$ (k even). However, recently Weissauer (1993) - using the trace formula in the context of Shimura varieties - proved the conjecture if $g=2$ and F is not a Saito-Kurokawa lift.

For arbitrary $g \geq 2$, one knows according to Duke-Howe-Li (1992) that

$$\lambda_p \ll_\varepsilon p^{\frac{gk}{2}-d_g+\varepsilon} \qquad (\varepsilon>0).$$

where one can take

$$d_g := \begin{cases} g(g+1)/12 & (g \geq 2) \\ g(g+1)/8 & (g=2^\nu,\ \nu \geq 1) \\ 1 & (g=2). \end{cases}$$

The proof uses local representation theory.

Also, for arbitrary $g \geq 2$ one knows according to Faltings-Chai that $\left|\prod_{\nu=1}^{g} \alpha_{\nu,g}\right| = 1$.

Concerning B, our knowledge so far is still extremely limited. Of course, one always has the Hecke bound

$$a(T) \ll_F (\det T)^{k/2}.$$

Using some arguments from analytic number theory and assuming in addition that the Fourier coefficients are "equidistributed" in a certain sense, one may hope for the truth of the following

<u>Conjecture</u> (Resnikoff-Saldaña, 1974). One should have

$$a(T) \ll_{F,\varepsilon} (\det T)^{\frac{k}{2} - \frac{g+1}{4} + \varepsilon} \qquad (\varepsilon > 0).$$

So far, this conjecture is not known in a single case for $g \geq 2$, and it is wrong if $g = 2$ and F is a Saito-Kurokawa lift.

(Slight) improvements upon Hecke's bound have been obtained by several authors, including Böcherer-Raghavan (1988), Fomenko (1988), Raghavan-Weissauer (1989), Kitaoka ($g=2$; 1984), Böcherer-K. (1993), K. ($g=2$; 1992/93), Breulmann ($g=3$; 1996). For arbitrary $g \geq 2$ (and $k > g+1$), the best result proved so far seems to be

$$a(T) \ll_{F,\varepsilon} (\det T)^{k/2 - 1/2g - (1-1/g)\beta_g + \varepsilon} \qquad (\varepsilon > 0)$$

with

$$\beta_g^{-1} := 4(g-1) + 4\left[\frac{g-1}{2}\right] + \frac{2}{g+2}$$

(Böcherer, K., 1993).

The proof uses Fourier-Jacobi expansions and the theory of Jacobi forms.

Note that in the limiting case $g \to \infty$ the above estimate is not better than Hecke's bound.

However, one can show (K., 1994/95) that for each fixed $g \equiv 0 \pmod 4$, there is an infinite sequence of weights $k \to \infty$ and for each such k a non-zero form $F_k \in S_k(\Gamma_g)$ whose Fourier coefficients satisfy the bound

$$a_k(T) \ll_{F_k,\varepsilon} (\det T)^{k/2 - 1/2 + \varepsilon} \qquad (\varepsilon > 0).$$

The functions F_k are constructed explicitly as theta series attached to certain positive definite unimodular lattices of rank $2g$ and spherical harmonics.

Concerning C, Duke-Howe-Li (1992) showed (using simple results on the action of T_p on Fourier coefficients) that

$$\lambda_p a(T) - a(pT) \ll_F (\det T)^{k/2} p^{\frac{gk}{2} - 1}$$

Supposing that $a(T) \ll_F (\det T)^{k/2 - \alpha}$ $\forall T > 0$ for some fixed $\alpha \geq 0$, one immediately gets from this

$$\lambda_p \ll_F p^{\frac{gk}{2} - \min\{1, \alpha g\}}$$

However, going more deeply into the Hecke theory, one can show (K., 1995) the following: suppose that $a(T) \ll_F (\det T)^{k/2 - \alpha}$ $(\alpha \geq 0)$ holds $\forall T > 0$. Then

$$\lambda_p a(T) - a(pT) \ll_F (\det T)^{k/2 - \alpha} p^{\frac{gk}{2} - (g\alpha - \gamma_\alpha)} \qquad (p \nmid 2\det(T))$$

where

$$\gamma_\alpha := \begin{cases} 0 & (0 \leq \alpha \leq 1) \\ 2\alpha - 2 & (1 \leq \alpha \leq 3/2) \\ (\alpha - \frac{1}{2})^2 & (\alpha \geq 3/2). \end{cases}$$

(Note that for $g \geq 2$ the function $\alpha \mapsto g\alpha - \gamma_\alpha$ $(0 \leq \alpha \leq \frac{g+1}{4})$ is non-negative and non-decreasing.) In particular, one has that

$$\lambda_p \ll_F p^{\frac{gk}{2} - (g\alpha - \gamma_\alpha)};$$

if $g \geq 3$, then the conjecture of Resnikoff-Saldaña implies the Ramanujan-Petersson conjecture.

References.

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Title: Torsion of elliptic curves over number fields

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TORSION OF ELLIPTIC CURVES OVER NUMBER FIELDS

Let K be a number field, let d denote its degree and let E be an elliptic curve defined over K. The theorem of Mordell-Weil states that the group of rational points E(K) is finitely generated; in particular its torsion subgroup $E(K)_{tors}$ is finite. The uniform boundedness conjecture asks whether the order of $E(K)_{tors}$ is bounded by a constant depending only on d. This statement is now proved.

The first result in this direction was obtained in 1977 by B. Mazur, who treated the case $d = 1$: he proved that for an elliptic curve over $\mathbf{Q}$, the order of $E(\mathbf{Q})_{tors}$ is bounded by 16, and even gave a list of all possible isomorphism classes of such groups.

The next step is due to S. Kamienny: he proved in 1988 that for $d = 2$ the order of $E(K)_{tors}$ is bounded by 24; M. Kenku and F. Momose gave a list of the possible torsion subgroups in this case.

In 1993, S. Kamienny and B. Mazur succeeded to treat the cases where $d \leq 8$, but without giving an explicit bound for $|E(K)_{tors}|$. Soon after, their result was extended to $d \leq 14$ by D. Abramovich.

In 1994, L. Merel proved that the primes dividing $|E(K)_{tors}|$ are bounded by a constant depending only on d. It was shown by J. Oesterlé that this constant can be taken equal to $(3^{d/2}+1)^2$. As was already remarked previously by S. Kamienny and B. Mazur, the theorem of Merel implies the uniform boundedness conjecture; however their argument uses the Mordell conjecture proved by Faltings, and therefore does not yield an effective bound of $|E(K)_{tors}|$.

Quite recently, P. Parent generalized in his (still unpublished) thesis the method of Merel method and obtained an explicit bound for the powers of a prime dividing $|E(K)_{tors}|$. One easily deduces from his result that $|E(K)_{tors}|$ is bounded by $\exp(2^{14}d^6)$. We can hope in a near future a drastic improvement of this bound by a direct study of $|E(K)_{tors}|$ rather than of its prime power divisors.

The previous works build in some sense a pyramid, since each of them heavily relies on the ideas and techniques introduced in the preceding ones. We have tried in the lecture to sketch the final proof and to trace the history of its main ingredients.

Title: Applications of Diagonal Quotient Surfaces to a Problem of Mazur
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The "problem of Mazur" may be formulated as follows.

Question. Let E/K be an elliptic curve over a number field K. To what extent is the isogeny class of E/K determined by its Galois representation

$$\rho_{E,N} : G_K = \mathrm{Gal}(\bar{K}/K) \to \mathrm{Aut}(E[N]) \simeq GL_2(\mathbb{Z}/N\mathbb{Z})$$

for one (large) integer N?

In 1994, H. Darmon proposed the following two conjectures.

Conjecture 1: (Darmon) For each number field K, there is a constant M_K such that

$$\rho_{E,N} \simeq \rho_{E',N} \text{ for one } N \geq M_K \Rightarrow E \sim E'.$$

Conjecture 2. (Darmon) Let

$$\mathcal{S}_{N,K} = \{ (E, E')/_K : \rho_{E,N} \simeq \rho_{E',N} \ \&\ E \not\sim E' \}.$$

non-isogeneous

Then there is an absolute constant M such that

$$\# \mathcal{S}_{N,K} < \infty, \quad \text{for all } N \geq M \text{ and all } K.$$

Remarks: 1) Conjecture 2 + "$\mathcal{S}_{N,K} = \emptyset$ for $N \underset{K}{>>} 0$" $\Rightarrow$ Conjecture 1.

2) Darmon remarked that "it would be very interesting to formulate a convincing guess about the precise value of M".

Here I would like to suggest the following:

<u>Conjecture 3</u>. Conjecture 2 is true with $H = 13$, if one assumes in addition that N a prime.

In fact, a stronger conjecture should be true. To formulate it, consider the following terminology.

<u>Definition</u>. A G_K-isomorphism $\psi : E[N] \xrightarrow{\sim} E'[N]$ is called <u>trivial</u> if there exists a cyclic isogeny $f : E \to E'$ of very small degree d (i.e. of degree $d \leq 27$ and $d \neq 22, 23, 26$) such that

$$f|_{E[N]} = k\psi, \quad \text{for some } k \text{ with } (k,N)=1.$$

Put

$$\mathcal{J}_{N,K} = \{(E,E')_{/K} : \exists \text{ trivial } \psi : E[N] \xrightarrow{\sim} E'[N]\}$$

$$\mathcal{J}'_{N,K} = \{(E,E')_{/K} : \exists \text{ non-trivial } \psi : E[N] \xrightarrow{\sim} E'[N]\}$$

<u>Conjecture 4</u>. For any number field K we have

$$\# \mathcal{J}'_{N,K} < \infty, \quad \text{if } N \geq 13 \text{ and } N \text{ is prime}$$

Note that Conjecture 4 $\Rightarrow$ Conjecture 3 since $\mathcal{J}'_{N,K} \supset \mathcal{S}_{N,K}$. However, the analogue of Conj. 1 cannot be true for the $\mathcal{J}'_{N,K}$'s for we have $\mathcal{J}'_{N,K} \neq \emptyset$ for infinitely many N's.

The reason for proposing Conjecture 4 is that it has a very natural geometric interpretation in terms of:

<u>Diagonal</u> <u>Quotient</u> <u>Surfaces</u>: – these are defined as follows.

<u>Given</u>: a curve X

$G \le \mathrm{Aut}(X)$ finite group of auto's

$\alpha \in \mathrm{Aut}(G)$,

<u>Let</u>: $Y = X \times X$ product surface

$\Delta_\alpha = \{(g, \alpha(g))\} \le G \times G$

$Z = Z_{X,G,\alpha} = \Delta_\alpha \backslash Y$ quotient surface

This quotient surface (and/or its desingularization $\tilde{Z}_{X,G,\alpha}$) will be called a (twisted) <u>diagonal</u> <u>quotient</u> <u>surface</u>.

These are relevant here because:

<u>Proposition 1</u>. The functor $Z_{N,\varepsilon}$, defined by

defined via Weil N-pairing

$$Z_{N,\varepsilon}(K) = \{(E_1, E_2, \psi)\}/\simeq, \text{ where } E_i/K \text{ ell. curves}, \ \psi : E_1[N] \xrightarrow{\sim} E_2[N], \text{[illegible]}$$

is coarsely representable by (an open subset of) the diagonal quotient surface $Z_{N,\varepsilon} = Z_{X_N, G_N, \alpha}$ where

$$X_N = X(N), \quad G_N = SL_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}, \quad \alpha_\varepsilon : g \mapsto Q_\varepsilon g Q_\varepsilon^{-1}, \quad Q_\varepsilon = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}.$$

<u>Remarks</u>: 1) These "modular d.q.s." are analogous to <u>Hilbert</u> <u>modular</u> <u>surfaces</u> (with $\Delta = N^2$).

2) These surfaces have canonical models $/\mathbb{Q}$.

