Boundary Value Problems in Boutet de Monvel's Algebra for Manifolds with Conical Singularities I

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Abstract

In the present article and a subsequent paper we shall develop a pseudodifferential calculus for boundary value problems on manifolds with finitely many conical singularities.
Outside the singular set we use Boutet de Monvel's calculus. Near a singularity, we identify the manifold with $X \times [0, \infty)/X \times \{0\}$, where X is a smooth compact manifold with boundary, and use operators of Mellin type on $\mathbf{R}_+$ with values in Boutet de Monvel's algebra on X. To this end, the present part provides a parameter-dependent version of Boutet de Monvel's calculus and a class of weighted Sobolev spaces with discrete asymptotics based on the Mellin transform.
Moreover, we introduce the Green operators, the residual operators in the calculus, and the smoothing Mellin operators with asymptotics, a class of operators which is regularizing but in general non-compact.

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Introduction

One large program of contemporary analysis of which the present paper is a part, has as its goal the establishment of an index theory for elliptic operators on singular manifolds. In analogy with the classical theory, this program has two phases: first and foremost, the construction of a natural algebra of pseudodifferential operators, and second, a thorough exploration of parametrices to elliptic elements at the symbol level.
Moreover, for any such approach to be useful in practice, it should be 'iterative'. That is, whenever one kind of singularity has been treated successfully and the corresponding algebra has been constructed, then it should be possible to also treat the 'cone' which has the present singularity as its base and then the edge over this cone. The previously constructed operator algebras should always serve as the basis for the following ones.
This is the first of two papers devoted to the former part of this program, namely the construction of an algebra of pseudodifferential operators for manifolds with boundary and conical singularities. Such manifolds are smooth outside a finite set of so-called 'singularities', where they locally have the structure of a cone $X \times [0,1)/X \times \{0\}$. The base of the cone, X, is a compact manifold with boundary. Notice that this already gives elements of a corner theory since we have two 'singular directions', that normal to the boundary and that along the axis of the cone, coming together at the tip of the cone.
In the larger program of analysis on singular manifolds there always is a certain freedom in the choice of the algebra one intends to work with. In view of a wide range of applications in mathematical physics, where the primary interest is in *differential* boundary value problems, it seemed natural for us to rely here on Boutet de Monvel's calculus, even though from the analytical point of view a more general concept, avoiding the transmission condition, might have been desirable.
Similarly as in Boutet de Monvel's approach to the case of manifolds with smooth boundary, the algebra of differential operators will be completed to an algebra of *pseudodifferential* operators; the present situation, however, requires the introduction of additional new elements, namely the analogues of the Mellin and Green operators. These played an important role already in the analysis on singular manifolds without boundary.
What we eventually would like to have is an algebra of operators with a symbolic structure that

(i) contains the classical boundary value problems

(ii) gives asymptotic expansion formulae for the symbols of compositions and adjoints, and

(iii) provides a notion of ellipticity in terms of (principal) symbols that allows the construction of parametrices to elliptic elements within the calculus; furthermore these parametrices should also be Fredholm inverses.

It is well-known that the solutions to classical boundary value problems on manifolds with conical singularities have particular asymptotics close to the singular points, cf. Kondrat'ev [14]. An additional task therefore is to provide suitable classes of spaces that contain the typical asymptotics and also are mapped continuously into each other by the operators.
Our method is the following: On the regular part of the manifold we use Boutet de Monvel's calculus in its standard form. Near the singularities we work with the cylinder

$X \times \mathbf{R}_+$. We denote by x the coordinate in X, by t that in $\mathbf{R}_+$. The operators we shall be dealing with are Mellin operators with respect to t with values in Boutet de Monvel's algebra over X. In order to make this rigorous we need the concept of meromorphic functions with values in Boutet de Monvel's algebra and additionally a parameter-dependent version of Boutet de Monvel's calculus. It is given in Section 2 of the present paper which is of independent interest. It provides a self-contained introduction to Boutet de Monvel's calculus with and without parameters based on the concept of operator-valued symbols on spaces with group actions. In our set-up, the parameter plays the role of an additional covariable, the parameter-dependence therefore is slightly less general than that in Grubb [9]. On the other hand, the new concept yields a very fast approach.
Section 3 starts with the definition of the Sobolev spaces $\mathcal{H}^{s,\gamma}, s, \gamma \in \mathbf{R}$, the operators are acting on. Outside the singular set, $\mathcal{H}^{s,\gamma}$ coincides with the standard Sobolev space H^s. On the cylinder $X \times \mathbf{R}_+$ it is defined in terms of an intertwined action of the Mellin transform with respect to the $t-$variable and order-reducing operators on X, combined with a weight function of the form $t^\gamma, \gamma \in \mathbf{R}$. For $s = 0, 1, 2, \ldots$, we may characterize $\mathcal{H}^{s,\gamma}$ in local coordinates as the space of all functions f such that $t^{\frac{n}{2}-\gamma}(t\partial_t)^k \partial_x^\alpha f \in L^2$ for $k + |\alpha| \leq s$. Moreover, we introduce asymptotics of the form $\sum_{j=0}^{\infty} \sum_{k=0}^{m_j} a_{jk}(x)\, t^{-p_j} \ln^k t\, \omega(t)$. Here $p_j \in \mathbf{C}, \mathrm{Re}\, p_j \to -\infty$ as $j \to \infty, m_j \in \mathbf{N}$, $a_{jk} \in C^\infty(X)$, and ω is a cut-off function near zero.
We may then introduce the Green operators, the residual operators with respect to the calculus: essentially, they are described by the fact that they map all the above spaces to spaces of smooth functions with asymptotics. For $\dim X = 0$ and Taylor asymptotics near $t = 0$ they coincide with Boutet de Monvel's singular Green operators of type zero.
In Section 4 we develop the theory of $t-$independent Mellin symbols with values in Boutet de Monvel's algebra. We study their mapping properties and their relation to the Green operators. We conclude this paper with the analysis of the algebra $C_{M+G}(X \times \mathbf{R}_+, g)$ consisting of all operators of the form $A = \sum t^j A_j + G$, with smoothing Mellin operators A_j and a Green operator G. This algebra is of particular interest since it will turn out to be an ideal in the final operator algebra.
Differential boundary value problems for manifolds with conical singularities and especially ellipticticity, regularity, and asymptotics have been studied in great detail by Kondrat'ev [14]. Also the concept of Plamenevskij [20], developed originally for closed manifolds, allows generalizations to manifolds with boundary. His techniques as well as his objects, however, are quite different from ours.
The present article focuses on the concept of ellipticity and the construction of parametrices in terms of symbols for the full *pseudodifferential* algebra with a very small class of residual elements.
There are formal analogies to the construction of the pseudodifferential calculus on manifolds with conical singularities without boundary, cf. Schulze [27], [30]. Our approach continues the analysis of Boutet de Monvel [2], Vishik & Eskin [35], Eskin [7], Plamenevskij [20], Rempel & Schulze [21], [22], and Schulze [27], [29]. In order to further pursue the program initiated by Schulze in [29], [30] it will be necessary to also consider boundary value problems without the transmission property and eventually their edge theory; this, however, will be the subject of a future paper by the authors.

1 Manifolds with Conical Singularities

1.1 Notation

An $n-$dimensional manifold with boundary is a topologigal (second countable) Hausdorff space M such that each point $m \in M$ has a neighborhood which is diffeomorphic to either $\mathbf{R}^n$ or the closed half-space $\overline{\mathbf{R}^n_+}$. The former points are called the interior points of M, the latter the boundary points. We will use the standard notation $\operatorname{int} M$ and ∂M.

1.1.1 Definition. A *manifold with boundary and conical singularities D of dimension n* is a topological (second countable) Hausdorff space with a finite subset $\Sigma \subset D$ ("singularities") such that $D \backslash \Sigma$ is an n-dimensional manifold with boundary, and for every $v \in \Sigma$ there is

- an open neighborhood U of v.
- a compact manifold with boundary X of dimension $n-1$.
- a system $\mathcal{F} \neq \emptyset$ of mappings with the following properties
  (1) For all $\phi \in \mathcal{F}$
  $$\phi : U \to X \times [0,1)/X \times \{0\}$$
  is a homeomorphism with
  $$\phi(v) = X \times 0/X \times \{0\}.$$
  (2) Given $\phi_1, \phi_2 \in \mathcal{F}$, the restriction
  $$\phi_1\phi_2^{-1} : X \times (0,1) \to X \times (0,1)$$
  extends to a *diffeomorphism*
  $$X \times (-1,1) \to X \times (-1,1).$$
  (3) The charts $\phi \in \mathcal{F}$ are compatible with the charts for the manifold for $D \backslash \Sigma$: The restriction $\phi : U \backslash \{v\} \to X \times (0,1)$ is a diffeomorphism.

If there is no fear of confusion, we will simply speak of a *manifold with conical singularities.*

1.1.2 Remark. We can and will assume that for each singularity $v \in \Sigma$, the system $\mathcal{F}$ is maximal with respect to the properties (1), (2), and (3) in Definition 1.1.1.

1.1.3 Definition and Remark. Let D be a manifold with boundary and conical singularities. By assumption, $D\backslash\Sigma$ is a manifold with boundary. Properties 1.1.1(1) and (2) imply that any neighborhood of a point $v \in \Sigma$ contains points of the topological boundary of $D\backslash\Sigma$, namely of $\partial X \times (0,1)$.
We may therefore define the interior and the boundary of D just as usual: $x \in D$ is an *interior point of D* if there is an open neighborhood of x which is homeomorphic to an open ball in $\mathbf{R}^n$, and $\operatorname{int} D$ is the collection of all interior points; $\partial D = D\backslash rmint\, D$ is the boundary of D. We always have $\Sigma \subset \partial D$.

1.1.4 Lemma. *Let D be a manifold with boundary and conical singularities. Then the topological boundary ∂D of D is a (boundaryless) manifold with conical singularities in the sense of* [29], 1.1.2 Definition 10.

Proof. By Definition 1.1.3, $\partial D\backslash\Sigma = (D\backslash\Sigma)\backslash\operatorname{int}(D\backslash\Sigma)$ is the boundary of a manifold with boundary, thus a manifold.
Let $v \in \Sigma$, and let U, X, ϕ be as in 1.1.1. Then $U \cap \partial D$ is an open neighborhood of v in ∂D, and

$$\phi|_{U\cap\partial D} : U \cap \partial D \to \partial X \times [0,1)/\partial X \times \{0\}$$

is a homeomorphism for every $\phi \in \mathcal{F}$: Injectivity and continuity are trivial; it remains to show that ϕ maps indeed to the right hand side and that it is surjective.
By assumption, $\phi(v) = X \times \{0\}/X \times \{0\} \equiv \partial X \times \{0\}/\partial X \times \{0\}$. Since $\phi|_{U\backslash\{v\}} \to X \times (0,1)$ is a homeomorphism, it maps boundary points to boundary points, so ϕ maps $(U\backslash\{v\}) \cap \partial D = (U \cap \partial D)\backslash\{v\}$ to $\partial X \times (0,1)$. The same argument now applies to ϕ^{-1} and shows that $\phi|_{U\cap\partial D}$ is surjective.
Therefore, ∂D satisfies the conditions in [29] 1.1.2, Definition 10. ◁

1.1.5 Remark. We even have somewhat more in 1.1.4, namely

$$\phi_1\phi_2^{-1} : \partial X \times (0,1) \to \partial X \times (0,1) \tag{1}$$

has an extension to a diffeomorphism

$$\partial X \times (-1,1) \to \partial X \times (-1,1). \tag{2}$$

In fact [29] 1.1.2 Definition 10 should be modified in the sense that (1) and (2) are required to hold.

1.1.6 Definition. Let D be a manifold with conical singularities. By $\mathbb{D}$ denote the topological space constructed by replacing for every singularity v the neighborhood U in Definition 1.1.1 by $X \times (0,1)$ via glueing with any one of the diffeomorphisms ϕ.
$\mathbb{D}$ is called the *stretched object associated with D*. Note that at the same time this procedure defines a stretched object $\mathbb{B}$ associated with $B = \partial D$.

1.1.7 Notation and Assumptions. Throughout this article we will keep the following notation fixed.

- D is a manifold with conical singularities of dimension $n+1$ with singularity set Σ.
- $I\!D$ is the associated $(n+1)-$dimensional stretched object defined in 1.1.6.
- $B = \partial D$ is the boundary of D, cf. 1.1.3, it is of dimension n and a manifold with conical singularities (without boundary).
- $I\!B$ is the corresponding stretched boundary object defined in 1.1.6.

In a neighborhood of one of the singularities,

- X will denote the cross-section as in 1.1.1; by definition, X is a manifold with boundary of dimension n, in particular, X contains its boundary. For practical purposes, this is often inconvenient. We shall therefore agree to denote by X the open interior, and by $\overline{X}$ the manifold including the boundary.
- $X^\wedge = X \times \mathbf{R}_+; \overline{X}^\wedge = \overline{X} \times \mathbf{R}_+$.
- $Y = \partial X$ is the topological boundary of $X; Y$ is a closed manifold of dimension $n-1$.
- $Y^\wedge = Y \times \mathbf{R}_+$.

We will assume that

- X is endowed with a Riemannian metric, and embedded in a closed Riemannian manifold Ω.
- $I\!D$ has a Riemannian metric which coincides with the canonical (cylindrical) metric on $X \times (0,1)$ near each singularity.

1.2 Motivation: Operators of Fuchs Type

Working on manifolds with conical singularities, one usually concentrates on a particular class of operators, the so-called *totally characteristic operators* or *operators of Fuchs type.*

1.2.1 Definition. As in 1.1.7 let X be a compact manifold with boundary Y, and denote by $\text{Diff}^k(X)$ the differential operators of order k on X. A boundary value problem of Fuchs type on the cylinder $X^\wedge = X \times \mathbf{R}_+$ is a system $(P, T_1, \ldots, T_\nu), \nu \in \mathbf{N}$, consisting of a differential operator P of order μ that can be written

$$P(x,t,D_x,D_t) = t^{-\mu} \sum_{j=0}^{\mu} c_j(t)(-t\partial_t)^j \tag{1}$$

with $c_j \in C^\infty(\overline{\mathbf{R}}_+, \text{Diff}^{\mu-j}(X))$ and boundary operators T_k of order μ_k given by

$$T_k(x,t,D_x,D_t) = \sum_{j=0}^{\mu_k} P_{jk}\, t^{-j}\, \gamma_j. \tag{2}$$

Here, the P_{jk} are operators of Fuchs type of order $\mu_k - j$ on $Y^\wedge$, i.e.

$$P_{jk} = t^{-\mu_k+j} \sum_{l=0}^{\mu_k-j} d_{jkl}(t)(-t\partial_t)^l$$

with suitable $d_{jkl} \in C^\infty(\overline{\mathbf{R}}_+, \mathrm{Diff}^{\mu_k-j-l}(Y))$, and γ_j is the evaluation operator at the boundary. Introducing normal coordinates (y, x_n) on X, where x_n denotes the direction normal to the boundary of X and $y \in Y^\wedge$, we may write

$$(\gamma_j f)(y,t) = \lim_{x_n \to 0} \partial_{x_n}^j f(y, x_n, t). \tag{3}$$

Operators of Fuchs type are also called *totally characteristic operators.*

1.2.2 Remark. In practice it is very inconvenient to have different orders appearing in one boundary value problem. Fortunately, there are order-reducing operators for (boundaryless) manifolds with conical singularities, cf. [27] and [26]. We may employ them to make all orders equal. This is why in Sections 3 and 4, we will deal with one order only for the operators on both the manifold and the boundary. If one is interested, however, in the asymptotics of solutions for a concrete problem, then it can be advisable to return to the original problem, because the order reducing operators contribute additional asymptotic data.

In order to motivate the choices in 1.2.1 it is instructive to compute the following almost trivial example.

1.2.3 Polar Coordinates and Differential Operators. Introduce polar coordinates in $\mathbf{R}^n$, i.e. write

$$x = |x| \cdot \frac{x}{|x|} = r\, S(\phi_1, \ldots, \phi_{n-1}) = x(r, \phi).$$

Here, S is a smooth function from an interval $D \subset \mathbf{R}^{n-1}$ to S^{n-1}. For a differentiable function f the chain rule gives

$$\begin{aligned} \frac{\partial f \circ x}{\partial r}(r,\phi) &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x(r,\phi))\, \frac{\partial x}{\partial r}(r,\phi) \quad \text{and} \\ \frac{\partial f \circ x}{\partial \phi_j}(r,\phi) &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x(r,\phi)) \frac{\partial x}{\partial \phi_j}(r,\phi). \end{aligned}$$

In other words,

$$\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) = \left(\frac{\partial f \circ x}{\partial r}, \frac{\partial f \circ x}{\partial \phi_1}, \ldots, \frac{\partial f}{\partial \phi_{n-1}}\right) \left[\frac{\partial x}{\partial(r,\phi)}\right]^{-1}.$$

Now

$$\frac{\partial x}{\partial(r,\phi)} = \begin{bmatrix} S_1(\phi) & \\ \vdots & r\frac{\partial S_j(\phi)}{\partial \phi_k} \\ S_n(\phi) & \end{bmatrix}.$$

By Cramer's rule, the inverse is of the form $(z_1(\phi), z_2(\phi)/r, \ldots, z_n(\phi)/r)^T$, where the $z_j, j = 1, \ldots, n$ are row vectors depending only on ϕ (and T stands for the transpose). Correspondingly we may write

$$\frac{\partial f}{\partial x_j} = d_j(\phi)\frac{\partial f \circ x}{\partial r} + \frac{1}{r}\sum_{k=1}^{n-1} c_{kj}\frac{\partial f \circ x}{\partial \phi_k}, \tag{1}$$

and a differential operator $P(x, D_x) = \sum_{|\alpha|\le\mu} a_\alpha(x) D_x^\alpha$ transforms into an operator of the form

$$\tilde{P}(\phi, r, D_\phi, D_r) = \sum_{|\beta|+k\le\mu} b_{\beta,k}(\phi, r)\frac{1}{r^{|\beta|}} D_\phi^\beta D_r^k \tag{2}$$

on the cylinder $D \times \mathbf{R}_+$. The coefficients $b_{\beta,k}(\phi, r)$ can be computed from (1) and the a_α. Notice that they are smooth up to $r = 0$. In particular, we can also write

$$\tilde{P}(\phi, r, D_\phi, D_r) = r^{-\mu}\sum_{k=1}^{\mu} C_k(\phi, r, D_\phi)(-r\partial_r)^k \tag{3}$$

with differential operators C_k of order $\le \mu - k$, depending smoothly on r up to $r = 0$, so that $\tilde{P}$ is an operator of Fuchs type.
In a similar way we may look at the function spaces. One of the easiest examples is $L^2(\mathbf{R}^n)$. Suppose the function f is measurable in $\mathbf{R}^n$. Consider a neighborhood of the origin, say $U = \{x : |x| < c\}$. Then

$$\int_U |f(x)|^2 dx = \int_0^c \int_{S^{n-1}} |(f \circ x)(r, \phi)|^2 r^{n-1} d\sigma(\phi) dr \tag{4}$$

with the sphere S^{n-1} and the surface measure $d\sigma$. So $f \in L^2(U)$ if and only if the function $F = f \circ x$ satisfies $F(r, \phi) r^{(n-1)/2} \in L^2((0, c) \times S^{n-1})$.

1.2.4 Boundary Value Problems. Now suppose M is a smooth n-dimensional manifold with boundary and A is the operator corresponding to a differential boundary value problem on M. Then we can pick an arbitrary point in the boundary of M and make it an artificial conical point simply by introducing polar coordinates in a neighborhood of this point. Instead of the variable r usually used for the Euclidean distance, we shall in this situation employ the variable t to denote the distance from the singularity.
Let us see what happens to A. Since the problem is purely local, we may as well assume that the manifold is $\mathbf{R}_+^{n+1} = \{\tilde{x} \in \mathbf{R}^{n+1} : \tilde{x}_{n+1} > 0\}$ and that the point is the origin. The operator A can be written in the form $A = (P, T_1, \ldots, T_\nu)$ with a differential operator $P = \sum_{\alpha\le\mu} a_\alpha(\tilde{x}) D_{\tilde{x}}^\alpha$ on $\mathbf{R}_+^{n+1}$ and a vector $(T_1, \ldots, T_\nu)$ of boundary operators on $\partial\mathbf{R}_+^{n+1} = \mathbf{R}^n$. Each of them is of the form $T_j = \gamma_0 B_j(\tilde{x}, D_{\tilde{x}})$ with a differential operator B_j of order μ_j and the evaluation operator at the boundary γ_0.
Now introduce polar coordinates. According to 1.2.3, P and the B_j transform to Fuchs type operators $\tilde{P}$ and $\tilde{B}_j$ on the cylinder $D \times \mathbf{R}_+$, where $D \subseteq \overline{\mathbf{R}_+^n}$ is relatively open. The operators $\tilde{B}_j$ have the particular form of 1.2.3(2), and the introduction of normal coordinates on the base D – which in this case reduces to using the standard Euclidean coordinates – will leave it invariant. Suppose that ϕ_n is the normal coordinate in D. Then

$\tilde{B}_j$ has the form

$$\begin{aligned}\tilde{B}_j(\phi, t, D_\phi, D_t) &= \sum_{|\beta|+k+l\le\mu_j} b_{j,k,l}(\phi,t)t^{-|\beta|-l}D^\beta_{\phi'}D^l_{\phi_n}D^k_t \\ &= \sum_{l=0}^{\mu_j}\left[\sum_{|\beta|+k\le\mu_j-l} b_{j,k,l}(\phi,t)t^{-|\beta|}D^\beta_{\phi'}D^k_t\right] t^{-l}\,D^l_{\phi_n}.\end{aligned}$$

Since γ_0 commutes with the differentiations along $\partial\mathbf{R}^n_+ = \mathbf{R}^{n-1}$, we may write $\tilde{B}_j$ in the form 1.2.1 (2).

Let us now look at a particular example.

1.2.5 Example. Let X denote a smooth compact n-dimensional manifold with boundary Y. Suppose that X is embedded in a smooth closed manifold Ω, also of dimension n, and let $\{h(t) : 0 \le t \le 1\}$ be a smooth family of Riemannian metrics on Ω. Consider the cone $C = X \times \overline{\mathbf{R}}_+/X \times \{0\}$ as the Riemannian manifold $X \times \overline{\mathbf{R}}_+$ with the degenerate metric g given locally on $X \times \{t\}$ by the tensor

$$g(t) = (\,g_{ij}(t)\,)_{i,j=1,\dots,n+1} = \begin{bmatrix} t^2h(t) & 0 \\ 0 & 1\end{bmatrix}.$$

In this example we are denoting the coordinates by $(x_1,\dots,x_{n+1})$, identifying x_{n+1} and t. Let Δ denote the Laplace-Beltrami operator on C associated to the metric g. Write $\Delta_{X,t}$ for the Laplace-Beltrami operator on X with respect to $h(t)$. In order to compute Δ, we note that the determinant of $g(t)$ is $t^{2n}\det h(t)$, and that the inverse of the matrix $(\,g_{ij}(t)\,)$ is the matrix

$$(\,g^{ij}(t)\,) = \begin{bmatrix} t^{-2}h^{-1}(t) & 0 \\ 0 & 1\end{bmatrix}.$$

The Laplace-Beltrami operator then is given in local coordinates as

$$\begin{aligned}\Delta &= [\det g(t)]^{-\frac{1}{2}}\sum_{i,j=1}^{n+1}\partial_{x_i}[\det g(t)]^{\frac{1}{2}}g^{ij}(t)\partial_{x_j} \\ &= t^{-n}[\det h(t)]^{-\frac{1}{2}}\sum_{i,j=1}^{n}\partial_{x_i}t^n[\det h(t)]^{\frac{1}{2}}t^{-2}h^{ij}(t)\partial_{x_j} \\ &\quad +t^{-n}[\det h(t)]^{-\frac{1}{2}}\partial_t t^n[\det h(t)]^{\frac{1}{2}}\partial_t + t^{-n}[\det h(t)]^{-\frac{1}{2}}\partial_t t^n[[\det h(t)]^{\frac{1}{2}}]'\partial_t \\ &= t^{-2}\Delta_{X,t} + nt^{-1}\partial_t + \frac{1}{2}[\det h(t)]'/\det h(t)\,\partial_t + \partial_t^2 \\ &= t^{-2}\left(\Delta_{X,t} + [n-1+f(t)](t\partial_t) + (t\partial_t)^2\right), \qquad (1)\end{aligned}$$

where we have denoted $f(t) = \frac{1}{2}t\,[\det h(t)]'/\det h(t)$. Note that this is a smooth function up to $t = 0$. In order to obtain a good boundary value problem, we add Dirichlet boundary conditions at Y. The evaluation operator γ_0 has the same form in these cordinates.
From this example we learn that if we want to establish a notion of ellipticity on a manifold with a conical singularity, then the natural candidates for elliptic symbols are those that are degenerate close to the singularity in a form similar to (1). Instead of asking that the symbol be elliptic in the usual sense, we will look for those symbols $p(x,t,\xi,\tau)$, where $q(x,t,\xi,\tau) = p(x,t,\xi,t\tau)$ is elliptic.
Similarly, 1.2.3(4) suggests on which kind of spaces we should consider these operators.

2 A Short Description of Boutet de Monvel's Algebra with and without Parameters

2.1 Symbol Spaces

It is the aim of this section to give an introduction to Boutet de Monvel's algebra with parameters. At the same time we take the opportunity to present the standard algebra in a new and simpler way.
In Section 4 we will consider Mellin symbols with values in Boutet de Monvel's algebra. This requires a topology on Boutet de Monvel's algebra, and we take some time to explain the topologies on the various spaces.
As before, X will be a compact $n-$dimensional manifold with smooth boundary Y. In a collar neighborhood of the boundary we introduce normal coordinates. A point there can be written $x = (y, r)$ with $y \in Y, r \geq 0$. Coordinates in $\Omega' \times \mathbf{R}_+$ with an open subset $\Omega' \subseteq \mathbf{R}^{n-1}$ will also be denoted by (x', r) or likewise $(x', x_n), x' \in \Omega'$.
For functions or distributions on $Y \times \mathbf{R}$ let r^+ denote restriction to $Y \times \mathbf{R}_+$; for functions on $Y \times \mathbf{R}_+$, e^+ denotes extension (by zero) to $Y \times \mathbf{R}$.

2.1.1 Definition. (a) Let $\Omega \subseteq \mathbf{R}^k$ be open, $\mu \in \mathbf{R}$. Then $S^\mu(\Omega, \mathbf{R}^n)$ is the space of all smooth functions p such that for every $K \subset\subset \Omega$,

$$|D_\xi^\alpha D_x^\beta p(x,\xi)| \leq C_{K,\alpha,\beta} \langle\xi\rangle^{\mu-|\alpha|} \tag{1}$$

for all $x \in K, \xi \in \mathbf{R}^n$, with constants $C_{K,\alpha,\beta}$. The Fréchet topology on $S^\mu(\Omega, \mathbf{R}^n)$ is given by the choice of the best constants in (1).
(b) By $S^\mu(\mathbf{R}^n)_{\mathrm{const}}$ denote the subspace of functions p independent of x, topologized correspondingly by the best constants in

$$|D_\xi^\alpha p(\xi)| \leq C_\alpha \langle\xi\rangle^{\mu-|\alpha|} \tag{2}$$

(c) We will sometimes also need the uniform version of the symbol classes: $S^\mu_{1,0}(\Omega, \mathbf{R}^n)$ is the space of all $p \in S^\mu(\Omega, \mathbf{R}^n)$, where the constant $C_{K,\alpha,\beta}$ is independent of K.
(d) A symbol $p \in S^\mu(\Omega, \mathbf{R}^n)$ is said to be *classical* (write $p \in S^\mu_{cl}(\Omega, \mathbf{R}^n)$) if it has an asymptotic expansion into symbols which are homogeneous in ξ for $|\xi| \geq 1$, i.e. there are symbols $p_j \in S^{\mu-j}(\Omega, \mathbf{R}^n), j = 0, 1, \ldots$ such that

$$p_j(x, \lambda\xi) = \lambda^{\mu-j} p_j(x,\xi),$$

for $\lambda \geq 1, |\xi| \geq 1$, and $p \sim \sum_{j=0}^\infty p_j$.

2.1.2 Remark. (a) In the same way, we can define symbol spaces where p takes values in matrices.
(b) The topology on $S^\mu(\Omega, \mathbf{R}^n)$ coincides with that of $C^\infty(\Omega, S^\mu(\mathbf{R}^n)_{\mathrm{const}})$. In view of the nuclearity of $C^\infty(\Omega)$ we therefore have

$$S^\mu(\Omega, \mathbf{R}^n) = C^\infty(\Omega)\hat{\otimes}_\pi S^\mu(\mathbf{R}^n)_{\mathrm{const}}.$$

2.1.3 Definition. Let $H^+ = \{(e^+f)^\wedge : f \in \mathcal{S}(\mathbf{R}_+)\}, H_0^- = \{(e^-f)^\wedge : f \in \mathcal{S}(\mathbf{R}_-)\}$, where the hat $\hat{}$ indicates the Fourier transform on $\mathbf{R}$. H' denotes the space of all polynomials. Then let

$$H = H^+ \oplus H_0^- \oplus H'.$$

Write $H_d, d \in \mathbf{N}$, for the subspace of all functions $f \in H$ with $f(\nu) = O(\langle\nu\rangle^{d-1})$.

2.1.4 Lemma. *Let $x = (x', r) \in \Omega = \Omega' \times \mathbf{R}, \Omega' \subseteq \mathbf{R}^{n-1}$ open, and let $p \sim \sum_{j=0}^{\infty} p_j \in S^\mu_{cl}(\Omega, \mathbf{R}^n)$ as in 2.1.1(c). Then for each fixed $(x', \xi') \in \Omega' \times \mathbf{R}^{n-1}$, the symbol $q \in S^\mu(\mathbf{R}, \mathbf{R})$ defined by*

$$q(r, \rho) = p(x', r, \xi', \rho)$$

is classical. In fact,

$$q(r,\rho) \sim \sum_j \sum_\alpha \frac{1}{\alpha!} \partial^\alpha_{\xi'} p_j(x', r, 0, \pm 1) \xi'^\alpha |\rho|^{\mu - j - |\alpha|}, \qquad \rho \to \pm\infty.$$

Proof. Without loss of generality assume that p is homogeneous of degree μ for $|\xi| \geq 1$. Then $q(r, \rho) = p(x', r, \frac{\xi'}{|\rho|}, \pm 1)|\rho|^\mu$. Now let $\tau = |\rho|^{-1}, x = (x', r)$ and consider $p(x, \rho\xi', \pm 1)$. By Taylor's formula,

$$p(x, \tau\xi', \pm 1) = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \partial^\alpha_{\xi'} p(x, 0, \pm 1) \xi'^\alpha \tau^{|\alpha|} + r_N(x, \xi', \tau)$$

with $|r_N(x, \xi', \tau)| \leq C_{x,\xi'} \tau^{N+1}$ as $\tau \to 0^+$, and C a continuous function of x and ξ'. ◁

2.1.5 Definition. Let $\Omega = \Omega' \times \mathbf{R}, \Omega' \subseteq \mathbf{R}^{n-1}$ open. A symbol $p \in S^\mu(\Omega, \mathbf{R}^n)$ has the *transmission property at* $r = 0$ if for every $k \in \mathbf{N}$

$$D^k_r p(x', r, \xi', \langle\xi'\rangle \rho)|_{r=0} \in S^\mu(\Omega'_{x'}, \mathbf{R}^{n-1}_{\xi'}) \hat{\otimes}_\pi H_{d,\rho}, \tag{1}$$

where $d = \text{entier}(\mu) + 1$. Write $p \in S^\mu_{tr}(\Omega, \mathbf{R}^n), p \in S^\mu_{cl,tr}(\Omega, \mathbf{R}^n)$, etc. We shall also say that p has the *transmission property with respect to* (r, ρ).

There is an easier formulation for classical symbols p. Let $\mu \in \mathbf{Z}$, $p \in S^\mu_{cl}(\Omega, \mathbf{R}^n)$, and

$$p \sim \sum_{j=0}^{\infty} p_j$$

with p_j homogeneous of degree $\mu - j$ for $|\xi| \geq 1$. Then we ask that for all k, α

$$D^k_r D^\alpha_{\xi'} p_j(x', 0, 0, 1) = (-1)^{\mu - j} D^k_r D^\alpha_{\xi'} p_j(x', 0, 0, -1), \tag{2}$$

cf. [21], 2.2.2.3, Proposition 1.

A third variant is the following. By 2.1.4 we know that for every fixed (x', ξ'),

$$q(r, \rho) = p(x', r, \xi', \rho) \in S^\mu_{cl}(\mathbf{R} \times \mathbf{R}),$$

and $q(r,\rho) \sim \sum_{j=0}^{\infty} a_j^{\pm}(r,x',\xi')\rho^{\mu-j}$ as $\rho \to \pm\infty$ for suitable $a_j^{\pm}$.
Then we have to ask that for all k, j, x', ξ'

$$D_r^k \left(a_j^+(r,x',\xi') - a_j^-(r,x',\xi')\right)|_{r=0} = 0, \tag{3}$$

cf. [29], 2.1.12(2).
Now suppose that $\Omega = \Omega_1 \times \mathbf{R} \times \Omega_2 \times \mathbf{R}$ with open subsets $\Omega_1, \Omega_2 \subseteq \mathbf{R}^{n-1}$ and $p \in S^\mu(\Omega, \mathbf{R}^n)$ is a 'double' symbol. Then p is said to have the transmission property if for all $k, l \in \mathbf{N}$,

$$D_r^k D_s^l p(x', r, y', s, \xi', \langle\xi'\rangle\, \rho)|_{r=s=0} \in S^\mu(\Omega_1 \times \Omega_2, \mathbf{R}^{n-1})\hat{\otimes}_\pi H_{d,\rho}, \tag{4}$$

where $d = \text{entier}(\mu) + 1$.

2.1.6 Symbols with parameters. (a) A smooth function p on $\Omega \times \mathbf{R}^n \times \mathbf{R}^l$ is called a *parameter-dependent symbol of order* $\mu \in \mathbf{R}$, *with parameter* $\lambda \in \mathbf{R}^l$, if

$$p(x,\xi,\lambda) \in S^\mu(\Omega_x, \mathbf{R}^n_\xi \times \mathbf{R}^l_\lambda).$$

Write $p \in S^\mu(\Omega, \mathbf{R}^n; \mathbf{R}^l)$.
(b) It is called *classical* in $S^\mu(\Omega, \mathbf{R}^n; \mathbf{R}^l)$, if it belongs to $S^\mu_{cl}(\Omega, \mathbf{R}^n \times \mathbf{R}^l)$; write $p \in S^\mu_{cl}(\Omega, \mathbf{R}^n; \mathbf{R}^l)$.
(c) Let $\Omega = \Omega' \times \mathbf{R}, \Omega' \subseteq \mathbf{R}^{n-1}$ open. Then a symbol $p \in S^\mu(\Omega' \times \mathbf{R}_r, \mathbf{R}^{n-1} \times \mathbf{R}_\rho \times \mathbf{R}^l_\lambda)$ is said to have the *transmission property* (*with parameter*), if it has the transmission property with respect to (r, ρ); similarly for 'double' symbols.
(d) A symbol $a \in S^\mu(\Omega, \mathbf{R}^n; \mathbf{R}^l)$ is called *parameter-elliptic* of order μ, if there is a $b \in S^{-\mu}(\Omega, \mathbf{R}^n; \mathbf{R}^l)$ such that $ab - 1$ and $ba - 1$ belong to $S^{-1}(\Omega, \mathbf{R}^n; \mathbf{R}^l)$.

2.1.7 Operator-valued symbols.
Let E, F be Banach spaces with strongly continuous group actions $\kappa_\lambda, \tilde{\kappa}_\lambda, \lambda \in \mathbf{R}_+$, i.e. $\lambda \mapsto \kappa_\lambda \in C(\mathbf{R}_+, \mathcal{L}_\sigma(E))$, $\lambda \mapsto \tilde{\kappa}_\lambda \in C(\mathbf{R}_+, \mathcal{L}_\sigma(F))$, and $\kappa_\lambda\kappa_\mu = \kappa_{\lambda\mu}, \tilde{\kappa}_\lambda\tilde{\kappa}_\mu = \tilde{\kappa}_{\lambda\mu}$.
Let $\Omega \subseteq \mathbf{R}^k$ and $p \in C^\infty(\Omega \times \mathbf{R}^n, \mathcal{L}(E,F)), \mu \in \mathbf{R}$. We shall write

$$p \in S^\mu(\Omega, \mathbf{R}^n; E, F),$$

provided that for every $K \subseteq \Omega$ and all multi-indices α, β, there is a constant $C = C(K, \alpha, \beta)$ with

$$\|\tilde{\kappa}_{\langle\eta\rangle^{-1}} D^\alpha_\eta D^\beta_y p(y,\eta) \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,F)} \le C\, \langle\eta\rangle^{\mu-|\alpha|}, \tag{1}$$

cf. [27] 3.2.1, Definition 1. The space $S^\mu(\Omega, \mathbf{R}^n; E, F)$ is Fréchet topologized by the choice of the best constants C.
A symbol $p \in S^\mu(\Omega, \mathbf{R}^n; E, F)$ is said to be *classical,* if it has an asymptotic expansion $p \sim \sum_{j=0}^\infty p_j$ with $p_j \in S^{\mu-j}(\Omega, \mathbf{R}^n; E, F)$ satisfying the homogeneity relation

$$p_j(y, \lambda\eta) = \lambda^{\mu-j}\tilde{\kappa}_\lambda\, p_j(y,\eta)\, \kappa_{\lambda^{-1}}$$

for all $\lambda \ge 1, |\eta| \ge 1$.