Thus: the problem of classifying isomorphisms between the $\rho_{E,N}$'s becomes (essentially) the diophantine problem of finding rational points on $Z_{N,\varepsilon}$; in fact:

$$(*) \qquad Z_{N,\varepsilon}(K) \text{ "=" } J_{N,K,\varepsilon} \,\dot\cup\, J'_{N,K,\varepsilon} \,\dot\cup\, \mathrm{cusps}(K)$$

Since one expects surfaces of general type to have fewer rational points, it becomes important to know when $Z_{N,\varepsilon}$ has general type.

<u>Theorem 1.</u> (C.F. Hermann, K.-Schanz) The rough classification type of $Z_{N,\varepsilon}$ may be determined explicitly and depends only on p_g ($= p_a$). In particular:
$Z_{N,\varepsilon}$ is of general type $\Leftrightarrow$ $p_g \geq 3$.

<u>Cor.</u> $Z_{N,\varepsilon}$ is of general type ($\forall\varepsilon$) $\Leftrightarrow$ $N \geq 13$.

To understand the meaning of the sets $J_{N,K,\varepsilon}$ and $J'_{N,K,\varepsilon}$, we require the <u>Hecke correspondences</u> on $Z_{N,\varepsilon}$:

Let T_n denote Hecke correspondence on $X(N)\times X(N)$, given by a diagram: $X(N) \xleftarrow{T_n} \rightarrow X(N)$ (for $(n,N)=1$)

$T_{n,k}$ its "twist" by $\langle k\rangle \times id$ ($\langle k\rangle \in G_N$, $k \in (\mathbb{Z}/N\mathbb{Z})^*$.)

$\overline{T}_{n,k}$ the image of $T_{n,k}$ via projection $X(N)\times X(N) \to Z_{N,\varepsilon}$

Then: $\overline{T}_{n,k} \overset{birat}{\sim} X_0(n) \Longleftrightarrow k^2 n\varepsilon \equiv 1\,(N)$,

and one has:

$$J_{N,K,\varepsilon} \text{ "=" } \bigcup_{\substack{n\leq 27\\ n\neq 22,23,26\\ k^2 n\varepsilon \equiv 1\,(N)}} \overline{T}_{n,k}(K) = \bigcup_{\substack{n\\ g(\overline{T}_{n,k})\leq 1}} \overline{T}_{n,k}(K).$$

This leads to:

Conjecture 5. If $N \geq 13$ is prime, then every curve $C \subset Z_{N,\varepsilon}$ with $g(C) \leq 1$ is a Hecke correspondence, i.e. $C = T_{n,k}$ (with n small) etc.

Remark: Clearly: Conjecture 4 $\Rightarrow$ Conjecture 5; the converse holds if one has that Lang's Conjecture is true for $Z_{N,\varepsilon}$:

Conjecture (Lang) If Z is a surface of general type, and $Z_{exc} = \bigcup_{\substack{g(C)\leq 1 \\ C \subset Z}} C$, then $Z' = Z \setminus Z_{exc}$ is "Mordellic", i.e. $\#Z'(K) < \infty$ for every K.

Evidence for Conjecture 5:
a) Every $C \subset Z$ with $g(C) \leq 1$ "look like" a $T_{n,k}$.
b) Conjecture 5 implies

Conjecture 6 (Hermann) The minimal model $\tilde{Z}^{min}_{N,\varepsilon}$ is obtained from $\tilde{Z}_{N,\varepsilon}$ by blowing down "known curves".
- this is analogous to a conjecture of Hirzebruch for Hilbert modular surfaces (which was proven by Hermann in certain cases).

Theorem 2 (Hermann) Conjecture 6 is true if $N \equiv 7 (8)$ (N prime) and $\varepsilon \equiv -1 (N)$.

Title: λ-structure in arithmetic K_0-theory
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An arithmetic variety X is a scheme that is quasi-projective and generically smooth over $\mathbf{Z}$;

$X(\mathbf{C})$ is the manifold of complex points of X;

F_∞ is the complex conjugation on $X(\mathbf{C})$;

$A^{p,p}(X)$ as the set of differential forms ω of type p,p on $X(\mathbf{C})$, that satisfy the equation $F_\infty^* \omega = (-1)^p \omega$;

$\tilde{A}(X) := \bigoplus_{p\geq 0}(A^{p,p}(X)/(Im\partial + Im\bar{\partial}))$;

A hermitian bundle $\overline{E} = (E, h)$ is a vector bundle E on X, endowed with a hermitian metric h, which is invariant under F_∞, on the holomorphic bundle $E_{\mathbf{C}}$ on $X(\mathbf{C})$, which is associated to E;

$ch(\overline{E})$ is the representative of the Chern character associated by the formulas of Chern-Weil to the hermitian holomorphic connection defined by h.

If

$$\mathcal{E} : 0 \to E' \to E \to E'' \to 0$$

is an exact sequence of vector bundles on X, we shall write $\overline{\mathcal{E}}$ for the sequence $\mathcal{E}$ and hermitian metrics on $E'_{\mathbf{C}}$, $E_{\mathbf{C}}$ and $E''_{\mathbf{C}}$ (invariant under F_∞). To $\overline{\mathcal{E}}$ is associated a secondary Bott-Chern class $\tilde{ch}(\overline{\mathcal{E}}) \in \tilde{A}(X)$. It satisfies the equation

$$dd^c \tilde{ch}(\overline{\mathcal{E}}) = ch(\overline{E}' \oplus \overline{E}'') - ch(\overline{E})$$

Arithmetic K_0-theory

Definition 0.1 *The arithmetic Grothendieck group $\widehat{K}_0(X)$ associated to X is the group generated by $\tilde{A}(X)$ and the isometry classes of hermitian bundles on X, with the relations*

(a) *For every exact sequence $\overline{\mathcal{E}}$ as above, we have $\tilde{ch}(\overline{\mathcal{E}}) = \overline{E}' - \overline{E} + \overline{E}''$;*

(b) *If $\eta \in \tilde{A}(X)$ is the sum of two elements η' and η'', then $\eta = \eta' + \eta''$ in $\widehat{K}_0(X)$.*

λ-rings

Definition 0.2 *A pre-λ-ring is a unital commutative ring R with operations $\lambda^k : R \to R$ $(k \geq 0)$, such that*

(i) $\lambda^0 = 1,\ \lambda^1 = Id$;

(ii) $\lambda^k(x+y) = \sum_{i=0}^k \lambda^i(x).\lambda^{k-i}(y)$

for all $k \geq 1$ and for all $x, y \in R$.
A λ-ring is a pre-λ-ring satisfying the following additional conditions

(iii) $\lambda^k(xy) = P_k(\lambda^1(x), \dots \lambda^k(x), \lambda^1(y), \dots \lambda^k(y))$;

(iv) $\lambda^k(\lambda^l(x)) = P_{k,l}(\lambda^1(x), \dots \lambda^{kl}(x))$

where $P_k, P_{k,l}$ are polynomials with integer coefficients, which are independent of the ring R.

Let $\lambda_t(x) : R \to 1 + R[[t]]$ be defined as $\lambda_t(x) = 1 + \sum_{k=1}^{\infty} \lambda^k(x)t^k$, where $1 + R[[t]]$ is the multiplicative subgroup of the ring of formal power series $R[[t]]$ consisting of power series with constant coefficient 1. A λ-ring structure on R gives rise to a collection of group homomorphisms, called Adams operations.

Definition 0.3 *The Adams operations $\psi^k : R \to R$ are defined by the equations*

$$\psi_{-t}(x) := \frac{-t.d\lambda_t(x)/dt}{\lambda_t(x)}; \quad \psi_t(x) =: \sum_{k \geq 1} \psi^k(x)t^k$$

Proposition 0.4 (Gillet-Soulé) *Let $\overline{E}$ be a hermitian bundle on X and $\eta \in \tilde{A}(X)$. There is a unique pre-λ-ring structure λ^k on $\widehat{K}_0(X)$, such that:*
(a) The element $\lambda^k(\overline{E})$ is represented by the k-th exterior power of E, endowed with the k-th exterior power metric.
(b) The equality

$$\psi^k(\eta) = \sum_{p \geq 0} k^{p+1}\eta_p$$

holds.

Proposition 0.5 (R.) *The pre-λ-ring structure of $\widehat{K}_0(X)$ is a λ-structure.*

Functoriality

Let η be an element of $\tilde{A}(Y)$ and (E, h) a hermitian bundle on Y, acyclic relatively to g. The sheaf of modules f_*E, which is the direct image of E, is then locally free and we write g_*h for the metric it inherits from E by integration on the fibers. We write $T(h_Y, h^E)$ for the higher analytic torsion of (E, h) relatively to the Kähler fibration defined by g and h_Y. We write Tg for the tangent bundle relative to g.

Proposition 0.6 *There is a unique group morphism*

$$g_* : \widehat{K}_0(Y) \to \widehat{K}_0(B)$$

such that

$$g_*((E,h)+\eta) = (g_*E, g_*h) - T(h_Y, h^E) + \int_{Y/B} Td(\overline{Tg_{\mathbf{C}}})\eta$$

for all (E, h) and η as above.

The Riemann-Roch theorem for the Adams operations

Definition 0.7 *The R genus is the unique additive characteristic class defined for a line bundle L by the formula*

$$R(L) = \sum_{m\ odd, \geq 1} (2\zeta'(-m) + \zeta(-m)(1 + \frac{1}{2} + \ldots + \frac{1}{m}))c_1(L)^m/m!$$

where $\zeta(s)$ is the Riemann zeta function.

For any special λ-ring A, denote by A_{fin} its subset of elements of finite λ-dimension. For each $k \geq$, the Bott cannibalistic class θ^k is uniquely defined by the following properties:

(a) For every special λ-ring A, θ^k maps A_{fin} into A and the equation $\theta^k(a+b) = \theta^k(a)\theta^k(b)$ holds for all $a, b \in A_{fin}$;

(b) The map θ^k is functorial with respect to λ-ring morphisms;

(c) If e is an element of λ-dimension 1, then $\theta^k(e) = \sum_{i=0}^{k-1} e^i$.

Theorem 0.8 (R.) *Let $g : Y \to B$ be an projective, flat, generically smooth local complete intersection morphism of arithmetic varieties. For each $k \geq 0$, let*

$$\theta_A^k(\overline{T}^\vee g)^{-1} =$$

$$\theta^k(\overline{T}^\vee g)^{-1}.(1 + R(Tg) - k.\phi^k(R(Tg))).$$

Then for the map $g_* : \widehat{K}_0(Y) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}] \to \widehat{K}_0(B) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}]$, the equality

$$\psi^k(g_*(y)) = g_*(\theta_A^k(\overline{T}^\vee g)^{-1}.\psi^k(y))$$

holds for all $k \geq 1$ and $y \in \widehat{K}_0(Y) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}]$.

The Riemann-Roch theorem for the Chern character

Let $rg : \widehat{K}_0(Y) \to \mathbf{Z}$ be the augmentation defined by the formula $rg : (\overline{E}+\eta) \mapsto rank(E)$.

Theorem 0.9 (R.) *(a) The γ-filtration $F^i\widehat{K}_0(Y)$ of $\widehat{K}_0(Y)$ is locally nilpotent.*

(b) The inclusion $g_* F^i\widehat{K}_0(Y)_{\mathbf{Q}} \subseteq F^{i+dim(B)-dim(Y)}\widehat{K}_0(B)_{\mathbf{Q}}$ holds.

Let $Gr\widehat{K}_0(Y) := \bigoplus_{i\geq 0} F^i\widehat{K}_0(Y)/F^{i+1}\widehat{K}_0Y$.

Let $ch_{Gr} : \widehat{K}_0(Y) \to Gr\widehat{K}_0(Y)_{\mathbf{Q}}$ be the Chern character. The formula

$$ch_{Gr}(g_*(y)) = g_*(Todd(\overline{Tg})(1 - R(Tg)).ch_{Gr}(y))$$

holds.

The behaviour of Adams operations under immersions

Let X be an arithmetic variety and $f : X \to B$ a projective, flat, generically smooth morphism. Let $i : Y \to X$ be a regular immersion of arithmetic varieties, such that $g = f \circ i$. Suppose that X is endowed with a Kähler metric and that Y carries the metric induced by i. Let $\overline{N}_{X/Y}$ be the normal bundle of i, endowed with the quotient metric. Let

$$0 \to \xi_m \to \xi_{m-1} \to \ldots \to \xi_0 \to i_*\eta \to 0$$

be a resolution on by vector bundles $\xi.$ on X of a vector bundle η on Y. Suppose that the ξ_i and η are endowed with hermitian metrics satisfying Bimsut's condition (A). Let $\overline{\alpha}$ be a hermitian bundle on X.

Theorem 0.10 (R.) *The equality*

$$g_*(\theta^k(\overline{N}^\vee_{X/Y})\psi^k(\overline{\eta})\overline{\alpha}) - \sum_{i=0}^{m}(-1)^i f_*(\psi^k(\overline{\xi}_i)\overline{\alpha}) =$$

$$\int_{Y(\mathbf{C})/B(\mathbf{C})} Todd(Tg_{\mathbf{C}})ch(\psi^k(\overline{\eta})\theta^k(\overline{N}^\vee_{X/Y}))R(N_{X(\mathbf{C})/Y(\mathbf{C})})ch(\alpha)+$$

$$\int_{X(\mathbf{C})/B(\mathbf{C})} Todd(\overline{Tg_{\mathbf{C}}})k.\phi^k(T(h^{\xi}))ch(\overline{\alpha})+$$

$$\int_{Y(\mathbf{C})/B(\mathbf{C})} ch(\psi^k(\overline{\eta})\theta^k(\overline{N}^\vee_{X/Y}))Todd^{-1}(\overline{N}_{X(\mathbf{C})/Y(\mathbf{C})})\widetilde{Todd}(\overline{N}_{\mathbf{C}}(g/f))ch(\overline{\alpha}).$$

holds for all $k \geq 1$. N.B. $k.\phi^k = \psi^k$

Title: Higher analytic torsion forms in real and complex geometry

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Page: 1

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The purpose of the talk is to illustrate the role of higher analytic torsion forms in real and complex geometry.

(1) The complex geometric setting.

Let $\pi: X \longrightarrow S$ be a holomorphic fibration with compact fibre Z. Let ξ be a holomorphic vector bundle on X. Assume $R\pi_* \xi$ is locally free.

Then by RRG,

$$ch(R\pi_* \xi) = \int_Z Td(TX/S)\, ch(\xi).$$

Introduce metrics g^{TZ}, g^ξ. Equip $R\pi_* \xi$ by the Hodge metric obtained by identifying $R\pi_* \xi$ to the fibrewise harmonic forms.

We can then represent cohomology classes by differential forms using Chern-Weil theory and the corresponding holomorphic Hermitian connections

One does not expect to have

$$ch(\xi, R\pi_* \xi) = \pi_*\left[Td(TZ_{/Y}^{T^2}) ch(\xi_{/Y} \xi)\right].$$

However if S is compact-Kähler, one can assert that there is T such that

$$(1) \quad \frac{\bar{\partial}\partial}{2i\pi} T = ch(\xi, R\pi_* \xi) - \pi_*\left[Td(TZ_{/Y}^{T^2}) ch(\xi_{/Y} \xi)\right].$$

Still this construction would be global on S, and not universal.

In degree 2, (1) says that

$$(2) \quad c_1\left(\det R\pi_*\xi, |\ |_{\det R\pi_*\xi}\right) - \frac{\bar{\partial}\partial}{2i\pi} T = \pi_*\left[Td(TZ_{/Y}^{T^2}) ch(\xi_{/Y} \xi)\right]$$

(here $|\ |_{\det R\pi_*\xi}$ is the L_2 metric on $\det R\pi_* \xi$).

Then if

$$(3) \quad \|\ \|_{\det R\pi_*\xi} = |\ |_{\det R\pi_*\xi} \exp\left(\frac{T}{2}\right).$$

we find that

$$(4) \quad c_1\left(\det R\pi_*\xi, \|\ \|_{\det R\pi_*\xi}\right) = \pi_*\left[Td(TZ_{Y}^{T^2}) ch(\xi, g^{\xi})\right]^{(4)}$$

In degree 0, by the curvature theorem of Gillet-Soulé-B., $T^{(0)}$ = the Ray-Singer torsion just verifies (1). The question is then to know whether there is a universal analytic machine which will produce such T.

This is in fact possible using results of Gillet-Soulé-B. and Köhler-B.

The second question is to find if such T verify natural functorial properties. This is made an important question especially in view of the program of Gillet-Soulé to prove a formula of RR in Arakelov theory, where such T appears explicitly.

We will explain results on functorial properties of T w.r.t. projections and immersions.

(1) The real setting.

Let $\pi : X \to S$ be a fibration of real manifolds with compact fiber Z. Let F be a flat complex bundle. Then $R\pi_* F$ is a flat bundle.

These flat bundles have classes in $H^{odd}(\mathbb{C}/\mathbb{Z})$: they are the differential characters of Cheeger-Simons.

Theorem (Lott, B.) The following identity holds
+B.

(5) $\hat{c}_\cdot(R\pi_* F) = \pi_*\left[\hat{e}(TZ)\cdot\hat{c}(F)\right]$ in $H^\cdot(S, \mathbb{C}/\mathbb{Q})$

for any odd Chern class $\hat{c}$

Remark: Since $TZ = TX/S$ not flat, the product $\hat{e}(TZ)\hat{c}(F)$ is taken in the sense of differential characters.

By splitting $\mathbb{C}/\mathbb{Q} = \mathbb{R} \oplus \mathbb{R}/\mathbb{Q}$, (5) gives an identity of cohomology classes in $H(S, \mathbb{R})$. Again, there is a Chern Weil theory for these R-classes, which can be explicitly represented by explicit differential forms attached to metrics. Again one can try to solve naturally the equation

$$d T = c^R(R\pi_*F, g^{R\pi_*F}) - \pi_*\left[e(TZ, \nabla^{TZ})\, c(F, \nabla^F)\right]$$

These T are analytic torsion forms constructed by Lott. B.
If $Z = S_1$, F is the flat bundle associated to a root of unity, T can be naturally expressed in terms of polylogarithms.

Title: An arithmetic formula for a topological invariant of Siegel modular varieties.

Page: 1

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This is a report on joint work with J. Schwermer [3].

We consider a totally real number field F, $[F:\mathbb{Q}]=N$. Let $\mathcal{O}_F$ be the ring of integers of F and fix an integral ideal $\mathfrak{a}=\prod_v \mathfrak{p}_v^{s_v}$ of $\mathcal{O}_F$. Denote by Sp_n the symplectic group and select $\mathfrak{a}$, such that the full congruence subgroup $\Gamma=\Gamma(\mathfrak{a})$ mod $\mathfrak{a}$ of $Sp_n(\mathcal{O}_F)$ is torsion free. Let $\mathfrak{h}_n$ be the Siegel upper half space. Then $\Gamma(\mathfrak{a})$ acts on $\mathfrak{h}_n \times \cdots \times \mathfrak{h}_n =: X$ (N-factors) and $\Gamma(\mathfrak{a})\backslash X$ is a locally symmetric manifold. Let θ be the standard Cartan involution of Sp_n and denote by $\rho : Sp_n(\mathbb{C})^N \longrightarrow GL_n(V)$ a representation defined over $\mathbb{Q}$ on which θ acts ie $\theta(\rho(g)v)=\rho(\theta(g))\,\theta(v)$. Then θ acts on $\Gamma(\mathfrak{a})\backslash X$ and on the locally constant sheaf $\tilde{V}$ attached to ρ

on $\Gamma(\sigma)\backslash X$. Hence we have an induced action on the cohomology $H^*(\Gamma(\sigma)\backslash X, \tilde{V})$ and a Lefschetz number

$$L(\Gamma, \theta, V) := \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}\left(\theta^i \mid H^i(\Gamma(\sigma)\backslash X, \tilde{V})\right) \in \mathbb{Z}$$

is defined.

We give an explicit formula for this number. Here we need the following notation: $N(\mathfrak{p}_v) = |\mathcal{O}_v/\mathfrak{p}_v|$; $\zeta_F(s)$ Zeta function of F at $s \in \mathbb{C}$, $L_F(\chi, s)$ L-function of the nontrivial character of $F(\sqrt{-1})/F$ at $s \in \mathbb{C}$, $|D_F|$ absolute value of the discriminant of F, $\operatorname{tr}(\theta|E)$ trace of θ acting on V; e_v ramification index of the place v of F over $\mathbb{Q}$.

<u>Theorem</u> Let $\mathfrak{n} = \prod_v \mathfrak{p}_v^{s_v}$, $s_v \geq 2e_v+1$ if $v|2$.

then

$$L(\Gamma(\mathfrak{n}),\theta,V) = 2^{2nN} |D_F|^{n^2/2} \operatorname{tr}(\theta|V) \left(\prod_{\nu=1}^{n} \frac{2\pi}{(\nu-1)!} \right)^{-N} \times$$

$$\times \prod_{\nu=1}^{[n/2]} \zeta_F(2\nu) \prod_{\nu=0}^{[\frac{n-1}{2}]} L_F(\chi, 2\nu+1) \cdot C(\mathfrak{n})$$

where

$$C(\mathfrak{n}) = \prod_{\mathfrak{p}_v|\mathfrak{n}} \mathfrak{N}(\mathfrak{p}_v)^{s_v n^2} \prod_{\nu=1}^{[n/2]} \left(1 - \mathfrak{N}(\mathfrak{p}_v)^{-2\nu}\right) \prod_{\nu=0}^{[\frac{n-1}{2}]} \left(1 - \chi(\mathfrak{p}_v)\mathfrak{N}(\mathfrak{p}_v)^{-2\nu-1}\right).$$

In the proof we use the method given in [2]. At first one has to prove a Lefschetz fixed point formula which identifies $L(\Gamma(\mathfrak{n}),\theta,V)$ with a sum of Euler-Poincaré characteristics of fixed point components parametrized by the first non abelian cohomology $H^1(\Theta, \Gamma)$ of the Θ-action on Γ. The Euler characteristics of the components are computed essentially as a volume with the help of Harder's Gauss Bonnet formula [1]. It turns out that the components are all locally symmetric spaces attached to unitary groups. This explains the occurrence and the form of the Euler-factors in the theorem. The summation of the fixed point contributions is carried out in the adelic language. One uses Ono's formulas for Tamagawa numbers, the Hasse principle, and a Θ-twisted version of Kottwitz' method to stabilize the trace formula. The assumption $s_v \geq 2e_v+1$ simplifies the computations at the place 2.

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Title: Local heights of subvarieties and formal geometry

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1. Introduction

K number field, X projective variety over K, $L_0, \ldots, L_t$ line bundles on X, Z t-dim. cycle on X.

Faltings defined the height of Z wrt. $L_0, \ldots, L_t$ by choosing models over the integers and metrics at ∞:

$$h(Z) := \hat{c}_1(\bar{\mathcal{L}}_0) \ldots \hat{c}_1(\bar{\mathcal{L}}_t) . \hat{Z}$$

Here $\hat{Z}$ is the Arakelov cycle for Z and $h(Z)$ is an arithmetic intersection number. There is an obvious decomposition

$$h(Z) = \sum_v \lambda(Z, v)$$

where the local height $\lambda(Z,v)$ at the place v is given by

$$\begin{cases} \text{a } *\text{-product of Green currents} & \text{if } v \mid \infty \\ \text{an intersection number of Cartier divisors with } \bar{Z}. & \text{if } v \nmid \infty \end{cases}$$

The local heights depend on the choice of meromorphic sections of $L_0, \ldots, L_t$.