For the usual or weighted Sobolev spaces on $\mathbf{R}_+$, we will always use the group action

$$[\kappa_\lambda f](r) = \lambda^{\frac{1}{2}} f(\lambda r). \tag{2}$$

On $E = \mathbf{C}$ use the trivial group action $\kappa_\lambda = id$.
If $F_1 \hookleftarrow F_2 \hookleftarrow \ldots$ is a sequence of Banach spaces with the same group action, and F is the Fréchet space given as the projective limit of the F_k, then let

$$S^\mu(\Omega, \mathbf{R}^n; E, F) = \text{proj} - \lim_k S^\mu(\Omega, \mathbf{R}^n; E, F_k). \tag{3}$$

Vice versa, if E is the inductive limit of the Banach spaces $E_1 \hookrightarrow E_2 \hookrightarrow \ldots$ with the same group action, then

$$S^\mu(\Omega, \mathbf{R}^n; E, F) = \text{ind} - \lim_k S^\mu(\Omega, \mathbf{R}^n; E_k, F), \tag{4}$$

Finally, a symbol p belongs to $S^\mu(\Omega, \mathbf{R}^n; E, F), E = \text{ind} - \lim E_k, F = \text{proj} - \lim F_l$, if the group actions coincide on the E_k and F_l, respectively, and $p \in S^\mu(\Omega, \mathbf{R}^n; E_k, F_l)$ for all k and l. We give it the topology induced by all the topologies of the spaces $S^\mu(\Omega, \mathbf{R}^n; E_k, F_l)$.

2.1.8 Remark. Note that

$$\begin{aligned} \mathcal{S}(\mathbf{R}_+) &= \text{proj} - \lim_{\sigma,\tau \in \mathbf{N}} H^{\sigma,\tau}(\mathbf{R}_+), \\ \mathcal{S}'(\mathbf{R}_+) &= \text{ind} - \lim_{\sigma,\tau \in \mathbf{N}} H_0^{-\sigma,-\tau}(\mathbf{R}_+), \end{aligned}$$

where $H^{\sigma,\tau}(\mathbf{R}_+), H_0^{\sigma,\tau}(\mathbf{R}_+)$ are the weighted Sobolev spaces defined by

$$\begin{aligned} H_0^{\sigma,\tau}(\mathbf{R}_+) &= \{\langle r\rangle^{-\tau} u : u \in H_0^\sigma(\mathbf{R}_{+,r})\}, \\ H^{\sigma,\tau}(\mathbf{R}_+) &= \{\langle r\rangle^{-\tau} u : u \in H^\sigma(\mathbf{R}_{+,r})\}. \end{aligned}$$

2.1.9 Remark. We will, in particular, deal with the spaces $S^\mu(\Omega, \mathbf{R}^n; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$. For the inductive and projective limit constructions in 2.1.7 (3), (4) we will then use the description of $\mathcal{S}'(\mathbf{R}_+)$ and $\mathcal{S}(\mathbf{R}_+)$, respectively, given in 2.1.8.

2.1.10 Lemma. *For $p \in S^\mu(\Omega, \mathbf{R}^n; E, F)$ and $q \in S^\nu(\Omega, \mathbf{R}^n; F, G)$, the symbol r defined by $r(y,\eta) = q(y,\eta)p(y,\eta)$ (point-wise composition of operators) belongs to $S^{\mu+\nu}(\Omega, \mathbf{R}^n; E, G)$, and $D_\eta^\alpha D_y^\beta p$ belongs to $S^{\mu-|\alpha|}(\Omega, \mathbf{R}^n; E, F)$.*

Proof. See [27] Section 3.2.1, Proposition 2. ◁

2.1.11 Definition. Let $\Omega = \Omega_1 \times \Omega_2 \subseteq \mathbf{R}^n \times \mathbf{R}^n$ be open and $p \in S^\mu(\Omega, \mathbf{R}^n; E, F)$ an operator-valued symbol. Then $\text{op}\, p$ is defined as usual by

$$[\text{op}\, p(f)](y) = (2\pi)^{-n} \int \int_{\Omega_2} e^{i(y-\tilde{y})\eta} p(y, \tilde{y}, \eta) f(\tilde{y}) d\tilde{y} d\eta, \tag{1}$$

$f \in C_0^\infty(\Omega_2, E), y \in \Omega_1$. This reduces to the usual

$$[\mathrm{op}\, p(f)](y) = (2\pi)^{-\frac{n}{2}} \int e^{iy\eta} p(y,\eta) \hat{f}(\eta) d\eta, \tag{2}$$

for 'simple' symbols. Here, $\hat{f}(\eta) = \mathcal{F}_{y\to\eta} f(\eta) = (2\pi)^{-n/2} \int e^{-iy\eta} f(y) dy$ is the vector-valued Fourier transform of f.
We may also consider the case that part of the covariables serve as parameters: For $\Omega \subseteq \mathbf{R}^n$ open, $p \in S^\mu(\Omega_y, \mathbf{R}^n_\eta \times \mathbf{R}^l_\lambda; E, F)$ then defines a parameter-dependent operator

$$[\mathrm{op}\, p(\lambda) f](y) = (2\pi)^{-n/2} \int e^{iy\eta} p(y, \eta, \lambda) \hat{f}(\eta) d\eta, \tag{3}$$

$f \in C_0^\infty(\Omega, E)$, similarly for 'double' symbols.
A subscript, say t, associated with the 'op' notation will indicate that we only let the operator act with respect to the variable t and the corresponding covariable. We will employ this notation particularly for operators acting with respect to the normal variable only.

2.1.12 Definition. Let E, κ_λ be as in 2.1.7, $q \in \mathbf{N}, s \in \mathbf{R}$. The *wedge Sobolev space* $\mathcal{W}^s(\mathbf{R}^q, E)$ is the completion of $\mathcal{S}(\mathbf{R}^q, E) = \mathcal{S}(\mathbf{R}^q) \hat{\otimes}_\pi E$ in the norm

$$\|u\|_{\mathcal{W}^s(\mathbf{R}^q, E)} = \left(\int \langle \eta \rangle^{2s} \| \kappa_{\langle \eta \rangle^{-1}} \mathcal{F}_{y\to\eta} u(\eta) \|_E^2 d\eta \right)^{\frac{1}{2}},$$

cf. [27], Section 3.1. $\mathcal{W}^s(\mathbf{R}^q, E)$ is a subset of $\mathcal{S}'(\mathbf{R}^q, E) := \mathcal{L}(\mathcal{S}(\mathbf{R}^q), E)$.
Suppose $\{E_k\}$ is a sequence of Banach spaces, $E_{k+1} \hookrightarrow E_k$, $E = \mathrm{proj} - \lim E_k$, and the group action coincides on all spaces. Then

$$\mathcal{W}^s(\mathbf{R}^q, E) = \mathrm{proj} - \lim \mathcal{W}^s(\mathbf{R}^q, E_k).$$

Vice versa, if $E_k \hookrightarrow E_{k+1}$, $E = \mathrm{ind} - \lim E_k$, and the group action is the same for all spaces, then

$$\mathcal{W}^s(\mathbf{R}^q, E) = \mathrm{ind} - \lim \mathcal{W}^s(\mathbf{R}^q, E_k).$$

We shall write $u \in \mathcal{W}^s_{comp}(\mathbf{R}^q, E)$, if there is a function $\phi \in C_0^\infty(\mathbf{R}^q)$ such that $u = \phi u$. Similarly, for $u \in \mathcal{S}'(\mathbf{R}^q, E)$, write $u \in \mathcal{W}^s_{loc}(\mathbf{R}^q, E)$, if for arbitrary $\phi \in C_0^\infty(\mathbf{R}^q)$, $\phi u \in \mathcal{W}^s(\mathbf{R}^q, E)$, cf. Hirschmann [11]. It will also be useful to define the weighted wedge Sobolev spaces

$$\mathcal{W}^{k,l}(\mathbf{R}^q, E) = \{ \langle y \rangle^{-l} u : u \in \mathcal{W}^k(\mathbf{R}^q, E) \}.$$

2.1.13 Elementary properties of wedge Sobolev spaces.
(a) $\mathcal{W}^s(\mathbf{R}^q, H^s(\mathbf{R}_+)) = H^s(\mathbf{R}^{q+1}_+), s \geq 0$.
(b) $\mathcal{W}^s(\mathbf{R}^q, H^s_0(\mathbf{R}_+)) = H^s_0(\mathbf{R}^{q+1}_+), s \leq 0$.
(c) $\mathrm{proj} - \lim_{k,l,\sigma,\tau\to\infty} \mathcal{W}^{k,l}(\mathbf{R}^q, H^{\sigma,\tau}(\mathbf{R}_+)) = \mathcal{S}(\mathbf{R}^{q+1}_+)$.
(d) $\mathrm{ind} - \lim_{k,l,\sigma,\tau\to\infty} \mathcal{W}^{-k,-l}(\mathbf{R}^q, H_0^{-\sigma,-\tau}(\mathbf{R}_+)) = \mathcal{S}'(\mathbf{R}^{q+1}_+)$.
(e) $\mathcal{W}^s(\mathbf{R}^q, \mathbf{C}) = H^s(\mathbf{R}^q)$, using the trivial group action $\kappa_\lambda = id$.

2.1.14 Theorem. *Let E, F be Banach spaces as in 2.1.7, $s, \mu \in \mathbf{R}$, and $a \in S^\mu(\mathbf{R}^q_y, \mathbf{R}^q_\eta \times \mathbf{R}^l_\lambda; E, F)$ or $a \in S^\mu(\mathbf{R}^q_y \times \mathbf{R}^q_{\tilde{y}}, \mathbf{R}^q_\eta \times \mathbf{R}^l_\lambda; E, F)$. Then for every $\lambda \in \mathbf{R}^l$*

$$\operatorname{op} a(\lambda) : \mathcal{W}^s_{comp}(\mathbf{R}^q, E) \longrightarrow \mathcal{W}^{s-\mu}_{loc}(\mathbf{R}^q, F)$$

is bounded. If a is independent of y and $\tilde{y}$, then we may omit the subscripts 'comp' and 'loc'.
The mapping op : symbol $\mapsto$ operator is continuous in the corresponding topologies.

A proof may be found in [27] Section 3.2.1. ◁

2.1.15 Remark. In fact, 2.1.14 is Theorem 6 in Section 3.2.1 of [27]. There, the additional assumption is made that $C_0^\infty(\mathbf{R}^q)$ acts continuously on $\mathcal{W}^s(\mathbf{R}^q, E)$ and $\mathcal{W}^s(\mathbf{R}^q, F)$. Hirschmann has meanwhile shown that this assumption is always fulfilled [11], Theorem 3.2.

2.1.16 Definition. Let $\Omega \subseteq \mathbf{R}^k$ be open. By $C_0^\infty(\Omega \times \overline{\mathbf{R}}_+)$ denote the space of all $f \in C^\infty(\Omega \times \mathbf{R}_+)$ which are restrictions of functions $f \in C_0^\infty(\Omega \times \mathbf{R})$.

2.1.17 Singular Green Operators. Let $\Omega' \subseteq \mathbf{R}^{n-1}$ be open, $\Omega = \Omega' \times \mathbf{R}_+, \mu \in \mathbf{R}, d, l \in \mathbf{N}$.
(a) A family $\{G_0(\lambda) : \lambda \in \mathbf{R}^l\}$ of operators

$$G_0(\lambda) : C_0^\infty(\Omega' \times \overline{\mathbf{R}}_+) \to \mathcal{D}'(\Omega)$$

is a *parameter-dependent regularizing singular Green operator of type d on Ω*, if G_0 can be written as an operator of the following form

$$[G_0(\lambda)f](x) = \sum_{j=0}^{d} \int_\Omega \phi_j(x, y; \lambda) \frac{\partial^j}{\partial y_n^j} f(y)dy, \tag{1}$$

where $\phi_j \in \mathcal{S}(\mathbf{R}^l, C^\infty(\Omega_0 \times \Omega_0)), \Omega_0 = \Omega' \times \overline{\mathbf{R}}_+$. Write $G_0 \in \mathbf{G}^{-\infty,d}(\Omega; \mathbf{R}^l)$.
We topologize this space as the Cartesian product of $d+1$ copies of $\mathcal{S}(\mathbf{R}^l, C^\infty(\Omega_0 \times \Omega_0))$ modulo the quotient of functions inducing the same operators. It is then a Fréchet space.
(b) A family $\{G(\lambda) : \lambda \in \mathbf{R}^l\}$ of operators

$$G(\lambda) : C_0^\infty(\Omega' \times \overline{\mathbf{R}}_+) \longrightarrow \mathcal{D}'(\Omega)$$

is a *parameter-dependent singular Green operator of order μ and type d*, if it can be written

$$G(\lambda) = \sum_{j=0}^{d} \operatorname{op} g_j(\lambda) \circ \partial^j_{x_n} + G_0(\lambda), \tag{2}$$

where $g_j \in S^{\mu-j}(\Omega' \times \Omega', \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$, and G_0 is a parameter-dependent regularizing singular Green operator. Write $G \in \mathbf{G}^{\mu,d}(\Omega; \mathbf{R}^l)$.

Notice that if all g_j in (2) belong to $S^{-\infty}(\Omega' \times \Omega', \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$, then G is parameter-dependent regularizing. This is a consequence of the mapping properties which we will establish in Theorem 2.2.1.
We shall call the (operator-valued) symbol

$$g(x', y', \xi', \lambda) = \sum_{j=0}^{d} g_j(x', y', \xi', \lambda) \circ \partial_{x_n}^j \tag{3}$$

the *singular Green symbol* of G. It is well-defined as an equivalence class of tuples $(g_0, \dots, g_d)$, $g_j \in S^{\mu-j}(\Omega' \times \Omega', \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$ with the property that

$$G(\cdot) - \text{op} \sum_{j=0}^{d} g_j(\cdot) \circ \partial_{x_n}^j \in \mathbf{G}^{-\infty,d}(\Omega; \mathbf{R}^l).$$

Like in (a), the Fréchet topology in $\mathbf{G}^{\mu,d}$ is induced by the representation (2) via the topologies on the symbol spaces and that on $\mathbf{G}^{-\infty,d}$.
In order to avoid additional notation, we have given these definitions for the scalar case. In general, all symbols or kernel functions will take values in $n_1 \times n_2$-matrices, $n_1, n_2 \in \mathbf{N}$.

2.1.18 Remark. Compare this with the usual situation, cf. [21], [9]. There, the operator-valued symbol $g \in S^{\mu}(\Omega' \times \Omega', \mathbf{R}^{n-1}; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$ of 2.1.17 is replaced by a so-called "singular Green symbol kernel" $\tilde{g}$ of order $\mu - 1$ satisfying the estimates

$$\begin{aligned} &\|x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} D_\eta^\alpha D_y^\beta D_{\tilde{y}}^\gamma \tilde{g}(y, \tilde{y}, \eta, x_n, y_n)\|_{L^2(\mathbf{R}^2_{++})} \\ \le\; &C_{K,k,k',m,m',\alpha,\beta,\gamma} \langle \eta \rangle^{\mu - |\alpha| - k + k' - m + m'} \end{aligned} \tag{1}$$

for every subset $K \subset\subset \Omega, y, \tilde{y} \in K$, all $k, k', m, m' \in \mathbf{N}$ and all multi-indices α, β, γ. We are using the notation $\mathbf{R}^2_{++} = \mathbf{R}_+ \times \mathbf{R}_+$.
These symbol kernels act as integral operators on $\mathbf{R}_+$; they induce operators $g(y, \tilde{y}, \eta, D_n)$ by

$$[g(y, \tilde{y}, \eta, D_n) f](x_n) = \int_0^\infty \tilde{g}(y, \tilde{y}, \eta, x_n, y_n) f(y_n) dy_n \tag{2}$$

for $f \in \mathcal{S}(\mathbf{R}_+)$.

The present definition has been established by Schulze in [29], vol. VIII.
At first, it is surprising that in 2.1.17(1) and (2) we have partial derivatives of orders $0, \dots, d$ to the right of the kernels and symbols instead of evaluations of derivatives of orders $0, \dots, d-1$ at the boundary like in the usual set-up. The explanation, however, is simple: Integrating by parts we have

$$\int_0^\infty \gamma(x_n, y_n) \partial_{y_n} f(y_n) dy_n = \gamma(x_n, 0) f(0) - \int_0^\infty \partial_{y_n} \gamma(x_n, y_n) f(y_n) dy_n \tag{3}$$

for $\gamma \in \mathcal{S}(\mathbf{R}^2_{++})$. By iteration, one obtains the 'standard' representation, cf. also 2.2.13, below, for a detailed exposition.

The only point to clarify is whether for type zero the usual definition coincides with that given in 2.1.17. This is the contents of Theorem 2.1.19, below.

For the formulation and the proof we may omit the parameters, since they only play the role of an additional covariable, and we can confine ourselves to symbols independent of $\tilde{y}$.

2.1.19 Theorem. *Let $\Omega' \subseteq \mathbf{R}^{n-1}$ be open, and suppose that for all $y \in \Omega', \eta \in \mathbf{R}^{n-1}, g(y,\eta) \in \mathcal{L}(L^2(\mathbf{R}_+))$. Then the following is equivalent:*

(a) $g(y,\eta) = \tilde{g}(y,\eta,D_n)$ *for a singular Green symbol kernel* $\tilde{g}$ *satisfying the estimates* 2.1.18(1).

(b) $g \in S^\mu(\Omega', \mathbf{R}^{n-1}; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$.

(c) $g \in S^\mu(\Omega', \mathbf{R}^{n-1}; L^2(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$, *and the formal x_n-adjoint, point-wise defined by* $g^*(y,\eta) = g(y,\eta)^*$, *with respect to the inner product in* $L^2(\mathbf{R}_+)$, *also belongs to* $S^\mu(\Omega', \mathbf{R}^{n-1}; L^2(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$.

Proof. (a) $\Rightarrow$ (b). It is easy to check that $\kappa_{\langle\eta\rangle^{-1}} D^\alpha_\eta D^\beta_y g(y,\eta)\kappa_{\langle\eta\rangle}$ is the integral operator with the symbol kernel

$$h_{\alpha,\beta}(y,\eta,x_n,y_n) = [D^\alpha_\eta D^\beta_y \tilde{g}](y,\eta,\langle\eta\rangle^{-1} x_n, \langle\eta\rangle^{-1} y_n)\,\langle\eta\rangle^{-1}.$$

The fact that $\tilde{g}$ satisfies the estimates of 2.1.18 implies that

$$\|x_n^k D^{k'}_{x_n} y_n^m D^{m'}_{y_n} h_{\alpha,\beta}(y,\eta,x_n,y_n)\|_{L^2(\mathbf{R}^2_{++})} \le C\, \langle\eta\rangle^{\mu-|\alpha|}.$$

So $h_{\alpha,\beta}(y,\eta,\cdot,\cdot)$ is a function in $\mathcal{S}(\mathbf{R}^2_{++})$, and all its semi-norms are $O(\langle\eta\rangle^{\mu-|\alpha|})$. This implies (b), for a rapidly decreasing kernel yields an operator from $\mathcal{S}'(\mathbf{R}_+)$ to $\mathcal{S}(\mathbf{R}_+)$. Moreover, for all choices of $E \in \{H_0^{-\sigma,-\tau}(\mathbf{R}_+) : \sigma,\tau \ge 0\}, F \in \{H^{\sigma,\tau}(\mathbf{R}_+) : \sigma,\tau \ge 0\}$,

$$\|\kappa_{\langle\eta\rangle^{-1}} D^\alpha_\eta D^\beta_y g(y,\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,F)} = \|h_{\alpha,\beta}(y,\eta,D_n)\|_{\mathcal{L}(E,F)}$$

can be estimated by finitely many of the above semi-norms which are all $O(\langle\eta\rangle^{\mu-|\alpha|})$.
(b) $\Rightarrow$ (c). If $g \in S^\mu(\Omega', \mathbf{R}^{n-1}; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$, then $g(y,\eta)$ has a rapidly decreasing integral kernel on $\mathbf{R}_+ \times \mathbf{R}_+$, say $\tilde{g}(y,\eta,\cdot,\cdot)$ for every fixed choice of y,η. This is a consequence of the continuity of $\kappa_{\langle\eta\rangle^{-1}}$ and $\kappa_{\langle\eta\rangle}$ on $\mathcal{S}(\mathbf{R}_+)$ and $\mathcal{S}'(\mathbf{R}_+)$, respectively.
The adjoint $g^*(y,\eta) = g(y,\eta)^*$ thus is the integral operator with the adjoint kernel $\tilde{h}(y,\eta,x_n,y_n) = \overline{\tilde{g}}(y,\eta,x_n,y_n)$, while finally $\kappa_{\langle\eta\rangle^{-1}} g^*(y,\eta)\kappa_{\langle\eta\rangle}$ is the integral operator with the kernel

$$\tilde{h}(y,\eta,\langle\eta\rangle^{-1} x_n, \langle\eta\rangle^{-1} y_n)\,\langle\eta\rangle^{-1}.$$

Now pick $k,k',m,m' \in \mathbf{N}$, and show that

$$\|x_n^k D^{k'}_{x_n} y_n^m D^{m'}_{y_n} \tilde{h}(y,\eta,x_n,y_n)\|_{\sup_{\mathbf{R}^2_{++}}} = O(\langle\eta\rangle^{\mu}). \tag{1}$$

Since the same considerations can be applied to $D^\alpha_\eta D^\beta_y g(y,\eta)$, we obtain the assertion.
So let us show (1). We have

$$\begin{aligned} & |x_n^k D^{k'}_{x_n} y_n^m D^{m'}_{y_n} \tilde{h}(y,\eta,\langle\eta\rangle^{-1} x_n, \langle\eta\rangle^{-1} y_n)\,\langle\eta\rangle^{-1}| \\ = \; & |x_n^k y_n^m \,\langle\eta\rangle^{-k'-m'} [D^{k'}_{x_n} D^{m'}_{y_n} \tilde{g}](y,\eta,\langle\eta\rangle^{-1} y_n, \langle\eta\rangle^{-1} x_n)\,\langle\eta\rangle^{-1}|. \end{aligned}$$

Letting $u = \langle\eta\rangle^{-1} x_n, v = \langle\eta\rangle^{-1} y_n, E = H^{\sigma,\tau}(\mathbf{R}_+)$ for some choice of σ,τ (large), $E' = H_0^{-\sigma,-\tau}(\mathbf{R}_+)$ its dual space, we can rewrite this last expression as

$$\langle\eta\rangle^{k-k'+m-m'-1}\; u^k D^{k'}_u v^m D^{m'}_v \tilde{g}(y,\eta,u,v).$$

We now use the fact that the integral kernel of the operator $g(y,\eta) : E \to E'$ can be written as $\tilde{g}(y,\eta,u,v) = \langle g(y,\eta)\delta_v, \delta_u\rangle_{E,E'}$ with the translation of Dirac's function $\delta_u : f \mapsto f(u)$. Correspondingly, $u^k v^m D_u^{k'} D_v^{m'} \tilde{g}(y,\eta,u,v)$ is the integral kernel of the operator $w^k D_w^{k'} g(y,\eta) w^m D_w^{m'}$. Notice the distinction between the variable, namely w, and the points where we evaluate, namely u and v; here w^k and w^m are to be understood as multiplication operators. Therefore,

$$\begin{aligned}
& |u^k v^m D_u^{k'} D_v^{m'} \tilde{g}(y,\eta,u,v)| \\
= & \left|\left\langle w^k D_w^{k'} g(y,\eta) w^m D_w^{m'} \delta_v, \delta_u \right\rangle_{E,E'}\right| \\
= & \left|\left\langle g(y,\eta) w^m D_w^{m'} \delta_v, D^{k'} w^k \delta_u \right\rangle_{E,E'}\right| \\
= & \left|\left\langle \kappa_{\langle\eta\rangle^{-1}} g(y,\eta) \kappa_{\langle\eta\rangle} \kappa_{\langle\eta\rangle^{-1}}(w^m D_w^{m'} \delta_v), \kappa_{\langle\eta\rangle^{-1}} D_w^{k'} w^k \delta_u \right\rangle_{E,E'}\right| \\
\leq & \|\kappa_{\langle\eta\rangle^{-1}} g(y,\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,E')} \, \|\kappa_{\langle\eta\rangle^{-1}} w^m D_w^{m'} \delta_v\|_{E'} \, \|\kappa_{\langle\eta\rangle^{-1}} D_w^{k'} w^k \delta_u\|_{E'}.
\end{aligned}$$

Now $\kappa_{\langle\eta\rangle^{-1}}(w^m D^{m'}\delta_v) = \langle\eta\rangle^{-m+m'} \kappa_{\langle\eta\rangle^{-1}}\delta_v$, while $(\kappa_{\langle\eta\rangle^{-1}}\delta_v)f = \delta_v(\kappa_{\langle\eta\rangle} f) = \langle\eta\rangle^{\frac{1}{2}} f(\langle\eta\rangle\, v)$ for $f \in \mathcal{S}(\mathbf{R}_+)$. Therefore, $\langle\eta\rangle^{k-k'+m-m'-1}\, u^k D_u^{k'} v^m D_v^{m'} \tilde{g}(y,\eta,u,v) = \mathrm{O}(\langle\eta\rangle^{\mu})$. This yields the assertion.

(c) $\Rightarrow$ (a). $\kappa_{\langle\eta\rangle^{-1}} g(y,\eta)\kappa_{\langle\eta\rangle} : L^2(\mathbf{R}_+) \to \mathcal{S}(\mathbf{R}_+)$ is continuous. In particular, it is a Hilbert-Schmidt operator on $L^2(\mathbf{R}_+)$ and thus has an integral kernel $h_1(y,\eta,\cdot,\cdot) \in L^2(\mathbf{R}^2_{++})$, and

$$\|h_1(y,\eta,x_n,y_n)\|_{L^2(\mathbf{R}^2_{++}))} = \|\kappa_{\langle\eta\rangle^{-1}} g(y,\eta)\kappa_{\langle\eta\rangle}\|_{HS(L^2(\mathbf{R}_+))}. \tag{2}$$

By a direct calculation, the operator $g(y,\eta)$ then has the integral kernel

$$\tilde{g}_1(y,\eta,x_n,y_n) = h_1(y,\eta,\langle\eta\rangle\, x_n, \langle\eta\rangle\, y_n)\,\langle\eta\rangle\,. \tag{3}$$

Correspondingly, the operator $\kappa_{\langle\eta\rangle^{-1}} g^*(y,\eta)\kappa_{\langle\eta\rangle}$ has an integral kernel $h_2(y,\eta,x_n,y_n)$, and

$$h_1(y,\eta,x_n,y_n) = \overline{h_2}(y,\eta,y_n,x_n). \tag{4}$$

The mapping $x_n^k D_{x_n}^{k'} \kappa_{\langle\eta\rangle^{-1}} D_\eta^\alpha D_y^\beta g(y,\eta)\kappa_{\langle\eta\rangle}$: $L^2(\mathbf{R}_+) \to \mathcal{S}(\mathbf{R}_+)$ also is continuous. Therefore, as in (2)

$$\|x_n^k D_{x_n}^{k'} D_\eta^\alpha D_y^\beta h_1(y,\eta,x_n,y_n)\|_{L^2(\mathbf{R}^2_{++})} = \mathrm{O}(\langle\eta\rangle^{\mu-|\alpha|}). \tag{5}$$

Using relation (4) we also have

$$\|y_n^m D_{y_n}^{m'} D_\eta^\alpha D_y^\beta h_1(y,\eta,x_n,y_n)\|_{L^2(\mathbf{R}^2_{++})} = \mathrm{O}(\langle\eta\rangle^{\mu-|\alpha|}). \tag{6}$$

Together, the estimates (5) and (6) show that

$$\|x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} D_\eta^\alpha D_y^\beta h_1(y,\eta,x_n,y_n)\|_{L^2(\mathbf{R}^2_{++})} = \mathrm{O}(\langle\eta\rangle^{\mu-|\alpha|}). \tag{7}$$

For a proof see [22], 1.2.2 Proposition 10. By combining (7) with (3), we obtain (a). $\lhd$

2.2 Mapping Properties. Boutet de Monvel's Algebra on the Half-Space

2.2.1 Theorem. *Let $\Omega' \subseteq \mathbf{R}^{n-1}$ be open, $s \in \mathbf{R}$, and let G be a parameter-dependent singular Green operator of order μ and type zero on $\Omega = \Omega' \times \mathbf{R}_+$. Then*

$$G(\lambda) : \mathcal{W}^s_{comp}(\Omega', \mathcal{S}'(\mathbf{R}_+)) \longrightarrow \mathcal{W}^{s-\mu}_{loc}(\Omega', \mathcal{S}(\mathbf{R}_+)) \qquad (1)$$

is continuous for all $\lambda \in \mathbf{R}^l$.
Note: *Since the symbol topology is stronger than the operator topology we may estimate the operator norm in terms of λ. In particular, if $g \in S^{-\infty}(\Omega' \times \Omega', \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$, then $\operatorname{op} g$ is an integral operator with a kernel function in $\mathcal{S}(\mathbf{R}^l, C^\infty(\Omega_0 \times \Omega_0)), \Omega_0 = \Omega' \times \overline{\mathbf{R}}_+$.*

Proof. This is an immediate consequence of the definition of the singular Green operators and Theorem 2.1.14.
The application to symbols of order $-\infty$ follows from the fact that for all α, β, μ, the operator $\lambda^\alpha D^\beta_\lambda \operatorname{op} g(\lambda)$ has property (1), uniformly in λ, in connection with 2.1.13(c),(d). ◁

2.2.2 Theorem. *For an open subset Ω' of $\mathbf{R}^{n-1}$ let G be a parameter-dependent singular Green operator of order μ and type d on $\Omega' \times \mathbf{R}_+$. Let $s \in \mathbf{R}, \sigma = (\sigma_1, \sigma_2) \in \mathbf{R}^2, \sigma_1 > d - \frac{1}{2}$. Then*

$$G(\lambda) : \mathcal{W}^s_{comp}(\Omega', H^\sigma(\mathbf{R}_+)) \longrightarrow \mathcal{W}^{s-\mu}_{loc}(\Omega', \mathcal{S}(\mathbf{R}_+))$$

is continuous for all $\lambda \in \mathbf{R}^l$, and we may estimate the operator norm in terms of λ.

Proof. This follows from the definition of the singular Green operators, Theorem 2.1.14, and Lemma 2.2.3(a), below. The proof of 2.2.3 is immediate from the definition of the norms in the wedge Sobolev spaces. ◁

2.2.3 Lemma. *Let $s \in \mathbf{R}, \sigma = (\sigma_1, \sigma_2) \in \mathbf{R}^2$.*
(a)

$$D_n : \mathcal{W}^s(\mathbf{R}^{n-1}, H^\sigma(\mathbf{R}_+)) \longrightarrow \mathcal{W}^{s-1}(\mathbf{R}^{n-1}, H^{\sigma-(1,0)}(\mathbf{R}_+))$$

is bounded.
(b) *Multiplication by x_n,*

$$x_n : \mathcal{W}^s(\mathbf{R}^{n-1}, H^\sigma(\mathbf{R}_+)) \longrightarrow \mathcal{W}^{s+1}(\mathbf{R}^{n-1}, H^{\sigma-(0,1)}(\mathbf{R}_+))$$

is bounded.

2.2.4 Definition. Let $\Omega' \subseteq \mathbf{R}^{n-1}$ be open, $\Omega = \Omega' \times \mathbf{R}$, and let $p \in S^\mu(\Omega, \mathbf{R}^n; \mathbf{R}^l)$. For fixed $(x', \xi', \lambda) \in \Omega' \times \mathbf{R}^{n-1} \times \mathbf{R}^l$ let

$$\operatorname{op}^+_{x_n} p(x', \xi', D_n, \lambda) = \mathrm{r}^+ \operatorname{op}_{x_n} p(x', x_n, \xi', \xi_n, \lambda) \mathrm{e}^+, \qquad (1)$$

where the action in op_{x_n} is with respect to x_n and the covariable ξ_n.
The operator in (1) is well-defined on $H^s(\mathbf{R}_+), s > -\frac{1}{2}$, since then extension by zero makes sense. More generally, given an operator T on distributions over $\Omega' \times \mathbf{R}$ define the operator T_+ on sufficiently smooth distributions over $\Omega' \times \mathbf{R}_+$ by $T_+ = \mathrm{r}^+ T \mathrm{e}^+$.

2.2.5 Theorem. *Let $\mu, \nu \in \mathbf{R}, \Omega, \Omega'$ be as in 2.2.4. Moreover, let $p \in S^\mu_{tr}(\Omega, \mathbf{R}^n; \mathbf{R}^l), q \in S^\nu_{tr}(\Omega, \mathbf{R}^n; \mathbf{R}^l)$, and suppose that $p(x,\xi,\lambda)$ or $q(x,\xi,\lambda)$ vanishes for x_n outside a compact set. Then*

$$\operatorname{op}^+_{x_n} p \circ_n \operatorname{op}^+_{x_n} q - (\operatorname{op}_{x_n} p \circ_n \operatorname{op}_{x_n} q)_+$$

induces a parameter-dependent singular Green operator of order $\mu + \nu$ and type $d = \max(\operatorname{entier}(\nu), 0)$.

For completeness the proof will be given in Appendix 2. ◁

2.2.6 Lemma. *Let Ω, Ω' be as in 2.2.4, and let $p \in S^\mu_{tr}(\Omega, \mathbf{R}^n; \mathbf{R}^l)$. Then for fixed $(x', \xi', \lambda) \in \Omega' \times \mathbf{R}^n \times \mathbf{R}^l$,*

$$\kappa_{\langle\xi',\lambda\rangle^{-1}} \operatorname{op}_{x_n} p(x', x_n, \xi', \xi_n, \lambda) \kappa_{\langle\xi',\lambda\rangle} = \operatorname{op}_{x_n}(x', \frac{x_n}{\langle\xi';\lambda\rangle}, \xi', \langle\xi',\lambda\rangle \xi_n, \lambda)$$

Similarly, for a 'double' symbol $q = q(x,y,\xi,\lambda)$, the transformed symbol has the form

$$q(x', \frac{x_n}{\langle\xi',\lambda\rangle}, y', \frac{y_n}{\langle\xi',\lambda\rangle}, \xi', \langle\xi',\lambda\rangle \xi_n, \lambda).$$

Proof. Let $f \in \mathcal{S}(\mathbf{R})$. Then for $\rho = \langle\xi',\lambda\rangle$, $(\kappa_\rho f)\hat{}(\xi_n) = \rho^{-\frac{1}{2}} \hat{f}(\frac{\xi_n}{\rho})$, and

$$p(x', x_n, \xi, D_n, \lambda)(\kappa_\rho f)(x_n) = (2\pi)^{-\frac{1}{2}} \int e^{x_n \rho \eta_n} \rho^{\frac{1}{2}} p(x', x_n, \xi', \rho\eta_n, \lambda) \hat{f}(\eta_n) d\eta_n.$$

This gives the assertion. The argument also applies to 'double' symbols. ◁

2.2.7 Lemma. *Let $\mu \in \mathbf{R}$. Then the symbol*

$$r^\mu(\xi,\lambda) = \langle\xi,\lambda\rangle^\mu \in S^\mu(\mathbf{R}^n, \mathbf{R}^n; \mathbf{R}^l)$$

induces the operator-valued symbol

$$r^\mu(\xi', D_n, \lambda) = \operatorname{op}_{x_n} r^\mu \in S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1}; H^s(\mathbf{R}), H^{s-\mu}(\mathbf{R}))$$

for arbitrary $s \in \mathbf{R}$.

Proof. We have to consider

$$\kappa_{\langle\xi',\lambda\rangle^{-1}} \mathrm{op}_{x_n} D^\alpha_{\xi'} D^\beta_\lambda \langle\xi',\xi_n,\lambda\rangle^\mu \kappa_{\langle\xi',\lambda\rangle}.$$

Now $\partial_{\xi_j}\langle\xi',\lambda\rangle^\mu = \mu\xi_j\langle\xi',\lambda\rangle^{\mu-2}$, so by induction $D^\alpha_{\xi',\lambda} r^\mu$ is a linear combination of terms of the form $\xi'^\beta \langle\xi',\lambda\rangle^{\mu-k}$ with $k-|\beta|\ge|\alpha|$. From Lemma 2.2.6 we conclude that $\kappa_{\langle\xi',\lambda\rangle^{-1}} \mathrm{op}_{x_n} D^\alpha_{\xi'} \langle\xi',\xi_n,\lambda\rangle^\mu \kappa_{\langle\xi',\lambda\rangle}$ is a linear combination of terms of the form

$$\xi'^\beta \mathrm{op}_{x_n} \langle\xi',\langle\xi',\lambda\rangle\xi_n,\lambda\rangle^{\mu-k} = \xi'^\beta \langle\xi',\lambda\rangle^{\mu-k} \mathrm{op}_{x_n} \langle\xi_n\rangle^{\mu-k},$$

since $k-|\beta|\ge|\alpha|$, $\xi'^\beta\langle\xi',\lambda\rangle^{\mu-k} = \mathrm{O}(\langle\xi',\lambda\rangle^{\mu-|\alpha|})$. Moreover, $\mathrm{op}_{x_n}\langle\xi_n\rangle^{\mu-k} : H^s(\mathbf{R}) \to H^{s-\mu}(\mathbf{R})$ is bounded. This completes the proof. ◁

2.2.8 Theorem. *Let $\Omega' \subseteq \mathbf{R}^{n-1}$ be open, $\Omega = \Omega'\times\mathbf{R}$, $p\in S^\mu(\Omega,\mathbf{R}^n_\xi;\mathbf{R}^l_\lambda)$. Assume that p is independent of x_n or that it vanishes for x_n outside a compact set.*
Then

$$\mathrm{op}_{x_n} p(x,\xi,D_n,\lambda) \in S^\mu(\Omega',\mathbf{R}^{n-1}\times\mathbf{R}^l, H^\sigma(\mathbf{R}), H^{\sigma-\mu}(\mathbf{R}))$$

for every $\sigma\in\mathbf{R}$.