Now let $(K, | \ |_v)$ be a field with a complete absolute value. If $v \mid \infty$ or v is discrete, then we can use the above to define local heights of cycles. If $v \nmid \infty$ is non-discrete, then there is no intersection product on the models over the valuation

ring since the local rings are no longer noetherian.
In the following, a normalized intersection product on formal models is introduced. Replacing algebraic models by formal models, we get local heights of cycles with the usual properties.

2. Intersection product on rigid varieties

$(K, | \ |_v)$ field with a complete, non-archimedean, non-trivial absolute value, X rigid variety $/K$, D Cartier divisor on X, Z cycle on X such that D and Z intersect properly.
The definition of the cycle $D.Z$ is a local question. We may assume that X is an affinoid variety given by the affinoid algebra $\mathfrak{A}$. Note that $\mathfrak{A}$ is noetherian. There is a one-to-one correspondence between subvarieties of X and subschemes of $\operatorname{Spec} \mathfrak{A}$ given by the ideal of vanishing. Since $D.Z$ is defined on $\operatorname{Spec} \mathfrak{A}$, we get an intersection product $D.Z$ on X.

3. Intersection product on formal models

K as above with valuation ring K° and residue field $\tilde{K}$.
$\mathfrak{X}$ is called a K°-model of X if $\mathfrak{X}$ is a flat formal scheme of topologically finite type over K° with generic fibre X.
$\tilde{\mathfrak{X}} := \mathfrak{X} \otimes_{K^\circ} \tilde{K}$ is a variety over $\tilde{K}$ called the special fibre of $\mathfrak{X}$.
A cycle on $\mathfrak{X}$ is a formal sum $\mathfrak{Z} = Z + Z_v$, where Z is a cycle on X and Z_v is a cycle on $\tilde{\mathfrak{X}}$.
The coefficients of the cycles on $\tilde{\mathfrak{X}}$ are with real coefficients.

Let $\mathcal{D}$ be a Cartier divisor on $\mathfrak{X}$ such that $D = \mathcal{D}|_X$ and Z intersect properly in $\mathfrak{X}$. We want to define

$$\mathcal{D}.\mathfrak{Z} = \underbrace{\mathcal{D}.Z}_{= Y + Y_v} + \underbrace{\mathcal{D}.Z_v}_{\text{well known on } \tilde{\mathfrak{X}}}$$

By §2, $Y = D.Z$ is well-defined on X and it remains to define Y_v. For the latter, it is enough to define the multiplicity of $\mathcal{D}$ in an irreducible component W of the special fibre $\tilde{\mathfrak{X}}$ (restrict to the closure of Z in $\mathfrak{X}$). We can make the following simplifications:

- base change $\rightsquigarrow$ K algebraically closed
- localization $\rightsquigarrow$ X affinoid variety with affinoid algebra $\mathcal{A}$ and $\mathcal{D}$ given by $a \in \mathcal{A}$.
- Linearity $\rightsquigarrow$ $\mathcal{A}$ integral domain
- Projection formula $\rightsquigarrow$ $\mathfrak{X}$ formal affine scheme with algebra $\mathcal{A}^\circ := \{a \in \mathcal{A} \mid |a(x)| \leq 1 \ \forall x \in X\}$.

Let $\pi: X \to \tilde{\mathfrak{X}}$ be the reduction map and let $W = W_0, W_1, \dots, W_r$ be the irreducible components of $\tilde{\mathfrak{X}}$.

$$|a(W)| := \sup_{x \in X,\ \pi(x) \notin \bigcup_{j \neq 0} W_j} |a(x)|_v$$

This is a multiplicative semi-norm on $\mathcal{A}$ and may be viewed as a point in the Berkovich-compactification of X.

$$\operatorname{ord}(D, W) := -\log |a(W)|$$

Summarizing the above, we have defined $\mathcal{D}.\mathfrak{Z}$ as the formal sum of a cycle on X and a rational equivalence class on $\tilde{\mathfrak{X}}$.

This product has the following properties

- Projection formula holds
- commutativity for Cartier divisors properly intersecting in the generic fibre.

Since it is not possible to translate the proofs from algebraic geometry, we scetch the arguments in §5 and 6.

4. Discrete valuation case

If v is a discrete valuation such that $-\log|\pi|_v = 1$ for uniform parameters 1, then the intersection product on algebraic models is compatible with the intersection product on the corresponding formal models obtained by completing the algebraic model along the special fibre.

5. Proof of projection formula

As in §2, §3, we consider a field K with a complete non-archimedean non-trivial absolute value $|\ |_v$. By base change, we may assume K algebraically closed. Using Stein factorization, it is possible to reduce the proof of the projection formula to

Lemma 1 Let $\mathcal{A}'$ be a finite $\mathcal{A}$-algebra where $\mathcal{A}$ is a K-affinoid algebra. $\mathcal{A}'$ and $\mathcal{A}$ are assumed to be integral domains. Let X' be the affinoid ~~alge~~ variety associated to $\mathcal{A}'$ and let $\mathcal{X}, \mathcal{X}'$ be the formal schemes corresponding to $\mathcal{A}^\circ, (\mathcal{A}')^\circ$. Then we have a finite morphism $\varphi: \mathcal{X}' \to \mathcal{X}$ induced by $\mathcal{A} \to \mathcal{A}'$ and, for $a \in \mathcal{A}\setminus\{0\}$, we have $\varphi_*(\varphi^*\mathrm{div}(a).X') = \mathrm{div}(a).\varphi_*X'$ on $\mathcal{X}$.

Proof: The considerations of §2 show the claim for horizontal components. So it is enough to check the projection formula in an irreducible component W of $\mathcal{X}$. Clearly, we may assume $W=\tilde{\mathcal{X}}$. An irreducible component V of $\mathcal{X}'$ is mapped onto W and we have $|a(V)| = |a(W)|$. We conclude

$$\varphi_*(\varphi^* \operatorname{div}(a).\ X')_v = \tilde{\varphi}_*\left(-\sum_V \log|a(V)|\, V\right) = -\log|a(W)| \sum_V [V:W]\, W.$$

Because of

$$(D.\ \varphi_*(X'))_v = -\log|a(W)|\,[X':X],$$

the claim follows from

<u>Lemma 2</u> If K is stable and $\mathcal{A}$ is as above, then $\mathrm{Quot}(\mathcal{A})$ is stable.

(K stable :$\Leftrightarrow$ $\forall\, L/K$ finite field extension, we have $\sum_i e_i f_i = [L:K]$, on $\mathrm{Quot}(\mathcal{A})$ we use the absolute value induced by the supremum norm) ramification index residue degree

Note:

If K is algebraically closed, then it is stable. There is a one-to-one correspondence between irreducible components of $\mathcal{X}'$ and absolute values on $\mathrm{Quot}(\mathcal{A}')$ extending the supremum norm on $\mathcal{A}$. Moreover, all ramification indices are 1. Therefore Lemma 2 implies Lemma 1.

6. Proof of Commutativity

By base change, we may assume that X is a curve. Using projection formula and methods of Bosch, Lütkebohmert, we may assume that X is a projective non-singular curve. By the semistable reduction theorem, we reduce the problem to the case of a disc or an annulus. In both cases, we can check the claim directly.

7. Formal metrics

Let X be a quasicompact and quasiseparated rigid variety over K with formal model $\mathfrak{X}$ over K°. A Cartier divisor $\mathscr{D}$ on $\mathfrak{X}$ with $D := \mathscr{D}|_X$ induces a metric $\| \ \|$ on $L = O(D)$ called a formal metric. If γ is a local equation of $s_{\mathscr{D}}$ on $\mathfrak{X}$, then the metric is given by

$$\| s_D(x) \| = |\gamma(x)| .$$

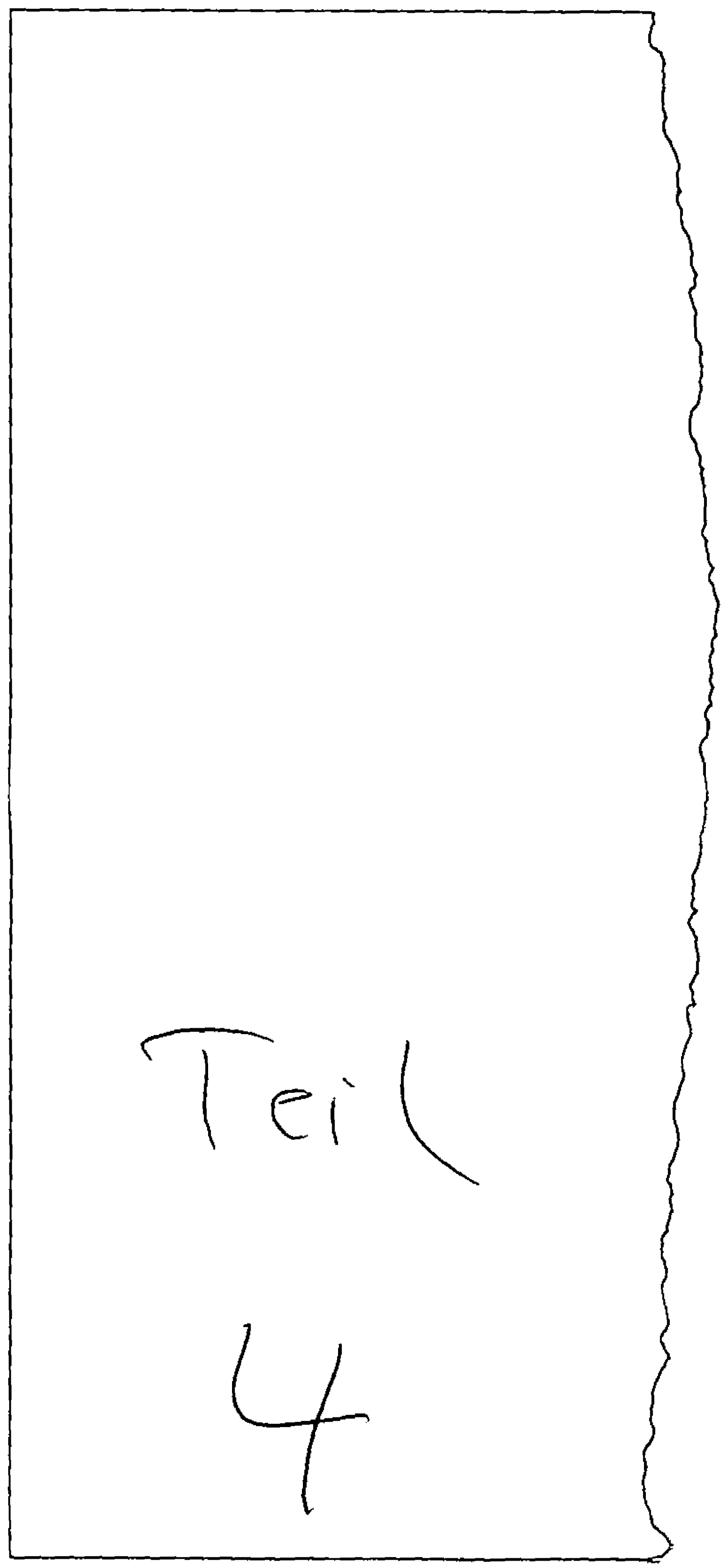
Formal metrics are closed under $\otimes$, duality, maximum, minimum. Every line bundle on X has a formal metric. Moreover, formal metrics separate points on X. By the theorem of Stone-Weierstrass, we get

Proposition 3 (For metrics on L,) There is a one-to-one correspondence of roots of formal metrics ~~on L~~ and metrics ~~on L~~ extending continuously to the Berkovich compactification of X.

As in non-Archimedean Arakelov theory of Bloch-Gillet-Soulé, one can consider the projective limit over all formal K°-models of $\mathfrak{X}$. In this sense, we have

Proposition 4 The dependence of the product $\mathscr{D}.\mathfrak{Z}$ on the choice of K°-models is measured by the formal metric $\| \ \|$ on $O(D)$.