Proof. Consider the symbols r^ν introduced in 2.2.7. For every ν, the operator $r^\nu(\xi',D_n,\lambda)$ is invertible with inverse $r^{-\nu}(\xi',D_n,\lambda)$.
Applying Lemma 2.1.10 it is sufficent to prove that

$$\begin{aligned} q(x,\xi',D_n,\lambda) &= r^{s-\mu}(\xi',D_n,\lambda)\circ_n p(x,\xi',D_n,\lambda)\circ_n r^{-s}(\xi',D_n,\lambda) \\ &\in S^0(\Omega',\mathbf{R}^{n-1}\times\mathbf{R}^l; L^2(\mathbf{R}),L^2(\mathbf{R})). \end{aligned} \tag{1}$$

By the standard calculus, $q(x,\xi',D_n,\lambda)$ is a pseudodifferential operator with a symbol $q\in S^0(\Omega,\mathbf{R}^n\times\mathbf{R}^l)$. For fixed (x',ξ'), we have $q(x',x_n,\xi',\xi_n,\lambda)\in S^0(\mathbf{R}_{x_n},\mathbf{R}_{\xi_n}\times\mathbf{R}^l_\lambda)_{\text{unif}}$ by assumption, and all symbol semi-norms depend continuously on x'. For all multi-indices α

$$\langle\xi',\lambda\rangle^{|\alpha|} |D^{\alpha'}_{\xi'} D^{\beta'}_{x'} q(x',x_n,\xi',\xi_n,\lambda)| \le c_\alpha(x').$$

with a continuous function c_α. By Lemma 2.2.6

$$\begin{aligned} &\kappa_{\langle\xi',\lambda\rangle^{-1}} \langle\xi',\lambda\rangle^{|\alpha|} D^\alpha_{\xi'} D^\beta_{x'} q(x,\xi',D_n,\lambda)\kappa_{\langle\xi',\lambda\rangle} \\ =\ & \mathrm{op}_{x_n} \langle\xi',\lambda\rangle^{|\alpha|} (D^\alpha_{\xi'} D^\beta_{x'} q)(x',\frac{x_n}{\langle\xi',\lambda\rangle},\xi',\langle\xi',\lambda\rangle\xi_n,\lambda). \end{aligned}$$

Lemma 2.2.9, below, shows that the norm of this family of operators is bounded by a continuous function $\tilde c(x')$ as (x',ξ',λ) varies over $\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}\times\mathbf{R}^l$.
This gives the desired result. ◁

2.2.9 Lemma. *cf. Coifman & Meyer* [4], Chapter II, Lemma 1. *Let $p\in S^0_{0,0}(\mathbf{R}^n\times\mathbf{R}^n)_{\text{unif}}$. Then for $\sigma\in\mathbf{R}$, the mapping*

$$\sigma \mapsto \left\| \mathrm{op}\, p(\frac{x}{\sigma},\sigma\xi) \right\|_{\mathcal{L}(L^2(\mathbf{R}^n))}$$

is a constant.
Note that the boundedness of the right hand side is a consequence of Calderón and Vaillancourt's theorem.

2.2.10 Lemma. *Let $\mu \in \mathbf{Z}$ and choose a function $\chi \in \mathcal{S}(\mathbf{R})$ with* $\operatorname{supp}\mathcal{F}^{-1}\chi \subseteq \mathbf{R}_-$ *and* $\chi(0) = 1$. *On $\mathbf{R}^n \times \mathbf{R}^l$ define the function r_-^μ by*

$$r_-^\mu(\xi,\lambda) = \left(\chi\left(\frac{\xi_n}{a\langle\xi',\lambda\rangle}\right)\langle\xi',\lambda\rangle - i\xi_n\right)^\mu .$$

Here, a is a real parameter with $a \gg \|\chi'\|_{\sup}$; χ' is the first derivative of χ. Moreover let

$$r_+^\mu(\xi,\lambda) = \overline{r_-^\mu(\xi,\lambda)}.$$

be the complex conjugate of r_-^μ. Then

(a) *$r_\pm^\mu$ belongs to $S^\mu(\mathbf{R}^n, \mathbf{R}^n; \mathbf{R}^l)$ and is parameter-elliptic.*

(b) $\operatorname{op}^+_{x_n} r_-^\mu \in S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \times \mathbf{R}^l; H^\sigma(\mathbf{R}_+), H^{\sigma-\mu}(\mathbf{R}_+))$ *for all $\sigma > -\frac{1}{2}$;*

we even have

(c) $\operatorname{op}^+_{x_n} r_-^\mu \in S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \times \mathbf{R}^l; H^{\sigma,\tau}(\mathbf{R}_+), H^{\sigma-\mu,\tau}(\mathbf{R}_+))$ *for all $\sigma > -\frac{1}{2}, \tau \in \mathbf{R}$.*

(d) $\operatorname{op}^+_{x_n} r_+^\mu \in S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \times \mathbf{R}^l; H_0^{\sigma,\tau}(\mathbf{R}_+), H_0^{\sigma-\mu,\tau}(\mathbf{R}_+))$ *for all $\sigma, \tau \in \mathbf{R}$. Here, e^+ is regarded as a trivial action on $H_0^{\sigma,\tau}(\mathbf{R}_+)$.*

(e) *Let $\nu \in \mathbf{Z}$, and assume that $\sigma - \mu > -\frac{1}{2}$. Then $\operatorname{op}^+_{x_n} r_-^\nu \circ_n \operatorname{op}^+_{x_n} r_-^\mu = \operatorname{op}^+_{x_n} r_-^{\nu+\mu}$; in particular, $\operatorname{op}^+_{x_n} r_-^\mu \circ_n \operatorname{op}^+_{x_n} r_-^{-\mu} = id$ on $H^{\sigma,\tau}(\mathbf{R}_+)$.*

Note: *In* (b) *and* (c) *the operator e^+ a priori requires the regularity $\sigma > -\frac{1}{2}$. On the other hand, the proof of* (b) *will show that*

$$\mathrm{r}^+\operatorname{op}_{x_n} r_-^\mu \mathrm{e}^+ f = \mathrm{r}^+\operatorname{op}_{x_n} r_-^\mu E f$$

for any extension operator E, whenever $\sigma > -\frac{1}{2}$. We therefore have the results of (b) *and* (c) *for all σ, provided we replace the extension e^+ by an arbitrary extension operator $H^\sigma(\mathbf{R}_+) \to H^\sigma(\mathbf{R})$.*

Proof. (a) First note that

$$\begin{aligned} \frac{\chi\left(\frac{\xi_n}{a\langle\xi',\lambda\rangle}\right)\langle\xi',\lambda\rangle - i\xi_n}{\langle\xi',\lambda\rangle - i\xi_n} &= 1 + \langle\xi',\lambda\rangle \frac{\chi\left(\frac{\xi_n}{a\langle\xi',\lambda\rangle}\right) - \chi(0)}{\langle\xi',\lambda\rangle - i\xi_n} \\ &= 1 + r, \end{aligned} \tag{1}$$

where $|r| \le \|\chi'\|_{\sup}(\langle\xi',\lambda\rangle \frac{|\xi_n|}{a\langle\xi',\lambda\rangle})/\langle\xi,\lambda\rangle \le \|\chi'\|_{\sup}/a \ll 1$. In particular, $|r_-^\mu(\xi,\lambda)| \ge c\langle\xi,\lambda\rangle$ for some $c > 0$; this implies parameter-ellipticity of both, r_+^μ and r_-^μ.
Using (1), it is easily checked that $D^\alpha_{\xi,\lambda} r_-^\mu(\xi,\lambda) = O(\langle\xi,\lambda\rangle^{\mu-|\alpha|})$, just as asserted.
(b) Together with (a), 2.2.8 shows that

$$\operatorname{op} r_-^\mu \in S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \times \mathbf{R}^l; H^\sigma(\mathbf{R}), H^{\sigma-\mu}(\mathbf{R}))$$

for arbitrary σ.

As a function of $\xi_n, r_- = r_-^1$ belongs to H^-, since supp $\mathcal{F}^{-1}\chi \subseteq \mathbf{R}_-$. Also r_-^{-1} belongs to H^-, by [21], Section 2.1.1.1, Corollary 2, for it has an analytic continuation to the upper half plane $\{\operatorname{Im} z \geq 0\}$, and it has an asymptotic expansion into negative powers of ξ_n. Since H^- is an algebra, r_-^μ belongs to H^- for every $\mu \in \mathbf{Z}$.
Now let $\sigma > -\frac{1}{2}$ and $E = E(\sigma)$ be an extension operator from $H^\sigma(\mathbf{R}_+)$ to $H^\sigma(\mathbf{R})$. Given $f \in H^\sigma(\mathbf{R}_+), Ef - \mathrm{e}^+ f$ is a distribution belonging to $H^{\tilde{\sigma}}, \tilde{\sigma} = \min\{\sigma, \frac{1}{2} - \epsilon\}$ for all $\epsilon > 0$; moreover, it is zero on $\mathbf{R}_+$.
Therefore, $\mathrm{r}^+ \mathrm{op}_{x_n} r_-^\mu (Ef - \mathrm{e}^+ f) = \mathrm{r}^+ \mathcal{F}^{-1}[r_-^\mu \mathcal{F}(Ef - \mathrm{e}^+ f)] = \mathrm{r}^+[\mathcal{F}^{-1} r_-^\mu * (Ef - \mathrm{e}^+ f)] = 0$, since both, $\mathcal{F}^{-1} r_-^\mu$ and $(Ef - \mathrm{e}^+ f)$ vanish on $\mathbf{R}_+$.
We conclude that

$$\mathrm{op}_{x_n}^+ r_-^\mu f = \mathrm{r}^+ \mathrm{op}\, r_-^\mu \circ Ef. \tag{2}$$

Both E and r^+ are bounded operators, and the norm of $\kappa_{\langle \xi', \lambda \rangle^{-1}} D_{\xi'}^\alpha D_\lambda^\beta \mathrm{op}_{x_n} r_-^\mu \kappa_{\langle \xi', \lambda \rangle}$ in $\mathcal{L}(H^\sigma(\mathbf{R}), H^{\sigma-\mu}(\mathbf{R}))$ is $\mathrm{O}(\langle \xi', \lambda \rangle^{\mu - |\alpha| - |\beta|})$ by (a) and 2.2.8 Hence we obtain the assertion.
(c) Without loss of generality assume that $\tau \in \mathbf{Z}$. The norm of $\kappa_{\langle \xi', \lambda \rangle^{-1}} D_{\xi'}^\alpha D_\lambda^\beta \mathrm{op}_{x_n}^+ r_-^\mu \kappa_{\langle \xi', \lambda \rangle}$ in $\mathcal{L}(H^{\sigma,\tau}(\mathbf{R}), H^{\sigma-\mu,\tau}(\mathbf{R}))$ equals the norm of $\langle x_n \rangle^\tau \kappa_{\langle \xi', \lambda \rangle^{-1}} D_{\xi'}^\alpha D_\lambda^\beta \mathrm{op}_{x_n} r_-^\mu \kappa_{\langle \xi', \lambda \rangle} \langle x_n \rangle^{-\tau}$ in $\mathcal{L}(H^\sigma(\mathbf{R}), H^{\sigma-\mu}(\mathbf{R}))$. One of the multiplication operators is a polynomial in x_n. Both commute with the group action. Moreover, we may use the rule $x_n \mathrm{op}_{x_n}^+ r_-^\mu = ([x_n, \mathrm{op}_{x_n}^+ r_-^\mu])_+ + \mathrm{op}_{x_n}^+ r_-^\mu x_n = \mathrm{op}_{x_n}^+(-D_{\xi_n} r_-^\mu) + \mathrm{op}_{x_n}^+ r_-^\mu x_n$ to move the polynomial part to the other side. Since $x_n^k \langle x_n \rangle^{-\tau}$ is a bounded operator for $k \leq \tau$, and since we know already that $D_{\xi_n} r_-^\mu$ has the desired mapping properties, this completes the proof.
(d) In view of the fact that e^+ is a trivial action, (a) in connection with 2.2.8 implies that $\mathrm{op}_{x_n} \mathrm{e}^+ r_+^\mu \in S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1}; H_0^{\sigma,\tau}(\mathbf{R}_+), H_0^{\sigma-\mu,\tau}(\mathbf{R}_+))$.
All we have to show is that for $v \in \mathcal{S}(\mathbf{R}_+), \mathrm{op}_{x_n} \mathrm{e}^+ r_+^\mu v = 0$ on $\mathbf{R}_-$. This, however, is easy: r_+^μ is the sum of a polynomial and a function in H^+, so $r_+^\mu \mathcal{F} \mathrm{e}^+ v \in H^+ \cdot \{\mathbf{C}[\xi_n] \oplus H^+\}$, and the inverse Fourier transform vanishes on $\mathbf{R}_-$.
(e) Since

$$\mathrm{op}_{x_n}^+ r_-^{\mu+\nu} = \mathrm{r}^+ \mathrm{op}_{x_n} r_-^{\mu+\nu} \mathrm{e}^+ = [\mathrm{r}^+ \mathrm{op}_{x_n} r_-^\mu \mathrm{e}^+][\mathrm{r}^+ \mathrm{op}_{x_n} r_-^\nu \mathrm{e}^+] + \mathrm{r}^+ \mathrm{op}_{x_n} r_-^\mu \mathrm{e}^- \mathrm{r}^- \mathrm{op}_{x_n} r_-^\nu \mathrm{e}^+,$$

whenever the compositions make sense, equation (2) gives the assertion. ◁

2.2.11 Theorem. *Let $p \in S_{tr}^\mu(\mathbf{R}^n, \mathbf{R}^n; \mathbf{R}^l), \mu \in \mathbf{Z}$, and assume that p is independent of x_n, or $p(x, \xi, \lambda) = 0$ for x outside a compact set. Then*

$$\mathrm{op}_{x_n}^+ p \in S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \times \mathbf{R}^l; H^{\sigma,\tau}(\mathbf{R}_+), H^{\sigma-\mu,\tau}(\mathbf{R}_+)) \tag{1}$$

for all $\sigma > -\frac{1}{2}, \tau \in \mathbf{R}$. Moreover, if $\sigma < 0$, then

$$\mathrm{op}_{x_n}^+ p \in S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \times \mathbf{R}^l; H_0^{\sigma,\tau}(\mathbf{R}_+), H^{\sigma-\mu,\tau}(\mathbf{R}_+)) \tag{2}$$

for $\sigma - \mu \geq 0$, and

$$\mathrm{op}_{x_n}^+ p \in S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \times \mathbf{R}^l; H_0^{\sigma,\tau}(\mathbf{R}_+), H_0^{\sigma-\mu,\tau}(\mathbf{R}_+)) \tag{3}$$

whenever $\sigma - \mu \leq 0$.

Proof. Let us first prove (1). For $-\frac{1}{2} < \sigma < \frac{1}{2}$ there is nothing to show, since then extension by zero is continuous $H^{\sigma,\tau}(\mathbf{R}_+) \to H^{\sigma,\tau}(\mathbf{R})$. Using interpolation, we may assume that $\sigma \in \mathbf{N}, \tau \in 2\mathbf{Z}$. Now (1) is equivalent to having

$$\operatorname{op}^+_{x_n} r_-^{\sigma-\mu} \circ_n \langle x_n\rangle^\tau \operatorname{op}^+_{x_n} p \langle x_n\rangle^{-\tau} \circ_n \operatorname{op}^+_{x_n} r_-^{-\sigma} \in S^0(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \times \mathbf{R}^l; L^2(\mathbf{R}_+), L^2(\mathbf{R}_+)).$$

First note that $\langle x_n\rangle^\tau \operatorname{op}^+_{x_n} p \langle x_n\rangle^{-\tau} = \operatorname{op}^+_{x_n} q$ for some $q \in S^\mu_{1,0,tr}(\mathbf{R}^n, \mathbf{R}^n; \mathbf{R}^l)$ which is either independent of x_n or vanishes for x_n outside the above compact set: The proof is the same as that of 2.2.10(c).
By 2.2.10(2), $\operatorname{op}_{x_n} r_-^{\sigma-\mu} \circ_n \operatorname{op}^+_{x_n} q = \operatorname{op}^+_{x_n}(r_-^{\sigma-\mu} \#_n q)$ with the Leibniz product $\#_n$ in x_n-direction. Since $\sigma \in \mathbf{N}$, composition of this operator with $\operatorname{op}^+_{x_n} r_-^{-\sigma}$ produces

$$\left[\operatorname{op}_{x_n} r_-^{\sigma-\mu} \circ_n \operatorname{op}_{x_n} q \circ_n \operatorname{op}_{x_n} r_-^{-\sigma}\right]_+ + g.$$

Here g is a singular Green symbol of order and type zero (cf. 2.2.5, noting that $\sigma \geq 0$), modulo remainders that induce parameter-dependent regularizing singular Green operators of type zero.
The term inside the brackets is $\operatorname{op} q_1$ with some $q_1 \in S^0_{1,0,tr}$. Moreover, either q is independent of x_n, then also q_1 is, or p vanishes for x_n outside a compact set. In that case, we may use the asymptotic expansion formula for the composition of $\operatorname{op} r_-^{\sigma-\mu}$ with a multiplication operator $\phi(x_n), \phi \in \mathrm{C}_0^\infty(\mathbf{R})$ to see that the term inside the brackets is the sum of a pseudodifferential operator with compact x_n–support and a singular Green operator of order and type zero. Now 2.2.1 yields the assertion.
The proof of (2) and (3) is similar, using r_+^μ instead of r_-^μ. ◁

2.2.12 Definition. Let $\Omega' \subseteq \mathbf{R}^{n-1}$ be open, $\Omega = \Omega' \times \mathbf{R}_+$, and $\Omega_0 = \Omega' \times \overline{\mathbf{R}}_+$. Moreover, let $\mu \in \mathbf{R}, d \in \mathbf{N}$.
(a) A *parameter-dependent trace operator of order* μ *and type* d on Ω is a family $\{T(\lambda) : \lambda \in \mathbf{R}^l\}$ of operators $T(\lambda) : C_0^\infty(\Omega_0) \to \mathcal{D}'(\Omega')$ of the form

$$T(\lambda) = \sum_{j=0}^{d} \operatorname{op} t_j(\lambda)\partial^j_{x_n} + T_0(\lambda) \tag{1}$$

with $t_j \in S^{\mu-j}(\Omega' \times \Omega', \mathbf{R}^{n-1}; \mathcal{S}'(\mathbf{R}_+), \mathbf{C})$ and T_0 a *parameter-dependent regularizing trace operator of type* d, which we define to be a family $\{T_0(\lambda) : \lambda \in \mathbf{R}^l\}$ of operators $T_0(\lambda) : C_0^\infty(\Omega_0) \to \mathcal{D}'(\Omega')$ of the form

$$T_0(\lambda)f(x') = \sum_{j=0}^{d} \int_{\Omega'} \int_0^\infty \tau_j(x', y', y_n, \lambda)\partial^j_{y_n} f(y', y_n)dy_n dy' \tag{2}$$

with $\tau_j \in \mathcal{S}(\mathbf{R}^l, C^\infty(\Omega' \times \Omega_0))$.
Notice: If all t_j in (1) belong to $S^{-\infty}(\Omega' \times \Omega', \mathbf{R}^{n-1}; \mathcal{S}'(\mathbf{R}_+), \mathbf{C})$, then T is parameter-dependent regularizing. This follows from the mapping properties in 2.2.15(c), below, in connection with 2.1.13.
The topology on the space of all trace operators of order μ and type d is defined via the natural Fréchet topologies on $\tau_j \in \mathcal{S}(\mathbf{R}^l, C^\infty(\Omega' \times \Omega_0))$ and those on the symbol spaces via the representations (1) and (2).

The parameter-dependent and operator-valued symbol

$$\sum_{j=0}^{d} \operatorname{op} t_j(\lambda)\partial_{x_n}^j \in S^\mu(\Omega' \times \Omega', \mathbf{R}^{n-1} \times \mathbf{R}^l; H^\sigma(\mathbf{R}_+), \mathbf{C}),$$

$\sigma \in \mathbf{R}^2, \sigma_1 > d - \frac{1}{2}$ is called a trace symbol for T. As in the case of singular Green symbols, it is not uniquely defined; we obtain an equivalence class of tuples $(t_0, \dots, t_d), t_j \in S^{\mu-j}(\Omega' \times \Omega', \mathbf{R}^{n-1} \times \mathbf{R}^l; H^\sigma(\mathbf{R}_+), \mathbf{C})$, with the property that

$$T(\cdot) - \operatorname{op} \sum t_j(\cdot)$$

is a regularizing parameter-dependent trace operator of type d.
(b) A *parameter-dependent Poisson or potential operator K of order μ on Ω'* is a family $\{K(\lambda) : \lambda \in \mathbf{R}^l\}$ of operators $K(\lambda) : C_0^\infty(\Omega') \to \mathcal{D}'(\Omega)$ of the form

$$K(\lambda) = \operatorname{op} k(\lambda) + K_0(\lambda) \tag{3}$$

with a symbol $k \in S^\mu(\Omega' \times \Omega', \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathbf{C}, \mathcal{S}(\mathbf{R}_+))$ and a *regularizing parameter-dependent Poisson or potential operator,* i.e. a family $\{K_0(\lambda) : \lambda \in \mathbf{R}^l\}$ of operators $K_0(\lambda) : C_0^\infty(\Omega') \to \mathcal{D}'(\Omega)$ of the form

$$K_0(\lambda)f(x', x_n) = \int_{\Omega'} \kappa_0(x', x_n, y', \lambda) f(y')dy' \tag{4}$$

with a function $\kappa_0 \in \mathcal{S}(\mathbf{R}^l, C^\infty(\Omega_0 \times \Omega'))$.
If the symbol k in (3) belongs to $S^{-\infty}(\Omega' \times \Omega', \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathbf{C}, \mathcal{S}(\mathbf{R}_+))$, then K is parameter-dependent regularizing in view of the mapping properties in 2.2.15(d), below.
The representations (3) and (4) together with the topologies on the symbol spaces and the space $\mathcal{S}(\mathbf{R}^l, C^\infty(\Omega_0 \times \Omega'))$ give a Fréchet topology for the potential operators of order μ.
Call k in (3) a potential symbol for K. Again, it is unique up to symbols inducing regularizing potential operators.
In general all symbols will take values in matrices.

2.2.13 Remark. Like in Theorem 2.1.19 one can check that the usual definition of a trace operator of order $\mu - \frac{1}{2}$ and type d coincides with that of a trace operator of order μ and type d in this set-up.
In particular, the standard trace operators $\gamma_j : \mathcal{S}(\mathbf{R}^n_+) \to \mathcal{S}(\mathbf{R}^{n-1})$ defined by

$$\gamma_j f(x') = \lim\nolimits_{t\to 0}(\partial_{x_n}^j f)(x', t)$$

for $j \in \mathbf{N}$ are trace operators of order and type $j+1$ in the usual set-up; they are of order $j + \frac{1}{2}$ and type $j+1$ here. Similarly, a usual potential operator of order $\mu + \frac{1}{2}$ corresponds to a potential operator of order μ in the sense of 2.2.12(b).

Let us now check that we have the usual representation also for a parameter-dependent singular Green operator of order μ and type d:

2.2.14 Lemma. *Let $\Omega, \Omega', \Omega_0$ be as in 2.2.12, $\mu \in \mathbf{R}, d \in \mathbf{N}$. A family $\{G(\lambda) : \lambda \in \mathbf{R}^l\}$ of operators*

$$G(\lambda) : C_0^\infty(\Omega_0) \to \mathcal{D}'(\Omega)$$

is a parameter-dependent singular Green operator of order μ and type d if and only if it can be written in the form

$$G(\lambda) = \sum_{j=0}^{d-1} K_j(\lambda)\gamma_j + G^0(\lambda) \tag{1}$$

with parameter-dependent potential operators K_j of order $\mu - j - \frac{1}{2}$ and a parameter-dependent singular Green operator G^0 of type 0.

Note: *Already in Theorem 2.1.19 we saw that our definition of singular Green operators of type zero coincides with the usual one. Together with Remark 2.2.13 we have therefore checked that both concepts coincide.*

Proof. First part. Suppose an operator of the form (1) is given. For simplicity assume that $d = 1$, i.e. $G(\lambda) = K(\lambda)\gamma_0 + G^0(\lambda)$ with a parameter-dependent potential operator K of order $\mu - \frac{1}{2}$. We can then write

$$K(\lambda) = \operatorname{op} k(\lambda) + K^0(\lambda)$$

with a regularizing parameter-dependent operator K^0 and a potential symbol of order $\mu - \frac{1}{2}$. The operator $\operatorname{op} k$ can also be given by a symbol kernel $k(x', \xi', x_n, \lambda)$ satisfying

$$\|{x_n}^k D_{x_n}^{k'} D_{\xi'}^\alpha D_{x'}^\beta D_\lambda^\gamma k(x', \xi', x_n, \lambda)\|_{L^2(\mathbf{R}_+)} = \mathrm{O}(\langle\xi'\rangle^{\mu - \frac{1}{2} - |\alpha| - |\gamma| - k + k'})$$

and K^0 has an integral kernel $\tilde{k} = \tilde{k}(x, y', \lambda)$ in $\mathcal{S}(\mathbf{R}^l, C^\infty(\Omega_0 \times \Omega'))$. We will now make use of the simple integration by parts identity $\int_a^b fg' + \int_a^b f'g = fg|_a^b$: Choose a function $\phi \in \mathcal{S}(\mathbf{R}_+)$ with $\phi(0) = 1$. Then for $f \in C_0^\infty(\Omega_0)$

$$\begin{aligned} K^0(\lambda)\gamma_0 f &= \int_{\Omega'} \tilde{k}(x, y', \lambda) f(y', 0) dy' \\ &= -\int_{\Omega_0} \tilde{k}(x, y', \lambda)\phi(y_n)\partial_{y_n} f(y) dy \\ &\quad -\int_{\Omega_0} \tilde{k}(x, y', \lambda)\phi'(y_n) f(y) dy \end{aligned}$$

Therefore $K^0(\cdot)\gamma_0$ is a regularizing parameter-dependent singular Green operator of type 1. Now consider $\operatorname{op} k(\cdot)\gamma_0$. We have for $f \in C_0^\infty(\Omega_0)$

$$\begin{aligned} \gamma_0 f(x') &= f(x', 0) \\ &= (2\pi)^{-\frac{n-1}{2}} \int e^{ix'\xi'} \mathcal{F}_{y' \to \xi'} f(\xi', 0) d\xi' \\ &= -(2\pi)^{-\frac{n-1}{2}} \int e^{ix'\xi'} \int_0^\infty \phi(\langle\xi', \lambda\rangle\, y_n) \mathcal{F}_{y' \to \xi'} \partial_{y_n} f(\xi', y_n) dy_n d\xi' \\ &\quad -(2\pi)^{-\frac{n-1}{2}} \int e^{ix'\xi'} \int_0^\infty \langle\xi', \lambda\rangle\, \phi'(\langle\xi', \lambda\rangle\, y_n) \mathcal{F}_{y' \to \xi'} f(\xi', y_n) dy_n d\xi'. \end{aligned}$$

We can therefore write $\operatorname{op} k(\lambda)\gamma_0 f = \operatorname{op} g_0(\lambda) f + \operatorname{op} g_1(\lambda)\partial_{x_n} f$, where $\operatorname{op} g_j, j = 0, 1$, are the operators with the symbol kernels $g_0(x', \xi', x_n, y_n, \lambda) = k(x', \xi', x_n, \lambda)\,\langle\xi', \lambda\rangle\,\phi'(\langle\xi', \lambda\rangle\, y_n)$ and $g_1(x', \xi', x_n, y_n, \lambda) = k(x', \xi', x_n, \lambda)\phi(\langle\xi', \lambda\rangle\, y_n)$. Cauchy-Schwarz' inequality gives

$$\|g_1(x', \xi', \cdot, \cdot, \lambda)\|_{L^2(\mathbf{R}^2_{++})} \leq \|k(x', \xi', \cdot, \lambda)\|_{L^2(\mathbf{R}_+)} \|\phi(\langle\xi', \lambda\rangle\, \cdot)\|_{L^2(\mathbf{R}_+)} = \mathrm{O}(\langle\xi', \lambda\rangle^{\mu - \frac{1}{2} - \frac{1}{2}}),$$

where the O denotes a constant depending continuously on x' and y'. This immediately leads to the desired estimate

$$\|{x_n}^k D_{x_n}^{k'} {y_n}^m D_{y_n}^{m'} D_{\xi'}^{\alpha} D_{x'}^{\beta} D_{\lambda}^{\gamma} g_1(x',\xi',\cdot,\cdot,\lambda)\|_{L^2(\mathbf{R}^2_{++})} = \mathrm{O}(\langle \xi',\lambda\rangle^{\mu-1-|\alpha|-|\gamma|-k+k'-m+m'});$$

similarly for g_0 with $\mu-1$ replaced by μ. In view of Theorem 2.1.19, $\operatorname{op} g_0$ is a parameter-dependent singular Green operator of order μ and type 0, while $\operatorname{op} g_1$ is of order $\mu-1$ and type 0. Now the case $d>1$ follows by iteration.
Second part. For simplicity suppose again that $d=1$ and

$$G(\lambda) = \operatorname{op} g_0(\lambda) + \operatorname{op} g_1(\lambda)\partial_{x_n} + \tilde{G}_0(\lambda) + \tilde{G}_1(\lambda)\partial_{x_n},$$

where g_0 is a parameter-dependent singular Green symbol of order μ and type zero, given by a symbol kernel $g_0(x',\xi',x_n,y_n,\lambda)$, g_1 is a parameter-dependent of order $\mu-1$ and type 0, and $\tilde{G}_0), \tilde{G}_1$ are regularizing parameter-dependent singular Green operators of type zero.
Then $\operatorname{op} g_0 + \tilde{G}_0$ is already of the right form, while an integration by parts yields for $f \in C_0^\infty(\Omega_0)$

$$\begin{aligned} &(2\pi)^{-\frac{n-1}{2}} \int e^{ix'\xi'} \int_0^\infty g_1(x',\xi',x_n,y_n,\lambda)\mathcal{F}_{y'\to\xi'}\partial_{y_n} f(\xi',y_n)dy_n d\xi' \\ = \;&(2\pi)^{-\frac{n-1}{2}} \int e^{ix'\xi'} g_1(x',\xi',x_n,0,\lambda)\mathcal{F}_{y'\to\xi'} f(\xi',0)d\xi' \\ &-(2\pi)^{-\frac{n-1}{2}} \int e^{ix'\xi'} \int_0^\infty \partial_{y_n} g_1(x',\xi',x_n,0,\lambda)\mathcal{F}_{y'\to\xi'} f(\xi',y_n)dy_n d\xi'. \end{aligned}$$

Now the inequality $|\phi(0)|^2 \le 2\|\phi\|_{L^2(\mathbf{R}_+)}\|\phi'\|_{L^2(\mathbf{R}_+)}$, valid for $\phi \in \mathcal{S}(\mathbf{R}_+)$, together with the symbol kernel estimates for g_1, cf. 2.1.18(1), implies that

$$\|{x_n}^k D_{x_n}^{k'} D_{\xi'}^{\alpha} D_{x'}^{\beta} D_{\lambda}^{\gamma} g_1(x',\xi',\cdot,0,\lambda)\|_{L^2(\mathbf{R}_+)} = \mathrm{O}(\langle \xi',\lambda\rangle^{\mu-\frac{1}{2}-|\alpha|-|\gamma|-k+k'})$$

so that $g_1(x',\xi',x_n,0,\lambda)$ induces a parameter-dependent potential operator of order $\mu-\frac{1}{2}$. The symbol kernel $\partial_{y_n} g_1$ induces a parameter-dependent singular Green operator of order μ and type 0. With a similar procedure we may write $\tilde{G}_1(\lambda)\partial_{x_n} = \tilde{K}(\lambda)\gamma_0 + \tilde{G}_2(\lambda)$, where $\tilde{K}$ is a regularizing parameter-dependent potential operator and $\tilde{G}_2$ is a regularizing singular Green operator of type zero. Hence $G(\lambda) = \operatorname{op} k(\lambda)\gamma_0 + G^0(\lambda)$ with a potential operator of order $\mu-\frac{1}{2}$ and a singular Green operator of order μ and type 0. ◁

2.2.15 Theorem. *For parameter-dependent trace and potential operators we have the following mapping properties. Let Ω, Ω' be as in* 2.2.12, $s \in \mathbf{R}, \sigma \in \mathbf{R}^2, \mu \in \mathbf{R}, d \in \mathbf{N}$.
(a) *Let T be a parameter-dependent trace operator of order μ and type d on Ω. If $\sigma_1 > d-\frac{1}{2}$, then*

$$T(\lambda): \mathcal{W}^s_{comp}(\Omega', H^\sigma(\mathbf{R}_+)) \to \mathcal{W}^{s-\mu}_{loc}(\Omega', \mathbf{C}) \tag{1}$$

is bounded for every λ.
In particular, 2.1.13 *implies that*

$$T(\lambda): H^s_{comp}(\Omega) \to H^{s-\mu}_{loc}(\Omega') \tag{2}$$

is bounded for all $s > d-\frac{1}{2}$.

If $d = 0$, then

$$T(\lambda) : \mathcal{W}^s_{comp}(\Omega', \mathcal{S}'(\mathbf{R}_+)) \to \mathcal{W}^{s-\mu}_{loc}(\Omega', \mathbf{C}) \tag{3}$$

is continuous.
(b) *Let K be a parameter-dependent potential operator of order μ. Then*

$$K(\lambda) : \mathcal{W}^s_{comp}(\Omega', \mathbf{C}) \to \mathcal{W}^{s-\mu}_{loc}(\Omega', \mathcal{S}(\mathbf{R}_+)) \tag{4}$$

is continuous for all λ.
In particular,

$$K(\lambda) : H^s_{comp}(\Omega') \to H^{s-\mu}_{loc}(\Omega) \tag{5}$$

is continuous.
(c) *If T is a regularizing trace operator of type d, then $\lambda^\alpha D^\beta_\lambda T(\lambda)$ has property (1) for arbitrary μ, α, β, uniformly in λ. If T even is of type zero, then we have property (3) for all choices of the parameters.*
(d) *If K is a regularizing potential operator then $\lambda^\alpha D^\beta_\lambda K(\lambda)$ has property (4) for every μ, α, β, uniformly in λ.*

Proof. This is a consequence of the definition in connection with Theorem 2.1.14. For (c) and (d) use the fact that the kernels are rapidly decreasing with respect to λ. ◁

2.2.16 Theorem. *Let Ω, Ω' be as in 2.2.12. Let G, K, T be parameter-dependent singular Green , potential, and trace operators of order μ and denote the type of G and T by d.*
Choose a function $\phi \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\phi \equiv 1$ near zero. Then
(a) *$(1-\phi)K$ is a regularizing potential operator.*
(b) *$T(1-\phi)$ is a regularizing trace operator of type 0.*
(c) *$G(1-\phi)$ is a regularizing singular Green operator of type 0.*
(d) *$(1-\phi)G$ is a regularizing singular Green operator of type d.*

Proof. (a) We start with the following observation.
Let $k \in S^\mu(\Omega', \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathbf{C}, \mathcal{S}(\mathbf{R}_+))$, i.e. for all $\sigma \in \mathbf{R}^2, \sigma_1 \geq 0$,

$$\|\kappa_{\langle\xi',\lambda\rangle^{-1}} D^\alpha_{\xi'} D^\beta_{x'} k(x', \xi', \lambda)\|_{\mathcal{L}(\mathbf{C}, H^\sigma(\mathbf{R}_+))} \leq C \langle \xi', \lambda\rangle^{\mu-|\alpha|}. \tag{1}$$

For $r \in \mathbf{N}$ consider x_n^r as the multiplication operator on $H^\sigma(\mathbf{R}_+)$. Since $x_n^r : H^\sigma(\mathbf{R}_+) \to H^{\sigma-(0,r)}(\mathbf{R}_+)$ is bounded, and since

$$\kappa_{\langle\xi',\lambda\rangle^{-1}} x_n^r k(x', \xi', \lambda) = \langle \xi', \lambda\rangle^{-r} x_n^r \kappa_{\langle\xi',\lambda\rangle^{-1}} k(x', \xi', \lambda), \tag{2}$$

we have

$$x_n^r k(x', \xi', \lambda) \in S^{\mu-r}(\Omega', \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathbf{C}, \mathcal{S}(\mathbf{R}_+)). \tag{3}$$

This yields the assertion: Choose any $r \in \mathbf{N}$. Since $\phi \equiv 1$ near zero, $(1-\phi)x_n^{-r}$ is bounded in all derivatives, and we may write

$$(1-\phi)K = [(1-\phi)x_n^{-r}]x_n^r K,$$

which is a potential operator of order $\mu - r$.

The proof of (b), (c), and (d) is similar. For (b) and (c) note that the type can be reduced to zero by writing e.g. $T(1-\phi) = Ty_n^r[y_n^{-r}(1-\phi)]$, $r > d$ and integration by parts as in 2.1.18(3). ◁

2.2.17 Definition. Let $\Omega' \subseteq \mathbf{R}^{n-1}$ be open, $\Omega = \Omega' \times \mathbf{R}_+$, and $\Omega_0 = \Omega' \times \overline{\mathbf{R}}_+$. A *parameter-dependent operator of order* $\mu \in \mathbf{R}$ *and type* $d \in \mathbf{N}$ *in Boutet de Monvel's calculus* on Ω is a family $\{A(\lambda) : \lambda \in \mathbf{R}^l\}$ of operators

$$A(\lambda) = \begin{bmatrix} P_+(\lambda) + G(\lambda) & K(\lambda) \\ T(\lambda) & S(\lambda) \end{bmatrix} : \begin{matrix} C_0^\infty(\Omega_0) \\ \oplus \\ C_0^\infty(\Omega') \end{matrix} \to \begin{matrix} C^\infty(\Omega_0) \\ \oplus \\ C^\infty(\Omega') \end{matrix}, \qquad (1)$$

where

- $P(\cdot)$ $= \operatorname{op} p(\cdot)$ with $p \in S^\mu_{tr}(\Omega \times \Omega, \mathbf{R}^n; \mathbf{R}^l)$, $P_+ = \mathrm{r}^+ P \mathrm{e}^+$,
- $G(\cdot)$ is a parameter-dependent singular Green operator of order μ and type d,
- $K(\cdot)$ is a parameter-dependent potential operator of order μ,
- $T(\cdot)$ is a parameter-dependent trace operator of order μ and type d,
- $S(\cdot)$ is a parameter-dependent pseudodifferential operator of order μ on Ω'.