Therefore we get local heights of cycles on X with respect to formally metrized line bundles. They have the usual properties.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Title: NON-VANISHING OF AUTOMORPHIC L-FUNCTIONS

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Page: 1

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Let π be a cuspidal automorphic representation of $GL_{2/\mathbb{Q}}$. For a fundamental discriminant d, denote by χ_d the character $\left(\frac{d}{\cdot}\right)$. We report on joint work with T. Stefanicki.

THEOREM 1. Let $s_0 \in \mathbb{C}$ with $\mathrm{Re}\, s_0 \in [\frac{1}{2}, 1]$ and $\varepsilon > 0$. Then there are infinitely many fundamental discriminants d such that

$$L(\pi \otimes \chi_d, s) \neq 0 \quad \text{in the disc} \quad |s - s_0| < \frac{1}{(\log |d|)^{1+\varepsilon}}$$

This follows from the following result.

THEOREM 2. Let s_0 be as above. Then

$$\sum_{\substack{d \equiv a\, (4 f_\pi) \\ d \text{ fundamental}}} L(\pi \otimes \chi_d, s_d) F\left(\frac{|d|}{Y}\right) = cY + O_\varepsilon\left(Y^{\frac{3}{2} - \mathrm{Re}\, s_0} (\log Y)^{-\varepsilon}\right)$$

where (a) $F : \mathbb{R}^+ \to \mathbb{R}$ is a smooth compactly supported function with $\int_0^\infty F(t)\,dt = 1$

(b) for each d, the point s_d is in the disc $|s_d - s_0| < \frac{1}{(\log Y)^{1+\varepsilon}}$

(c) $a \in \mathbb{Z}$: $(a, F_\pi) = 1$, $a \equiv 1 \pmod 4$

(d) $c = c(\pi, F, s_0, a) \neq 0$

A quantitative version of Theorem 1 can be obtained using the following.

THEOREM 3 With notation as above

$$\sum_{\substack{d \equiv a \pmod{4F_\pi} \\ d \text{ fundamental} \\ |d| \ll Y}} |L(\pi \otimes \chi_d, s_d)|^4 \ll Y^{\frac{19}{7}+\varepsilon}.$$

Title: Special values of theta functions with CM

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Page: 1

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(joint work with E. Goren)

§1. Motivation

Our motivating problem is the search for "abelian units" – special units in abelian extensions of CM fields.

Recall: K – quadratic imaginary
H – its Hilbert class field
Δ – Ramanujan's delta function

Then

- $$u(\mathfrak{a}) = \frac{\Delta(\mathfrak{a}^{-1})}{\Delta(\mathcal{O}_K)} \in H^\times$$
- $u(\mathfrak{a}) \sim \mathfrak{a}^{12}$
- $u(\mathfrak{a}\mathfrak{b}) = u(\mathfrak{a})^{\sigma_\mathfrak{b}} u(\mathfrak{b}) = u(\mathfrak{b})^{\sigma_\mathfrak{a}} u(\mathfrak{a})$
  (here $\sigma_\mathfrak{a} = (\mathfrak{a}, H/K)$)
- $u(\mathfrak{a},\mathfrak{b}) = u(\mathfrak{a})u(\mathfrak{b})/u(\mathfrak{a}\mathfrak{b}) \in \mathcal{O}_H^\times$
  and depends only on $[\mathfrak{a}], [\mathfrak{b}] \in Cl_K$

(Siegel units).

§2. Consider the following set-up

K = quartic CM field
H = its Hilbert class field

$$\begin{array}{ccccc} K & \overset{h_K^-}{\text{———}} & H^- & \overset{h_K^+}{\text{———}} & H \\ {\scriptstyle 2}| & & & & \\ F = K \cap \mathbb{R} & & & & \\ {\scriptstyle 2}| & & & & \\ \mathbb{Q} & & & & \end{array}$$

• Assumptions (simplifying):

$$\left[\begin{array}{l} Gal(K/\mathbb{Q}) \text{ cyclic } = \langle \sigma \rangle \\ \mathbb{N}_{F/\mathbb{Q}}(\text{fund. unit}) = -1, \quad h_K \text{ odd} \\ \mathcal{D}iff_{K/\mathbb{Q}} = (\delta) \quad \bar{\delta} = -\delta \end{array}\right.$$

Fix $\Phi = (\text{W.L.O.G.}) \{1, \sigma\}$ a CM type
$\Phi' = \{1, \sigma^3\}$ the reflex CM type $(K = K')$

The class group decomposes $Cl_K = Cl_K^+ \times Cl_K^-$

• Observe $Cl_K^- = \mathrm{Ker}(1+\sigma^2) = \mathrm{Im}(1-\sigma^2)$
$= \{[\mathfrak{a}] \in Cl_K \mid \mathfrak{a}\bar{\mathfrak{a}} = (a),\ a \in \mathbb{Q}^\times\}$
$= \{[\mathbb{N}_{\Phi'}\mathcal{A}]\} \quad (= \mathrm{Im}(1+\sigma^3))$

• Put $\mathbb{Z}[Cl_K^-]_\infty = \left\{ \sum n_i[\mathfrak{a}_i] \;\middle|\; \sum n_i = 0,\ \prod [\mathfrak{a}_i]^{n_i} = 1 \right\}$
$\cong \mathbb{Z}^{h_K^- - 1}$

• Adjusting δ by a unit may assume $\mathrm{Im}\, \varphi(\delta) > 0$ for $\varphi \in \Phi = \{1, \sigma\}$

§3. Construction of certain invariants

$G = GSp(4) = \{\alpha \in GL(4) \mid \alpha J\, {}^t\alpha = \nu(\alpha) J\}$ $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\mathfrak{h}_2$ = Siegel space of genus 2

For $\tau \in \mathfrak{h}_2$ $u \in \mathbb{C}^2$ $r, s \in \mathbb{R}^2$

$$\theta\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right](u,\tau) = \sum_{n\in\mathbb{Z}^2} e^{2\pi i\left\{\frac{1}{2}{}^t(n+r)\tau(n+r) + {}^t(n+r)(u+s)\right\}}$$

- The theta characteristics $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]$ are called <u>integral</u> if $r, s \in \frac{1}{2}\mathbb{Z}^2$, <u>even</u> if ${}^t r\cdot s \in \frac{1}{2}\mathbb{Z}$. Up to change of θ by $\pm$ r, s may be taken then mod $\mathbb{Z}^2$. There are <u>10</u> even theta char, 6 odd integral characteristics (even/odd is also the parity of θ under $u \longmapsto -u$).
- $\theta_{ev}(\tau) = \prod\limits_{(10\text{ even ones})} \theta\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right](0,\tau)$ $\quad$ ($=$ Igusa's χ_{10})

$\theta_{ev}^2 \in \mathcal{M}_{10}(Sp(4,\mathbb{Z}), \mathbb{Q})$ $\quad$ $\left\{\begin{matrix} q\text{-expansion rat'l} \\ \text{over } \mathbb{Q} \end{matrix}\right.$

$$\begin{aligned} \mathrm{div}(\theta_{ev}) = H_1 &= Sp(4,\mathbb{Z})\cdot\left\{\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right) \middle|\ \tau_i \in \mathfrak{H}_1\right\} \\ &= \{\tau \mid (A_\tau, E_\tau) \text{ splits as } \mathcal{E}_1\times\mathcal{E}_2 \text{ (with polarization), i.e. is } \underline{\text{not}} \text{ a Jacobian}\} \end{aligned}$$

- Given $\underline{K, \Phi, \delta}$ and an ideal $\underline{\mathfrak{a}}$ s.t. $\mathfrak{a}\bar{\mathfrak{a}} = (a)$, $0 < a \in \mathbb{Q}^\times$, we get two point in moduli space over $\mathbb{C}$:

$$(\Phi(\mathcal{O}_K), E_\delta) \quad\text{and}\quad (\Phi(\mathfrak{a}), \tfrac{1}{a}E_\delta)$$

where $\quad E_\delta(\Phi(u), \Phi(v)) = \mathrm{Tr}_{K/\mathbb{Q}}(\delta^{-1}\bar{u}v)$

is a principal polarization by assumption on $\mathfrak{D}iff_K$.

- For any principally polarized lattice (L, E) (E = the Riemann form) choose a symplectic basis $(\omega_1, \omega_2) = \Omega$ (i.e. $E(\Omega x, \Omega y) = {}^t x J y$) and then
$$\Delta_2(L,E) = \det(\omega_2^{-10}) \cdot \theta_{ev}^2(\omega_2^{-1}\omega_1)$$
is well defined. In our case, by the assumption that $N_{F/\mathbb{Q}}(\text{fund. unit}) = -1$, <u>every</u> polarization compatible with the complex multiplications on $\Phi(\mathcal{O}_K)$ is $E_{\lambda^2\delta}$ for $\lambda \in \mathcal{O}_F^\times$, changing Δ_2 by $N_\Phi(\lambda)^{-10} = 1$, hence $\Delta_2(\Phi(\mathcal{O}_K), E_\delta)$ is independent of the (principal) polarization, and similarly for $\Phi(\mathfrak{a})$.

- <u>Def^n</u>: $u(\mathfrak{a}) = u(\Phi, \mathfrak{a}) = \dfrac{\Delta_2(\Phi(\mathfrak{a}^{-1}))}{\Delta_2(\Phi\mathcal{O}_K)}$

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§4. <u>Known properties of the invariants</u>

– (1) $u(\mathfrak{a}) \neq 0, \infty$ depends only on $(K, \Phi, \mathfrak{a})$

– (2) $u(\Phi\tau, \tau^{-1}(\mathfrak{a})) = u(\Phi, \mathfrak{a})$ for $\tau \in Gal(K/\mathbb{Q})$
so it is enough to consider one Φ (the Galois group acts transitively on the CM types)

– (3) $u(\mathfrak{a}) \in H^-$ (the fixed field of Cl_K^+)

$\sqrt{u(\mathfrak{a})} \in K^{ab}$

– (4) $u(\mathfrak{a})^{(\mathcal{C}, H/K)} = u(\mathfrak{a}\mathfrak{c})/u(\mathfrak{c})$

$\mathfrak{c} = \mathbb{N}_{\Phi'}(\mathcal{C})$ ("cocycle condition")

Both (3) and (4) follow from Shimura's reciprocity law.

– (5) For $\lambda \in K^\times$ $u(\lambda\mathfrak{a}) = \mathbb{N}_\Phi(\lambda)^{10} \cdot u(\mathfrak{a})$

– (6) $(h_K^-, 5) = 1 \Rightarrow$ the $u(\mathfrak{a})$ generate H^-; in particular they are non trivial!

This follows from (4). It is expected without the restriction $(h_K^-, 5) = 1$.

- If $z = \sum' n_i[\mathfrak{a}_i] \in \mathbb{Z}[Cl_K^-]_{00}$, and $\prod \mathfrak{a}_i^{n_i} = (\lambda)$ define a homomorphism

  - $u(z) = u(\Phi, z) = \prod u(\mathfrak{a}_i)^{n_i} \mathbb{N}_{\Phi}(\lambda)^{-10}$.

  By (5) this is well defined, (4) $\Rightarrow$

  - $u([\mathfrak{a}]z) = u(z)^{\sigma_A}$ if $\mathfrak{a} = \mathbb{N}_{\Phi'}(A)$

– (7) $\mathbb{N}_{H^-/K}\, u(z) = 1$

– (8) Assume $h_K^- = \text{prime} \neq 5$; then u is

injective, so $\mathrm{rk}\; u(\mathbb{Z}[Cl_K^-]_{00}) = h_K^- - 1$.
(This follows from (6) and the fact that $\mathbb{Q}[Cl_K^-]_0$ is an irreducible rep$^{\underline{n}}$ of Cl_K^-).
Is u injective in general ??

§5. Integrality questions.

Prop$^{\underline{n}}$: The following are equivalent

$(*)$ $\forall \mathfrak{a} \subset \mathcal{O}_K$ (integral) $u(\mathfrak{a}) \in \mathcal{O}_{H^-}$ (integral)

$(**)$ $(u(\mathfrak{a})) = N_\Phi(\mathfrak{a})^{10}$ (realizing the Hauptidealsatz)

$(***)$ $(u(\mathfrak{a}))$ is $Gal(H^-/K)$-invariant

- If these conditions hold, the $u(z)$'s are units for $z \in \mathbb{Z}[Cl_K^-]_{00}$.
- Studying the geometry of the Siegel moduli space over $\mathbb{Z}$ we know

Prop$^{\underline{n}}$: If $(p, \mathfrak{a}) = 1$ and p is either <u>totally decomposed</u> or <u>inert</u> in K, then $u(\mathfrak{a})$ is a unit above p.

- The case $p = \mathfrak{p}_1\mathfrak{p}_2$ in K seems to be difficult!

Title: Heights for line bundles on arithmetic varieties

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1. Arithmetic Intersection Theory (Gillet / Soulé)

X arithmetic variety: regular, projective, flat scheme $/\mathbb{Z}$
$\pi: X \to \operatorname{Spec} \mathbb{Z}$

<u>Definition</u>: $\widehat{CH}^i(X) := \widehat{Z}^i(X) / \widehat{R}^i(X)$

$\widehat{Z}^i(X) := \{ (Z, g_Z) \mid Z \in Z^i(X), g_Z \in D^{i-1,i-1}(X(\mathbb{C}))$ real current, invariant under complex conjugation F_∞, $dd^c g_Z + \delta_Z = \omega_Z$ smooth form$\}$

$\widehat{R}^i(X)$ generated by $(\operatorname{div} f, -\log|f|^2)$, where $f \in k(y)^*$, $y \in X^{(i-1)}$ and $(0, \partial u + \bar{\partial} v)$

<u>Theorem</u>: $\exists$ intersection product $\widehat{CH}^p(X) \times \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X)_{\mathbb{Q}}$ such that
1. $\bigoplus_{p \geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}$ graded commutative $\mathbb{Q}$-algebra with 1,
2. $\zeta: \widehat{CH}^p(X)_{\mathbb{Q}} \to CH^p(X)$, $(Z, g_Z) \mapsto Z$ and $\omega: \widehat{CH}^p(X) \to Z^{pp}(X(\mathbb{C}))$, $(Z, g_Z) \mapsto \omega_Z = dd^c g_Z + \delta_Z$

are homomorphisms of $\mathbb{Q}$-algebras

$(W, g_W) \cdot (Z, g_Z) = (W \cdot Z, g_W \delta_Z + g_Z \cdot \omega_W)$ - heuristic formula

<u>Theorem</u>: $f: X \to Y$ morphism of arithmetic varieties

a) $f^*: \widehat{CH}^i(Y) \to \widehat{CH}^i(X)$

b) $f_*: \widehat{CH}^i(X) \to \widehat{CH}^{i-d}(Y)$, $(Z, g_Z) \mapsto (f_* Z, f_* g_Z)$,

when f proper, generically smooth of relative dimension d.

$f^*(\alpha \cdot \beta) = f^*(\alpha) \cdot f^*(\beta)$, $f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)$, $(fg)^* = g^* f^*$, $(fg)_* = f_* g_*$

<u>Fact</u> (first Chern class): $\exists$ natural isomorphism

$\hat{c}_1: \widehat{Pic}(X) \xrightarrow{\sim} \widehat{CH}^1(X)$

$(L, \|.\|) \mapsto [(div(s), -\log|s|^2)]$, where s rational section

<u>Fact</u> (arithmetic degree): $\exists$ isomorphism

$\widehat{deg}: \widehat{CH}^1(\operatorname{Spec} \mathbb{Z}) \to \mathbb{R}$

$(\sum_p n_p \cdot p, \lambda) \mapsto \sum_p n_p \cdot \log p + \frac{1}{2}\lambda$

2. Heights for effective cycles (Faltings - Bost / Gillet / Soulé)

Fact: $\exists$ pairing $\widehat{CH}^i(X) \times Z_i(X) \to \mathbb{R}$
$$((W, g_W), Z) \mapsto \widehat{\deg}\, \pi_*(W \cdot Z, g_W \cdot \delta_Z)$$
(restriction to a cycle)

Definition: $\overline{\mathcal{L}} \in \widehat{Pic}(X)$, $Z \in Z_i(X)$. The height of Z with respect to $\overline{\mathcal{L}}$ is given by
$$h_{\overline{\mathcal{L}}}(Z) := (\hat{c}_1(\overline{\mathcal{L}})^i \mid Z) .$$

philosophy: heights - objects in arithmetic geometry analogous to degrees in algebraic geometry over a ground field

Theorem (Bost / Gillet / Soulé): $\mathcal{L} \in Pic(X)$ ample, $A \in \mathbb{R}$. Then there are only finitely many effective cycles $Z \in Z_i(X)$ such that $\deg_{\mathcal{L}}(Z_{\mathbb{Q}}) \leq A$ and
$$h_{\overline{\mathcal{L}}}(Z) \leq A.$$

Idea: Induction on i

For $i = 1$ one recovers the "naive" height for rational points and the degree for curves over a finite field.

3. Line bundles.

They have degrees in algebraic geometry over a field; they should have heights.

One would like to define $h_{\bar{\mathcal{L}}}(\mathcal{E}) := \widehat{\deg}\, \pi_*(\hat{c}_1(\bar{\mathcal{E}}) \cdot \hat{c}_1(\bar{\mathcal{L}})^d)$, when $\dim X = d+1$.

problem: need a „natural" hermitian metric on $\mathcal{E}$

One way:

1. Fix a Kähler metric ω on $X(\mathbb{C})$, invariant under F_∞ (as Arakelov did) and require
$c_1(\mathcal{E}, \|.\|)$ $(= -dd^c \log \|s\|^2 + \delta_{\mathrm{div}(s)}$ for a rational section $s)$ to be harmonic.

2. require $\widehat{\deg}(\det R\pi_* \mathcal{E}, h_Q) = 0$. ($h_Q$ is Quillen's metric)

Fact: If $\chi(\mathcal{E}_{\mathbb{Q}}) \neq 0$ such distinguished metrics exist. They determine $\hat{c}_1(\mathcal{E}, \|.\|)$ up to numerical equivalence.

Definition: $h_{\omega, \bar{\mathcal{L}}}(\mathcal{E}) := \widehat{\deg}\, \pi_*(\hat{c}_1(\mathcal{E}, \|.\|) \cdot \hat{c}_1(\bar{\mathcal{L}})^d)$, where $\|.\|$ is a distinguished metric on $\mathcal{E}$, is said to be the height of $\mathcal{E}$ with respect to $\bar{\mathcal{L}}$ (and ω).

Theorem 1 (finiteness): K number field, $\mathcal{O}_K$ ring of integers, $X/\mathcal{O}_K$ arithmetic variety with $X(K) \neq \emptyset$ and X_K connected. E algebraic equivalence class of line bundles on X_K such that

$\mathcal{P}$ on $X_K \times \mathrm{Pic}^E(X_K)$ tautological line bundle such that

(*) $\mathcal{P}|_{\{\cdot\} \times \mathrm{Pic}^E(X_K)} \overset{alg}{\sim} 0$. Then

$$(\det R\pi_{2*}\mathcal{P})^\vee$$

is ample on $\mathrm{Pic}^E(X_K)$.

Further assume $\chi(\mathcal{E}) > 0$ for $\mathcal{E} \in E$ and $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(X)$ ample.

Then $\forall A \in \mathbb{R}$ there are only finitely many $\mathcal{E} \in \mathrm{Pic}(X)$ with $\mathcal{E}_K \in E$, $|\deg_{\mathcal{L}} \mathcal{E}|_{X_{\mathfrak{p}i}}| < H$ on components of special fibers and

$$h_{\overline{\mathcal{L}}}(\mathcal{E}) < H.$$

Proposition: K field, R/K scheme, smooth, projective and connected with $R(K) \neq \emptyset$. E - algebraic equivalence class of line bundles on R. $\mathcal{P}$ - tautological line bundle on $R \times \mathrm{Pic}^E(R)$ such that $\mathcal{P}|_{\{\cdot\} \times \mathrm{Pic}^E(R)} \overset{alg}{\sim} 0$. Then $(\det R\pi_{2*}\mathcal{P})^\vee$ is ample, if

a) R is a curve,

b) R is abelian variety, E ample class,

c) (R arbitrary), E' any, A ample class; for $E = E' + nA$, $n \gg 0$.

Idea: a) $\det R\pi_{2*}\mathcal{P} \sim \mathcal{O}(-\Theta)$

b) Computation using Grothendieck - Riemann - Roch

c) Induction on $\dim R$

<u>Idea of proof of Theorem 1</u>:

- $\mathcal{P} \otimes \pi_2^* \mathcal{J}$ still tautological, $\det R\pi_{2*}(\mathcal{P} \otimes \pi_2^* \mathcal{J}) = \det R\pi_{2*}\mathcal{P} \otimes \mathcal{J}^{\otimes \chi(\mathcal{E})}$
- make this a <u>trivial</u> line bundle (possible after replacing $\mathrm{Pic}^E(R)$ by an étale cover)
- metrize $\mathcal{P}'$ such that it becomes distinguished fiber-by-fiber (uses continuity of Quillen's metric)
- show that $\mathcal{P}'^{\otimes n}$ has for some $n>0$ a rational sections that does not vanish or become undefined in a whole fiber, $s \in \Gamma(\mathcal{O}(\pi_1^* D))$ for some divisor D
- Compare with heights of pole-zero-divisors of $s|_{X \times \{\cdot\}}$ (use a theorem of Stoll on continuity of fiber integrals).

4. Special case - arithmetic surfaces

<u>Theorem</u>: $\mathcal{C}/\mathcal{O}_K$ arithmetic variety, $\dim \mathcal{C} = 2$. $C := \mathcal{C}_K$ curve (connected) of genus g, $x \in C(K)$. Θ: Θ-divisor on $\mathrm{Pic}^g(C)$ (defined using x), ω Kähler metric on $C(\mathbb{C})$. $H \in \mathbb{R}$

Then, for line bundles $\mathcal{E} \in \mathrm{Pic}(\mathcal{C})$, fiber-by-fiber of degree g and satisfying $|\deg_{\mathcal{L}} \mathcal{E}|_{X_{\mathfrak{p},i}}| < H$ on components of special fibers

$$h_{\omega, \overline{\mathcal{O}(x)}}(\mathcal{E}) = h_\Theta(\mathcal{E}_K) + O(1).$$

Theorem 3 (asymptotic behaviour): $\mathcal{C}$ as above, $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(X)$, $\mathcal{E}, \mathcal{F} \in \mathrm{Pic}(X)$, metrize $\mathcal{F}$. Then

$$h_{\overline{\mathcal{L}}}(\mathcal{E} \otimes \mathcal{F}^{\otimes n}) = h_{\overline{\mathcal{L}}}(\mathcal{E}) + n\, \widehat{\deg}\, \pi_*(\hat{c}_1(\overline{\mathcal{L}})\hat{c}_1(\overline{\mathcal{F}})) - \frac{A \cdot n^2 + \frac{1}{2} B \cdot n + h_{\overline{\mathcal{F}}}(\mathcal{E}) \cdot n}{\chi_K(\mathcal{E} \otimes \mathcal{F}^{\otimes n})} \cdot \deg \mathcal{L}_C\,,$$

where $A := \widehat{\deg}\, \pi_*(\hat{c}_1(\overline{\mathcal{F}})^2)$,

$B := \widehat{\deg}\, \pi_*(\hat{c}_1(\overline{\mathcal{F}})\hat{c}_1(T\pi))$.

$T\pi$ is the relative tangent bundle, metrized by ω.

Idea: Apply the arithmetic Riemann-Roch Theorem to $(\det R\pi_*(.), h_Q)$.