We shall write $A \in \mathcal{B}^{\mu,d}(\Omega; \mathbf{R}^l)$. The topology on this space is that of a non-direct sum of Fréchet spaces induced by (1) and the topologies on the spaces of pseudodifferential, singular Green , trace, and potential operators.
A *parameter-dependent regularizing operator* A *of type* d *in Boutet de Monvel's calculus on* Ω is one that can be written in the form (1) with all entries being regularizing operators. Write $A \in \mathcal{B}^{-\infty,d}(\Omega; \mathbf{R}^l)$, and give this space the obvious Fréchet topology.
It is a consequence of 2.2.15, 2.2.10, 2.2.1, and 2.1.14 that the operators in (1) indeed have the desired mapping properties.

In general, all entries will be matrix-valued: given $n_1, n_2, n_3, n_4 \in \mathbf{N}$, P and G will be $n_2 \times n_1$ matrices, K will be $n_2 \times n_3$, T of size $n_4 \times n_1$, and S of size $n_4 \times n_3$. For shortness call this an $(n_2, n_4) \times (n_1, n_3)$ matrix.
We may define a family $\{a(\lambda) : \lambda \in \mathbf{R}^l\}$ of parameter-dependent operator-valued symbols for the family $\{A(\lambda)\}$ by letting

$$a(x', \xi', \lambda) = \begin{bmatrix} \operatorname{op}^+_{x_n} p(x, \xi, \lambda) + g(x', \xi', \lambda) & k(x', \xi'\lambda) \\ t(x', \xi', \lambda) & s(x', \xi', \lambda) \end{bmatrix} : \begin{matrix} C_0^\infty(\overline{\mathbf{R}}_+)^{n_1} \\ \oplus \\ \mathbf{C}^{n_3} \end{matrix} \to \begin{matrix} C^\infty(\overline{\mathbf{R}}_+)^{n_2} \\ \oplus \\ \mathbf{C}^{n_4} \end{matrix},$$

where p, g, t, k, s are symbols of P, G, T, K, and S, respectively. We understand the symbol a as an equivalence class of tuples in the corresponding symbol classes with the property that

$$A - \operatorname{op} a \in \mathcal{B}^{-\infty,d}(\Omega; \mathbf{R}^l);$$

i.e. $a_1 \sim a_2$ iff $\operatorname{op} a_1 - \operatorname{op} a_2 \in \mathcal{B}^{-\infty,d}(\Omega; \mathbf{R}^l)$.
Within this equivalence class, we may always find a representative which is properly supported, cf. [27], p.296.

2.2.18 Theorem. *Let Ω, Ω' be as in 2.2.17, $A \in \mathcal{B}^{\mu,d}(\Omega; \mathbf{R}^l)$ be an $(n_2, n_4) \times (n_1, n_3)$ matrix, and $B \in \mathcal{B}^{\mu',d'}(\Omega; \mathbf{R}^l)$ an $(n_1, n_3) \times (n_5, n_6)$ matrix. Suppose that for one of them, the pseudodifferential symbol vanishes for x_n outside a compact set and that A or B is properly supported.*
Then the composition AB is defined; it is of size $(n_2, n_4) \times (n_5, n_6)$ and belongs to $\mathcal{B}^{\mu'',d''}(\Omega; \mathbf{R}^l)$ with $\mu'' = \mu + \mu'$ and $d'' = \max\{\mu' + d, d'\}$.

Proof. Choose symbols a, b such that $A = \operatorname{op} a + A_0, B = \operatorname{op} b + B_0$ with A_0, B_0 regularizing and a, b properly supported. Then the assertion is a consequence of the composition formulas for properly supported operator-valued symbols [27] Section 3.2.2 Theorem 14, and Theorem 2.2.5. For the composition of regularizing operators with others use the mapping properties in 2.2.15(c), (d) in order to show that the result also is regularizing. $\triangleleft$

For convenience, the following theorem will be formulated for a $(1,1) \times (1,1)$ matrix. The case of arbitrary matrix sizes n_i causes an evident modification.

2.2.19 Theorem. *Let Ω, Ω' be as in 2.2.17, $A \in \mathcal{B}^{\mu,d}(\Omega; \mathbf{R}^l)$. Then for all $\lambda \in \mathbf{R}^l$, $s, \sigma \in \mathbf{R}, \sigma > d - \frac{1}{2}$*

$$A(\lambda): \begin{array}{c} \mathcal{W}^s_{comp}(\Omega', H^\sigma(\mathbf{R}_+)) \\ \oplus \\ \mathcal{W}^s_{comp}(\Omega', \mathbf{C}) \end{array} \rightarrow \begin{array}{c} \mathcal{W}^{s-\mu}_{loc}(\Omega', H^{\sigma-\mu}(\mathbf{R}_+)) \\ \oplus \\ \mathcal{W}^{s-\mu}_{loc}(\Omega', \mathbf{C}) \end{array}$$

is bounded. In particular, if $s > d - \frac{1}{2}$, then

$$A(\lambda): \begin{array}{c} H^s_{comp}(\Omega) \\ \oplus \\ H^s_{comp}(\Omega') \end{array} \rightarrow \begin{array}{c} H^{s-\mu}_{loc}(\Omega) \\ \oplus \\ H^{s-\mu}_{loc}(\Omega') \end{array}$$

is bounded.
If $d = 0$, then we additionally have the bounded extensions

$$A(\lambda): \begin{array}{c} \mathcal{W}^s_{comp}(\Omega', H^\sigma_0(\mathbf{R}_+)) \\ \oplus \\ \mathcal{W}^s_{comp}(\Omega', \mathbf{C}) \end{array} \rightarrow \begin{array}{c} \mathcal{W}^{s-\mu}_{loc}(\Omega', H^{\sigma-\mu}_{\{0\}}(\mathbf{R}_+)) \\ \oplus \\ \mathcal{W}^{s-\mu}_{loc}(\Omega', \mathbf{C}) \end{array}$$

for $\sigma < 0$. Here, $H^{\sigma-\mu}_{\{0\}}(\mathbf{R}_+)$ denotes the space $H^{\sigma-\mu}_0(\mathbf{R}_+)$ for $\sigma - \mu < 0$ and $H^{\sigma-\mu}(\mathbf{R}_+)$ for $\sigma - \mu \geq 0$.
In all cases, the symbol topology is stronger than the operator topology.

Proof. This is immediate from the continuity properties of the various components, cf. 2.2.15, 2.2.10, 2.2.1, and 2.1.14. $\triangleleft$

2.2.20 Theorem. *Let Ω, Ω' be as in 2.2.17, $A \in \mathcal{B}^{0,0}(\Omega; \mathbf{R}^l)$. Then the adjoint A^* with respect to the extension of the L^2 inner product to the spaces in 2.2.19 belongs to $\mathcal{B}^{0,0}(\Omega; \mathbf{R}^l)$.*

Proof. This follows from the corresponding result for all entries of A. For a pseudodifferential operator P of order zero, we have $P_+{}^* = P^*{}_+$; while for the singular Green , potential and trace operators the assertion relies on the fact that the adjoint of an operator-valued symbol belongs to the calculus, cf. [27], Section 3.2.2, Theorem 15. $\triangleleft$

2.2.21 Definition. Let Ω be as before, $\mu \in \mathbf{Z}, d \in \mathbf{N}$ and $d \leq \mu_+ = \max\{\mu, 0\}$.
An operator $A \in \mathcal{B}^{\mu,d}(\Omega; \mathbf{R}^l)$ with a symbol a is called *parameter–elliptic*, if there is an operator $B \in \mathcal{B}^{-\mu,d'}, d' = (-\mu)_+$, with symbol b such that

$$a(\lambda)b(\lambda) - id_{\mathcal{S}(\mathbf{R}_+)^{n_2} \oplus \mathbf{C}^{n_4}} = c_1(\lambda) \tag{1}$$

and

$$b(\lambda)a(\lambda) - id_{\mathcal{S}(\mathbf{R}_+)^{n_1} \oplus \mathbf{C}^{n_3}} = c_2(\lambda), \tag{2}$$

where c_1 and c_2 are symbols of operators of order -1 and types $d_1 = (-\mu)_+, d_2 = \mu_+$.

Clearly, this definition is independent of the particular choice of the symbols a and b. We shall also say that the symbol is parameter–elliptic.

2.2.22 Theorem. *Let Ω, Ω' be as in 2.2.17, and let $A \in \mathcal{B}^{\mu,d}(\Omega; \mathbf{R}^l), d \leq \mu_+$, be parameter–elliptic. Then there is a $B \in \mathcal{B}^{-\mu,d'}(\Omega,\ \mathbf{R}^l), d' = (-\mu)_+$ such that the operators*

$$R_1 = AB - I,$$

and

$$R_2 = BA - I$$

belong to $\mathcal{B}^{-\infty,d_1}(\Omega; \mathbf{R}^l)$, and $\mathcal{B}^{-\infty,d_2}(\Omega; \mathbf{R}^l)$, respectively, with $d_1 = (-\mu)_+$, and $d_2 = \mu_+$.

Proof. This is immediate from the usual Neumann series argument together with the fact that operator-valued symbols can be summed up asymptotically, [27], Section 3.2.2 Theorem 4. $\triangleleft$

2.2.23 Classical elements. Let Ω, Ω' be as in 2.2.17. An operator $A = \operatorname{op} a + A_0 \in \mathcal{B}^{\mu,d}(\Omega; \mathbf{R}^l)$ with A_0 regularizing is called *classical,* if all entries of a can be chosen to be classical elements in the sense of 2.1.7.

2.3 The Manifold Case

2.3.1 Definition. (a) Let X be an n-dimensional compact C^∞ manifold with boundary Y, embedded in a compact n-dimensional manifold Ω without boundary. In order to fix the notation let $\{\Omega_j\}$ denote a finite open covering of Ω and suppose that the coordinate charts map $X \cap \Omega_j$ to $U_j \subset \mathbf{R}^n_+$ and $Y \cap \Omega_j$ to $\mathbf{R}^{n-1} \times \{0\}$. We may identify a neighborhood $Y_{(1)}$ of Y with $Y \times (-1, 1)$ and assume that this neighborhood is covered by open sets

Ω_k of the form $\Omega_k = \Omega'_k \times (-1,1)$, where the sets Ω'_k form an open covering of Y by coordinate neighborhoods.
Then let $Y_{[\frac{1}{2}]}$ denote the neighborhood of Y identified with $Y \times [-\frac{1}{2}, \frac{1}{2}]$ and choose coordinate neighborhoods for the remaining part of Ω that do not intersect $Y_{[\frac{1}{2}]}$.
For a partition of unity $\{\phi_j : j = 1, \dots, J\}$ and cut-off functions $\{\psi_j : j = 1, \dots, J\}$ with $\phi_j \psi_j = \phi_j$ subordinate to the above covering of Ω, write Φ_j for the multiplication operator with the matrix

$$\begin{bmatrix} \phi_j & 0 \\ 0 & \phi_j|_Y \end{bmatrix};$$

correspondingly use the notation Ψ_j for multiplications with ψ_j.

(b) The results of Theorem 2.2.16 now allow us to introduce Boutet de Monvel's calculus on X:
Suppose V_1, V_2 are finite-dimensional vector bundles over X and V_3, V_4 are finite-dimensional vector bundles over Y and all are trivial over the above coordinate patches.
We will write $A \in \mathcal{B}^{\mu,d}(X; \mathbf{R}^l)$, if

$$A(\lambda): \begin{matrix} C^\infty(\overline{X}, V_1) \\ \oplus \\ C^\infty(Y, V_3) \end{matrix} \to \begin{matrix} C^\infty(\overline{X}, V_2) \\ \oplus \\ C^\infty(Y, V_4) \end{matrix}, \tag{1}$$

is an operator with the following properties: Writing

$$A = \sum_{j=1}^{J} \Phi_j A \Psi_j + \sum_{j=1}^{J} \Phi_j A(1 - \Psi_j),$$

we ask that

(i) For every j, the operator A_j induced by $\Phi_j A \Psi_j$ via the coordinate charts belongs to $\mathcal{B}^{\mu,d}(U_j; \mathbf{R}^l)$.

If the coordiante chart does not intersect the boundary, then we will assume that – except for the pseudodifferential part – all entries in the matrix A_j vanish; this is motivated by Theorem 2.2.16 and (ii), below.

(ii) The remaining sum $\sum_{j=1}^{J} \Phi_j A(1 - \Psi_j)$ is induced by an integral operator from $C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, V_3)$ to $C^\infty(\overline{X}, V_2) \oplus C^\infty(Y, V_4)$ depending on the parameter $\lambda \in \mathbf{R}^l$. Its kernel density is C^∞ and a rapidly decreasing function of λ in all seminorms defining the Fréchet topology of the smooth densities.

$\mathcal{G}^{\mu,d}(X; \mathbf{R}^l)$ is the subspace of all elements in $\mathcal{B}^{\mu,d}(X; \mathbf{R}^l)$ where the pseudodifferential part can be taken to be zero.
In order to keep notation at a low level, we will not indicate the vector bundles A is acting on, understanding that this has to be made clear in the context unless their choice is completely arbitrary.

(c) In each coordinate patch U_j intersecting the boundary we may associate a symbol with the operator A induced by asking that

$$A_j = \tilde{\Phi}_j \operatorname{op} a_j \tilde{\Psi}_j + A_{j_0} \tag{2}$$

with a symbol a_j of order μ and type d and regularizing A_{j_0}. Here, we have written $\tilde{\Phi}_j, \tilde{\Psi}_j$ for the multiplication operators Φ_j, Ψ_j in local coordinates.
In an interior chart, only the pseudodifferential part in the matrix for A is non-zero; it has a symbol p_j. Letting $a_j = \begin{bmatrix} p_j & 0 \\ 0 & 0 \end{bmatrix}$, we also obtain relation (2). We shall call the tuple $(a_1, \ldots, a_J)$ a symbol for A.

(d) Call A classical, if all the operators A_j are classical, i.e. if the pseudodifferential part of A is classical, and if in all coordinate neighborhoods intersecting the boundary, the operators A_j are classical in the sense of 2.2.23. Write $A \in \mathcal{B}^{\mu,d}_{cl}(X; \mathbf{R}^l)$. The operator A then has:

- a principal pseudodifferential symbol, $\sigma^\mu_\psi(A) = \sigma^\mu_\psi(A)(x, \xi, \lambda)$, well-defined as a function on $(T^*X \times \mathbf{R}^l)\backslash 0$, (where 0 denotes the zero-section in the sense that $(\xi, \lambda) = 0$, with values in $\mathcal{L}(V_1, V_2)$, and
- a principal boundary symbol, operator-valued, $\sigma^\mu_\wedge(A) = \sigma^\mu_\wedge(A)(x', \xi', \lambda)$, defined on $(T^*Y \times \mathbf{R}^l)\backslash 0$ with values in $\mathcal{L}(\pi^*V_{1+} \oplus \pi^*V_3, \pi^*V_{2+} \oplus \pi^*V_4)$. Here, $\pi : (T^*Y \times \mathbf{R}^l)\backslash 0 \to Y$ is the canonical projection, $V_{1+} = V_1|_Y \otimes H^+, V_{2+} = V_2|_Y \otimes H^+$, cf. [21], Section 3.1.1.1.

2.3.2 Definition. We will say that $A \in \mathcal{B}^{\mu,d}(X; \mathbf{R}^l), d \le \mu_+$ is *parameter-elliptic* if there is an operator $B \in \mathcal{B}^{-\mu,d}(X; \mathbf{R}^l), d \le (-\mu)_+$ such that

- for each interior coordinate chart, the local pseudodifferential components p_j, q_j of the symbols A and B, respectively, satisfy the relations
$$p_j q_j - 1, q_j p_j - 1 \in S^{-1}(U_j \times U_j, \mathbf{R}^n; \mathbf{R}^l), \text{and} \tag{1}$$
- for each boundary chart, the corresponding boundary symbols a_j, b_j satisfy the ellipticity relations
$$\tilde{\Phi}_j a_j b_j \tilde{\Psi}_j - \tilde{\Phi}_j I = c_1 \tag{2}$$
$$\tilde{\Phi}_j b_j a_j \tilde{\Psi}_j - \tilde{\Phi}_j I = c_2 \tag{3}$$
with parameter-dependent symbols c_1, c_2 of order -1 and types $d_1 = (-\mu)_+, d_2 = \mu_+$. Like in 2.3.1(c), the tilde denotes the function in local coordinates, I is the identity.

With the same argument as in 2.2.22 we may then construct a parametrix B for A :

2.3.3 Theorem. *Let $A \in \mathcal{B}^{\mu,d}(X; \mathbf{R}^l)$ be parameter-elliptic, $d \le \mu_+$. Then there is an operator $B \in \mathcal{B}^{-\mu,d'}(X; \mathbf{R}^l), d' = (-\mu)_+$ such that*
$$R_1 = AB - I \in \mathcal{B}^{-\infty,d_1}(X; \mathbf{R}^l) \text{ and } R_2 = BA - I \in \mathcal{B}^{-\infty,d_2}(X; \mathbf{R}^l),$$
where $d_1 = (-\mu)_+, d_2 = \mu_+$. In particular, in the notation of 2.3.1:

$$A(\lambda): \begin{matrix} H^s(X,V_1) \\ \oplus \\ H^s(Y,V_3) \end{matrix} \rightarrow \begin{matrix} H^{s-\mu}(X,V_2) \\ \oplus \\ H^{s-\mu}(Y,V_4) \end{matrix}$$

is a Fredholm operator for $s, s-\mu > -\frac{1}{2}$.

2.3.4 Remark. (a) Vice versa, the existence of a parametrix as in Theorem 2.3.4 implies the ellipticity of the operator A.
(b) From (a) we conclude that it is sufficient to ask that the symbols c_1 and c_2 in 2.3.2(2),(3) are of order $-\epsilon, \epsilon > 0$ arbitrary.

2.3.5 Theorem. *Let* $A \in \mathcal{B}^{\mu,d}_{cl}(X;\mathbf{R}^l)$. *Then* A *is parameter-elliptic if and only if*

(i) *The principal pseudodifferential symbol is invertible for all* $(x,\xi,\lambda) \in (T^*X \times \mathbf{R}^l)\backslash 0$, *and*

(ii) *for all* $(x',\xi',\lambda) \in (T^*Y \times \mathbf{R}^l)\backslash 0$, *the principal boundary symbol is an isomorphism.*

2.3.6 Lemma. *A family of operators* $\{G(\lambda) : \lambda \in \mathbf{R}^l\}$ *acting on vector bundles as in 2.3.1(1) is an element of* $\mathcal{B}^{-\infty,0}(X;\mathbf{R}^l)$ *if and only if for all multi-indices* α, β *and all* $N \in \mathbf{N}$ *the extension*

$$\lambda^\alpha D^\beta_\lambda G(\lambda) : H^{-N}_0(X,V_1) \oplus H^{-N}(Y,V_3) \rightarrow H^N(X,V_2) \oplus H^N(Y,V_4)$$

exists and is uniformly bounded with respect to λ.

Proof. By definition, $\{G(\lambda) : \lambda \in \mathbf{R}^l\} \in \mathcal{B}^{-\infty,0}(X;\mathbf{R}^l)$ if and only if it is an integral operator with a smooth kernel density, $\gamma(x,\tilde{x},\lambda)$ such that $\lambda \mapsto \gamma(\cdot,\cdot,\lambda)$ is rapidly decreasing with respect to all C^∞ semi-norms. In the proof of 2.1.19, on the other hand, we have seen how the kernel semi-norms can be controlled in terms of the mapping properties. $\triangleleft$

2.3.7 Theorem. *Let* $A \in \mathcal{B}^{-\infty,0}(X;\mathbf{R}^l)$. *Then for all* $s \in \mathbf{R}$,

$$I + A(\lambda): \begin{matrix} H^s(X,V_1) \\ \oplus \\ H^s(Y,V_3) \end{matrix} \rightarrow \begin{matrix} H^s(X,V_1) \\ \oplus \\ H^s(Y,V_3) \end{matrix} \qquad (1)$$

is invertible for large λ, *and* $(I + A(\lambda))^{-1} = I + B(\lambda)$ *for some* $B \in \mathcal{B}^{-\infty,0}(X;\mathbf{R}^l)$.

Proof. Since $\|A(\lambda)\|_s = o(\lambda), I + A(\lambda)$ is invertible for large $|\lambda|$; here the index s of the norm refers to the situation in (1). By replacing $A(\lambda)$ by $\phi(\lambda)A(\lambda)$ for a smooth function $0 \le \phi \le 1$, vanishing in a sufficiently large ball around zero and equal to 1 near ∞, we may assume that $A(\lambda)$ is invertible for all λ and $\|A(\lambda)\|_s < \frac{1}{2}$.

Now, $(I+A)^{-1} = I - A + A(I+A)^{-1}A$. So all we have to check is that

$$\lambda^\alpha D_\lambda^\beta[A(\lambda)[I+A(\lambda)]^{-1}A(\lambda)] : H_0^{-N}(X,V_1)\oplus H^{-N}(Y,V_3) \to H^N(X,V_1)\oplus H^N(Y,V_3)$$

is bounded. This however, is immediate from the fact that $\|[I+A(\lambda)]^{-1}\|_s \le \sum_j \|A(\lambda)\|_s^j < 2$, the differentiation rules and the corresponding properties of $A(\lambda)$. ◁

2.3.8 Theorem. *Let $G \in \mathcal{B}^{-\infty,d}(X;\mathbf{R}^l)$, and suppose that for given $s \in \mathbf{R}, s > d - \frac{1}{2}$,*

$$I + G(\lambda) : H^s(X,V_1)\oplus H^s(Y,V_3) \to H^s(X,V_1)\oplus H^s(Y,V_3)$$

is invertible for all λ. Then there is an $H \in \mathcal{B}^{-\infty,d}(X;\mathbf{R}^l)$ such that

$$(I+G)^{-1} = I + H.$$

Proof. For simplicity consider the case where G consists only of the singular Green part, i.e. $V_3 = 0$; moreover, we will assume that G is scalar, i.e. $V_1 = \mathbf{C}$.
Wirte $G = \sum_{j=0}^d G_j \partial_r^j$, where $G_j \in \mathcal{B}^{-\infty,0}(X;\mathbf{R}^l)$ and ∂_r denotes the normal derivative, defined in a neighborhood of the boundary. We know that the norm of $G(\lambda)$ on $H^s(X)$ tends to zero as $|\lambda|$ tends to infinity. We may thus replace $G(\lambda)$ by $\phi(\lambda)G(\lambda)$ where ϕ is an excision function as in the proof of 2.3.7. We now use the fact that

$$\begin{aligned} [I+G]^{-1} &= I - G + G[I+G]^{-1}G \\ &= I - \sum_{j=0}^d (G_j - G[I+G]^{-1}G_j)\partial_r^j. \end{aligned}$$

In view of 2.3.6, all we have to check is that for all α, β, N

$$\lambda^\alpha D_\lambda^\beta(G_j(\lambda) - G(\lambda)[I+G(\lambda)]^{-1}G_j(\lambda)) : H_0^{-N}(X) \to H^N(X)$$

is uniformly bounded. This, however, is immediate from the corresponding properties of the G_j. ◁

2.3.9 Corollary. Let $A \in \mathcal{B}^{\mu,d}(X;\mathbf{R}^l), d = \mu_+$ be parameter-elliptic. Then

$$A(\lambda) : \begin{array}{c} H^s(X,V_1) \\ \oplus \\ H^s(Y,V_3) \end{array} \to \begin{array}{c} H^{s-\mu}(X,V_1) \\ \oplus \\ H^{s-\mu}(Y,V_3) \end{array}$$

in invertible for large $|\lambda|$, and $A(\lambda)^{-1} = C(\lambda)$ for some $C \in \mathcal{B}^{-\mu,d'}(X;\mathbf{R}^l), d' = (-\mu)_+$.

Proof. By 2.3.3, there is a parametrix $B \in \mathcal{B}^{-\mu,d'}(X;\mathbf{R}^l)$ such that $AB - I = R_1 \in \mathcal{B}^{-\infty,d'}(X;\mathbf{R}^l), BA - I = R_2 \in \mathcal{B}^{-\infty,d}(X;\mathbf{R}^l)$. The operators $I + R_j(\lambda), j = 1,2$, are invertible for large $|\lambda|$. By multiplying with an excision function on a large ball as in the proof of 2.3.7 we may assume that they are invertible everywhere. By 2.3.8 the inverses are of the same kind, and $C = B(I+R_1)^{-1} \in \mathcal{B}^{-\mu,d'}(X;\mathbf{R}^l)$. ◁

2.3.10 Reduction of the Order. Let Ω, X, Y be as in 2.3.1, and $\mu \in \mathbf{Z}$. Assume V is a smooth vector bundle over Ω, trivial over the chosen coordinate charts. Then there exists a pseudodifferential operator with the transmission property $R_-^\mu(\lambda) = \operatorname{op}(r_-^\mu(\lambda))$ with the following properties

(i) $r_-^\mu \in S^\mu_{1,0,tr}(\Omega, \mathbf{R}^n; \mathbf{R}^l)$ is parameter-elliptic.

(ii) $[R_-^\mu(\lambda)]_+ : H^s(X,V) \to H^{s-\mu}(X,V)$ is a topological isomorphism for all $s \in \mathbf{R}, |\lambda|$ large; its inverse also is pseudodifferential.

Similarly, there is a parameter-dependent pseudodifferential operator with the transmission property, $R_+^\mu = \operatorname{op} r_+^\mu$, with

(iii) $r_+^\mu \in S^\mu_{1,0,tr}(\Omega, \mathbf{R}^n; \mathbf{R}^l)$ is parameter-elliptic, and

(iv) $[R_+^\mu(\lambda)]_+ : H_0^s(X,V) \to H_0^{s-\mu}(X,V)$ is a topological isomorphism for all s and large $|\lambda|$.

Proof. Choose a global normal coordinate in a neighborhood of the boundary. Then pick a function $\tau \in C_0^\infty(\mathbf{R}), 0 \le \tau \le 1, \tau \equiv 1$ on $[-\frac{1}{4}, \frac{1}{4}], \tau \equiv 0$ outside $[-\frac{1}{3}, \frac{1}{3}]$. Let χ be as in Lemma 2.2.10. In the boundary charts define the symbols

$$\left[\chi\left(\frac{\xi_n}{\langle \xi', \lambda\rangle}\right)\langle \xi', \lambda\rangle - i\xi_n\right]^{\mu\tau(x_n)} \langle \xi', \lambda\rangle^{\mu(1-\tau(x_n))} I_V. \tag{1}$$

assuming that the boundary neighborhoods have the properties of 2.3.1(a).
In the interior charts, define the symbols

$$\langle \xi', \lambda\rangle^\mu I_V. \tag{2}$$

Then form the corresponding pseudodifferential operators. Transport them to the manifold via the coordinate charts, and patch them together with a partition of unity and cut-off functions as in 2.3.1. Call this operator $R_-^\mu(\lambda)$; its symbol (in the sense of a tuple of complete local symbols) is denoted by $r_-^\mu(\lambda)$. It is then straightforward to check (i).
Property (i) implies that for all $s \in \mathbf{R}$,

$$R_-^\mu(\lambda) : H^s(\Omega, V) \to H^{s-\mu}(\Omega, V) \tag{3}$$

is a topological isomorphism provided $|\lambda|$ is large.
In a neighborhood of the boundary, the symbol $r_-^\mu(x,\xi,\lambda)$ is an H^-–function of ξ_n, up to regularizing pseudodifferential terms. This allows to construct a left and right parameter-dependent parametrix $P(\lambda)$ with the same properties. Both left-over terms, $L(P(\lambda), R_-^\mu(\lambda)) = [P(\lambda)]_+[R_-^\mu(\lambda)]_+ - [P(\lambda)R_-^\mu(\lambda)]_+$ and $L(R_-^\mu(\lambda), P(\lambda)) = [R_-^\mu(\lambda)]_+[P(\lambda)]_+ - [R_-^\mu(\lambda)P(\lambda)]_+$ are then regularizing parameter-dependent singular Green operators, thus also regularizing parameter-dependent pseudodifferential operators. This implies (ii) for all $s > -\frac{1}{2}$.
In order to obtain statements (iii) and (iv) start with the complex conjugates of the symbols in (1) and (2) and repeat the above process. This yields a parameter-elliptic symbol $r_+^\mu = r_+^\mu(x,\xi,\lambda)$ which is in H^+ as a function of ξ_n, up to regularizing pseudodifferential terms.

Now we can conclude the proof of (ii): Let $X_- = \Omega\backslash\overline{X}$; then X_- is a manifold with boundary Y. With respect to X_- the construction in (1) defines a symbol in H^+. Therefore the operator $[R_-^\mu(\lambda)]_- = \mathrm{r}^- R_-^\mu(\lambda)\mathrm{e}^-$ according to (iv) extends to an isomorphism $H_0^s(X_-, V) \to H_0^{s-\mu}(X_-, V)$ for all $s \in \mathbf{R}$. On the other hand, e^- and r^- are trivial operations on these spaces. So, in view of (3), $R_-^\mu(\lambda)$ gives an isomorphism $H^s(\Omega, V)/H_0^s(X_-, V) = H^s(X, V) \to H^{s-\mu}(\Omega, V)/H_0^{s-\mu}(X_-, V) = H^{s-\mu}(X, V)$. ◁

3 Weighted Sobolev Spaces with Asymptotics

3.1 Sobolev Spaces Based on the Mellin Transform

Let $\mathbb{D}, \mathbb{B}, X, Y$ be as in Section 1, and assume that X is embedded in a compact manifold Ω without boundary, V is a vector bundle over Ω.

3.1.1 Proposition. *For $\mu \in \mathbf{R}, l \in \mathbf{N}$ there is a parameter-elliptic pseudodifferential operator $\Lambda^\mu \in \operatorname{op} S^\mu(\Omega, \mathbf{R}^n; \mathbf{R}^l)$ such that*

$$\Lambda^\mu(\tau) : H^s(\Omega, V) \to H^{s-\mu}(\Omega, V)$$

is an isomorphism for all $\tau \in \mathbf{R}^l$.

Proof. Choose an arbitrary parameter-elliptic pseudodifferential operator with symbol $\lambda^\mu \in S^\mu(\Omega, \mathbf{R}^n; \mathbf{R}^{l+1})$, cf. 2.1.6(a); the parameter is $(\tau, \sigma) \in \mathbf{R}^l \times \mathbf{R}$ (more about this in Remark 3.1.2, below). Then there is an operator with symbol $q^{-\mu} \in S^{-\mu}(\Omega, \mathbf{R}^n; \mathbf{R}^{l+1})$ with

$$\operatorname{op} \lambda^\mu \operatorname{op} q^{-\mu} - I = \operatorname{op} r$$

and $r \in S^{-1}(\Omega, \mathbf{R}^n; \mathbf{R}^{l+1})$. Since the norm of $\operatorname{op} r(\tau, \sigma)$ is $o(1)$, a simplified version of the argument in 2.3.9 yields a right inverse of $\operatorname{op} \lambda^\mu(\tau, \sigma)$, provided $|\tau, \sigma| \geq C$. Similarly we obtain a left inverse on the same set.
Now we simply let for $\tau \in \mathbf{R}^l$

$$\Lambda^\mu(\tau) = \operatorname{op} \lambda^\mu(\tau, C).$$

$\triangleleft$

3.1.2 Remark. In order to obtain a parameter-elliptic pseudodifferential operator as it is needed in the proof of Proposition 3.1.1, one can e.g. start with symbols of the form $\langle \xi, (\tau, \sigma) \rangle^\mu \in S^\mu(\mathbf{R}^n, \mathbf{R}^n; \mathbf{R}^{l+1})$ and patch them together to an operator on the manifold Ω with the help of a partition of unity $\{\phi_j\}$ and cut-off functions $\{\psi_j\}$ as in 2.3.1(a).
Alternatively, one can choose a Hermitean connection on V and consider the operator $(C + |\sigma|^2 - \Delta)^{\frac{\mu}{2}}$, where Δ denotes the connection Laplacean, and C is a large positive constant.
Then $C + |\sigma|^2 - \Delta$ is a parameter-dependent differential operator with principal symbol $(C + |\sigma|^2 + |\xi|^2) I_V$.
Using a construction by Seeley we may form the powers $(C + |\sigma|^2 - \Delta)^{\frac{\mu}{2}}$, and they are parameter-elliptic pseudodifferential operators of order μ.
In the following we will suppose we are given a fixed family $\{\Lambda^\mu : \mu \in \mathbf{R}\}$ of pseudodifferential symbols with parameter-elliptic symbols of order μ, depending on a parameter $\tau \in \mathbf{R}$.
For this family we will define the spaces $\mathcal{H}^{s,\gamma}, s, \gamma \in \mathbf{R}$. It is easily seen that the spaces do not depend on the particular choice of this family. However, it will often be helpful to know that we have for the above special families an additional parameter, namely the constant C, to influence the behavior of the family.

3.1.3 Definition. (a) Let $\{\Lambda^\mu(\tau) : \tau \in \mathbf{R}^l\}$ be a family of pseudodifferential operators as in 3.1.1. For $s, \gamma \in \mathbf{R}$, the space $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$ was introduced in [29], Section 1.1.1, and in [27], Section 2.1.1, as the closure of $C_0^\infty(\Omega^\wedge)$ in the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\Omega^\wedge)} = \left\{ \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|\Lambda^s(\operatorname{Im} z) Mu(z)\|^2_{L^2(\Omega)} \, |dz| \right\}^{\frac{1}{2}}. \tag{1}$$

Recall that n is the dimension of X and Ω. As usual, $\Gamma_\beta = \{z \in \mathbf{C} : \operatorname{Re} z = \beta\}$.
(b) Now let r^+ denote restriction to X, and let

$$\mathcal{H}^{s,\gamma}(X^\wedge) = \{\mathrm{r}^+ f : f \in \mathcal{H}^{s,\gamma}(\Omega^\wedge)\}.$$

The space $\mathcal{H}^{s,\gamma}(X^\wedge)$ carries the quotient norm:

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \inf\{\|f\|_{\mathcal{H}^{s,\gamma}(\Omega^\wedge)} : f \in \mathcal{H}^{s,\gamma}(\Omega^\wedge), \mathrm{r}^+ f = u\}.$$

(c) $\mathcal{H}_0^{s,\gamma}(X^\wedge)$ is the space of all distributions in $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$ with support in $\overline{X}^\wedge = \overline{X} \times \overline{\mathbf{R}}_+$. Since, by definition, $C_0^\infty(\Omega^\wedge)$ is dense in $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$, $\mathcal{H}_0^{s,\gamma}(X^\wedge)$ is the closure of $C_0^\infty(X^\wedge)$ in the topology of $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$.

3.1.4 Remark. (a) Suppose $u \in C_0^\infty(\Omega^\wedge)$. For fixed $x \in \Omega, u(x,\cdot) \in C_0^\infty(\mathbf{R}_+)$, so it has a Mellin transform which is holomorphic in the whole plane. Moreover,

$$z \mapsto Mu(\cdot, z) \in \mathcal{A}(\mathbf{C}, C^\infty(\Omega)),$$

the space of entire functions with values in $C^\infty(\Omega)$, and it is rapidly decreasing on all lines Γ_β, uniformly for β in compact intervals. Therefore the integral in 3.1.3(1) makes sense. It turns out that $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$ is a subspace of $\mathcal{D}'(\Omega^\wedge)$, cf. 3.1.7, hence $\mathcal{H}^{s,\gamma}(X^\wedge) \subset \mathcal{D}'(X^\wedge)$. In order to evaluate the integral in 3.1.3(1) it is in fact sufficient to know Mu only on the line $\Gamma_{\frac{n+1}{2}-\gamma}$. We can therefore extend the concrete definition of the norm in 3.1.3(a) to a larger space of functions by replacing M by the weighted Mellin transform $M_{\gamma-\frac{n}{2}}$: $t^{\gamma-\frac{n}{2}} L^2(\mathbf{R}_+) \to L^2(\Gamma_{\frac{n+1}{2}-\gamma})$, cf. 5.1.5.
(b) For $s = l \in \mathbf{N}$ we obtain the alternative description

$$u \in \mathcal{H}^{l,\gamma}(\Omega^\wedge) \quad \text{iff} \quad t^{\frac{n}{2}-\gamma}(t\partial_t)^{\alpha_0}\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u(x,t) \in L^2(\Omega^\wedge)$$

for all $\alpha_0 + \alpha_1 + \ldots \alpha_n \le l$, cf. [27], Section 2.1.1, Proposition 2.

The well-known properties of the space $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$ immediately imply the statements of the lemma, below.

3.1.5 Lemma. (a) *The space $\mathcal{H}^{s,\gamma}(X^\wedge)$ is independent of the particular choice of the order-reducing family.*
(b) $\mathcal{H}^{s,\gamma}(X^\wedge) \subseteq H^s_{loc}(X^\wedge)$.
(c) $\mathcal{H}^{s,\gamma}(X^\wedge) = t^\gamma \mathcal{H}^{s,0}(X^\wedge)$.

(d) $\mathcal{H}^{0,0}(X^\wedge) = t^{-n/2}L^2(X^\wedge)$.
(e) $\mathcal{H}^{0,0}(X^\wedge)$ *has a natural inner product*

$$(u,v)_{\mathcal{H}^{0,0}(X^\wedge)} = \frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}}} (Mu(z), Mv(z))_{L^2(X)}\,dz.$$

(f) *Let* $\phi \in C_0^\infty(\overline{\mathbf{R}}_+)$, *and denote for the moment by* M_ϕ *the operator of multiplication by* ϕ. *Then* M_ϕ *induces continuous operators*

$$M_\phi : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge)$$

for all $s,\gamma \in \mathbf{R}$. *Moreover,* $\phi \mapsto M_\phi$ *induces a continuous embedding*

$$C_0^\infty(\overline{\mathbf{R}}_+) \to \bigcap_{s,\gamma} \mathcal{L}(\mathcal{H}^{s,\gamma}(X^\wedge)).$$

(g) *Similarly, if* ϕ *is the restriction to* $X^\wedge$ *of a function in* $C_0^\infty(\Omega \times \mathbf{R})$, *then*

$$M_\phi : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge)$$

is bounded for all $s,\gamma \in \mathbf{R}$, *and the mapping* $\phi \mapsto M_\phi$ *is continuous in the corresponding topology.*

Notice that (a) is a simple consequence of the fact that if $\{\Lambda^\mu : \mu \in \mathbf{R}\}$ and $\{\tilde{\Lambda}^\mu : \mu \in \mathbf{R}\}$ are two order-reducing families, then for each μ, the operator $\Lambda^\mu\tilde{\Lambda}^{-\mu}$ is parameter-elliptic of order zero. This yields the corollary, below.

3.1.6 Corollary. Suppose that $\{\Omega_j : j = 1,\dots,J\}$ is an open covering of Ω, and $\{\phi_j\}$ is a subordinate partition of unity.
Moreover, let $\{R^\mu : \mu \in \mathbf{R}\}$ be an order-reducing family on $\mathbf{R}^n$, and $\|\cdot\|_{\mathcal{H}^{s,\gamma}(\mathbf{R}^n\times\mathbf{R}_+)}$ the corresponding norm. Then

$$|||u|||_{s,\gamma} = \left(\sum_{j=1}^{J} \|(\phi_j u)_*\|^2_{\mathcal{H}^{s,\gamma}(\mathbf{R}^n\times\mathbf{R}_+)}\right)^{\frac{1}{2}} \tag{1}$$

furnishes an equivalent norm on $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$. Here, $(\phi_j u)_*$ is the distribution induced on $\mathbf{R}^n \times \mathbf{R}_+$ via the coordinate functions.

Proof. We have

$$\|u\|^2_{\mathcal{H}^{s,\gamma}(\Omega^\wedge)} \equiv \sum_{j=1}^{J}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|\phi_j\Lambda^s(\operatorname{Im} z)Mu(z)\|^2_{L^2(\Omega)}\,|dz| \tag{2}$$

in terms of equivalent norms. Choose cut-off functions ψ_j supported in Ω_j with $\phi_j\psi_j = \phi_j$. Again in terms of norms, expression (2) is equivalent to

$$\sum_{j=1}^{J}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|\phi_j\Lambda^s(\operatorname{Im} z)M(\psi_j u)(z)\|^2_{L^2(\Omega)}\,|dz|, \tag{3}$$

provided the parameter C in the choice of the order reduction, cf. 3.1.2, is chosen sufficiently large. The reason is the following. Since $\phi_j\Lambda^s(1-\psi_j)$ is parameter-dependent

regularizing, its norm becomes rapidly small as C tends to infinity. In particular, its norm is small compared to those of $\phi_j\Lambda^s(\operatorname{Im} z)\psi_j$ and $\phi_j\Lambda^s(\operatorname{Im} z)$.
The continuity of the transport of functions to Euclidean space shows that (3) in turn is equivalent to the expression

$$\sum_{j=1}^{J}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|(\phi_j\Lambda^s(\operatorname{Im} z)M(\psi_j u))_*(z)\|^2_{L^2(\mathbf{R}^n)}\,|dz|; \tag{4}$$

here the asterisk indicates that the corresponding function is taken in local coordinates. Now there is a parameter-elliptic operator Λ^s_{j*} on Euclidean space such that

$$(\phi_j\Lambda^s(\operatorname{Im} z)\psi_j M(u))_* = \phi_{j*}\Lambda^s_{j*}(\operatorname{Im} z)\psi_{j*}Mu_*(z).$$

We can write

$$\phi_{j*}\Lambda^s_{j*}(\operatorname{Im} z)\psi_{j*} - \Lambda^s_{j*}(\operatorname{Im} z)\phi_{j*} = -\phi_{j*}\Lambda^s_{j*}(\operatorname{Im} z)(1-\psi_{j*}) + [\Lambda^s_{j*}(\operatorname{Im} z),\phi_{j*}].$$

For the first operator on the right hand side we apply the same argument we have used above. The commutator $[\Lambda^s_{j*}(\tau),\phi_{j*}]$ is a parameter-dependent operator of order $s-1$, so its norm is $O(\langle\tau,C\rangle^{s-1})$ while the norm of $\Lambda_{j*}(\tau)$ is $\geq$ const $\langle\tau,C\rangle^s$. By making C larger, we can make the quotient arbitrarily small; i.e. the norm of the commutator is negligible with respect to that of the two operators on the left hand side. We obtain the assertion from 3.1.5(a). ◁

3.1.7 Remark. On $\mathbf{R}^n$ we may choose a particularly simple order reduction, namely $\Lambda^\mu(\tau) = \operatorname{op}\langle\xi,\tau\rangle$. Using the transformation $\Phi_{n,\gamma}$ defined by

$$\Phi_{n,\gamma}v(r) = \exp(r(\frac{n+1}{2}-\gamma))v(e^r) = \left(t^{\frac{n+1}{2}-\gamma}v(t)\right)|_{t=e^r}$$

one can then check that

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbf{R}^n\times\mathbf{R}_+)} = \|\Phi_{n,\gamma}u\|_{H^s(\mathbf{R}^n\times\mathbf{R})};$$

in other words,

$$\mathcal{H}^{s,\gamma}(\mathbf{R}^n\times\mathbf{R}_+) = \{t^{-\frac{n+1}{2}+\gamma}u(x,\ln t) : u\in H^s(\mathbf{R}^n\times\mathbf{R})\}, \tag{1}$$

cf. [27], 2.1.6(4).
For $X=\mathbf{R}^n_+$ in $\Omega=\mathbf{R}^n$ we obtain

$$\mathcal{H}^{s,\gamma}(\mathbf{R}^n_+\times\mathbf{R}_+) = \{t^{-\frac{n+1}{2}+\gamma}u(x,\ln t) : u\in H^s(\mathbf{R}^n_+\times\mathbf{R})\}, \tag{2}$$

$$\mathcal{H}^{s,\gamma}_0(\mathbf{R}^n_+\times\mathbf{R}_+) = \{t^{-\frac{n+1}{2}+\gamma}u(x,\ln t) : u\in H^s_0(\mathbf{R}^n_+\times\mathbf{R})\}. \tag{3}$$

Moreover, it is easily checked that, in the notation of (2) and (3), we have the following relation between the Fourier and the weighted Mellin transform:

$$[M_{\gamma-\frac{n}{2}}f(x,\cdot)](\frac{n+1}{2}-\gamma+i\tau) = [\mathcal{F}u(x,\cdot)](\tau).$$

As the notation indicates, both transforms act with respect to the last variable only. Therefore

$$M_{\gamma-\frac{n}{2}}\mathcal{H}^{s,\gamma}(\mathbf{R}^n_+\times\mathbf{R}_+) \text{ is isomorphic to } \mathcal{F}_{s\to t}H^s(\mathbf{R}^n_+\times\mathbf{R})$$

if we identify the lines $\Gamma_{\frac{n+1}{2}-\gamma}$ and $\mathbf{R}$.

3.1.8 Remark. The well-known fact that for $-\frac{1}{2} < s < \frac{1}{2}$ we have $H_0^s(\mathbf{R}_+^n) = H^s(\mathbf{R}_+^n)$ together with 3.1.7 then implies that

$$\mathcal{H}^{s,\gamma}(X^\wedge) = \mathcal{H}_0^{s,\gamma}(X^\wedge), \qquad -\frac{1}{2} < s < \frac{1}{2}. \tag{1}$$

Moreover, using a partition of unity, we conclude from Remark 3.1.7 that

$$M_{\gamma-\frac{n}{2}}\mathcal{H}^{s,\gamma}(X^\wedge) = \mathcal{F}H^s(X \times \mathbf{R}), \tag{2}$$

where the action of both transformations is with respect to the last variable and we identify $\Gamma_{\frac{n+1}{2}-\gamma}$ and $\mathbf{R}$.
Here we define the norm in $H^s(X \times \mathbf{R})$ in the canonical way: for a finite partition of unity $\{\phi_j\}$ on $\overline{X}$ subordinate to the coordinate charts we let $\|u\|_{H^s(X\times\mathbf{R})} = \left(\sum \|\phi_j u\|_{H^s(\mathbf{R}_+^n\times\mathbf{R})}\right)^{\frac{1}{2}}$. $H^s(X^\wedge)$ is the space of restrictions of distributions in $H^s(X \times \mathbf{R})$ to $X \times \mathbf{R}_+$.

3.1.9 Proposition. (a) *For* $s \in \mathbf{Z}, \gamma \in \mathbf{R}$, $\mathcal{H}^{s,\gamma}(X^\wedge)$ *is the completion of* $C_0^\infty(\overline{X}^\wedge), \overline{X}^\wedge = \overline{X} \times \mathbf{R}_+$ *with respect to the norm*

$$\|u\|'_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \left\{\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|[R_-^s(\operatorname{Im} z)]_+ Mu(z)\|^2_{L^2(X)}\,|dz|\right\}^{\frac{1}{2}}, \tag{1}$$

where $\{R_-^\mu : \mu \in \mathbf{Z}\}$ *is the special parameter-dependent reduction of the order with the transmission property of* 2.3.10.
Recall that the operators R_-^s are defined by patching together corresponding operators on Euclidean space with a partition of unity and cut-off functions:

$$R_-^s f = \sum \phi_{j*} \operatorname{op} r_{-*}^s \psi_{j*},$$

where the r_-^s are as in 2.2.10.
We may replace the operator e^+ *in the definition of* $[R_-^s(\cdot)]_+$ *by any other extension operator modulo equivalent norms. Therefore expression (1) makes sense for all* $s \in \mathbf{Z}$, *not just* $s \in \mathbf{N}$.
Similarly

$$\|u\|'_{\mathcal{H}_0^{s,\gamma}(X^\wedge)} = \left\{\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|[R_+^s(\operatorname{Im} z)]_+ Mu(z)\|^2_{L^2(X)}\,|dz|\right\}^{\frac{1}{2}}, \tag{2}$$

with the corresponding family $\{R_+^\mu : \mu \in \mathbf{Z}\}$ *gives an equivalent norm on* $\mathcal{H}_0^{s,\gamma}(X^\wedge)$.
We may also replace the operators $[R_+^s(\cdot)]_+$ *in (2) by* $[R_-^{-s}]_+^{-1*}$, *consisting of the inverses of the formal* L^2 *adjoints of the operators* $[R_-^{-\mu}]_+$. *As we have noted in Remark* 3.1.4(a) *we could replace* M *in both cases by the weighted Mellin transform* $M_{\gamma-\frac{n}{2}}$.
(b) *The inner product in* 3.1.5(e) *extends from* $C_0^\infty(\overline{X}^\wedge) \times C_0^\infty(\overline{X}^\wedge)$ *to a non-degenerate sesquilinear form*

$$\mathcal{H}^{s,\gamma}(X^\wedge) \times \mathcal{H}_0^{-s,-\gamma}(X^\wedge) \to \mathbf{C}.$$

This admits the identification $\mathcal{H}_0^{-s,-\gamma}(X^\wedge) \cong (\mathcal{H}^{s,\gamma}(X^\wedge))'$. *Moreover,*

$$\|f\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \sup\left\{|(f,v)_{\mathcal{H}^{0,0}(X^\wedge)}| : \|v\|_{\mathcal{H}_0^{-s,-\gamma}(X^\wedge)} = 1\right\}$$

furnishes another equivalent norm on $\mathcal{H}^{s,\gamma}(X^\wedge)$.

Proof. (a) Since the symbol r_-^μ of R_-^μ belongs to H^- as a function of ξ_n, we have for any extension U of u

$$[R_-^\mu(\lambda)]_+ u \equiv \mathrm{r}^+ R_-^\mu(\lambda) U$$

modulo perturbations that are operators with norm $= O(\langle\lambda\rangle^{-N})$, where $N \in \mathbf{N}$ is arbitrarily large, cf. (2) in the proof of 2.2.10. Therefore the corresponding norms will be equivalent, provided we choose an additional parameter sufficiently large, cf. the argument in the proof of 3.1.6. Note that we have such a parameter by by 2.3.10.

Trivially, $\|[R_-^s(\mathrm{Im}\, z)]_+ Mu\|_{L^2(X)} \le \|R_-^s(\mathrm{Im}\, z)\mathrm{e}^+ Mu\|_{L^2(\Omega)}$. On the other hand, the calculus shows that, up to reflection at the boundary and regularizing terms, $\mathrm{r}^- R_-^s(\cdot)\mathrm{e}^+$ is a parameter-dependent singular Green operator in Boutet de Monvel's calculus of order s and type zero, cf. [9], Theorem 2.7.6. Hence $\mathrm{r}^- R_-^s(\cdot)\mathrm{e}^+ [R_-^s(\cdot)]_+^{-1}$ is a singular Green operator of order zero and type $(-s)_+$, and we also get the converse inequality, up to a constant independent of $\mathrm{Im}\, z$.

Finally we may replace the family $\{R_+^\mu : \mu \in \mathbf{Z}\}$ by $\{[R_-^{-\mu}]^{-1*} : \mu \in \mathbf{Z}\}$, because the latter operators are also parameter-elliptic of order μ, and, by duality, $[R_-^\mu]_+^* : H_0^s(X, V) \to H_0^{s-\mu}(X, V)$ also is invertible. The operator $[R_-^\mu]_+^{-1}$ differs from $[(R_-^\mu)^{-1}]_+$ only by a regularizing parameter-dependent pseudodifferential operator, and this is also true for the adjoints.

(b) In view of the identity $M(t^\gamma u)(z) = (Mu)(z + \gamma)$ we may suppose that $\gamma = 0$. Now the result follows from the last statement in (a) and the fact that

$$\begin{aligned}(Mu(z), Mv(z))_{L^2(X)} &= \Big(Mu(z), [R_-^\mu(\mathrm{Im}\, z)]_+^{-1}[R_-^\mu(\mathrm{Im}\, z)]_+ Mv(z)\Big)_{L^2(X)} \\ &= \Big([R_-^\mu(\mathrm{Im}\, z)]_+^{-1*} Mu(z), [R_-^\mu(\mathrm{Im}\, z)]_+ Mv(z)\Big)_{L^2(X)} \\ &\le \|[R_-^\mu(\mathrm{Im}\, z)]_+^{-1*} Mu(z)\|_{L^2(X)} \|[R_-^\mu(\mathrm{Im}\, z)]_+ Mv(z)\|_{L^2(X)}.\end{aligned}$$

◁

3.1.10 Theorem. *Let $s > \frac{1}{2}, \gamma \in \mathbf{R}, u \in \mathcal{H}^{s,\gamma}(\Omega^\wedge)$. Then the restriction $\gamma_0 u = u|_{Y^\wedge}$ of u to $Y^\wedge$ is well-defined and belongs to $\mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(Y^\wedge)$; the mapping*

$$\gamma_0 : \mathcal{H}^{s,\gamma}(\Omega^\wedge) \to \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(Y^\wedge)$$

is continuous. Clearly, the same assertion holds if we replace $\Omega^\wedge$ by $X^\wedge$.

Proof. By Corollary 3.1.6 and Remark 3.1.7 we may assume that $\Omega = \mathbf{R}^n, Y = \{x_n = 0\} \cong \mathbf{R}^{n-1}$. Relation 3.1.7(1) gives the assertion, if we use the standard restriction theorem for Sobolev spaces. The shift in γ simply is due to the fact that the dimension of Y is $n-1$. ◁

3.1.11 Corollary. Let $s > j + \frac{1}{2}, j \in \mathbf{N}, \gamma \in \mathbf{R}, u \in C_0^\infty(\Omega \times \mathbf{R})$. By r denote the normal coordinate in a neighborhood of Y. Then the operators $\gamma_j : u \mapsto \partial_r^j u|_{Y^\wedge}$ define continuous mappings

$$\gamma_j : \mathcal{H}^{s,\gamma}(\Omega^\wedge) \to \mathcal{H}^{s-j-\frac{1}{2},\gamma-\frac{1}{2}}(Y^\wedge).$$

In view of the definition of the spaces and their topology, this result extends to $\gamma_j : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s-j-\frac{1}{2},\gamma-\frac{1}{2}}(X^\wedge)$.
Proof. This follows from 3.1.9 in connection with the lemma, below. ◁

3.1.12 Lemma. *Choose a smooth function ϕ equal to 1 in a neighborhood of Y and supported in the neighborhood of Y, where the normal derivative is defined. Then the operator $f \mapsto \partial_r(\phi f)$, defined for $f \in C^\infty(\Omega^\wedge)$ has a bounded extension to an operator*

$$\mathcal{H}^{s,\gamma}(\Omega^\wedge) \to \mathcal{H}^{s-1,\gamma}(\Omega^\wedge).$$

Proof. This is a local result; it follows from 3.1.7(1) together with the fact that multiplication by ϕ is continuous on $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$. ◁

The following theorem states that the spaces $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$ are invariant under changes of coordinates if we restrict ourselves to the subspaces of functions with support in a compact set $\Omega \times \{t : 0 \le t \le R\}$, and if we ask that the diffeomorphism, say Φ, *respects the set* $\{t = 0\}$, i.e. Φ is the restriction of a diffeomorphism of $\Omega \times \overline{\mathbf{R}}_+$. In particular, we then have $\Phi(x, 0) \in \Omega \times \{0\}$.
Alternatively, we might ask that there are neighborhoods U, U' of $\overline{\mathbf{R}}_+$ in $\mathbf{R}$ such that the diffeomorphism is the restriction of a diffeomorphism $\Omega \times U \to \Omega \times U'$.

3.1.13 Theorem. *Let Φ be a diffeomorphism on $\Omega \times \mathbf{R}_+$, respecting $\{t = 0\}$. Then the space*

$$\{u \in \mathcal{H}^{s,\gamma}(\Omega^\wedge) : u = 0 \text{ on } \{t > R\} \text{ for suitable } R\}$$

is invariant under the change of coordinates induced by Φ.

Proof. In view of Corollary 3.1.6 and Remark 3.1.7 we may assume that we are given a distribution $v \in \mathcal{H}^{s,\gamma}(\mathbf{R}^n \times \mathbf{R}_+)$ with support in a bounded set and that Φ is a diffeomorphism of bounded open sets in $\mathbf{R}^n \times \overline{\mathbf{R}}_+$. Moreover, we may assume $s \ge 0$ using the duality in 3.1.5. By 3.1.7 we have

$$v(x,t) = t^{-\frac{n+1}{2}+\gamma} u(x, \ln t)$$

for some $u \in H^s(\mathbf{R}^n \times \mathbf{R})$. Clearly,

$$u(x,r) = e^{(\frac{n+1}{2}-\gamma)r} v(x, e^r)$$

by letting $r = \ln t$. Now write $(\underline{x}, \underline{t}) = \Phi(x,t) = (\Phi_1(x,t), \Phi_2(x,t))$ so that the transformed function $\underline{v}$ is given by

$$\begin{aligned} \underline{v}(x,t) &= v(\Phi(x,t)) = v(\underline{x}, \underline{t}) = \underline{t}^{-\frac{n+1}{2}+\gamma} u(\underline{x}, \ln \underline{t}) \\ &= \Phi_2(x,t)^{-\frac{n+1}{2}+\gamma} u(\Phi_1(x,t), \ln \Phi_2(x,t)). \end{aligned}$$

In order to show the invariance under coordinate transforms, it is sufficient to show two facts. First, there is a function $\tilde{\Phi}_2(x,t) \in C_b^\infty(\Omega^\wedge)$ (i.e. all derivatives are bounded) such that

$$\Phi_2(x,t) = \tilde{\Phi}_2(x,t) \cdot t,$$

with

$$c \leq \tilde{\Phi}_2(x,t) \leq c^{-1} \tag{1}$$

with a constant $c > 0$ independent of x. Second, the function

$$\underline{u}(x,r) = u(\Phi_1(x,e^r), \ln \Phi_2(x,e^r))$$

belongs to $H^s(\mathbf{R}^{n+1})$.

Now it is well-known, cf. [15], that $H^s(\mathbf{R}^{n+1})$ is invariant under all coordinate transformations Ψ satisfying

$$\partial^\alpha \Psi \in C_b^\infty(\mathbf{R}^{n+1}), \qquad \alpha \neq 0 \tag{2}$$

and

$$c \leq |\det(D\Psi)| \leq c^{-1}, \qquad c > 0. \tag{3}$$

The change of coordinates we have to consider is

$$\Psi(x,r) = (\Phi_1(x,e^r), \ln \Phi_2(x,e^r)). \tag{4}$$

Let us check that it does satisfy conditions (1), (2), and (3). We may restrict our attention to bounded x and bounded t, equivalently $-\infty < r < c_0, c_0 \in \mathbf{R}$.

The diffeomorphism Φ respects $\{t = 0\}$. Therefore

$$\begin{aligned} \Phi_2(x,t) &= \Phi_2(x,t) - \Phi_2(x,0) \\ &= \int_0^1 \partial_t \Phi_2(x,\tau t) d\tau \cdot t \\ &= \tilde{\Phi}_2(x,t)t \end{aligned}$$

with a smooth function $\tilde{\Phi}_2(x,t)$, bounded in all derivatives, since our parameter space is bounded. By considering the inverse we see that $\tilde{\Phi}_2(x,t)$ also is bounded away from zero. This gives (1).

On the bounded parameter space, Φ_1 is bounded in all non-zero derivatives; therefore also $(x,r) \mapsto \Phi_1(x,e^r)$ is bounded in all derivatives. Let us show that the same is true for $\Psi_2 : (x,r) \mapsto \ln \Phi_2(x,e^r)$:

$$\partial_{x_j} \Psi_2(x,r) = \frac{\partial_{x_j} \Phi_2(x,e^r)}{\Phi_2(x,e^r)}.$$

This is bounded, since $\Phi_2(x,t) \geq ct$, while $\partial_{x_j} \Phi_2(x,t) = \partial_{x_j} \tilde{\Phi}_2(x,t)t$; similarly,

$$\partial_r \Psi_2(x,r) = \frac{\partial_t \Phi_2(x,e^r)e^r}{\Phi_2(x,e^r)}$$

is bounded.

In view of the quotient rule for differentiation, we conclude that all derivatives are bounded.

Finally, the Jacobian is

$$\begin{aligned} |\det D\Psi(x,r)| &= \begin{vmatrix} \partial_x \Phi_1(x,e^r) & \partial_t \Phi_1(x,e^r)e^r \\ \partial_x \Phi_2(x,e^r)\frac{1}{\Phi_2(x,e^r)} & \partial_t \Phi_2(x,e^r)\frac{e^r}{\Phi_2(x,e^r)} \end{vmatrix} \\ &= \left|\frac{e^r}{\Phi_2(x,e^r)}\right| |\det D\Phi(x,e^r)| \end{aligned}$$

which is both bounded and bounded away from zero because of property (1) and the fact that we are considering Φ on a bounded set. $\triangleleft$

We have not used that we have a diffeomorphism of the entire manifold $\Omega^\wedge$. The result is also true, if $\Phi : U_1 \to U_2$ is a diffeomorphism of open subsets $U_1, U_2 \subseteq \Omega^\wedge$ and respects $\{t = 0\}$ in the sense that Φ extends to a diffeomorphism of the closure of U_1 and U_2 in $\Omega \times \mathbf{R}$.
We say that a diffeomorphism Φ of $X^\wedge$ is *boundary-preserving* if there are open neighborhoods U_1, U_2 of $X^\wedge$ in $\Omega^\wedge$, and Φ extends to a diffeomorphism $\Phi : U_1 \to U_2$ respecting $\{t = 0\}$.

This immediately leads to the following corollary.

3.1.14 Corollary. Also the subspace of $\mathcal{H}^{s,\gamma}(X^\wedge)$ consisting of the distributions that vanish for large t is invariant under changes of coordinates induced by boundary-preserving diffeomorphisms.

3.1.15 Definition. Let $\mathcal{F}$ be a subspace of $\mathcal{D}'(X^\wedge)$ or $\mathcal{D}'(\Omega^\wedge)$ with a stronger topology. Suppose that ϕ is a smooth function on $\overline{\mathbf{R}}_+$ and that multiplication by ϕ is continuous on $\mathcal{F}$. Then $[\phi]\mathcal{F}$ denotes the closure of the space $\{\phi u : u \in \mathcal{F}\}$ in $\mathcal{F}$.

3.1.16 Theorem. *Let $\omega \in C_0^\infty(\overline{\mathbf{R}}_+), \omega \equiv 1$ near zero. Then for $s \geq s', \gamma \geq \gamma'$*

$$[\omega]\mathcal{H}^{s,\gamma}(X^\wedge) \hookrightarrow [\omega]\mathcal{H}^{s',\gamma'}(X^\wedge)$$

is continuous. For $s > s', \gamma > \gamma'$ the embedding

$$[\omega]\mathcal{H}^{s,\gamma}(X^\wedge) \hookrightarrow [\omega]\mathcal{H}^{s',\gamma'}(X^\wedge)$$

is compact.

Proof. This is immediate from 3.1.7(1) together with the embedding results for the Sobolev spaces. $\triangleleft$

3.1.17 Proposition. *Let $\{R_-^\mu : \mu \in \mathbf{Z}\}$ be the parameter-dependent order-reducing family in Boutet de Monvel's calculus in 2.3.10, $\gamma, s \in \mathbf{R}$. Denote by $M_{\gamma-\frac{n}{2}}$ the weighted Mellin transform of 5.1.5.*
Then the Mellin operator $\operatorname{op}_M^{\gamma-\frac{n}{2}}[R_-^\mu(\operatorname{Im} z)]_+ : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s-\mu,\gamma}(X^\wedge)$ given by

$$\operatorname{op}_M^{\gamma-\frac{n}{2}}[R_-^\mu(\operatorname{Im} z)]_+ f = M_{\gamma-\frac{n}{2}}^{-1}[R_-^\mu(\operatorname{Im} z)]_+ M_{\gamma-\frac{n}{2}} f$$

is an isomorphism.
Note: *More on Mellin operators in Section* 4.1.

Proof. By interpolation we can confine ourselves to the case where $s \in \mathbf{Z}$. We have according to 3.1.9

$$\begin{aligned}
&\|\operatorname{op}_M^{\gamma-\frac{n}{2}}[R_-^\mu(\operatorname{Im} z)]_+ f\|^2_{\mathcal{H}^{s-\mu,\gamma}(X^\wedge)} \\
= &\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|[R_-^{s-\mu}(\operatorname{Im} z)]_+ M_{\gamma-\frac{n}{2}} M^{-1}_{\gamma-\frac{n}{2}}[R_-^\mu(\operatorname{Im} z)]_+ M_{\gamma-\frac{n}{2}} f)\|^2_{L^2(X)}|dz| \\
= &\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|[R_-^{s-\mu}(\operatorname{Im} z)]_+[R_-^\mu(\operatorname{Im} z)]_+ M_{\gamma-\frac{n}{2}} f\|^2_{L^2(X)}|dz| \\
\le\ &C\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|[R_-^s(\operatorname{Im} z)]_+ M_{\gamma-\frac{n}{2}} f\|^2_{L^2(X)}|dz| = \|f\|_{\mathcal{H}^{s,\gamma}(X^\wedge)};
\end{aligned}$$

the constant in the final estimate is the norm of

$$[R_-^{s-\mu}(\operatorname{Im} z)]_+[R_-^\mu(\operatorname{Im} z)]_+[R_-^s(\operatorname{Im} z)]_+^{-1}$$

in $\mathcal{L}(L^2(X))$. ◁

3.1.18 Definition. For $s, \gamma \in \mathbf{R}, \omega$ a cut-off function on $\mathbf{R}_+$, let

$$\mathcal{K}^{s,\gamma}(X^\wedge) = \{u \in \mathcal{D}'(X^\wedge) : \omega u \in \mathcal{H}^{s,\gamma}(X^\wedge), (1-\omega)\, t^{\frac{n}{2}}\, u \in H^s(X^\wedge)\}. \tag{1}$$

Here, $H^s(X^\wedge)$ is as in 3.1.8. The definition is independent of the choice of ω by 3.1.5(b). In the notation of 3.1.15,

$$\mathcal{K}^{s,\gamma}(X^\wedge) = [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]\, t^{-\frac{n}{2}}\, H^s(X^\wedge); \tag{2}$$

similarly,

$$\mathcal{K}_0^{s,\gamma}(X^\wedge) = [\omega]\mathcal{H}_0^{s,\gamma}(X^\wedge) + [1-\omega]\, t^{-\frac{n}{2}}\, H_0^s(X^\wedge), \tag{3}$$

cf. 3.1.3(c). In fact, the left hand side clearly is contained in the sum of spaces on the right hand side of (2). On the other hand, if $u_n \in \omega\mathcal{H}^{s,\gamma}(X^\wedge)$ converges to u in $\mathcal{H}^{s,\gamma}(X^\wedge)$, then we have $\tilde{\omega}u = u$ for all cut-off functions $\tilde{\omega}$ equal to 1 in a sufficiently large neighborhood of $\{t = 0\}$; in particular, $\tilde{\omega}u = u \in \mathcal{H}^{s,\gamma}(X^\wedge)$, and $(1-\tilde{\omega})u = 0 \in H^s(X^\wedge)$, so $u \in \mathcal{K}^{s,\gamma}(X^\wedge)$. If $v_n \in (1-\omega)H^s(X^\wedge)$ tends to v in $H^s(X^\wedge)$, then a similar argument shows that $v \in \mathcal{K}^{s,\gamma}(X^\wedge)$. Therefore we have equality in (2).
We shall give $\mathcal{K}^{s,\gamma}(X^\wedge)$ the Banach topology induced by (2):

$$\|u\|_{\mathcal{K}^{s,\gamma}(X^\wedge)} = \|\omega u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} + \|(1-\omega)\, t^{\frac{n}{2}}\, u\|_{H^s(X^\wedge)}.$$

Notice that $\mathcal{K}^{0,\frac{n}{2}}(X^\wedge) = L^2(X^\wedge)$.
This also allows us to introduce the space $\mathcal{H}^{s,\gamma}(\mathbb{D})$: Near each singularity v, $\mathbb{D}$ is diffeomorphic to $X_v^\wedge$, with suitable X_v as in 1.1.1. We define $\mathcal{H}^{s,\gamma}(\mathbb{D})$ as the space of all distributions belonging to $\mathcal{H}^{s,\gamma}(X_v^\wedge)$ near a singularity v and belonging to $H^s(\mathbb{D})$ in the interior; for the precise construction use a cut-off function ω_v near each singularity v.

3.1.19 Remark. (Non-direct sums of Fréchet spaces) Let E, F be Fréchet spaces and suppose both are continuously embedded in the same Hausdorff vector space.
Then we may form the exterior direct sum $E \oplus F$, which is Fréchet and has the closed subspace $\Delta = \{(a, -a) : a \in E \cap F\}$. The non-direct sum of E and F then is the Fréchet space

$$E + F := E \oplus F/\Delta.$$

3.1.20 Definition. Let Θ be the interval $(\theta, 0], \theta < 0$, and let $s, \gamma \in \mathbf{R}$.
$\mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$ is defined as the intersection $\bigcap_{\epsilon>0} \mathcal{K}^{s,\gamma-\theta-\epsilon}(X^\wedge)$. We endow this space with the projective limit topology.
For $\Theta = (-\infty, 0]$ define $\mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$ as the intersection of all the above spaces for $\theta < 0$.

3.1.21 Remark. (a) Let $u \in \mathcal{K}^{s,\gamma}(X^\wedge), s > \frac{1}{2}$. Then the restriction $u|_Y$ belongs to $\mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(Y^\wedge)$: This is immediate from Corollary 3.1.11 and the definition.
(b) In view of 3.1.5(e) we obtain natural dualities

$$\mathcal{K}^{s,\gamma}(X^\wedge)' \cong \mathcal{K}_0^{-s,-\gamma}(X^\wedge)$$

and

$$\mathcal{K}_0^{s,\gamma}(X^\wedge)' \cong \mathcal{K}^{-s,-\gamma}(X^\wedge)$$

for all $s, \gamma \in \mathbf{R}$.
(c) Let ϕ be as in 3.1.5(g). Then the multiplication operator

$$M_\phi : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge)$$

and

$$M_\phi : \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) \to \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$$

is continuous.
(d) Of course, all these distributions may take values in finite-dimensional vector bundles with a Hermitean structure which are restrictions of smooth Hermitean bundles on $\Omega \times \mathbf{R}$.

3.2 Spaces with Asymptotics

Throughout this section, X and Y will denote the manifolds of Section 1.

3.2.1 Definition. cf. [29], 1.1.2, Definition 1.
(a) A *weight datum* $g = (\gamma, \Theta)$ consists of a number $\gamma \in \mathbf{R}$ and an interval $\Theta = (\theta, 0]$ with $-\infty \leq \theta < 0$.
(b) Given a weight datum $g = (\gamma, \theta), S_\Theta^\gamma$ denotes the strip

$$S_\Theta^\gamma = \{z \in \mathbf{C} : \frac{1}{2} - \gamma + \theta < \operatorname{Re} z \leq \frac{1}{2} - \gamma\}.$$

(c) The *collection of asymptotic types* $As(X,g)$ for a weight datum $g = (\gamma,(\theta,0])$ with $\theta > -\infty$ ("finite weight interval") is the set of all finite vectors

$$P = \{(p_j, m_j, L_j) : j = 0, \dots, N(P) \in \mathbf{N}\}$$

consisting of

(i) $p_j \in \operatorname{int} S_\Theta^{\gamma-\frac{n}{2}}$, where $n = \dim X$,

(ii) $m_j \in \mathbf{N}$, and

(iii) L_j a finite-dimensional subspace of $C^\infty(\overline{X})$.

The elements P of $As(X,g)$ are called *asymptotic types.*
If g is a weight datum with $\theta = -\infty$, ("infinite weight interval") then $As(X,g)$ is the family of all vectors $P = \{(p_j, m_j, L_j) : j = 0, \dots, N(P) \le \infty\}$ with the additional assumption that

(iv) $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$, whenever P is infinite.

By $\pi_{\mathbf{C}} P$ denote the set $\{p_j : j = 0, \dots, N(P)\}$.
Correspondingly, $As(Y,g)$ is the set of all $P = \{(p_j, m_j, L_j) : j \in \mathbf{N}\}$ with $p_j \in \operatorname{int} S_\Theta^{\gamma-\frac{n-1}{2}}, m_j \in \mathbf{N}$, and L_j a finite-dimensional subspace of $C^\infty(Y)$. As before we assume that $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ whenever P is infinite. Finally we let for $g = (\gamma, \Theta)$

$$As(X,Y,g) = \{P = (P_1, P_2) : P_1 \in As(X,g), P_2 \in As(Y, (\gamma - \frac{1}{2}, \Theta))\}.$$

(d) The space $\mathcal{K}_P^{s,\gamma}(X^\wedge)$, for $P = \{(p_j, m_j, L_j) : j = 0, \dots, N\} \in As(X,g)$ with finite weight interval consists of all $u = u(x,t) \in \mathcal{K}^{s,\gamma}(X^\wedge)$ such that for suitable $c_{jk} \in L_j, 0 \le j \le N, 0 \le k \le m_j$, and all cut-off functions near zero, ω,

$$u - \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk}(x) t^{-p_j} \ln^k t \, \omega(t) \in \mathcal{K}_\Theta^{s,\gamma}(X^\wedge);$$

cf. 3.1.20 for the definition of $\mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$. In the case of an infinite weight interval first let $g_k = (\gamma, (-k, 0]), k = 1, 2, \dots,$ and define $P_k \in As(X,g)$ by

$$P_k = \{(p_j, m_j, L_j) \in P : \frac{n+1}{2} - \gamma - k < \operatorname{Re} p_j \le \frac{n+1}{2} - \gamma\}.$$

Then let

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \bigcap_k \mathcal{K}_{P_k}^{s,\gamma}(X^\wedge). \tag{1}$$

$\mathcal{K}_P^{\infty,\gamma}(X^\wedge)$ is the intersection of all $\mathcal{K}_P^{s,\gamma}(X^\wedge), s \in \mathbf{R}$.
(e) For a finite weight interval g and $P \in As(X,g)$ let $\mathcal{E}_P(X^\wedge)$ be the space of all functions

$$\{u \in C^\infty(X^\wedge) : u(x,t) = \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk} t^{-p_j} \ln^k t : c_{jk} \in L_j\}.$$

3.2.2 Remark. It is obvious from the considerations in 3.1.13 that the representation of a function in the form

$$u(x,t) = \sum_{j=0}^{N}\sum_{k=0}^{m_j} c_{jk}(x)\, t^{-p_j} \ln^k t\, \omega(t) + f(x,t) \tag{1}$$

with $f \in \mathcal{K}_{\Theta}^{s,\gamma}(X^\wedge)$ as in 3.2.1(d) depends on the particular choice of coordinates. In order for the definition to make sense we shall check that under a change of coordinates, the function $\sum_{j=0}^{N}\sum_{k=0}^{m_j} c_{jk}(x)\, t^{-p_j} \ln^k t\, \omega(t)$ transforms to a function $\sum_{j=0}^{N'}\sum_{k=0}^{m_j'} c_{jk}'(x)\, t^{-p_j'} \ln^k t\, \omega'(t)$ $+g(x,t)$ with $g \in \mathcal{K}^{\infty,M}(X^\wedge)$ for arbitrarily large M. As indicated by the use of N' and p_j', there may be more and different exponents in the resulting representation. We shall see, however, that all p_j' are of the form $p_k - l$, for a suitable p_k and $l \in \mathbf{N}$. Moreover, we shall check that if the c_{jk} vary over a finite-dimensional subspace of $C^\infty(X)$, then so will the c_{jk}'.
We will use the notation of 3.1.13. The change of coordinates is $(\underline{x},\underline{t}) = \Phi(x,t) = (\Phi_1(x,t), \Phi_2(x,t))$, where $\Phi_2(x,t) = \tilde{\Phi}_2(x,t)\cdot t$ with a function $\tilde{\Phi}_2(x,t) \in C_b^\infty(\Omega \times \overline{\mathbf{R}}_+)$, satisfying $c \le \tilde{\Phi}_2(x,t) \le c^{-1}$ for a constant $c > 0$. Recall that we are only interested in the case $t \in (0,T], T < \infty$. Now we consider the various terms separately. Fix an arbitrary $M \in \mathbf{N}$.