One interpretation: $\deg \mathcal{L}_C = 1$, $\deg \mathcal{E}_C = g$, $\deg \mathcal{F}_C = 0$ $\rightsquigarrow$ $-A \cdot n^2$ dominates for $n \gg 0$ (We recover the height for degree 0-line bundles due to Faltings - Hriljac.)

Title: Bounds for the size of the Tate Shafarevich group

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Page: 1

We consider the following conjectures (where k is a number field or a function field of one variable over a finite field $\mathbb{F}$ with q elements):

$(C\gamma)_k$ There exist a constant $\gamma(k)$ such that: For any elliptic curve E over k with conductor N one has for the Tate-Shafarevich group

$$\# | Ш (E, k) | \leq N^{\gamma(k)}$$

$(C\sigma)_k$ There exists a constant $\sigma(k)$ such that for any elliptic curve E over k with conductor N one has for the discriminant:

$$| \Delta_E | \leq N^{\sigma(k)}$$

We also consider the strong form of both: $\# | Ш | \underset{\epsilon}{\leq} N^{1/2+\epsilon}$ and $| \Delta | \underset{\epsilon}{\leq} N^{6+\epsilon}$

In all these considerations we suppose that the Birch-Swinnerton-Dyer conjecture is true ($k = \mathbb{Q}$) or the Artin-Tate conjecture is true (k function field).

We then prove the following:

Th 1 Suppose $k = \mathbb{Q}$ and all E are modular elliptic curve then $(C\gamma)_{\mathbb{Q}}$ is equivalent to $(C\sigma)_{\mathbb{Q}}$

Moreover if we assume the Riemann hypothesis for ~~the~~ Rankin-Selberg zeta functions

$$(C, \gamma = \tfrac{1}{2}+\epsilon)_{\mathbb{Q}} \Rightarrow (C, \sigma = 6+\epsilon)_{\mathbb{Q}}$$

I have proved the conjecture $(C, \sigma)_k$ for functions fields over finite fields* in 1980, and the same arguments as in Th 1 gives us:

Th 2 Let k be a function field over a finite field k. Then the conjecture $(C\gamma)_k$ is true for k with a computable γ.

One should note that in the analogy number fields $\longleftrightarrow$ function fields the

(*) with $\sigma = 6p^e$ where p^e is the inseparability degree of the j map cf [S].

role of the period $2\int_{E(\mathbb{R})} \frac{dx}{y} = \Omega$ is played by $q^{-\alpha}$ where α is an Euler-Poincaré characteristic. The relation of Ω with the discriminant was proved in [G], the relation of α with the degree of the discriminant is in [G, S] and [R].

[G] D. Goldfeld "Modular elliptic curves and diophantine problems" Proc Conf of Canada Number Theory Ass. Banff, C Alberta Canada 1988 p 157-176

[G, S] D. Goldfeld and L. Szpiro "Bounds for the order of the Tate-Shafarevich group" Compositio Math 97 (F. Oort volume) p. 71-87. 1995

[R] C.S Rajan "On the size of the Shafarevich-Tate group of elliptic curves over function fields" preprint 1995 Mac Gill University Canada

Author: L. Szpiro

Page: 4

[5] L. Szpiro "Propriétés numériques du dualisant relatif" in Pinceaux de courbes Astérisque 86 (1981) 44-78

Title: Oort's conjecture for $\mathbb{P}^1 \times \mathbb{P}^1$.

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The conjecture mentioned in the title is the following:

Conjecture (F. Oort, Y. André) Let $C \subset \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ be an irreducible curve dominating both $\mathbb{P}^1_{\mathbb{C}}$'s and containing infinitely many $(x,y) \in \mathbb{C}^2$ with x and y j-invariant of CM-elliptic curves. Then $\exists\, n \geq 1$ s.t. C is the image of

$$X_0(n)_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \quad (E \xrightarrow{\varphi} F) \mapsto (j(E), j(F)).$$

Remark. This is a special case of a conjecture of Oort, saying that ~~each~~ irred. subvar. of A_g in which the CM-points are Zariski dense is in fact a sub-Shimura-variety. The case where the subvariety is a curve was already conjectured by Y. André.

Results. 1. B. Moonen, Ph.D. thesis Utrecht, 1995. He proves that the conj. is true if $\exists$ prime p and infin. many such (x,y) that are "canonical" at p. (In fact, under this assumption, he proves the general case.) His method is to show that the completion of the subvariety along its ordinary locus in char. p is stable by the usual Frobenius lift; then, in Serre-Tate local coordinates, the subvar. is given by linear equations (multiplicatively) and from that he derives that it is a sub-Shimura-variety.

2. Y. André, dec. 1995. He shows the conjecture is true if the curve C meets the two-$\mathbb{P}^1_{\mathbb{C}}$'s at infinity only at ∞ and CM-points. His proof uses G-functions. In the case where C meets $(\{\infty\}\times\mathbb{P}^1_{\mathbb{C}})\cup(\mathbb{P}^1_{\mathbb{C}}\times\{\infty\})$ only at (∞,∞) he has a very elementary argument.

3. Myself, dec. 1995, independently. I showed that the conjecture is a consequence of GRH for quadratic imaginary fields. Moreover, analytic number theory is at a stage where this GRH is almost unnecessary (more about this later).

I will only discuss the proof of 3. First we need some notation and preliminaries on CM elliptic curves.

Let $K/\mathbb{Q}$ be quadr. imaginary, and fix $K\hookrightarrow\overline{\mathbb{Q}}$. For $f\geq 1$ let $O_{K,f} := \mathbb{Z}+f.O_K$, where $O_K\subset K$ is the ring of integers. Again for $f\geq 1$, let $S_{K,f}$ be the set of pairs $(E/\overline{\mathbb{Q}},\alpha)$ with E an elliptic curve over $\overline{\mathbb{Q}}$ and $\alpha: O_{K,f}\xrightarrow{\sim}\mathrm{End}(E)$ an isom. inducing the fixed $K\hookrightarrow\overline{\mathbb{Q}}$ via the action on $\mathrm{Lie}(E)$, up to isomorphism. On the set $S_{K,f}$ we have an action of $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ and an action of $\mathrm{Pic}(O_{K,f})$: $(E,[\mathcal{L}])\mapsto$ "$\mathcal{L}\otimes_{O_{K,f}} E$" (think of E as $\mathbb{C}/\Lambda$ or use a presentation of $\mathcal{L}$). The two actions commute with each other, and $S_{K,f}$ is a $\mathrm{Pic}(O_{K,f})$-torsor. It follows that G_K acts via a homomorphism, known to be surjective, $G_K\twoheadrightarrow\mathrm{Pic}(O_{K,f})$. Let $H_{K,f}/K$ be the corresponding field extension; it has Galois group $\mathrm{Pic}(O_{K,f})$. We will use that for all $(E/\overline{\mathbb{Q}},\alpha)$ in $S_{K,f}$ we have $H_{K,f} = K(j(E))$.

<u>First step in the proof of 3</u>. (this was also done by André).

Let $C_{\mathbb{C}}$ be as in the conjecture. Then $C_{\mathbb{C}}$ is defined over $\overline{\mathbb{Q}}$. Let C be the image of $C_{\mathbb{C}}$ in $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$. This C is an irreducible curve, maybe not geometrically irreducible. We will prove that C is the image of some $X_0(n)_{\mathbb{Q}}$. Let (d_1, d_2) be the bi-degree of C.

<u>Proposition</u>. $\exists$ infinitely many $(x_1,x_2) \in C(\overline{\mathbb{Q}})$, CM points, such that $\mathrm{End}^0(x_1) := \mathbb{Q} \otimes \mathrm{End}(x_1) = \mathrm{End}^0(x_2)$.

<u>Proof</u>. The idea is that for $(x_1,x_2) \in C(\overline{\mathbb{Q}})$ we have $[\mathbb{Q}(x_1,x_2):\mathbb{Q}(x_1)] \le d_2$ and $[\mathbb{Q}(x_1,x_2):\mathbb{Q}(x_2)] \le d_1$, or, equivalently $|G_{\mathbb{Q},x_1} \cdot x_2| \le d_2$ etc.

Suppose that $(x_1,x_2) \in C(\overline{\mathbb{Q}})$ is CM and that $K_1 := \mathrm{End}^0(x_1) \neq K_2$. Let f_1 and $f_2 \geqslant 1$ be given by: $\mathrm{End}(x_i) = O_{K_i,f_i}$. We want to show that $H_{K_1,f_1} \cap H_{K_2,f_2}$ is small.

<u>Lemma</u>. Write $\mathrm{Gal}(K_i/\mathbb{Q}) = \{1,\sigma_i\}$. Then σ_i acts on $\mathrm{Gal}(H_{K_i,f_i}/K_i)$ as -1.

<u>Proof</u>. $\mathrm{Gal}(H_{K_i,f_i}/K_i) = \mathrm{Pic}(O_{K_i,f_i}) \xrightarrow{1+\sigma_i} \mathrm{Pic}(O_{K_i,f_i})$. ⊠

$$\begin{tikzcd} \mathrm{Pic}(O_{K_i,f_i}) \arrow[rr, "1+\sigma_i"] \arrow[dr] & & \mathrm{Pic}(O_{K_i,f_i}) \\ & \mathrm{Pic}(\mathbb{Z}) \arrow[ur] & \end{tikzcd}$$

Define $K_i^{ab,-}/K_i$ by: $\mathrm{Gal}(K_i^{ab,-}/K_i) = \mathrm{Gal}(K_i^{ab}/K_i)/(1+\sigma_i)$.

We have $x_i \in K_i^{ab,-}$.

<u>Lemma</u>. $\mathrm{Gal}((K_1^{ab,-} \cap K_2^{ab,-}).K_1K_2 / K_1K_2)$ is killed by 2.

<u>Proof</u>: each of the σ_i acts simultaneously as 1 and as -1. ⊠

From this we conclude that it suffices to show that for quadratic imaginary K the group $\mathrm{Pic}(\mathcal{O}_{K,f})\otimes\mathbb{F}_2$ is small compared to $\mathrm{Pic}(\mathcal{O}_{K,f})$. Some genus theory shows that

$$\log_2 |\mathrm{Pic}(\mathcal{O}_{K,f})\otimes\mathbb{F}_2| \approx |\{p \mid p \text{ prime}, p \mid \mathrm{discr}(\mathcal{O}_{K,f})\}|.$$

the difference is at most 2 (or so).

We also have the following theorem:

<u>Thm.</u> (Siegel) $\log|\mathrm{Pic}(\mathcal{O}_{K,f})| = (\frac{1}{2}+o(1))\cdot\log|\mathrm{discr}(\mathcal{O}_{K,f})|$, as $|\mathrm{discr}(\mathcal{O}_{K,f})|\to\infty$.

Putting these two things together gives:

$$\frac{|\mathrm{Pic}(\mathcal{O}_{K,f})\otimes\mathbb{F}_2|}{|\mathrm{Pic}(\mathcal{O}_{K,f})|} \to 0 \quad \text{as } |\mathrm{discr}(\mathcal{O}_{K,f})|\to\infty.$$

As a consequence, we get bounds for the $|\mathrm{discr}(\mathcal{O}_{K_i,f_i})|$ in terms of the d_i. Hence only finitely many of the CM-points $(x_1,x_2)\in C(\overline{\mathbb{Q}})$ can have $K_1\neq K_2$. $\boxtimes$ (end of proof of Proposition).

<u>Remark.</u> One can even reduce to the case where there are infin. many $(x_1,x_2)\in C(\overline{\mathbb{Q}})$ s.t. $\mathrm{End}(x_1)=\mathrm{End}(x_2)$. We won't use this.