(i) We have $\underline{t}^{-p_j} = t^{-p_j}\tilde{\Phi}_2(x,t)^{-p_j}$. The second factor is a smooth function up to $t = 0$ thus has a Taylor expansion

$$\tilde{\Phi}_2(x,t)^{-p_j} = \sum_{k=0}^{M-1} d_{jk}(x)\, t^j + t^M\, d_j(x,t),$$

where $d_j \in C_b^\infty(X \times \overline{\mathbf{R}}_+)$.

(ii) Similarly, $\ln^k \underline{t} = [\ln t + \ln \tilde{\Phi}_2(x,t)]^k$. A Taylor expansion of $\ln \tilde{\Phi}_2(x,t)$ (which is smooth up to $t = 0$) then yields a linear combination of terms of the form $e_{jl}(x)t^j \ln^l t$ with $j = 0,\ldots,M-1, l = 0,\ldots,k$ and smooth e_{jl}, plus a remainder of the form $t^M e_k(x,t)$ with $e_k \in C_b^\infty(X \times \overline{\mathbf{R}}_+)$.

(iii) Finally, we use a Taylor expansion for $c_{jk}(\Phi_1(x,t))$ at $t = 0$ which yields a finite sum of terms of the form $f_{jkl}(x)t^l$ with $l = 0,\ldots,M-1$, smooth f_{jkl}, and a remainder of the form $t^M f_{jk}(x,t)$ with $f_{jk} \in C_b^\infty(X \times \overline{\mathbf{R}}_+)$.

Writing out the product, we obtain the assertion. Notice that the finite-dimensional spaces in the asymptotic type can be replaced by other finite-dimensional spaces; the corresponding changes can be read off from the above Taylor expansions.
Spaces with asymptotics are therefore well-defined if we either keep coordinates fixed or else interpret the subscript P associated with an asymptotic type P as an equivalence class of of possible asymptotic types. This is the sense in which all the notation involving asymptotic types should be understood.

3.2.3 Lemma. (a) *The space $\mathcal{E}_P(X^\wedge)$ in 3.2.1(e) (finite weight interval) is finite-dimensional, and we have*

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \mathcal{K}_{\Theta}^{s,\gamma}(X^\wedge) + [\omega]\mathcal{E}_P(X^\wedge) \tag{1}$$

for every cut-off function ω near zero.
(b) *For a finite weight interval we may make $\mathcal{K}_P^{s,\gamma}(X^\wedge)$ a Fréchet space by endowing it with the topology of the sum of Fréchet spaces in* (1).
(c) *For an infinite weight interval $\mathcal{K}_P^{s,\gamma}(X^\wedge)$ becomes a Fréchet space, if we give it the projective topology induced by* (b) *and* 3.2.1(1)

Proof. This is immediate from the definition. ◁

3.2.4 Theorem. *Let $P = \{(p_j, m_j, L_j)\}_{j=0}^{\infty} \in As(X, (\gamma, (-\infty, 0]))$, and let $c_{jk} \in L_j, j \in \mathbf{N}, k = 0, \ldots, m_j$. Then there is a distribution $u \in \mathcal{K}_P^{\infty,\gamma}(X^\wedge)$ with the following property. For every $\theta < 0$ there is an $N = N(\theta)$ such that*

$$u - \sum_{j=0}^{N} c_{jk}(x) t^{-p_j} \ln^k t\, \omega(t) \in \mathcal{K}_\Theta^{s,\gamma}(X^\wedge),$$

$\Theta = (\theta, 0]$. In particular,

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \mathcal{K}_{(-\infty,0]}^{s,\gamma}(X^\wedge) + \mathcal{K}_P^{\infty,\gamma}(X^\wedge).$$

For the proof of Theorem 3.2.4 we shall employ the following lemma.

3.2.5 Lemma. *Fix a cut-off function, ω, near zero, $p \in \mathbf{C}, k \in \mathbf{N}$. Consider the function*

$$f(\sigma, t) = t^{-p} \ln^k t\, \omega(\sigma t)$$

for $\sigma, t > 0$.
(a) *For every γ with $\operatorname{Re} p + \gamma - \frac{1}{2} < 0$ and every $s \geq 0$,*

$$\|f(\sigma, t)\|_{\mathcal{H}^{s,\gamma}(\mathbf{R}_+)} \to 0 \quad \text{as} \quad \sigma \to \infty.$$

(b) *If $c \in C^\infty(X)$ and $\operatorname{Re} p + \gamma - \frac{n+1}{2} < 0$, then*

$$\|c(x) f(\sigma, t)\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} \to 0 \quad \text{as} \quad \sigma \to \infty.$$

Proof. (a) By interpolation we only have to consider the case $s = l \in \mathbf{N}$; using 3.1.4(b), we simply have to estimate $t^{-\gamma}(t\partial_t)^j f(\sigma, t)$ in $L^2(\mathbf{R}_+)$ for $j = 0, \ldots, l$. This is elementary.
(b) The argument is almost the same; we now have to consider $t^{\frac{n}{2}-\gamma}(t\partial_t)^j f(\sigma, t) \partial_x^\alpha c$ in $L^2(X^\wedge)$ for $|\alpha| + j \leq l$. ◁

Proof of Theorem 3.2.4. Choose an increasing sequence $\gamma_j \to \infty$ with $\operatorname{Re} p_r + \gamma_j - \frac{n+1}{2} < 0$ for all $r \geq j$. Using Lemma 3.2.5 we may choose $\sigma_{jk} > 1$ such that

$$\|c_{jk} t^{-p_j} \ln^k t\, \omega(\sigma_{jk} t)\|_{\mathcal{H}^{j,\gamma_j}(X^\wedge)} \leq 2^{-j}/(m_j + 1). \tag{1}$$

Let us check that

$$\sum_{j=0}^{\infty}\sum_{k=0}^{m_j} c_{jk}t^{-p_j}\ln^k t\,\omega(\sigma_{jk}t) \tag{2}$$

converges in $\mathcal{K}_P^{\infty,\gamma}(X^\wedge)$, i.e. in each $\mathcal{K}_P^{s,\gamma}(X^\wedge), s \in \mathbf{R}$. This means we have to show convergence in $\mathcal{K}_{(\theta,0]}^{s,\gamma}(X^\wedge) + [\omega]\mathcal{E}_{P_\theta}(X^\wedge)$ for all $\theta < 0$. Here, P_θ is the set

$$\{(p_j, m_j, L_j) : \frac{n+1}{2} - \gamma + \theta < \operatorname{Re} p_j \leq \frac{n+1}{2} - \gamma\}.$$

Without loss of generality we may use the same ω as above.
To this end choose j_0 so large that $\gamma_{j_0} > \gamma - \theta$ and $j_0 > s$. Rewrite (2) as three sums:

$$\sum_{\{j<j_0:\operatorname{Re} p_j > \frac{n+1}{2}-\gamma+\theta\}} \ldots + \sum_{\{j<j_0:\operatorname{Re} p_j \leq \frac{n+1}{2}-\gamma+\theta\}} \ldots + \sum_{j\geq j_0} \ldots.$$

The finitely many terms in the first sum belong to $\mathcal{E}_{P_\theta}(X^\wedge)$, those in the (finite) second summation to $\mathcal{K}_{(\theta,0]}^{s,\gamma}(X^\wedge)$. For $j \geq j_0$, relation (1) implies that

$$\sum_{k=0}^{m_j} \|c_{jk}t^{-p_j}\ln^k t\,\omega(\sigma_{jk}t)\|_{\mathcal{K}^{s,\gamma-\theta}(X^\wedge)} \leq 2^{-j},$$

noting that, on these functions, the $\mathcal{H}-$norms and the $\mathcal{K}-$ norms coincide. Therefore, the third summation converges in $\mathcal{K}_{(\theta,0]}^{s,\gamma}(X^\wedge)$.
So, if $u \in \mathcal{K}_P^{s,\gamma}(X^\wedge)$, and $u_0 \in \mathcal{K}_P^{\infty,\gamma}(X^\wedge)$ is the function in (2), then $u - u_0 \in \mathcal{K}_{(\theta,0]}^{s,\gamma}(X^\wedge)$ for every $\theta < 0$. In fact, let $u_N = \sum_{j=0}^{N}\sum_{k=0}^{m_j} c_{jk}t^{-p_j}\ln^k t\,\omega(\sigma_{jk}t)$. Then $u - u_0 = (u - u_N) - (u_0 - u_N)$. For large N, the above argument shows that $u_0 - u_N \in \mathcal{K}_{(\theta,0]}^{s,\gamma}(X^\wedge)$, while $u - u_N \in \mathcal{K}_{(\theta,0]}^{s,\gamma}(X^\wedge)$ in view of Definition 3.2.3(d), since the finitely many terms of the form

$$c_{jk}t^{-p_j}\ln^k t\,(\omega(t) - \omega(\sigma_{jk}t))$$

belong to $\mathcal{K}^{\infty,\delta}(X^\wedge)$ for all δ. We obtain the assertion. ◁

3.2.6 Definition. cf. [29], 1.1.1, Definition 4. Let E be a Fréchet space.
(a) For an open subset U of $\mathbf{C}$ let $\mathcal{A}(U,E) = \mathcal{A}(U)\hat{\otimes}_\pi E$ denote all holomorphic functions on U with values in E.
(b) Let $g = (\gamma, \Theta)$ be a weight datum with finite or infinite weight interval and $P \in As(X, g)$.
Then $\mathcal{A}_P^{s,\gamma}(X^\wedge)$ is the space of all holomorphic functions f in the interior of $S_\Theta^{\gamma-\frac{n}{2}} \setminus \pi_{\mathbf{C}}P$ with values in the space $H^s(X)$ and the following properties

(i) In $p_j \in \pi_{\mathbf{C}}P, f$ has a pole of order $m_j + 1$ and a Laurent expansion

$$f(z) = \sum_{k=0}^{m_j} c_{jk}(z - p_j)^{-k-1} + \tilde{f}(z)$$

with $c_{jk} \in L_j$ and $\tilde{f}$ holomorphic near p_j.

(ii) For $\epsilon > 0$ choose a function $\chi_\epsilon \in C^\infty(\mathbf{C})$, vanishing in an $\epsilon-$neighborhood of $\pi_{\mathbf{C}}P$ and equal to 1 outside a $2\epsilon-$neighborhood of $\pi_{\mathbf{C}}P$.
For $\beta \in \mathbf{R}$ define the semi-norms

$$\|u\|_{s,\beta} = \|M^{-1}_{\beta-\frac{n}{2}}u\|_{\mathcal{H}^{s,\beta}(X^\wedge)} \qquad (1)$$

using the 'weighted' Mellin transform M_β, cf. 5.1.5.
We now ask that for every β with $\gamma \le \beta < \gamma - \theta$ and every $\epsilon > 0$,

$$\|\chi_\epsilon f\|_{s,\beta} < \infty, \qquad (2)$$

uniformly for β in compact intervals.

(c) Let $\mathcal{A}^{\infty,\gamma}_P(X^\wedge) = \bigcap_s \mathcal{A}^{s,\gamma}_P(X^\wedge)$.
(d) For a weight datum $g = (\gamma, \Theta)$ and an 'empty' vector $P = \emptyset$ let

$$\mathcal{A}^{s,\gamma}_\Theta(X^\wedge) = \mathcal{A}^{s,\gamma}_\emptyset(X^\wedge).$$

3.2.7 Lemma. (a) *$\mathcal{A}^{s,\gamma}_P(X^\wedge)$ is a Fréchet space with the topology induced from*

(i) *the topology of $\mathcal{A}(S^{\gamma-\frac{n}{2}}_\Theta \backslash \pi_{\mathbf{C}}P, H^s(X))$*

(ii) *the countable set of semi-norms induced by* 3.2.6(2).

For $\mathcal{A}^{\infty,\gamma}_P(X^\wedge)$ use the projective topology.
(b) *We then have*

$$\mathcal{A}^{s,\gamma}_P(X^\wedge) = \mathcal{A}^{s,\gamma}_\Theta(X^\wedge) + \mathcal{A}^{\infty,\gamma}_P(X^\wedge).$$

3.2.8 Theorem. *Let ω be a cut-off function near zero. The weighted Mellin transform $M_{\gamma-\frac{n}{2}} : C^\infty_0(X^\wedge) \to \mathcal{S}(\Gamma_{\frac{n+1}{2}-\gamma}, C^\infty(X))$, cf.* 5.1.5, *extends to continuous operators*

(i) $M_{\gamma-\frac{n}{2}} : [\omega]\mathcal{K}^{s,\gamma}_\Theta(X^\wedge) \to \mathcal{A}^{s,\gamma}_\Theta(X^\wedge)$,
(ii) $M_{\gamma-\frac{n}{2}} : [\omega]\mathcal{E}^\gamma_P(X^\wedge) \to \mathcal{A}^{\infty,\gamma}_P(X^\wedge)$,
(iii) $M_{\gamma-\frac{n}{2}} : [\omega]\mathcal{K}^{s,\gamma}_P(X^\wedge) \to \mathcal{A}^{s,\gamma}_P(X^\wedge)$.

Vice versa, if $u \in \mathcal{A}^{s,\gamma}_\Theta(X^\wedge)$, then 3.2.6(b) *implies that $M^{-1}_{\gamma-\frac{n}{2}}u \in \mathcal{H}^{s,\gamma-\delta}(X^\wedge), \theta < \delta \le 0$.*

Proof. cf. 5.1.6. (i) is just the definition, (ii) follows from the fact that the Mellin transform of the function

$$u(t) = \omega(t)t^{-p}\ln^k t$$

is meromorphic in the plane with a single pole in p of order $k+1$ and that $\chi_\epsilon(z)(1+|z|^2)^s Mu(z)$ is L^2 on the line Γ_β for all β and s.
Finally, (iii) follows from (ii) by linearity. ◁

3.2.9 Definition. Let $P \in As(X,g), g = (\gamma, \Theta)$. Then

$$\mathcal{S}^\gamma_P(X^\wedge) = [\omega]\mathcal{K}^{\infty,\gamma}_P(X^\wedge) + [1-\omega]\mathcal{S}(X^\wedge).$$

Remember that everything depends on the choice of Θ.

3.2.10 Lemma. *Let $\tilde{\phi} \in C^\infty(\Omega \times \mathbf{R}), \phi = \tilde{\phi}|_{X^\wedge}$. Then the multiplication operator*

$$M_\phi : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge)$$

is bounded. If $P \in As(X, g)$ satisfies the "shadow condition" (i.e. given a triple $(p, m, L) \in P$ and $j \in \mathbf{N}$, there is an element $(p - j, m(j), L(j)) \in P$ with $m(j) \geq m, L(j) \supseteq L$) then also

$$M_\phi : \mathcal{K}_P^{s,\gamma}(X^\wedge) \to \mathcal{K}_P^{s,\gamma}(X^\wedge)$$

is continuous.

Proof. The first part is immediate from 3.1.21(c). In order to obtain the second statement, use a Taylor expansion of ϕ at $t = 0$. ◁

3.2.11 Remark. Of course, all notions make sense for distributions with values in finite-dimensional Hermitean vector bundles which are smooth up to the boundary.

3.3 Green Operators. The Algebras $C_G(X^\wedge, g)$ and $C_G(\mathbb{D}, g)$

3.3.1 Definition. Let $g = (\gamma, \delta, \Theta)$ with $\gamma, \delta \in \mathbf{R}, \Theta = (\theta, 0], -\infty \leq \theta < 0; g$ is called a 'double' weight datum. Moreover, let P, Q be two asymptotic types, $P = (P_1, P_2) \in As(X, Y, (\delta, \Theta)), Q = (Q_1, Q_2) \in As(X, Y, (-\gamma, \Theta))$, and $V_1, V_2, \ldots$ smooth Hermitean vector bundles.

(a) Let

$$G \in \bigcap_s \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge, V_1) \oplus \mathcal{K}^{s,\gamma-\frac{1}{2}}(Y^\wedge, V_3), \mathcal{K}^{\infty,\delta}(X^\wedge, V_2) \oplus \mathcal{K}^{\infty,\delta-\frac{1}{2}}(Y^\wedge, V_4)).$$

We shall write $G \in C_G^0(X^\wedge, g)_{P,Q}$ if the following holds: for all $s \geq 0$

$$G = \begin{bmatrix} G_G & G_K \\ G_T & G_S \end{bmatrix} : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2}}(Y^\wedge, V_3) \end{array} \to \begin{array}{c} \mathcal{S}_{P_1}^{\delta}(X^\wedge, V_2) \\ \oplus \\ \mathcal{S}_{P_2}^{\delta-\frac{1}{2}}(Y^\wedge, V_4) \end{array} \tag{1}$$

and

$$G^* : \begin{array}{c} \mathcal{K}^{s,-\delta}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}^{s,-\delta-\frac{1}{2}}(Y^\wedge, V_4) \end{array} \to \begin{array}{c} \mathcal{S}_{Q_1}^{-\gamma}(X^\wedge, V_1) \\ \oplus \\ \mathcal{S}_{Q_2}^{-\gamma-\frac{1}{2}}(Y^\wedge, V_3) \end{array} \tag{2}$$

are continuous. In (2), G^* is the formal adjoint of G. It is defined from the duality between $\mathcal{K}^{s,\gamma}(X^\wedge, V_1) \oplus \mathcal{K}^{s,\gamma-\frac{1}{2}}(Y^\wedge, V_2)$ and $\mathcal{K}_0^{-s,-\gamma}(X^\wedge, V_1) \oplus \mathcal{K}^{-s,-\gamma-\frac{1}{2}}(Y^\wedge, V_2)$, which comes from an extension of the inner product

$$\begin{aligned} ((f_1, g_1), (f_2, g_2)) &= \int_{\Gamma_{\frac{n+1}{2}}} (Mf_1(z), Mf_2(z))_{L^2(X)} |dz| \\ &+ \int_{\Gamma_{\frac{n+1}{2}}} (Mg_1(z), Mg_2(z))_{L^2(Y)} |dz| \end{aligned}$$

on $\mathcal{H}^0(X^\wedge) \oplus \mathcal{H}^{0,-\frac{1}{2}}(Y^\wedge)$. Notice that the second term on the right hand side differs from the standard inner product on $\mathcal{H}^{s,\gamma}(Y^\wedge)$, where the integration is over $\Gamma_{\frac{n}{2}-\gamma}$, for $\dim Y = n-1$. Since $(Mu)(z+\frac{1}{2}) = M(t^{\frac{1}{2}}u)(z)$, this term yields a duality between $\mathcal{H}^{s,\gamma-\frac{1}{2}}(Y^\wedge)$ and $\mathcal{H}^{-s,-\gamma-\frac{1}{2}}(Y^\wedge)$. Clearly, (1) and (2) will be satisfied whenever they hold for $s=0$.
As before, we will not refer to the bundles in the notation.

(b) $C^0_G(\mathbb{D},g)_{P,Q}$ is the corresponding space with $X^\wedge$ replaced by $\mathbb{D}$ and the spaces $\mathcal{S}^\delta_{P_1}(X^\wedge,V_2),\ldots,\mathcal{S}^{-\gamma-\frac{1}{2}}_{Q_2}(Y^\wedge,V_3)$ by $\mathcal{H}^{\infty,\delta}_{P_1}(\mathbb{D},V_2),\ldots,\mathcal{H}^{\infty,-\gamma-\frac{1}{2}}_{Q_2}(\mathbb{B},V_3)$. We call the elements of $C^0_G(X^\wedge,g)_{P,Q}$ and $C^0(\mathbb{D},g)_{P,Q}$ the *Green operators of type zero* on $X^\wedge$ and $\mathbb{D}$, respectively.

(c) Let $k\in\mathbb{N}$. An operator $G = \begin{bmatrix} G_G & G_K \\ G_T & G_S \end{bmatrix}$ acting as in (1) is called a *Green operator of type* k, if it can be written

$$G = \sum_{j=0}^{k} G_j \begin{bmatrix} \partial_r^j & 0 \\ 0 & I \end{bmatrix} \tag{3}$$

with Green operators G_j of type zero. The order s in (1) then is assumed to be $\geq k$.
With the replacements in (b) we can use the same definition for operators acting on functions over $\mathbb{D}$. In (3), ∂_r denotes the normal derivative defined in a neighborhood of the boundary of the Riemannian manifolds $X^\wedge$ and $\mathbb{D}$, respectively, multiplied by a cut-off function, so that it makes sense everywhere.
We shall write

$$G\in C^k_G(X^\wedge,g)_{P,Q} \quad\text{and}\quad G\in C^k_G(\mathbb{D},g)_{P,Q},$$

respectively. Without loss of generality we assume that the asymptotic types P and Q in (1) and (2) are the same for all $G_j, j=0,\ldots,k$.

(d) The mapping properties (1) and (2) give a natural Fréchet topology for $C^0_G(X^\wedge,g)_{P,Q}$ and $C^0_G(\mathbb{D},g)_{P,Q}$. The spaces $C^k_G(X^\wedge,g)_{P,Q}$ and $C^k_G(\mathbb{D},g)_{P,Q}$ are topologized as non-direct sums of Fréchet spaces, cf. 3.1.18(a).
(e) We shall refer to the entries G_G, G_K, G_T, G_S of G as the proper Green, potential, trace, and boundary parts of G, respectively.

In the following, g, P, Q will denote an arbitrary weight datum and arbitrary asymptotic types. $V_1, V_2, \ldots$ are Hermitean vector bundles smooth up to the boundary.

3.3.2 Theorem.

$$C^0_G(X^\wedge,g)_{P,Q} \cong \left[\mathcal{S}^\delta_{P_1}(X^\wedge,V_1)\oplus\mathcal{S}^{\delta-\frac{1}{2}}_{P_2}(Y^\wedge,V_3)\right]\hat{\otimes}_\pi\left[\mathcal{S}^{-\gamma}_{\overline{Q}_1}(X^\wedge,V_2)\oplus\mathcal{S}^{-\gamma-\frac{1}{2}}_{\overline{Q}_2}(Y^\wedge,V_4)\right]. \tag{1}$$

The isomorphism is given by the mapping that associates with G its integral kernel. Here, $\overline{Q}=(\overline{Q}_1,\overline{Q}_2)$ is an asymptotic type in $As(X,Y,g)$. $\overline{Q}_k$ is constructed by replacing each element $(p,m,L)\in Q_k$ by the complex conjugate $(\overline{p},m,\overline{L})$, $k=1,2$. Similarly,

$$C^0_G(\mathbb{D},g)_{P,Q} \cong \left[\mathcal{H}^{\infty,\delta}_{P_1}(\mathbb{D},V_1)\oplus\mathcal{H}^{\infty,\delta-\frac{1}{2}}_{P_2}(\mathbb{B},V_3)\right]\hat{\otimes}_\pi\left[\mathcal{H}^{\infty,-\gamma}_{\overline{Q}_1}(\mathbb{D},V_2)\oplus\mathcal{H}^{\infty,-\gamma-\frac{1}{2}}_{\overline{Q}_2}(\mathbb{B},V_4)\right]. \tag{2}$$

Proof. This is the same as in the classical case, cf. [29], Volume I, p. 44. The change in the asymptotic type Q is, of course, due to the fact that the integral kernel of the adjoint of the operator with kernel $k(x,y)$ is $\overline{k(y,x)}$. $\triangleleft$

3.3.3 Corollary. (a) Let ϕ_1 and ϕ_2 be excision functions for the singular set of D, and let $G \in C_G^0(I\!D, g)$. Then $\phi_1 G \phi_2$ is a regularizing singular Green operator in Boutet de Monvel's calculus for $I\!D$ (we have defined these operators only for *compact* manifolds with boundary; however, since G vanishes near the singularities we can consider it as an operator on the manifold obtained by 'doubling' $I\!D$ along the cylinders $X_v \times (0,1)$).
(b) Let $G \in C_G^k(X^\wedge, g)_{P,Q}$. Then there are finite-dimensional operators $G^{[N]}, N \in \mathbf{N}$, with $G = \lim_{N\to\infty} G^{[N]}$ in the topology of $C_G^k(X^\wedge, g)_{P,Q}$.
The same is true for $G \in C_G^k(I\!D, g)_{P,Q}$.

Proof. (a) The mapping properties imply that both G and G^* map $L^2(I\!D)$ to $C^\infty(I\!D)$, so they have a smooth integral kernel.
(b) This follows immediately from the representation 3.3.1(3), Theorem 3.3.2, and the properties of the $\pi-$tensor product. $\triangleleft$

3.3.4 Lemma. *Let $G_1 \in C_G^0(X^\wedge, g)_{P,Q}$ and $G_2 \in C_G^0(I\!D, g)_{P,Q}$. Then the mappings*

$$G_1 : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2}}(Y^\wedge, V_3) \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{t,\delta}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}^{t,\delta-\frac{1}{2}}(Y^\wedge, V_4) \end{array}$$

and

$$G_2 : \begin{array}{c} \mathcal{H}^{s,\gamma}(I\!D, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma-\frac{1}{2}}(I\!B, V_3) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}^{t,\delta}(I\!D, V_2) \\ \oplus \\ \mathcal{H}^{t,\delta-\frac{1}{2}}(I\!B, V_4) \end{array}$$

are compact for every choice of $s, t \geq 0$.

Proof. Consider G_1. Its image is in fact contained in $\mathcal{K}^{\infty,\delta+\epsilon}(X^\wedge, V_2) \oplus \mathcal{K}^{\infty,\delta-\frac{1}{2}+\epsilon}(X^\wedge, V_4)$ for small ϵ, since $\mathcal{K}_{P_1}^{\tau,\delta+\epsilon}(X^\wedge, V_2) \subseteq \mathcal{K}^{\tau,\delta}(X^\wedge, V_2)$ and $\mathcal{K}_{P_2}^{\tau,\delta-\frac{1}{2}+\epsilon}(Y^\wedge, V_4) \subseteq \mathcal{K}^{\tau,\delta-\frac{1}{2}+\epsilon}(Y^\wedge, V_4)$ for all ϵ with $0 \leq \epsilon < \operatorname{dist}(\pi_{\mathbf{C}} P, \Gamma_{\frac{n+1}{2}-\delta})$. The assertion now follows from 3.1.16 and compactness of $[1-\omega]\mathcal{S}(X^\wedge)$ in $H^s(X^\wedge)$ and $[1-\omega]\mathcal{S}(Y^\wedge)$ in $H^s(Y^\wedge)$. The consideration for G_2 is analogous. $\triangleleft$

3.3.5 Theorem. *Let $g = (\gamma, \gamma, \Theta), G_1 \in C_G^0(X^\wedge, g)_{P,Q}$, and $G_2 \in C_G^0(I\!D, g)_{P,Q}$. Suppose that for some given $s_0 > -\frac{1}{2}$*

$$I + G_1 : \begin{array}{c} \mathcal{K}^{s_0,\gamma}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s_0,\gamma-\frac{1}{2}}(Y^\wedge, V_2) \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s_0,\gamma}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s_0,\gamma-\frac{1}{2}}(Y^\wedge, V_2) \end{array} \tag{1}$$

and

$$I + G_2 : \begin{array}{c} \mathcal{H}^{s_0,\gamma}(I\!D, V_1) \\ \oplus \\ \mathcal{H}^{s_0,\gamma-\frac{1}{2}}(I\!B, V_2) \end{array} \to \begin{array}{c} \mathcal{H}^{s_0,\gamma}(I\!D, V_1) \\ \oplus \\ \mathcal{H}^{s_0,\gamma-\frac{1}{2}}(I\!B, V_2) \end{array} \tag{2}$$

are invertible. Then there are $H_1 \in C_G^0(X^\wedge, g)_{P,Q}$ *and* $H_2 \in C_G^0(I\!D, g)_{P,Q}$ *with*

$$(I + G_1)^{-1} = I + H_1, \quad \text{and} \quad (I + G_2)^{-1} = I + H_2. \tag{3}$$

Proof. Using the identity $(1-x)^{-1} = 1 + x + x(1-x)^{-1}x$ we obtain the desired result from the fact that, for every s, $G_1(I+G_1)^{-1}G_1$ and $G_1^*[(I+G_1)^{-1}]^*G_1^*$ have the mapping properties of 3.3.1(1),(2). The argument for G_2 is the same. ◁

3.3.6 Corollary. The operators $I + G_1$ and $I + G_2$ in equations (1) and (2) of 3.3.5 are also invertible on $\mathcal{K}^{s,\gamma}(X^\wedge, V_1) \oplus \mathcal{K}^{s,\gamma-\frac{1}{2}}(Y^\wedge, V_2)$ and $\mathcal{H}^{s,\gamma}(I\!D, V_1) \oplus \mathcal{H}^{s,\gamma-\frac{1}{2}}(I\!B, V_2)$ for every choice of s.

Proof. Again consider only G_1. The operator H_1 of 3.3.5 has the mapping property (1) in 3.3.1. Therefore

$$(I + G_1)(I + H_1) = (I + H_1)(I + G_1) = I \tag{1}$$

on $\mathcal{S}_{P_1}^{\gamma}(X^\wedge, V_1) \oplus \mathcal{S}_{P_2}^{\gamma-\frac{1}{2}}(Y^\wedge, V_2)$. Since this space is dense in all the spaces $\mathcal{K}^{s,\gamma}(X^\wedge, V_1) \oplus \mathcal{K}^{s,\gamma-\frac{1}{2}}(Y^\wedge, V_2)$ identity (1) extends and shows that $I + H_1$ is an inverse to $I + G_1$ also in these spaces. ◁

3.3.7 Lemma. *A Green operator* $G \in C_G^k(X^\wedge, g)_{P,Q}$ *of type* k *can also be written*

$$G = \sum_{j=0}^{k-1} \begin{bmatrix} K_j\gamma_j & 0 \\ S_j\gamma_j & 0 \end{bmatrix} + G_0,$$

where

- K_j *are potential parts of suitable Green operators of type zero,*
- S_j *are boundary parts of suitable Green operators of type zero,*
- $\gamma_j : f \mapsto \partial_r^j f|_{Y^\wedge}$, *and*
- G_0 *is a Green operator of type zero.*

Similarly for $G \in C_G^k(I\!D, g)_{P,Q}$.

Proof. Consider the left upper corner. It is a sum of terms $H_j\partial_r^j, j = 0, \ldots, k$, where H_j are the proper Green parts of a Green operator of type zero. By 3.3.2 there are functions $h_j \in \mathcal{S}_P^\delta(X^\wedge, V_1)\hat{\otimes}_\pi \mathcal{S}_{\overline{Q}}^{-\gamma}(X^\wedge, V_2)$ such that

$$H_j\partial_r^j f(x) = \int_{X^\wedge} h_j(x, \tilde{x})\partial_r^j f(\tilde{x})d\tilde{x}.$$

Integrate by parts in the normal direction. This gives terms of the form $\int_{X^\wedge} \partial_r^l h_j(x,\tilde{x}) f(\tilde{x}) d\tilde{x}$ and of the form $\int_{Y^\wedge} k_{jl}(x,y)\gamma_m f(y)dy$, where $k_{jl}(x,y) = \partial_r^l h_j(x,\tilde{x})|_{\tilde{x}=y\in Y^\wedge}$ and $m+l \leq j-1$.
Applying 3.3.2 again, the former operators define the proper Green parts of suitable Green operators of type zero, while the integral operators with kernels k_{jl} give rise to potential parts of Green operators of type zero, since restriction to the boundary maps $\mathcal{S}_{\overline{Q}}^{-\gamma}(X^\wedge)$ to $\mathcal{S}_{\overline{Q}}^{-\gamma-\frac{1}{2}}(Y^\wedge)$. ◁

3.3.8 Lemma. *Let $g_1 = (\gamma,\delta,\Theta), g_2 = (\delta,\eta,\Theta)$ be weight data, P,Q,R asymptotic types, let $G_1 \in C_G^d(X^\wedge, g_1)_{P,Q}$, and $G_2 \in C_G^{d'}(X^\wedge, g_2)_{Q,R}$. Then*

$$G_2 G_1 \in C_G^d(X^\wedge, g_3)_{P,R'}$$

with $g_3 = (\gamma,\eta,\Theta)$ and a resulting asymptotic type R' depending on G_2 and R. We will have $R' = R$ for $d' = 0$.
We tacitly assume that G_1 and G_2 act on vector bundles so that the composition makes sense.
The corresponding result also holds with $X^\wedge$ replaced by $I\!D$.

Proof. For $d = d' = 0$, this follows immediately from the definition. In view of 3.3.1(3), we may assume that $d = 0$ and that the matrices for G_1 and G_2 only consist of the entry in the upper left corner. Using 3.3.7, we even may suppose that $G_1 = K\gamma_{d'}$, where K is the potential part of a Green operator of type zero. Now we apply 3.3.2, writing

$$\begin{aligned} K\gamma_{d'} G_2 f(x) &= \int k(x,y')\gamma_{d'} \int l(y,z) f(z)dz\,dy' \\ &= \int k(x,y') \int \partial^{d'} l(y,z)|_{y_n=0} f(z)dz\,dy' \end{aligned}$$

with corresponding kernels. A second application of 3.3.2 then yields the assertion. ◁

3.3.9 Definition and Remark. In view of the preceeding result the Green operators of arbitrary type form an algebra. In fact, $C_G^0(X^\wedge, g)_{P,P}$ is an algebra in the usual sense, while in general, the asymptotic types will change.
For $g = (\gamma,\gamma,\Theta)$ we let $C_G(X^\wedge, g)$ denote the space of all operators that belong to any one of the families $C^d(X^\wedge, g)_{P,Q}$ for arbitrary d, P, Q. In view of Lemma 3.3.8, the elements of $C_G(X^\wedge, g)$ that act on fitting vector bundles can be composed. In this more general sense, this space also is an algebra. The proof of 3.3.8 shows that the composition is continuous with respect to the corresponding topologies.

3.3.10 Theorem. *Theorem 3.3.5 and Corollary 3.3.6 on invertibility extend to the case where $G_1 \in C_G^k(X^\wedge, g)_{P,Q}$ and $G_2 \in C_G^k(I\!D, g)_{P,Q}, 0 \neq k \in \mathbb{N}$; we have to assume that $s_0, s \in \mathbb{N}$ are $\geq k$ in order to have all mappings well-defined. The corresponding operators H_1 and H_2 belong to $C_G^k(X^\wedge, g)_{\tilde{P},\tilde{Q}}$ and $C_G^k(I\!D, g)_{\tilde{P},\tilde{Q}}$, respectively for suitable asymptotic types $\tilde{P}$ and $\tilde{Q}$, respectively.*

Proof. Consider G_1. For simplicity assume that the bundles V_1 and V_2 are trivial and scalar. Introduce the Hilbert space $E = \mathcal{K}^{s_0,\gamma}(X^\wedge) \oplus \mathcal{K}^{s_0,\gamma-\frac{1}{2}}(Y^\wedge)$. According to 3.3.7 we can write

$$I + G_1 = I + H_0 + H_k;$$

here H_0 is of type zero, and $H_k = \sum_{j=0}^{k-1} A_j \begin{bmatrix} \gamma_j & 0 \\ 0 & 0 \end{bmatrix}$ with $A_j = \begin{bmatrix} R_j & 0 \\ S_j & 0 \end{bmatrix}$ consisting of a potential part R_j and a boundary part S_j of a suitable Green operator of type zero. We may rewrite H_k in the form $H_k = \sum_{j=0}^{k-1} K_j T_j$; here $T_j : E \to \mathcal{K}^{s_0,\gamma-\frac{1}{2}}(Y^\wedge)$ is given by $T_j(f_1 \oplus f_2) = \gamma_j f_1$ and $K_j : \mathcal{K}^{s_0,\gamma-\frac{1}{2}}(Y^\wedge) \to E$ by $K_j g = R_j g \oplus S_j g$.
Lemma 3.3.4 implies that the operator $I + H_0$ is a Fredholm operator on E of index zero. Choose bases $\{\phi_1, \ldots, \phi_J\}$ of its kernel and $\{\psi_1, \ldots, \psi_J\}$ of the orthogonal complement of its image. Define the operator

$$P : f \mapsto \sum_{j=0}^{J} (f, \phi_j)_E \psi_j.$$

Then $I + H_0 + P : E \to E$ is invertible. More is true. P even is a Green operator: First, $(I+H_0)\phi_j = 0$ implies that $\phi_j = -H_0\phi_j \in \mathcal{S}^{\gamma}_{P_1}(X^\wedge) \oplus \mathcal{S}^{\gamma-\frac{1}{2}}_{P_2}(Y^\wedge)$. Moreover, we may replace the functions ψ_j by functions $\tilde{\psi}_j \in C_0^\infty(X^\wedge) \oplus C_0^\infty(Y^\wedge)$ without losing the invertibility of $I + H_0 + P$ on E; this is a consequence of the fact that the compactly supported smooth functions are dense in E. In particular, Theorem 3.3.2 implies that $P \in C_G^0(X^\wedge, g)_{P,Q}$, since its integral kernel belongs to $[\mathcal{S}^{\gamma}_{P_1}(X^\wedge) \oplus \mathcal{S}^{\gamma-\frac{1}{2}}_{P_2}(Y^\wedge)] \otimes [\mathcal{S}^{-\gamma}_{\overline{Q_1}}(X^\wedge) \oplus \mathcal{S}^{-\gamma-\frac{1}{2}}_{\overline{Q_2}}(Y^\wedge)]$ Let us write $P = \sum_{j=0}^{J} L_j U_j$; here $U_j : E \to \mathbf{C}$ is defined by $U_j f = \langle f, \phi_j \rangle_E$, and $L_j : \mathbf{C} \to E$ by $L_j c = c\tilde{\psi}_j$.
Now we shall use an analog of the method in [29], 2.1.12, Proposition 24. Let $\mathcal{K} = \mathcal{K}^{s_0,\gamma-\frac{1}{2}}(Y^\wedge)$; denote by K, L the row vectors with components $K_1, \ldots, K_{k-1}$ and $L_1, \ldots, L_J$, respectively, by T, U the column vectors with entries $T_1, \ldots, T_{k-1}$ and $U_1, \ldots, U_J$.
Note that $I + G_1 : E \to E$ is an isomorphism iff

$$\begin{bmatrix} I + H_0 & -K & L \\ T & I & 0 \\ U & 0 & I \end{bmatrix} : \begin{matrix} E \\ \oplus \\ \mathcal{K}^k \\ \oplus \\ \mathbf{C}^J \end{matrix} \to \begin{matrix} E \\ \oplus \\ \mathcal{K}^k \\ \oplus \\ \mathbf{C}^J \end{matrix} \tag{1}$$

is an isomorphism. In fact, this is a consequence of the matrix identity below, noting that $I + G_1 = I + H_0 + P + \sum K_j T_j - \sum L_j U_j$:

$$\begin{bmatrix} I & K & -L \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I + H_0 + P & -K & L \\ T & I & 0 \\ U & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -T & I & 0 \\ -U & 0 & I \end{bmatrix} = \begin{bmatrix} I + G_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{2}$$

By construction, $I + H_0 + P : E \to E$ is an invertible Green operator of type zero. So by 3.1.5, its inverse is of the form $I + G, G \in C_G^0(X^\wedge, g)_{P,Q}$. Now

$$\begin{bmatrix} I & 0 & 0 \\ -T(I+G) & I & 0 \\ -U(I+G) & 0 & I \end{bmatrix} \begin{bmatrix} I + H_0 + P & -K & L \\ T & I & 0 \\ U & 0 & I \end{bmatrix} \begin{bmatrix} I & (I+G)K & -(I+G)L \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I + H_0 + P & 0 & 0 \\ 0 & I + T(I+G)K & -T(I+G)L \\ 0 & U(I+G)K & I - U(I+G)L \end{bmatrix} \tag{3}$$

The 2×2 matrix in the lower right corner is invertible, since the whole matrix is. Consider the operator $T(I+G)K : \mathcal{K}^k \to \mathcal{K}^k$. It is a Green operator of type zero on the surface, i.e. in $C_G^0(Y^\wedge, g)_{P,Q}$, since it has an integral kernel in $\mathcal{S}_{P'}^{\gamma-\frac{1}{2}}(Y^\wedge)\hat{\otimes}_\pi \mathcal{S}_{Q'}^{-\gamma-\frac{1}{2}}(Y^\wedge)$, computable from those of G and K with suitable asymptotic types P' and Q'.
Just like before we may determine operators $W_j : \mathcal{K}^k \to \mathbf{C}$ and $V_j : \mathbf{C} \to \mathcal{K}^k, j = 1, \ldots, \tilde{J}$ such that $I + T(I+G)K + \sum V_j W_j$ is invertible. With the notation and technique of before, cf. (2), the inverse to the matrix in the lower right corner of (3) can be computed from the inverse to

$$\begin{bmatrix} I + T(I+G)K + VW & -T(I+G)L & V \\ U(I+G)K & I - U(I+G)L & 0 \\ W & 0 & I \end{bmatrix} \tag{4}$$

in $\mathcal{L}(\mathcal{K}^k \oplus \mathbf{C}^J \oplus \mathbf{C}^{\tilde{J}})$. Since the upper left corner is invertible, we may apply a decomposition as in (3) leading to an invertible matrix of the form

$$\begin{bmatrix} I + T(I+G)K + VW & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}. \tag{5}$$

The lower right 2×2 matrix then is an invertible operator on $\mathcal{L}(\mathbf{C}^{J+\tilde{J}})$, and so is its inverse. This allows us to compute, step by step, the inverses of the matrices in (5), (4), (3), and (1). We obtain the inverse of $I + G_1$ in the desired form.
For G_2, the proof is similar. ◁

4 Mellin Symbols with Values in Boutet de Monvel's Algebra

4.1 The Spaces of Mellin Symbols with Asymptotics. Mapping Properties

4.1.1 Definition. (a) A Mellin asymptotic type is a sequence

$$P = \{(p_j, m_j, L_j)\}_{j\in\mathbf{Z}}$$

with $p_j \in \mathbf{C}$, $\operatorname{Re} p_j \to \pm\infty$ as $j \to \mp\infty$, $m_j \in \mathbf{N}$, and L_j a finite-dimensional subspace of finite-dimensional operators in $\mathcal{B}^{-\infty,d}(X)$.
We denote the collection of all these asymptotic types by $As\left(\mathcal{B}^{-\infty,d}(X)\right)$. Just like in 3.2.1, we let $\pi_{\mathbf{C}}P = \{p_j : j \in \mathbf{Z}\}$.

(b) Let $P \in As\left(\mathcal{B}^{-\infty,d}(X)\right)$, $\mu \in \mathbf{R}, d \in \mathbf{N}$. $M_P^{\mu,d}(X)$ denotes the space of all functions

$$a \in \mathcal{A}\left(\mathbf{C}\setminus\pi_{\mathbf{C}}P, \mathcal{B}^{\mu,d}(X)\right) \tag{1}$$

with the following properties

(i) in a neighborhood of $p_j \in \pi_{\mathbf{C}}P$

$$a(z) = \sum_{k=0}^{m_j} \nu_{jk}(z-p_j)^{-k-1} + a_0(z) \tag{2}$$

with $\nu_{jk} \in L_j$, $k = 0, \ldots, m_j$, and a_0 holomorphic near p_j.

(ii) Let $0 < \varepsilon_1 < \varepsilon_2$. For every function $\chi \in C^\infty(\mathbf{C})$ supported in $\{z : \operatorname{dist}(z, \pi_{\mathbf{C}}P) > \varepsilon_1\}$ and equal to 1 outside an ε_2-neighborhood of $\pi_{\mathbf{C}}P$, and for every $\beta \in \mathbf{R}$

$$(\chi a)(\beta + i\tau) \in \mathcal{B}^{\mu,d}(X; \mathbf{R}_\tau), \tag{3}$$

uniformly for $\beta \in [c_1, c_2]$, $c_1 < c_2 \in \mathbf{R}$.

We call the elements of $M_P^{\mu,d}(X)$ Mellin symbols of order μ, type d, with asymptotic type P.
Of course, we are assuming in (1) that the vector bundles $a(z)$ is acting on, cf. 2.3.1(1), are independent of z.

(c) $M_{P,cl}^{\mu,d}(X)$ is the corresponding space with $\mathcal{B}^{\mu,d}(X)$ replaced by $\mathcal{B}_{cl}^{\mu,d}(X)$.

(d) If $P = \emptyset$ then we shall write $M_O^{\mu,d}(X)$ and $M_{O,cl}^{\mu,d}(X)$.

4.1.2 Remark. The topology of $M_P^{\mu,d}(X)$ is given by three semi-norm systems

(i) that for the topology of $\mathcal{A}\left(\mathbf{C}\setminus\pi_{\mathbf{C}}P, \mathcal{B}^{\mu,d}(X)\right)$;

(ii) that induced by $a \mapsto \nu_{jk} \in L_j \subseteq \mathcal{B}^{-\infty,d}(X)$, where $a \in M_P^{\mu,d}(X)$ is as in 4.1.1(2), and the topology of $\mathcal{B}^{-\infty,d}(X)$;

(iii) that given by

$$q_{c_1,c_2,j}(a)(\beta+i\tau) = \sup_{c_1\le\beta\le c_2} r_j(\chi\, a)(\beta+i\tau), \qquad c_1, c_2 \in \mathbf{Z}, j \in \mathbf{N},$$

where $\{r_j : j \in \mathbf{N}\}$ is a semi-norm system for the topology of $\mathcal{B}^{\mu,d}(X;\mathbf{R}_\tau)$, and χ is fixed.

4.1.3 Remark. (a) $M_P^{\mu,d}(X)$ is a Fréchet space in the above topology.
(b) $M_P^{-\infty,d}(X) = \bigcap_\mu M_P^{\mu,d}(X)$ is a nuclear Fréchet space.

Proof. (a) is obvious from the definition.
(b) Follows from a representation of $M_P^{\mu,d}(X)$ as a projective limit of Hilbert spaces with nuclear embeddings. The construction is analogous to that in [27], 1.1.3, Proposition 6. $\triangleleft$

4.1.4 Examples. (a) Let $\mu, k \in \mathbf{N}$ and $A_k \in \mathcal{B}^{\mu-k,d}(X)$, $k = 0, \ldots, \mu$. Then

$$a(z) = \sum_{k=0}^{\mu} A_k z^k \in M_O^{\mu,d}(X).$$

(b) Let $\nu \in \mathcal{B}^{-\infty,d}(X)$ have finite-dimensional range. Moreover, let $p \in \mathbf{C}$ with $\mathrm{Re}\, p < \frac{1}{2}$, $k \in \mathbf{N}, \omega$ a cut-off function near zero. Then

$$b(z) = \nu M_{t\to z}(t^{-p} \ln^k t\, \omega(t)) \in M_P^{-\infty,d}(X)$$

whenever P is an asymptotic type that contains an entry (p, k, L) with $\nu \in L$.
(c) Under the same assumptions but with $\mathrm{Re}\, p > \frac{1}{2}$,

$$c(z) = \nu M_{t\to z}(t^{-p} \ln^k t\, \omega(1/t)) \in M_P^{-\infty,d}(X).$$

Proof. (a) Clearly, $a \in \mathcal{A}\left(\mathbf{C}, \mathcal{B}^{\mu,d}(X)\right)$. Since $A_k(\beta+i\tau)^k$ is a polynomial in β and τ of degree k, we have $a(\beta+i\tau) \in \mathcal{B}^{\mu,d}(X;\mathbf{R}_\tau)$, uniformly for β in compact sets.
(b) The function $M_{t\to z}(t^{-p} \ln^k t\, \omega(t))$ is meromorphic in $\mathbf{C}$ with a single pole of order $k+1$ in p, cf. 5.1.6, so the relations (1) and (2) in 4.1.1(b) are trivially fulfilled. If χ is a smooth function on $\mathbf{C}$ which is zero near p and 1 near infinity, then χu is rapidly decreasing on each line Γ_β, uniformly for β in compact intervals, cf. 5.1.6. Therefore χb satisfies the relation (3) in 4.1.1(b) for every $\mu \in \mathbf{R}$.
The proof of (c) is similar. $\triangleleft$

4.1.5 Theorem. *Let P be a Mellin asymptotic type, $\mu \in \mathbf{R}, d \in \mathbf{N}$. The function $a \in \mathcal{A}(\mathbf{C}\setminus\pi_{\mathbf{C}}P, \mathcal{B}^{\mu,d}(X))$ is a Mellin symbol in $M_P^{\mu,d}(X)$ if and only if it can be written*

$$a(z) = \sum_{k=0}^{d} a_k(z) \begin{bmatrix} \partial_\tau^k & 0 \\ 0 & I \end{bmatrix} \tag{1}$$

with $a_k \in M_Q^{\mu-k,0}(X)$. Here, ∂_τ stands for the operator given by the normal derivative in a neighborhood of the boundary, multiplied by a suitable cut-off function. Q is a slightly modified asymptotic type; it contains the same p_j and m_j, but the L_j are now finite-dimensional spaces of finite-dimensional operators in $\mathcal{B}^{\mu,0}(X)$.

Proof. Clearly a function with the representation (1) belongs to $M_P^{\mu,d}(X)$. So we only have to prove the converse. We write

$$a(z) = \begin{bmatrix} P(z) + G(z) & K(z) \\ T(z) & S(z) \end{bmatrix}$$

as a matrix in Boutet de Monvel's calculus depending on z.
For $P(z), K(z), S(z)$ there is no 'type', so we only have to consider $G(z)$ and $T(z)$. The proof is almost the same for both, so let us concentrate on $G(z)$.
Fixing $\beta \in \mathbf{R}$, and an excision function χ for the poles, we have

$$\chi(\beta + i\tau)G(\beta + i\tau) \in \mathcal{G}^{\mu,d}(X; \mathbf{R}_\tau),$$

uniformly for β in compact intervals (for $\mathcal{G}^{\mu,d}(X;\mathbf{R}_\tau)$ cf. Definition 2.3.1(b)). We may write in a unique way

$$G(\beta + i\tau) = \sum_{j=0}^{d-1} K_j(\beta + i\tau)\gamma_j + G^0(\beta + i\tau), \tag{2}$$

where $\chi(\beta + i\cdot)K_j(\beta + i\cdot)$ is a parameter-dependent potential operator of order $\mu - j - \frac{1}{2}$ and $G^0(\beta + i\tau) \in \mathcal{G}^{\mu,0}(X; \mathbf{R}_\tau)$, uniformly for β in compact intervals. On the other hand, we may write for each fixed z

$$G(z) = \sum_{j=0}^{d-1} \tilde{K}_j(z)\gamma_j + \tilde{G}^0(z) \tag{3}$$

with $\tilde{K}_j(z)$ a potential operator of order $\mu - j - \frac{1}{2}$ and $\tilde{G}^0(z) \in \mathcal{G}^{\mu,0}(X)$. The mapping $G(z) \mapsto \tilde{K}_j(z)$ is continuous in the symbol topology. Since the decomposition is unique in both (2) and (3), we have, fixing τ, $K_j(\beta + i\tau) = \tilde{K}_j(z)|_{z=\beta+i\tau}$. Moreover, the mapping $z \mapsto \tilde{K}_j(z)$ is a holomorphic function of z on $\mathbf{C}\setminus\pi_{\mathbf{C}}P$: Since $G(z) \in \mathcal{A}(\mathbf{C}\setminus\pi_{\mathbf{C}}P, \mathcal{G}^{\mu,d}(X)) = \mathcal{A}(\mathbf{C}\setminus\pi_{\mathbf{C}}P)\hat{\otimes}_\pi\mathcal{G}^{\mu,d}(X)$ we have

$$G(z) = \sum_{l=0}^{\infty} \lambda_l h_l(z) H_l \tag{4}$$

with $\{\lambda_l\} \in \ell^1$, and $\{h_l\} \subset \mathcal{A}(\mathbf{C}\setminus\pi_{\mathbf{C}}P)$ and $\{H_l\} \subset \mathcal{G}^{\mu,d}(X)$ null sequences. Again we may write in a unique way $H_l = \sum_{j=0}^{d-1} K_{lj}\gamma_j + H_l^0$. Interchanging the summation in (4) we see that

$$G(z) = \sum_{j=0}^{d-1}\Big(\sum_{l=0}^{\infty} \lambda_l h_l(z) K_{lj}\Big)\gamma_j + \sum_{l=0}^{\infty} \lambda_l h_l(z) H_l^0.$$

By uniqueness,

$$\sum_l \lambda_l h_l(z) K_{jl} = \tilde{K}_j(z) \quad \text{and} \quad \sum_l \lambda_l h_l(z) H_l^0 = \tilde{G}^0(z)$$

for each fixed z. Since the left hand side is holomorphic in z we see that $\tilde{K}_j$ and $\tilde{G}^0$ are holomorphic functions of z outside $\pi_{\mathbf{C}}P$. Multiplying by powers of $z - p_j$ we see that all singularities are poles of order m_j, just as before.
Now we may also fix a way to convert an expression of the (unique) form $\sum_{j=0}^{d-1} L_j\gamma_j + L^0$ with potential operators L_j of order $\mu - j - \frac{1}{2}$ and a type zero singular Green operator L^0 of order μ to the (non-unique) form $\sum_{j=0}^{d} H_j\partial_r^j$ with singular Green operators H_j of order $\mu - j$ and type zero, with or without parameters, cf. 2.2.14. Using (2) we may therefore write

$$G(\beta + i\tau) = \sum_{j=0}^{d} G_j(\beta + i\tau)\partial_r^j$$

with

$$\chi(\beta + i\tau)G_j(\beta + i\tau) \in \mathcal{G}^{\mu-j,0}(X; \mathbf{R}_\tau),$$

uniformy for β in compact intervals, and, applying (3),

$$G(z) = \sum_{j=0}^{d} \tilde{G}_j(z)\partial_r^j$$

with $G_j(\beta + i\tau) = \tilde{G}_j(z)|_{z=\beta+i\tau}$. In view of the considerations above, $\tilde{G}_j$ is a holomorphic function of z on $\mathbf{C}\backslash\pi_{\mathbf{C}}P$, all singularities are poles, namely of orders μ_j.
It remains to check that, near $p_j \in \pi_{\mathbf{C}}P$, we have

$$\tilde{G}_l(z) = \sum_{k=0}^{m_j} \nu_{jkl}(z - p_j)^{-k-1} + h_{jl}(z)$$

with suitable $\nu_{jkl} \in \mathcal{B}^{-\infty,0}(X)$ of finite-dimensional range. By definition, the coefficients of $(z - p_j)^{-k-1}, k = 0, \dots, m_j$ in the power series of $G(z)$ are finite rank operators in $\mathcal{B}^{-\infty,d}(X)$. The uniqueness of the representation (3) together with Lemma 4.1.6, below, then implies that the corresponding coefficients for all $\tilde{K}_j(z)$ are regularizing potential operators of finite rank. The pointwise conversion according to 2.2.14 preserves this property. This concludes the proof. $\triangleleft$

In the proof we have used the following lemma.

4.1.6 Lemma. *Let $s \in \mathbf{R}, d \in \mathbf{N}, \nu \in \mathcal{B}^{-\infty,d}(X)$, and suppose that for some $s > d - \frac{1}{2}$, $\nu : H^s(X) \to H^s(X)$ has finite rank. Then in the representation*

$$\nu = \sum_{j=0}^{d-1} K_j\gamma_j + \nu_0$$

with regularizing potential operators K_j and $\nu_0 \in \mathcal{B}^{-\infty,0}(X)$ the operators $\nu_0, K_0, \dots, K_{d-1}$ all have finite-dimensional range.

Proof. The operator ν_0 has an integral kernel in $C^\infty(\overline{X} \times \overline{X})$, hence extends to $L^2(X)$. Since $C_0^\infty(X)$ is dense in $L^2(X)$ we have

$$\nu_0(C_0^\infty(X)) \subseteq \nu_0(H^s(X)) \subseteq \nu_0(L^2(X))$$

with the first space dense in the third. If the second were infinite-dimensional, so were the first. But this is not the case, for on $C_0^\infty(X)$, ν and ν_0 coincide.
In the following we may therefore assume that $\nu_0 = 0$. Choose a smooth function ρ supported in a small neighborhood of the boundary and vanishing to first order on Y. Clearly, $K_j(C^\infty(Y))$ is dense in the range of K_j. For $\phi \in C^\infty(Y)$, however,

$$\nu(\rho^{d-1}\phi) = \sum_{j=0}^{d-1} K_j\gamma_j(\rho^{d-1}\phi) = K_{d-1}[\gamma_{d-1}\rho^{d-1} \cdot \phi].$$

As ϕ runs over $C^\infty(Y)$, $\gamma_{d-1}\rho^{d-1} \cdot \phi$ runs over $C^\infty(Y)$, for $\gamma_{d-1}\rho^{d-1}$ is a nowhere vanishing smooth function. By assumption, ν has finite-dimensional range, therefore also the range of K_{d-1} is finite-dimensional. Iteration completes the argument. $\triangleleft$

4.1.7 Proposition. *Let μ, $\mu' \in \mathbf{Z}$, d, $d' \in \mathbf{N}$, and let $P = \{(p_j, m_j, L_j)\}$, $P' = \{(p'_j, m'_j, L'_j)\}$ be two Mellin asymptotic types. For $a \in M_P^{\mu,d}(X)$ and $b \in M_{P'}^{\mu',d'}(X)$ the function*

$$c(z) = a(z)\, b(z) \tag{1}$$

belongs to $M_{P''}^{\mu'',d''}(X)$, where

- $\mu'' = \mu + \mu'$;
- $d'' = \max\{\mu' + d, d'\}$;
- *P'' is a suitable Mellin asymptotic type that can be determined from a and b; in particular, $\pi_{\mathbf{C}}P'' \subseteq \pi_{\mathbf{C}}P \cup \pi_{\mathbf{C}}P'$.*

We are tacitly assuming that the composition in (1) makes sense, i.e. $a(z)$ and $b(z)$ are acting on appropriately chosen bundles.

Proof. By 2.3.1 and 2.2.18, $a(z)b(z) \in \mathcal{A}(\mathbf{C}\setminus(\pi_{\mathbf{C}}P \cup \pi_{\mathbf{C}}P'), \mathcal{B}^{\mu'',d''}(X))$. Also, if χ is an excision function for $\pi_{\mathbf{C}}P \cup \pi_{\mathbf{C}}P'$ as in 4.1.1(b.ii), then we can find excision functions χ_a and χ_b for $\pi_{\mathbf{C}}P$ and $\pi_{\mathbf{C}}P'$, respectively, with $\chi = \chi\chi_a\chi_b$. Then $\chi(z)\, a(z)\, b(z) = \chi(z)\, \chi_a(z)\, a(z)\, \chi_b(z)\, b(z)$, and we obtain 4.1.1(b.ii). Finally, let $p \in \pi_{\mathbf{C}}P \cup \pi_{\mathbf{C}}P'$. Near p write, according to 4.1.1 (b.i),

$$a(z) = \sum_{k=0}^{m} \nu_k(z-p)^{-k-1} + \sum_{k=0}^{m'+1} \nu'_k(z-p)^k + (z-p)^{m'+2}a_h(z) \tag{2}$$

$$b(z) = \sum_{l=0}^{m'} \mu_l(z-p)^{-l-1} + \sum_{l=0}^{m+1} \mu'_l(z-p)^l + (z-p)^{m+2}b_h(z) \tag{3}$$

Here, the ν_k are finite-dimensional operators in $\mathcal{B}^{-\infty,d}(X)$, the μ_l are finite-dimensional operators in $\mathcal{B}^{-\infty,d'}(X)$, ν_k' and μ_l' are operators in $\mathcal{B}^{\mu,d}(X)$ and $\mathcal{B}^{\mu',d'}(X)$, respectively; a_h and b_h are holomorphic near p.
In view of the composition rules in Boutet de Monvel's calculus, the finitely many operators $\nu_k\mu_l$, $\nu_k'\mu_l$, $\nu_k\mu_l'$, $\nu_k'\mu_l'$ are all finite-dimensional operators in $\mathcal{B}^{-\infty,d''}(X)$, they generate a finite-dimensional subspace. ◁

4.1.8 Theorem.

$$M_P^{\mu,d}(X) = M_O^{\mu,d}(X) + M_P^{-\infty,d}(X).$$

Proof. The proof is similar to that of 3.2.4. Let $P = \{(p_j, m_j, L_j)\}$ and $\nu_{jk} \in L_j$ for $j \in \mathbf{Z}$, $k = 0, 1, ..., m_j$, and let $a \in M_P^{\mu,d}(X)$ have the form of 4.1.1(2).
Choose a line $\Gamma_\beta \subseteq \mathbf{C}$ that does not intersect $\pi_{\mathbf{C}}P$. Without loss of generality assume that the line is $\Gamma_{\frac{1}{2}}$ and that the enumeration in P is such that $\operatorname{Re} p_j < \frac{1}{2}$ for $j \geq 0$, $\operatorname{Re} p_j > \frac{1}{2}$ for $j < 0$.
Now fix a cut-off function ω near 0 and let

$$u(t) = \sum_{j=0}^{\infty}\sum_{k=0}^{m_j} \nu_{jk}\, t^{-p_j} \ln^k t\, \omega(c_j t) + \sum_{j=-\infty}^{-1}\sum_{k=0}^{m_j} \nu_{jk}\, t^{-p_j} \ln^k t\, \omega\left((c_j t)^{-1}\right).$$

Set $b = Mu$, the Mellin transform of u. For $c_j \to \infty$ sufficiently fast, the summation will converge in the semi-norms of $M_P^{-\infty,d}(X)$; the complete argument is given below. Moreover, b has poles of order m_{j+1} in p_j, the coefficients of $(z - p_j)^{-k-1}$ are ν_{jk}, just as for a. Thus $a - b \in M_O^{\mu,d}(X)$.
Now for the missing part of the argument. *A priori* the sum converges to a holomorphic function in the strip

$$\{z : \max_{j\geq 0} \operatorname{Re} p_j < \operatorname{Re} z < \min_{j<0} \operatorname{Re} p_j\}$$

with values in $\mathcal{B}^{-\infty,d}(X)$. In fact, if the c_j tend to infinity sufficiently fast, then the first summation will converge for $\operatorname{Re} z > \max_{j\geq 0} \operatorname{Re} p_j$, while the second will converge for $\operatorname{Re} z < \min_{j<0} \operatorname{Re} p_j$ as a consequence of Lemma 5.1.6.
Let us check that it converges indeed to a function in $\mathcal{A}(\mathbf{C}\backslash\pi_{\mathbf{C}}P, \mathcal{B}^{\mu',d}(X))$ for every $\mu' \in \mathbf{R}$:
Let $\alpha < \beta \in \mathbf{R}$ be given. For $N \in \mathbf{N}$ consider

$$b_N(z) = M_{t\to z}\left[\sum_{j=N}^{\infty}\sum_{k=0}^{m_j} \nu_{jk}\, t^{-p_j} \ln^k t\, \omega(c_j t) + \sum_{j=-\infty}^{-N}\sum_{k=0}^{m_j} \nu_{jk}\, t^{-p_j} \ln^k t\, \omega\left((c_j t)^{-1}\right)\right].$$

Just as before, the summation for b_N will converge to a holomorphic function in the strip $\{\alpha < \operatorname{Re} z < \beta\}$ provided N is sufficiently large. The difference $b - b_N$ on the other hand is a finite sum and meromorphic in the strip, again by 5.1.6.
In order to see that the convergence even is in $M_P^{-\infty,d}(X)$, cf. 4.1.2, we now choose a smooth function χ on $\mathbf{C}$, vanishing near $\pi_{\mathbf{C}}P$ and equal to 1 near infinity. By 5.1.6(c),

$$\chi(z) M_{t\to z}(t^{-p_j} \ln^k t\, \omega(c_j t))(z)$$

is rapidly decreasing on each line Γ_β, uniformly for β in compact intervals. Moreover, for $\mathrm{Re}\, p_j < \frac{1}{2}$ it tends to zero on $\{\mathrm{Re}\, z > \mathrm{Re}\, p_j\}$ as c_j tends to infinity. Applying the corresponding argument for $\mathrm{Re}\, p_j > \frac{1}{2}$, the summation for $(\chi\, b_N)(\beta + i\tau)$ will converge in the topology of $\mathcal{B}^{\mu',d}(X; \mathbf{R}_\tau)$, uniformly for β in compact intervals and arbitrary $\mu' \in \mathbf{R}$, provided the c_j tend to infinity sufficiently fast. Since there is nothing to check with respect to 4.1.2(ii), this shows the convergence in $M_P^{\mu',d}(X)$ for arbitrary μ', thus in $M_P^{-\infty,d}(X)$. $\lhd$

4.1.9 Definition. Let $\gamma \in \mathbf{R}$, E, F Hilbert spaces.
(a) If f is a function on $U \subseteq \mathbf{C}$, then let $(T^\gamma f)(z) = f(z+\gamma)$ whenever $z + \gamma \in U$.

(b) For a polynomially bounded function g on $\Gamma_{\frac{1}{2}}$ with values in $\mathcal{L}(E, F)$ let

$$\mathrm{op}_M g : C_0^\infty(\mathbf{R}_+, E) \to C^\infty(\mathbf{R}_+, F)$$

be defined by

$$(\mathrm{op}_M g)(u) = M^{-1} g\, M u$$

with the vector-valued Mellin transform $M : L^2(\mathbf{R}_+, E) \to L^2(\Gamma_{\frac{1}{2}}, E)$.
(c) For g defined on $\Gamma_{\frac{1}{2}-\gamma}, \gamma \in \mathbf{R}$, let

$$\mathrm{op}_M^\gamma g = t^\gamma\, \mathrm{op}_M \left(T^{-\gamma} g\right) t^{-\gamma} = M_\gamma^{-1} g M_\gamma,$$

with the weighted Mellin transform M_γ, cf. 5.1.5.

4.1.10 Lemma. *Let $a \in M_P^{\mu,d}(X)$, $\mu \in \mathbf{Z}$, $d \in \mathbf{N}$, $d \le \mu_+ = \max\{\mu, 0\}, \gamma \in \mathbf{R}$, P a Mellin asymptotic type with $\pi_{\mathbf{C}} P \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$. Suppose that for fixed z, $a(z) \in \mathcal{B}^{\mu,d}(X)$ acts on vector bundles as in* 2.3.1. *Then*

$$\mathrm{op}_M^\gamma a : \begin{matrix} C_0^\infty(\overline{X}^\wedge, V_1) \\ \oplus \\ C_0^\infty(Y^\wedge, V_3) \end{matrix} \longrightarrow \begin{matrix} C^\infty(\overline{X}^\wedge, V_2) \\ \oplus \\ C^\infty(Y^\wedge, V_4) \end{matrix} \tag{1}$$

is a continuous operator.

Proof. Without loss of generality assume $V_3 = V_4 = 0$, while V_1, V_2 are trivial 1-dimensional bundles, so we need not mention them.
If $f \in C_0^\infty(\overline{X}^\wedge)$, then so is $t^{-\gamma} f$. Therefore, $M_{t\to z}(t^{-\gamma} f)$ is rapidly decreasing on $\Gamma_{\frac{1}{2}}$. Since $\pi_{\mathbf{C}} P \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$, $T^{-\gamma} a$ is holomorphic in a neighborhood of $\Gamma_{\frac{1}{2}}$, and

$$(T^{-\gamma} a)\left(\frac{1}{2} + i\tau\right) \in \mathcal{B}^{\mu,d}(X; \mathbf{R}_\tau).$$

Consequently, given any semi-norm r for $\mathcal{B}^{\mu,d}(X)$,

$$r\left(T^{-\gamma} a\right)\left(\frac{1}{2} + i\tau\right) = \mathrm{O}\left(\langle\tau\rangle^\mu\right).$$

Hence $(T^{-\gamma}a)M_{t\to z}(t^{-\gamma}f)(z)$ decays rapidly on $\Gamma_{\frac{1}{2}}$; it has values in $C^\infty(\overline{X})$. Therefore $\mathrm{op}_M^\gamma a(f) \in C^\infty(\overline{X}^\wedge)$ by 5.1.2. Continuity follows from the continuity of the isomorphism $M : L^2(\mathbf{R}_+) \to L^2(\Gamma_{\frac{1}{2}})$ and the closed graph theorem. ◁

4.1.11 Theorem. *Under the assumptions of* 4.1.10, $\mathrm{op}_M^\gamma a$ *has a bounded extension*

$$\mathrm{op}_M^\gamma a : \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, V_3) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, V_4) \end{array}$$

for all $s \in \mathbf{R}$, $s > d - \frac{1}{2}$.

4.1.12 Corollary. Let $\omega, \omega' \in C_0^\infty(\overline{\mathbf{R}}_+)$. Under the assumptions of 4.1.10

$$\omega\, \mathrm{op}_M^\gamma(a)\, \omega' : \begin{array}{c} \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, V_3) \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, V_4) \end{array}$$

is bounded for all $s \in \mathbf{R}, s > d - \frac{1}{2}$.

Proof of 4.1.11. We make the same simplification concerning the vector bundles as in the proof of 4.1.10. On the line $\Gamma_{\frac{1}{2}-\gamma}$, $a(z)$ is a parameter-dependent operator in Boutet de Monvel's calculus. We can write

$$a(\frac{1}{2} + i\tau) = \sum_{j=0}^{d} a_j(\tau)\partial_r^j + \sum_{j=0}^{d} r_j(\tau)\partial_r^j,$$

where the a_j are local terms, given by symbols of order $\mu - j$ and type zero, and the second sum defines a global contribution: each $r_j(\tau)$ acts as an integral operator, and the associated kernel is a rapidly decreasing function of τ taking values in $C^\infty(\overline{X} \times \overline{X})$.
From 3.1.7 and 3.1.8 one concludes that the normal derivative ∂_r maps $\mathcal{H}^{s,\gamma}(X^\wedge)$ to $\mathcal{H}^{s-1,\gamma}(X^\wedge)$ for $s > \frac{1}{2}$. Moreover, the integral operators induced by the r_j are continuous on $\mathcal{H}^{s,\gamma}(X^\wedge)$.
So we can focus on the first sum. In view of its locality and the above considerations on the normal derivative we may assume that we are dealing with a single parameter-dependent operator $a = a(\tau)$ of order μ and type zero in Boutet de Monvel's algebra on $\mathbf{R}_+^n$, supported by a compact set, uniformly in τ. Now use the observation made in 3.1.8 that $M_{\gamma-\frac{n}{2}}\mathcal{H}^{s,\gamma}(\mathbf{R}_+^n \times \mathbf{R}_+) = \mathcal{F}_{n+1}H^s(\mathbf{R}_+^n \times \mathbf{R})$. The index $n+1$ for $\mathcal{F}$ indicates that the action is with respect to the last variable. Applying additionally the Fourier transform with respect to the first $n-1$ variables, $\mathcal{F}'$, the space $\mathcal{F}_{n+1}H^s(\mathbf{R}_+^n \times \mathbf{R})$ is mapped to $\mathcal{W}^s(\mathbf{R}^{n-1} \times \mathbf{R}, H^s(\mathbf{R}_+))$. From all this we conclude that

$$\mathrm{op}_M^{\gamma-\frac{n}{2}} a = M_{\gamma-\frac{n}{2}}^{-1}\mathcal{F}'^{-1}\mathrm{op}_{x_n} a\mathcal{F}' M_{\gamma-\frac{n}{2}} : \mathcal{H}^{s,\gamma}(\mathbf{R}_+^n \times \mathbf{R}_+) \to \mathcal{H}^{s-\mu,\gamma}(\mathbf{R}_+^n \times \mathbf{R}_+)$$

is continuous if and only if

$$(\mathcal{F}'\mathcal{F}_{n+1})^{-1}\, \mathrm{op}_{x_n} a\, (\mathcal{F}'\mathcal{F}_{n+1}) : \mathcal{W}^s(\mathbf{R}^n \times \mathbf{R}, H^s(\mathbf{R}_+)) \to \mathcal{W}^{s-\mu}(\mathbf{R}^n \times \mathbf{R}, H^{s-\mu}(\mathbf{R}_+))$$

is bounded. The latter fact, however, was proven in 2.2.19. Notice that we can omit the subscripts *comp* and *loc*, for $a(\tau)$ is compactly supported. $\triangleleft$

For completeness we note the lemma, below.

4.1.13 Lemma. *Use the notation of* 4.1.10 *and assume additionally that* $d = 0, s \geq 0$. *Then the operator* $A = \mathrm{op}_M^\gamma a$ *has a formal adjoint* A^* *with respect to the dualities*

$$\mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \oplus \mathcal{H}^{s,\gamma+\frac{n}{2}-\frac{1}{2}}(Y^\wedge, V_3), \mathcal{H}_0^{-s,-\gamma-\frac{n}{2}}(X^\wedge, V_1) \oplus \mathcal{H}^{-s,-\gamma-\frac{n}{2}-\frac{1}{2}}(Y^\wedge, V_3)$$

and

$$\mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \oplus \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}-\frac{1}{2}}(Y^\wedge, V_4), \mathcal{H}_{\{0\}}^{-s+\mu,-\gamma-\frac{n}{2}}(X^\wedge, V_2) \oplus \mathcal{H}^{-s+\mu,-\gamma-\frac{n}{2}-\frac{1}{2}}(Y^\wedge, V_4).$$

Here, the index $\{0\}$ *means that we use the* $\mathcal{H}_0-$*spaces for* $s - \mu \geq 0$ *and the usual* $\mathcal{H}-$*spaces otherwise, cf.* 3.3.1. *We have*

$$A^* = \mathrm{op}_M^{-\gamma-n} a^{(*)} \quad \text{with} \quad a^{(*)} = a(n+1-\bar{z})^*; \tag{1}$$

the last asterisk indicates the matrix adjoint. The fact that $a \in M_P^{\mu,0}(X)$ *implies that* $a^{(*)} \in M_Q^{\mu,0}(X)$ *for a resulting asymptotic type* Q.

Proof. Since the type is zero, this is easily deduced from the usual result, cf. [29] 1.1.4, Proposition 16. For completeness, the detailed proof is given in 5.1.10 $\triangleleft$

4.1.14 Theorem. *Let* $a \in M_P^{\mu,d}(X)$, *with* μ, d, P *as in* 4.1.10. *Moreover, let* $\omega, \omega' \in C_0^\infty(\overline{\mathbb{R}}_+)$ *and* $g = (\gamma + \frac{n}{2}, \Theta), \Theta = (\theta, 0]$ *be a weight datum.*
Then for every asymptotic type $Q = (Q_1, Q_2) \in As(X, Y, g)$ *there is an asymptotic type* $R = (R_1, R_2) \in As(X, Y, g)$ *such that*

$$\omega \,\mathrm{op}_M^\gamma(a)\, \omega' : \begin{matrix} \mathcal{K}_{Q_1}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}_{Q_2}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, V_3) \end{matrix} \longrightarrow \begin{matrix} \mathcal{K}_{R_1}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}_{R_2}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, V_4) \end{matrix}$$

is continuous for all $s \geq d$.