One is now very much tempted to do the following. Take a CM point $(x_1,x_2)\in C(\overline{\mathbb{Q}})$ such that $K_1=K_2=:K$. Let n be the minimal degree of an isogeny between the x_i, and consider the intersection of C with (the image of) $X_0(n)_{\overline{\mathbb{Q}}}$. This contains all the Galois conjugates of (x_1,x_2). Let f_i be given by $\mathrm{End}(x_i)=\mathcal{O}_{K,f_i}$. Then $|G_K\cdot(x_1,x_2)| = |\mathrm{Pic}(\mathcal{O}_{K,f})|$, with $f:=\mathrm{lcm}(f_1,f_2)$. If $|\mathrm{Pic}(\mathcal{O}_{K,f})|$

is $\gg$ the intersection number of C and $X_0(n)_{\mathbb{Q}}$, then we can conclude that $C = X_0(n)_{\mathbb{Q}}$. Unfortunately, the bound on n that one gets from Minkowski's theorem is just not good enough. For example, it says that $n \leq \text{const.} |\mathrm{discr}(\mathcal{O}_K)|^{1/2}$ if $f = 1$. So we have to try something else.

<u>Second step in the proof of 3.</u>

Take a CM-point $(x_1, x_2) \in C(\overline{\mathbb{Q}})$ with $K := K_1 = K_2$. Let $f := \mathrm{lcm}(f_1, f_2)$. Then $(x_1, x_2) \in C(H_{K,f})$ and $|G_K \cdot (x_1, x_2)| = |\mathrm{Pic}(\mathcal{O}_{K,f})|$.

Let p be a prime such that $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{O}_{K,f} \cong \mathbb{F}_p \times \mathbb{F}_p$. Choose one of the two $\mathfrak{p}$ over p. Then $[\mathfrak{p}] \in \mathrm{Pic}(\mathcal{O}_{K,f})$ acts on x_1 and x_2 by isogenies of degree p. Hence $[\mathfrak{p}](x_1, x_2)$ is in $(T_p \times T_p)(C)$, where $T_p \times T_p$ is the correspondence on $\mathbb{P}^1 \times \mathbb{P}^1$ given by:

$$\begin{tikzcd} & X_0(p) \times X_0(p) \arrow[dl, "s"'] \arrow[dl, shift left=2, "s"] \arrow[dr, "t"'] \arrow[dr, shift left=2, "t"] & \\ \mathbb{P}^1 \times \mathbb{P}^1 & & \mathbb{P}^1 \times \mathbb{P}^1 \end{tikzcd}$$

$\left(\text{so } (T_p \times T_p)(C) = (t,t)(s,s)^{-1} C\right)$.

The curve $(T_p \times T_p)(C)$ has bidegree $((p+1)^2 d_1, (p+1)^2 d_2)$, so $C \cdot (T_p \times T_p)(C) = 2 d_1 d_2 (p+1)^2$. So if $|\mathrm{Pic}(\mathcal{O}_{K,f})| \geq 2(p+1)^2 d_1 d_2$ we must have $C \subset (T_p \times T_p)(C)$. Recall that by Siegel we have $|\mathrm{Pic}(\mathcal{O}_{K,f})| = |\mathrm{discr}(\mathcal{O}_{K,f})|^{\frac{1}{2} + o(1)}$, so we would be happy if for some $\varepsilon > 0$ there exists, for $|\mathrm{discr}(\mathcal{O}_{K,f})| \gg 0$, a prime $p \leq |\mathrm{discr}(\mathcal{O}_{K,f})|^{\frac{1}{4} - \varepsilon}$ which is split in $\mathcal{O}_{K,f}$.

Now if one assumes GRH for quadr. imag. K, one gets such p that are at most $c(\log|\mathrm{discr}(\mathcal{O}_{K,f})|)^2$. What is actually proved in analytic number theory is that such primes exist with $p \leq |\mathrm{discr}(\mathcal{O}_{K,f})|^{\frac{1}{4}+\varepsilon}$ (for any $\varepsilon>0$)! (I am grateful to E. Fouvry to explain this to me.)

Assuming GRH, we get infinitely many primes p such that $C \subset (T_p \times T_p)(C)$. For what follows, this is the only property of C that we will use.

Third and final step in the proof of 3.

Let K be the function field of C. Choose $E/\mathbb{Q}(j)$ s.t. $j(E)=j$. Let $E_i := pr_i^{-1}E$ over K. In order to show 3, one has to show that E_1 and E_2 are isogeneous over $\overline{K}$.

Now for $p \gg 0$ such that $C \subset (T_p\times T_p)(C)$, one shows quite easily that $E_1[p](\overline{K}) \cong E_2[p](\overline{K})$ as G_K-modules, possibly after twisting E_2. From this one can conclude (c.f. Frey's talk) that E_1 and the twist of E_2 are isogeneous.

Another way to finish is to prove, using elementary topological group theory, that for any curve $C \hookrightarrow \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and p prime such that $C \subset (T_p\times T_p)(C)$ and $p > 5, \underbrace{d_1, d_2}_{\text{bidegree of } C}$, C has to be of the form $X_0(n)_{\mathbb{C}}$ for some $n \geq 1$.

Title: Modular forms and construction of unimodular lattices

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Page: 1

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My talk was devoted to the application of theta ~~se~~ series of lattices with spherical coefficients to classification of lattices.

As first exemple I discussed the known classification of 24-dimensional even unimodular lattices. Such a lattice is uniquely determined by its rootsystem Λ_2, which is completly characterized by the two property:

1) Λ_2 is either void or of maximal rank 24,

2) all irreducible components of Λ_2 have the same Coxeter number.

Theese two propertjes follow easly from the annulation of the theta serie with spherical coefficient of degree two. Details look in [1].

Second exemple was the classification with R. Scharlau [2] of 16 dimensional lattices of $\det = 2^8$ wich are doubly even. Here one has to consider the full root system of Λ wich is $\Lambda_2 \cup 2\Lambda^*_4$, again such a rootsystem is either trivial or of maximal rank, all components have the same Coxeter number and number of short roots equals the number of long roots. Theese properties are sufficient for the complete classification of ~~theeg~~ those lattices. There are exactly 24 such lattices. ~~Theese~~ The ~~pr~~ above properties follow again from the theta series with spherical coefficient of degree 2.

The third and main topic was the recent classification with R. Bacher [3] of 28 dimensional unimodular lattices without roots. There are exactly ~~3~~ 38 such lattices. Here the theta series with spherical coefficients permit to ~~a~~ have a control on the Kneser method of neighboring lattices, which gives this classification.

Bibliography: [1] Chapter 18 in J. H. Conway, N. J. A. Sloane Sphere Packings, Lattices and Groups. Springer Verlag 1993

[2] R. Scharlau, B. Venkov. The genus of the Barnes-Wall lattice, Comment. Math. Helv. 69 (1994), 322-333

[3] R. Bacher, B. Venkov. Reseaux entiers unimodulaires sans racine en dimension 27 et 28. Preprint N332, Grenoble 1996.

Title: Canonical flags on arithmetic Chevalley ~~schemes~~ schemes

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Let $\mathcal{O}$ be the ring of algebraic integers of an algebraic number field K. We consider semi simple group schemes $\mathcal{G} \to \mathrm{Spec}(\mathcal{O})$ whose generic fibre $\mathcal{G} \times_{\mathcal{O}} K = G$ is split. The easiest examples of such schemes are obtained from locally free modules $M/\mathcal{O}$ of finite rank, in this case $\mathcal{G} = SL(M)$ is such a scheme.

For any place $v \in S_\infty$ = set of places at infinity we can form $\mathcal{G} \times_{\mathcal{O}} K_v = G_v$ which we consider as an algebraic group over $\mathbb{R} \subset K_v$. An extension of $\mathcal{G} \to \mathrm{Spec}(\mathcal{O})$ to an arithmetic Chevalley scheme is the choice of a Cartan involution

$$\Theta = (\ldots \Theta_v \ldots)_{v \in S_\infty}$$

This involution can be used to define an euclidian metric $\langle\, ,\rangle_\Theta$ on the Lie algebra $\mathfrak{g}_\infty$ of $G_\infty = \prod_v G_v$.

Let $\mathcal{B} \subset \mathcal{G}$ be a Borel subgroup scheme. Its unipotent radical $\mathcal{U}$ has a filtration $\mathcal{U} \supset \mathcal{U}_1 \supset \mathcal{U}_2 \ldots \mathcal{U}_r = \{1\}$ where the succesive quotient $\mathcal{U}_i/\mathcal{U}_{i+1}$ are labeled by the positive roots α, we rewrite $\mathcal{U}_i/\mathcal{U}_{i+1} \simeq \mathcal{U}_\alpha$. Such a subquotient is a locally free module of rank 1. We use the metric $\langle\, ,\rangle_\Theta$ to induce an euclidian metric on $\mathcal{U}_{\alpha,\infty} = \prod_v \mathcal{U}_{\alpha,v}$ and define

$$n_\alpha(B,\Theta) = -\log \operatorname{vol}(\mathcal{U}_{\alpha,\infty}/\mathcal{U}(\mathcal{O})) - \frac{1}{2}\log(D_{K/\mathbb{Q}})$$

One verifies that $n_\alpha(B,\Theta) + n_\beta(B,\Theta) = n_{\alpha+\beta}(B,\Theta)$ whenever this makes sense.

We have also the dominant fundamental weights λ_α which are defined by

$$2\frac{\langle \lambda_{\alpha}, \beta\rangle}{\langle \beta, \beta\rangle} = \delta_{\alpha,\beta}$$

for all simple roots β. The can be expressed as

$$\lambda_\alpha = \sum c_{\alpha\beta}\, \beta$$

where $c_{\alpha,\beta}$ are rational and ≥ 0 and $\det(c_{\alpha\beta}) \neq 0$. We define $p_\alpha(B,\theta) = \sum c_{\alpha,\beta}\, n_\beta(B,\theta)$ and then we show that this number can be expressed in terms of the metric $\langle\, ,\rangle_\theta$ on the unipotent radical $R_u(P_\alpha)$ where P_α is the maximal parabolic corresponding to α: Its system of simple roots is the system of simple roots of G where we remove α. Hence

$$p_\alpha(B,\theta) = p_\alpha(P_\alpha,\theta) = \rho(P_\alpha,\theta).$$

The arithmetic scheme $(\mathcal{G}, \Theta)$ is called stable (resp. semi stable) if for all maximal parabolic subgroups $P_\alpha \subset \mathcal{G}$ we have

$$\rho(P_\alpha, \Theta) < 0 \quad (\text{resp. } \rho(P_\alpha, \Theta) \leq 0)$$

We can extend the notion of invariants of type $\rho(\ ,\ )$ to arbitrary parabolic subgroups, i.e. we can define $\rho(P, \theta)$ in terms of certain volumina on subquotients of $R_u(\underline{P})$ with respect to Θ.

We can use this to define invariants

$$n_{\tilde{\alpha}}(P, \Theta)$$

for the simple roots α which belong to the unipotent radical of P. To do this we write α in the form

$$\alpha = \sum_{\beta \notin \Pi_{\underline{P}}} a_{\alpha,\beta} \lambda_\beta + \sum_{\beta \in \Pi_{\underline{P}}} b_{\alpha,\beta} \beta$$

where $\Pi_{\underline{P}}$ is the set of simple roots of the reduction quotient $P/R_u(P) = M$. Then we drop the second summand and define $\tilde{\alpha} = \sum_{\beta \notin \Pi_M} a_{\alpha,\beta} \lambda_\beta$ and put

$$n_{\tilde{\alpha}}(P, \theta) = \sum a_{\alpha\beta}\, p\,(P_\beta, \theta)$$

Now we have the

Theorem For any arithmetic Chevalley-scheme $(\mathcal{G}, \theta) \to \mathrm{Spec}(\mathcal{O}) \cup \{\infty\}$ we have a unique parabolic subgroup scheme $\mathcal{P} \subset \mathcal{G}$ such that

$$(\mathcal{P}/R_u(\mathcal{P}), \theta_P) = (\mathcal{M}, \theta_{\mathcal{M}})$$

is semi stable and

$$n_{\tilde{\alpha}}(P, \theta) > 0$$

for all $\tilde{\alpha}$