Proof. For simplicity we may assume that V_1 and V_2 are one-dimensional and scalar, while $V_3 = V_4 = 0$ and that $\gamma = 0$. In view of the definition of the spaces $\mathcal{K}_P^{s,\gamma}(X^\wedge)$, cf. 3.1.18, we may also assume that $\Theta = (\theta, 0]$ is a finite weight interval. Supposing that Q_1 has the form $\{(q_j, n_j, N_j)\}$, write $u \in \mathcal{K}_{Q_1}^{s,\frac{n}{2}}(X^\wedge)$ in the form

$$u(x,t) = \sum_{j=0}^{J} \sum_{k=0}^{n_j} c_{jk}(x) t^{-q_j} \ln^k t\, \omega_1(t) + u_0(x,t) = u_1(x,t) + u_0(x,t)$$

with $c_{jk} \in N_j$, a cut-off function ω_1, and $u_0 \in \mathcal{K}_\Theta^{s,\frac{n}{2}}(X^\wedge)$.
Writing $P = \{(p_j, m_j, L_j)\}$ and supposing that $\pi_{\mathbb{C}} P \cap \left\{\frac{1}{2} + \theta < \operatorname{Re} z \leq \frac{1}{2}\right\} = \{p_{j_1}, \ldots, p_{j_2}\}$ we decompose

$$a(z) = a_1(z) + a_0(z),$$

where $a_0(z) \in M_P^{\mu,d}(X)$ is holomorphic in the strip $\left\{\frac{1}{2}+\theta < \operatorname{Re} z \leq \frac{1}{2}\right\}$, and

$$a_1(z) = \sum_{j=j_1}^{j_2} \nu_{jk} M_{t\to z}\left(t^{-p_j} \ln^k t\, \omega_2(t)\right)$$

with an arbitrary cut-off function ω_2 near zero and suitable ν_{jk}, cf. 4.1.4.
Now we consider the terms separately. By 4.1.12, $\omega \operatorname{op}_M^0(a)\omega' : \mathcal{K}_\Theta^{s,\frac{n}{2}}(X^\wedge) \to \mathcal{K}_\Theta^{s-\mu,\frac{n}{2}}(X^\wedge)$ is continuous. So we only have to consider the action of $\omega \operatorname{op}_M^0(a)\omega'$ on u_1. By linearity, it is sufficient to assume that

$$u_1 = c\,t^{-p} \ln^k t\, \omega_1(t) \tag{1}$$
$$a_1 = \nu\, M_{t\to z}(t^{-q} \ln^l t\, \omega_2(t)) \tag{2}$$

for fixed $p, q \in \mathbf{C}, k, l \in \mathbf{N}, \nu \in \mathcal{B}^{-\infty,d}(X), c \in C^\infty(X)$. By 5.1.6

$$M_{t\to z}(t^{-p} \ln^k t\, \omega_1(t)) = \frac{d^k}{dz^k}(z^{-1}\Phi(z))(z-p),$$

where Φ is the Mellin transform of a $C_0^\infty(\mathbf{R}_+)$ function. Applying the same argument to a_1, $a_1(Mu_1) = \nu(c)\psi(z)$, where ψ is a meromorphic function on $\mathbf{C}$ with poles at p and q, possibly $p = q$.
From Theorem 5.1.7 we know that ψ is rapidly decreasing outside the poles on all lines Γ_β. So we may choose coefficients $d_j, e_j, j = 1, \ldots, N$ such that

$$\psi(z) - \sum_{j=0}^{N} M_{t\to z}(d_j t^{-p} \ln^j t\, \omega_1(t) + e_j t^{-q} \ln^j t\, \omega_1(t)) \tag{3}$$

is entire and therefore satisfies the estimates 5.1.7(2). Hence it is the Mellin transform of a $C_0^\infty(\mathbf{R}_+)$ function, and $a_1(Mu_1) \in \mathcal{K}_{R_1}^{s-\mu,\frac{n}{2}}(X^\wedge)$ provided the asymptotic type takes care of the singularities arising from (3).
The argument for $\omega \operatorname{op}_M^0 a_0 \omega' u_1$ is similar: Let $\chi \in C^\infty(\mathbf{C})$ vanish near p and equal 1 outside a neighborhood of p. From the estimates in 4.1.1(3) for a_0 in connection with those for $M_{t\to z}u_1$ we conclude that

$$v(z) := a_0(z)c\chi(z)M_{t\to z}(t^{-p} \ln^k t\omega_1(t)) = O(\langle z\rangle^m)$$

for arbitrary m, uniformly on all lines $\Gamma_\beta, \frac{1}{2}+\theta < \beta \leq \frac{1}{2}$, and with respect to all semi-norms for the topology of $C^\infty(\overline{X})$. Therefore $M^{-1}v \in \mathcal{H}^{s-\mu,\frac{n}{2}-\delta}(X^\wedge), \theta < \delta \leq 0$. Near $z = p$, $a_0(z)c$ is a holomorphic function with values in $C^\infty(\overline{X})$; so we can find coeffcients $d_l \in C^\infty(\overline{X}), l = 0, \ldots, k$ such that

$$a_0(z)cM_{t\to z}(t^{-p} \ln^k t\, \omega_1(t)) - \sum_{l=0}^{k} d_l M_{t\to z}(t^{-p} \ln^l t\, \omega_1(t)) \tag{4}$$

is holomorphic near $z = p$. We know the behavior of the terms under the summation from 5.1.6, and conclude that

$$\omega M^{-1} a_0(z)cM_{t\to z}(t^{-p} \ln^k t\omega_1(t)) \in \mathcal{K}_R^{s-\mu,\frac{n}{2}}(X^\wedge),$$

provided the asymptotic type R contains entries corresponding to the singularities arising in (4). $\triangleleft$

4.2 Mellin Operators and Green Operators

The following lemma is elementary but useful.

4.2.1 Lemma. *Let f be meromorphic in an open set $U \subseteq \mathbf{C}$; let $p_1, \ldots, p_J$ be the poles of f with respective multiplicities $m_j + 1$. Choose a contour C in U around the poles with winding number 1 for all poles. Define a holomorphic functional ζ_f carried by $\{p_1, \ldots, p_J\}$ by letting*

$$\langle \zeta_f, h \rangle = \frac{1}{2\pi i} \int_C f(z) h(z)\, dz, \qquad h \in \mathcal{A}(\mathbf{C}).$$

Then, for $t > 0$,

$$\left\langle \zeta_f, t^{-z} \right\rangle = \sum_{j=0}^{J} \sum_{k=0}^{m_j} \frac{c_{jk}}{k!} t^{-p_j} \ln^k t.$$

Here, the $c_{jk}, k = 0, \ldots, m_j$ are the coefficients of $(z - p_j)^{-k-1}$ in the principal part of the Laurent expansion of f near p_j.

4.2.2 Proposition. *Let G be a regularizing singular Green operator of type zero with finite-dimensional range, $\gamma \in \mathbf{R}, k \in \mathbf{N}, \omega, \omega_1$ cut-off functions near $0 \in \overline{\mathbf{R}}_+$. Let C be a smooth contour in the half plane $\{\operatorname{Re} z > \frac{1}{2} - \gamma\}$, and suppose C has winding number 1 with respect to the point $p \in \mathbf{C}$.*
Then the operator A defined by

$$Au(t) = \frac{1}{2\pi i} \omega(t) \int_C t^{-z} G(z-p)^{-k-1} M(\omega_1 u)(z)\, dz$$

for $u \in C_0^\infty(\overline{X}^\wedge)$ maps any space $\mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge), s \geq 0$, to the finite-dimensional space of all functions of the form

$$v(x,t) = \sum_{j=0}^{k} c_j(x) t^{-p} \ln^j t\, \omega(t), \qquad c_j \in \operatorname{im} G.$$

Proof. Since $u \in \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge), M(\omega_1 u)$ is holomorphic in $\{\operatorname{Re} z > \frac{1}{2} - \gamma\}$ with values in $H^s(X)$. In particular, near p,

$$M(\omega_1 u)(z) = \sum_{\ell=0}^{k} d_\ell(x)(z-p)^\ell + r(z)$$

with $d_\ell \in H^s(X)$ and r a holomorphic function with a zero of order at least $k+1$ in p. By Lemma 4.2.1,

$$Au(x,t) = \omega(t) \sum_{\ell=0}^{k} G(d_\ell)(x) t^{-p} \ln^{k-\ell} t.$$

This proves the assertion. ◁

4.2.3 Theorem. *Let $a \in M_P^{\mu,d}(X), \mu \in \mathbf{Z}, d \in \mathbf{N}, P$ a Mellin asymptotic type. Moreover, let $\gamma \in \mathbf{R}, \beta \geq 0, \omega, \omega_1 \in C_0^\infty(\overline{\mathbf{R}}_+)$, and suppose that*

$$\pi_{\mathbf{C}} P \cap \Gamma_{\frac{1}{2}-\gamma} = \pi_{\mathbf{C}} P \cap \Gamma_{\frac{1}{2}-\gamma+\beta} = \emptyset.$$

Then

$$\omega\, t^\beta \operatorname{op}_M^\gamma(a)\,\omega_1 - \omega \operatorname{op}_M^\gamma(T^\beta a)\, t^\beta \omega_1 \in C_G^d(X^\wedge, g)_{Q,R} \tag{1}$$

for suitable asymptotic types $Q, R \in As(X,Y,g), g = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2}, (-\infty, 0])$, depending on P. The operator in (1) has finite-dimensional range. It is given as in 4.2.2 by a contour integral around the finitely many singularities of a in the strip between $\Gamma_{\frac{1}{2}-\gamma}$ and $\Gamma_{\frac{1}{2}-\gamma+\beta}$. Recall that $(T^\beta a)(z) = a(z+\beta)$.
For $\beta < 0$, the same is true with the weight datum $g = (\gamma + \frac{n}{2} - \beta, \gamma + \frac{n}{2} + \beta, (-\infty, 0])$.

Proof. (cf. [29], 1.1.4, Theorems 20, 21) Let us first consider the case $\beta \geq 0$. We may assume that the vector bundles a is acting on, cf. 4.1.10(1), are trivial one-dimensional over X and 0 over Y. Applying 4.1.5 and 4.1.8, we may write

$$a = a_0 + a_1 = a_0 + \sum_{l=0}^{d} b_l \partial_{x_n}^l$$

with $a_0 \in M_O^{\mu,d}(X), a_1 \in M_P^{-\infty,d}(X)$, and $b_l \in M_{\tilde{P}}^{-\infty,0}(X)$. Denote by A the operator on the left hand side of (1). Then $A = A_0 + A_1 = A_0 + \sum_{l=0}^{d} B_l \partial_{x_n}^l$ where A_j, B_l are the corresponding operators with a replaced by $a_j, j = 0,1$ and $b_l, l = 0,\dots,d$, respectively. Choose $u \in C_0^\infty(\overline{X}^\wedge)$, and let $v = (2\pi i)^{-1} M(\omega_1 u)$. By 5.1.9

$$\begin{aligned} A_j u(t) &= \omega t^\beta \int_{\Gamma_{\frac{1}{2}-\gamma}} t^{-z} a_j(z) v(z)\, dz - \omega \int_{\Gamma_{\frac{1}{2}-\gamma}} t^{-z} a_j(z+\beta) v(z+\beta)\, dz \\ &= \omega t^\beta \left(\int_{\Gamma_{\frac{1}{2}-\gamma}} t^{-z} a_j(z) v(z)\, dz - \int_{\Gamma_{\frac{1}{2}-\gamma+\beta}} t^{-z} a_j(z) v(z)\, dz \right). \end{aligned}$$

Since v decreases rapidly, Cauchy's integral formula implies that

$$A_j u(t) = \omega t^\beta \int_C t^{-z} a_j(z) v(z)\, dz, \tag{2}$$

where C is a smooth curve around the finitely many poles $p_0, \dots, p_J$ of a_j in the strip $\{\frac{1}{2} - \gamma < \operatorname{Re} z < \frac{1}{2} - \gamma + \beta\}$. Applying Lemma 4.2.1, $A_0 u = 0$, hence $A_0 = 0$.
Now let us show that A_1 is a Green operator of type d. Recall that $a_1 = \sum_{l=0}^{d} b_l \partial_{x_n}^l$. Write

$$b_l(z) = \sum_{j=0}^{J} \sum_{k=0}^{m_j} G_{jkl} (z - p_j)^{-k-1} + h(z),$$

where h is holomorphic in the strip, and G_{jkl} are regularizing singular Green operators of type zero.
Given $s \geq 0$, Proposition 4.2.2 shows that B_l maps $\mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge)$ to $\mathcal{S}_Q^{\gamma+\frac{n}{2}}(X^\wedge)$, continuously, whenever Q is an asymptotic type containing the above singularity data.

The operator $\operatorname{op}_M^\gamma b_l$ has a formal adjoint $\operatorname{op}_M^{-\gamma-n} b_l^{(*)}$ by 4.1.13. Now $b_l^{(*)}(z) = b_l(n+1-\bar{z})^* \in M_{\tilde{P}'}^{-\infty,0}$ with an asymptotic type $\tilde{P}'$ induced by the operation in 4.1.13(1). Moreover,

$$\begin{aligned} B_l^* &= \overline{\omega_1}[\operatorname{op}_M^\gamma b_l]^* t^\beta \overline{\omega} - \overline{\omega_1}\, t^\beta [\operatorname{op}_M^\gamma T^\beta b_l]^* \overline{\omega} \\ &= \overline{\omega_1}\operatorname{op}_M^{-\gamma-n}(b_l{}^{(*)}) t^\beta \overline{\omega} - \overline{\omega_1}\, t^\beta \operatorname{op}_M^{-\gamma-n}(T^{-\beta} b_l^{(*)})\overline{\omega} \\ &= \overline{\omega_1}\, t^\beta \operatorname{op}_M^{-\gamma-n}(c_l)\overline{\omega} - \overline{\omega_1}\operatorname{op}_M^{-\gamma-n}(T^\beta c_l)\overline{\omega} \end{aligned}$$

where $c_l = -T^{-\beta} b_l^{(*)}$. Using 4.2.2 we conclude as above that B_l^* maps $\mathcal{K}^{s,-\gamma-\frac{n}{2}}(X^\wedge)$ to $\mathcal{S}_R^{-\gamma-\frac{n}{2}}(X^\wedge)$ for a suitable asymptotic type R.
Now assume that $\beta < 0$. Write $b = T^\beta a$. Then by (1)

$$\omega\, t^\beta \operatorname{op}_M^\gamma(a)\,\omega_1 = \omega\, t^\beta \operatorname{op}_M^\gamma T^{-\beta}(b)\,\omega_1 = \omega \operatorname{op}_M^\gamma(b)\,\omega_1\, t^\beta + G t^\beta = \omega \operatorname{op}_M^\gamma(T^\beta a)\,\omega_1\, t^\beta + G t^\beta.$$

◁

4.2.4 Remark. In particular, the proof of 4.2.3 shows that the difference

$$\omega\, t^\beta \operatorname{op}_M^\gamma(a)\,\omega_1 - \omega \operatorname{op}_M^\gamma(T^\beta a)\, t^\beta\, \omega_1$$

is zero if a has no singularities in the strip $\{\frac{1}{2} - \gamma \le \operatorname{Re} z \le \frac{1}{2} - \gamma + \beta\}$.

4.2.5 Theorem. *Let $h \in M_P^{-\infty,d}(X)$, $\gamma \in \mathbf{R}$, $\pi_{\mathbf{C}} P \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$. Moreover, let $\omega, \omega_1, \omega_2, \omega_3, \omega_4$ be arbitrary cut-off functions near $0 \in \mathbf{R}$, and $\varphi \in C_0^\infty(\mathbf{R}_+)$. Then*

(a) $\omega \operatorname{op}_M^\gamma(h)\,\varphi \in C_G^0(X^\wedge, g)_{Q,O}$.

(b) $\varphi \operatorname{op}_M^\gamma(h)\,\omega \in C_G^d(X^\wedge, g)_{O,R}$.

(c) $\omega_1 \operatorname{op}_M^\gamma(h)\,\omega_2 - \omega_3 \operatorname{op}_M^\gamma(h)\,\omega_4 \in C_G^d(X^\wedge, g)_{Q,R}$.

In (a), (b) and (c), Q and R are suitable asymptotic types in $As(X,Y,g)$ that can be computed from P; O is the 'zero' asymptotic type, and g is the weight datum $g = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2}, (-\infty, 0])$.

Proof. (a) We may write $h = \sum_{\ell=0}^d h_\ell \partial_{x_n}^\ell$ with $h_\ell \in M_P^{-\infty,0}(X)$. Then

$$\omega \operatorname{op}_M^\gamma h\,\varphi = \sum_{\ell=0}^d \omega \operatorname{op}_M^\gamma h_\ell\, \varphi\, \partial_{x_n}^\ell, \tag{1}$$

so we may assume without loss of generality that $d = 0$. Multiplication by φ maps any space $\mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge)$ continuously into $\mathcal{K}_O^{s,\gamma+\frac{n}{2}}(X^\wedge)$; this in turn is mapped to $[\omega]\,\mathcal{K}_P^{\infty,\gamma+\frac{n}{2}}(X^\wedge) \hookrightarrow \mathcal{S}_P^{\gamma+\frac{n}{2}}(X^\wedge)$ by $\omega \operatorname{op}_M^\gamma h$. The adjoint of $\omega \operatorname{op}_M^\gamma h \varphi$ is $\overline{\varphi} \operatorname{op}_M^{-\gamma-n} h^{(*)} \overline{\omega}$ with $h^{(*)}$ as in 4.1.13. Since $h^{(*)} \in M_Q^{-\infty,0}(X)$, the adjoint maps $\mathcal{K}^{s,-\gamma-\frac{n}{2}}(X^\wedge)$ to $[\overline{\varphi}]\mathcal{K}_O^{\infty,-\gamma-\frac{n}{2}}(X^\wedge) \hookrightarrow \mathcal{S}_O^{-\gamma-\frac{n}{2}}(X^\wedge)$. This proves (a). Note that the type of the resulting operator is zero, since $G\varphi$ is of type zero, if G is a Green operator of type d.
(b) is proven in the same way. Now, the type will remain d.

(c) follows from (a) and (b) writing

$$\omega_1 \mathrm{op}_M^\gamma(h)\omega_2 - \omega_3 \mathrm{op}_M^\gamma(h)\omega_4 = (\omega_1-\omega_3)\mathrm{op}_M^\gamma(h)\omega_2 + \omega_3\mathrm{op}_M^\gamma(h)(\omega_2-\omega_4)$$

and noting that both $(\omega_1-\omega_3)$ and $(\omega_2-\omega_4)$ belong to $C_0^\infty(\mathbf{R}_+)$. ◁

4.2.6 Remark. (a) In the notation of 4.2.3, we have for $f \in C_0^\infty(\overline{X}^\wedge)$ and $\beta \in \mathbf{R}$

$$\begin{aligned} t^\beta \omega\, \mathrm{op}_M^{\gamma-\beta}(a)\,\omega_1 f &= t^\beta \omega \int_{\Gamma_{\frac{1}{2}-\gamma+\beta}} t^{-\zeta} a(\zeta)\, M(\omega_1 f)(\zeta) d\zeta \\ &= \omega \int_{\Gamma_{\frac{1}{2}-\gamma}} t^{-z} a(z+\beta) M(\omega_1 f)(z+\beta) dz \\ &= \omega\, \mathrm{op}_M^\gamma(T^\beta a)\,\omega_1\, t^\beta f. \end{aligned} \quad (1)$$

By 4.2.3, the last operator equals $\omega\, t^\beta \mathrm{op}_M^\gamma(a)\omega_1 f$ modulo a Green operator, say G. Here we have assumed that $\Gamma_{\frac{1}{2}-\gamma} \cap \pi_{\mathbf{C}} P = \emptyset = \Gamma_{\frac{1}{2}-\gamma+\beta} \cap \pi_{\mathbf{C}} P$. For every $j \geq \beta$ we therefore have

$$\omega\, t^j \mathrm{op}_M^\gamma(a)\,\omega_1 - \omega\, t^j \mathrm{op}_M^{\gamma-\beta}(a)\,\omega_1 = t^{j-\beta} G,$$

which also is a Green operator, namely with respect to $(\gamma+\frac{n}{2}, \gamma+\frac{n}{2}, (-\infty,0])$, even $(\gamma+\frac{n}{2}, \gamma+\frac{n}{2}+j-\beta, (-\infty,0])$ for $\beta \geq 0$ and with respec to $(\gamma+\frac{n}{2}-\beta, \gamma+\frac{n}{2}+j, (-\infty,0])$ for $\beta < 0$.

(b) In view of 4.2.4 and the discreteness of the singularity set we note the following consequence: If $j > 0$, $\Gamma_{\frac{1}{2}-\gamma} \cap \pi_{\mathbf{C}} P = \emptyset$, and $\epsilon > 0$ is sufficiently small, then on $C_0^\infty(\overline{X}^\wedge)$

$$\omega\, t^j \mathrm{op}_M^\gamma(a)\,\omega_1 \;=\; \omega\, t^j \mathrm{op}_M^{\gamma-\epsilon}(a)\,\omega_1. \quad (2)$$

Part (a) of this remark is the basis for the proposition below.

4.2.7 Proposition. *Let $\gamma \in \mathbf{R}, j > 0$, and $0 \leq \rho_k, \rho_k' \leq j, k = 1, \ldots, r$. Moreover, let P_k, P_k' be Mellin asymptotic types with $\pi_{\mathbf{C}} P_k \cap \Gamma_{\frac{1}{2}-\gamma+\rho_k} = \emptyset = \pi_{\mathbf{C}} P_k' \cap \Gamma_{\frac{1}{2}-\gamma+\rho_k'}$, and finally let $a_k \in M_{P_k}^{\mu,d}(X), a_k' \in M_{P_k'}^{\mu,d}(X)$. For $\omega, \omega_1 \in C_0^\infty(\overline{\mathbf{R}}_+)$ define*

$$\begin{aligned} A &= \omega\, t^j \sum_{k=1}^r \mathrm{op}_M^{\gamma-\rho_k}(a_k)\,\omega_1, \qquad \text{and} \\ A' &= \omega\, t^j \sum_{k=1}^r \mathrm{op}_M^{\gamma-\rho_k'}(a_k')\,\omega_1. \end{aligned}$$

Then $A - A' \in C_G^d(X^\wedge, g)_{Q,R}$, whenever $\sum_{k=1}^r a_k(z) = \sum_{k=1}^r a_k'(z)$ for all z.
Here, $g = (\gamma+\frac{n}{2}, \gamma+\frac{n}{2}, (-\infty,0])$; Q and R are resulting asymptotic types.

Proof. Choose any β with $0 \leq \beta \leq j$ such that $\pi_{\mathbf{C}} P_k \cap \Gamma_{\frac{1}{2}-\gamma+\beta} = \emptyset = \pi_{\mathbf{C}} P_k' \cap \Gamma_{\frac{1}{2}-\gamma+\beta}$. Consider the operators $\tilde{A} = \omega\, t^j \sum_{k=1}^r \mathrm{op}_M^{\gamma-\beta}(a_k)\,\omega_1$ and $\tilde{A}' = \omega\, t^j \sum_{k=1}^r \mathrm{op}_M^{\gamma-\beta}(a_k')\,\omega_1$. According to 4.2.6(a) we have

$$G_1 = A - \tilde{A} \in C_G^d(X^\wedge, g_1)_{P_1,Q_1} \quad \text{and} \quad G_2 = A' - \tilde{A}' \in C_G^d(X^\wedge, g_2)_{P_2,Q_2}$$

with suitable asymptotic types and weight data. Moreover, $\tilde{A}$ and $\tilde{A}'$ both define bounded maps from $\mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge)$ to $\mathcal{K}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge)$ for $s > d-\frac{1}{2}$, since $\mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma+\frac{n}{2}-\beta}(X^\wedge)$; the latter space is mapped continuously to $\mathcal{K}^{s,\gamma+\frac{n}{2}-\beta+j}(X^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge)$ by $\omega t^j \mathrm{op}_M^{\gamma-\beta}(a_k)\omega_1$ in view of 4.1.12. By 4.2.3 the Green operators G_1 and G_2 are given as contour integrals around finitely many of the singularities of the a_k and a'_k in the strip between $\Gamma_{\frac{1}{2}-\gamma}$ and $\Gamma_{\frac{1}{2}-\gamma+j}$. Writing $G_j = \sum_{k=0}^d G_{jk}\partial_r^k$ the same is true for the G_{jk}. The continuity of $A - \tilde{A}$ and $A' - \tilde{A}'$ on $\mathcal{K}^{s,\gamma+\frac{n}{2}}$ then implies that the weight data G_1 and G_2 can indeed be chosen to be $(\gamma + \frac{n}{2}, \gamma + \frac{n}{2}, (-\infty, 0])$. Noting that by assumption $\tilde{A} - \tilde{A}' = 0$ we obtain the assertion. $\triangleleft$

4.3 The Algebras $C_{M+G}(X^\wedge, g)$ and $C_{M+G}(\mathbb{D}, g)$.

4.3.1 Definition. Let $\mu, \nu \in \mathbf{R}, \mu - \nu \in \mathbf{N}, d \in \mathbf{N}$, and let $g = (\gamma, \gamma - \mu, \Theta)$ be a (double) weight datum, $\gamma \in \mathbf{R}$. We suppose that $\Theta = (-k, 0]$, for some $k \in \mathbf{N}\backslash\{0\}$.
For $d \in \mathbf{N}$ we let $C_{M+G}^{\nu,d}(X^\wedge, g)$ denote the space of all operators $A = A_M + A_G$, where

(i) A_M is a Mellin operator of the form $A_M = t^{-\nu}\sum_{j=0}^{k-1}\omega_j\, t^j\, \mathrm{op}_M^{\gamma_j}(h_j)\,\tilde{\omega}_j$ with

  (i.1) suitable cut-off functions $\omega_j, \tilde{\omega}_j$ near zero,
  (i.2) $\gamma - (\mu - \nu) - j - \frac{n}{2} \le \gamma_j \le \gamma - \frac{n}{2}$,
  (i.3) $h_j \in M_{P_j}^{-\infty,d}(X)$, and
  (i.4) Mellin asymptotic types P_j with $\pi_{\mathbf{C}}P_j \cap \Gamma_{\frac{1}{2}-\gamma_j} = \emptyset$.

(ii) A_G is a Green operator in $C_G^d(X^\wedge, g)_{P,Q}$ for suitable asymptotic types $P, Q \in As(X, Y, g)$.

Clearly, $C_{M+G}^{\nu,d}(X^\wedge, g) \subseteq C_{M+G}^{\mu,d}(X^\wedge, g)$, since

$$t^{-\nu}\sum_{j=0}^{k-1}\omega_j\, t^j\, \mathrm{op}_M^{\gamma_j}(h_j)\,\tilde{\omega}_j = t^{-\mu}\sum_{j=0}^{k-1}\omega_j\, t^{j+\mu-\nu}\, \mathrm{op}_M^{\gamma_j}(h_j)\,\tilde{\omega}_j$$

and $\mu - \nu \in \mathbf{N}$. $C_{M+G}^{\nu,d}(\mathbb{D}, g)$ is the corresponding space where in (ii) we replace $X^\wedge$ by $\mathbb{D}$, and in (i) we additionally make the support of $\omega_j, \tilde{\omega}_j$ so small that the operators are well-defined on the cylindrical parts of $\mathbb{D}$ close to the singularities. In view of 4.2.5 we might also ask that the cut-off functions ω_j and $\tilde{\omega}_j$ are independent of j.

In the following we will assume that $\gamma, \mu, \nu \in \mathbf{R}, d \in \mathbf{N}$, and the weight datum $g = (\gamma, \gamma - \mu, \Theta)$ are fixed with the properties in 4.3.1 unless specified otherwise. In order to also fix the notation suppose that A acts on vector bundles $V_1, \ldots, V_4$ in the following way:

$$A: \begin{array}{c} C_0^\infty(X^\wedge, V_1) \\ \oplus \\ C_0^\infty(Y^\wedge, V_3) \end{array} \to \begin{array}{c} C^\infty(X^\wedge, V_2) \\ \oplus \\ C^\infty(Y^\wedge, V_4). \end{array}$$

4.3.2 Remark. Using Theorem 4.1.5 and the definition of the Green operators, an operator $A \in C^{\nu,d}_{M+G}(X^\wedge, g)$ can be written

$$A = \sum_{j=0}^{d} A_j \begin{bmatrix} \partial_r^j & 0 \\ 0 & 1 \end{bmatrix}$$

with $A_j \in C^{\nu,0}_{M+G}(X^\wedge, g)$.

4.3.3 Theorem. *For operators $A \in C^{\nu,d}_{M+G}(X^\wedge, g)$ and $B \in C^{\nu,d}_{M+G}(\mathbb{D}, g)$ the mappings*

$$A : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2}}(Y^\wedge, V_3) \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}^{\infty,\gamma-\mu-\frac{1}{2}}(Y^\wedge, V_4) \end{array}$$

and

$$B : \begin{array}{c} \mathcal{H}^{s,\gamma}(\mathbb{D}, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma-\frac{1}{2}}(\mathbb{B}, V_3) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}^{\infty,\gamma-\mu}(\mathbb{D}, V_2) \\ \oplus \\ \mathcal{H}^{\infty,\gamma-\mu-\frac{1}{2}}(\mathbb{B}, V_4) \end{array}$$

are continuous for all $s > d - \frac{1}{2}$.
If $P = (P_1, P_2) \in As(X, Y, (\gamma, \Theta))$ is an asymptotic type, then there is a resulting asymptotic type $P' = (P_1', P_2') \in As(X, Y, (\gamma - \mu, \Theta))$ such that

$$A : \begin{array}{c} \mathcal{K}^{s,\gamma}_{P_1}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2}}_{P_2}(Y^\wedge, V_3) \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{\infty,\gamma-\mu}_{P_1'}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}^{\infty,\gamma-\mu-\frac{1}{2}}_{P_2'}(Y^\wedge, V_4) \end{array}$$

and

$$B : \begin{array}{c} \mathcal{H}^{s,\gamma}_{P_1}(\mathbb{D}, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma-\frac{1}{2}}_{P_2}(\mathbb{B}, V_3) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}^{\infty,\gamma-\mu}_{P_1'}(\mathbb{D}, V_2) \\ \oplus \\ \mathcal{H}^{\infty,\gamma-\mu-\frac{1}{2}}_{P_2'}(\mathbb{B}, V_4) \end{array}$$

are continuous for all $s > d - \frac{1}{2}$.
Note: *Since $\Theta = (-k, 0]$ is a finite weight interval, $\pi_{\mathbb{C}} P_1$ and $\pi_{\mathbb{C}} P_2$ are finite sets in the strip $\{\frac{n+1}{2} - \gamma - k < \operatorname{Re} z \leq \frac{n+1}{2} - \gamma\}$; $\pi_{\mathbb{C}} P_1'$ and $\pi_{\mathbb{C}} P_2$ are finite sets in the strip $\{\frac{n+1}{2} + \mu - \gamma - k < \operatorname{Re} z \leq \frac{n+1}{2} + \mu - \gamma\}$, cf.* 3.2.1.

Proof. This is immediate from the definition of the Green operators, see 3.3.1, and, moreover, the mapping properties for Mellin operators: In view of the fact that $\gamma_j \geq \gamma - n/2$, we have $\mathcal{K}^{s,\gamma}(X^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma_j+\frac{n}{2}}(X^\wedge)$; the latter space is mapped continuously to $\mathcal{K}^{\infty,\gamma_j+\frac{n}{2}+j-\nu}(X^\wedge) \hookrightarrow \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge)$ by $\omega t^{-\nu+j} \mathrm{op}_M^{\gamma_j}(h_j)\omega_1$ in view of 4.1.12. ◁

4.3.4 Lemma. *Let $A \in C^{\nu,d}_{M+G}(X^\wedge, g)$ be as above. Given $\alpha, \beta \geq 0$ with $\alpha + \beta \geq k$ we will have*

$$t^\alpha A t^\beta \in C^d_G(X^\wedge, g)_{P',Q'}$$

with resulting asymptotic types P' and Q'. In particular, $C^{\nu,d}_{M+G}(X^\wedge, g) \subset C^d_G(X^\wedge, g)$ for $\mu - \nu \geq k$.

Proof. For simplicity let us assume that we are dealing with a scalar bundle over $X^\wedge$ only. If G is a Green operator, then so is $t^\alpha G t^\beta$, and the type will not increase.
On the other hand suppose that $h_j, P_j, \omega_j, \tilde{\omega}_j$ are as in 4.3.1. By Theorem 4.2.3 we then have

$$t^{-\nu+\alpha}\,\omega_j\, t^j\, \mathrm{op}_M^{\gamma_j}(h_j)\,\tilde{\omega}_j t^\beta \equiv \omega_j\, t^{-\nu+\alpha+\beta+j}\mathrm{op}_M^{\gamma_j}(T^{-\beta}h_j)\,\tilde{\omega}_j \tag{1}$$

modulo $C_G^d(X^\wedge, g)$, provided that $\Gamma_{\frac{1}{2}-\gamma_j-\beta} \cap \pi_{\mathbf{C}} P_j = \emptyset$. Let us show that the operator on the right hand side of (1) also is a Green operator. Applying 4.1.5 and writing $T^{-\beta}h_j = \sum_{k=0}^d b_{jk}\partial_{x_n}^k$ with $b_{jk} \in M_{\tilde{P}_j}^{-\infty,0}(X)$ we see that it is no restriction to assume $d = 0$. For any $s \geq 0$, the right hand side then maps $\mathcal{K}^{s,\gamma}(X^\wedge)$ to $\omega_j\,\mathcal{K}^{\infty,\gamma-\nu+\alpha+\beta}(X^\wedge) \hookrightarrow \mathcal{S}_O^{\gamma-\mu}(X^\wedge)$. Here we employ the assumption that $\alpha+\beta \geq k$. Taking into account the Green operator omitted in (1) we will, however, in general have a non-trivial asymptotic type. The adjoint of the operator on the right hand side of (1) is, according to 4.1.13,

$$\overline{\tilde{\omega}}_j\, \mathrm{op}_M^{-\gamma_j-n}(T^{-\beta}h_j)^{(*)}\, t^{\alpha+\beta+j-\nu}\overline{\omega}_j = \overline{\tilde{\omega}}_j\, t^{\alpha+\beta}\mathrm{op}_M^{-\gamma_j}(T^{\alpha+j}h_j)^{(*)}\, t^{-\nu+j}\,\overline{\omega}_j + Gt^{-\nu+j}$$

with $C_G^0(X^\wedge, h)$, and $h = (-\gamma+\frac{n}{2}, -\gamma+\frac{n}{2}, \Theta)$, provided we again avoid the singularities. So its Mellin part maps $\mathcal{K}^{s,\mu-\gamma}(X^\wedge)$ to $\overline{\tilde{\omega}}_j\,\mathcal{K}^{\infty,-\gamma+\alpha+\beta}(X^\wedge) \hookrightarrow \mathcal{S}_O^{-\gamma}(X^\wedge)$. Here we are using the estimate $-\gamma+(\mu-\nu)+j \geq -\gamma_j - \frac{n}{2} \geq -\gamma$ of 4.3.1 and $\alpha+\beta \geq k$. Again, the Green part might generate non-trivial asymptotics.
In case we would hit the singularity set with either one of the above constructions we make the following consideration. Since the singularity set is discrete, we may commute any slightly smaller power to the left without problems. We then obtain (1) with an exponent $-\nu+\alpha+\beta+j-\epsilon$ for t on the right hand side; the Green operator by which it differs from the operator on the left hand side is given (as in the proof of 4.2.3) by an integration around the finitely many singularities of h_j in the strip between $\Gamma_{\frac{1}{2}-\gamma_j-\beta+\epsilon}$ and $\Gamma_{\frac{1}{2}-\gamma_j}$. For sufficiently small ϵ its range will be independent of ϵ and will be spanned by functions of the form $c(x)t^{-p-\mu}\ln^k t\,\omega(t)$ with $c \in C^\infty(X)$, ω a cut-off function near $0 \in \overline{\mathbf{R}}_+$, and p a singularity in the strip. The range of the operator on the right hand side of (1) will be contained in $\mathcal{S}_O^{\gamma-\mu-\epsilon}(X^\wedge)$. Since this is true for all $\epsilon > 0$ the range is a subset of $\mathcal{S}_O^{\gamma-\epsilon}(X^\wedge)$, cf. Definition 3.2.9. Adding the above Green operator we will have a finite asymptotic type. An analogous argument applies to the adjoint. $\triangleleft$

4.3.5 Definition. Let $A = A_M + A_G \in C_{M+G}^{\nu,d}(X^\wedge, g)$ be as in Definition 4.3.1. Define

$$\sigma_M^{\nu-j}(A) = h_j, j = 0, \ldots, k-(\mu-\nu)-1,$$

and call $\sigma_M^{\nu-j}(A)$ the conormal symbol of order $\nu - j$ of A.
Note that for $j \geq k-(\mu-\nu)$, the operators $\omega_j\, t^{-\nu+j}\,\mathrm{op}_M^{\gamma_j}(a_j)\,\tilde{\omega}_j$ are necessarily Green operators.

4.3.6 Remark. We know from Proposition 4.2.7 that two operators in $C_{M+G}^{\nu,d}(X^\wedge, g)$ which have the same conormal symbols of all order differ only by a Green operator, provided the weights γ_j are suitably chosen.
Vice versa, the conormal symbols $\sigma_M^{\nu-j}(A), j = 0, \ldots, k-(\mu-\nu)-1$, are also well-defined. This follows from the proposition, below, which is of independent interest.

4.3.7 Proposition. *The operator A in 4.3.3 is a Green operator, if and only if $\sigma_M^{\nu-j}(A) = 0, j = 0, \ldots, k-(\mu-\nu)-1$.*

Proof. In view of 4.2.7, we only have to show that the conormal symbols vanish for Green operators. In order to see this, we can essentially use the argument given in [27], Section 1.3.1, Theorem 4. For simplicity let us assume that we are dealing with a trivial scalar bundle over $X^\wedge$ only. Choose a cut-off function ω near zero and a function $\phi \in C_0^\infty(X)$. For $p \in \mathbf{C}$ with $\operatorname{Re} p < \frac{1}{2}$ let $u_p(t) = t^{-p}\omega(t)\cdot\phi(x)$. By 5.1.6(b) we have $M_{t\to z}u_p = v(z-p)$, where

$$v(z) = [M_{t\to z}\omega](z) \cdot \phi = (\frac{c}{z} + f(z)) \cdot \phi \tag{1}$$

with an entire function f and some $c \neq 0$. Choose an operator

$$A = t^{-\mu}\sum_{j=0}^{k-1}\omega_j\, t^j \operatorname{op}_M^{\gamma_j}(h_j)\,\tilde{\omega}_j + G$$

with the notation of 4.3.1. We will show that we can recover the functions h_j by considering $M_{t\to z}Au_p$.

We may choose ω with support very close to zero. Therefore it is no loss of generality to ask that $\tilde{\omega}_j\,\omega = \omega$; in other words, the functions $\tilde{\omega}_j$ can be ignored in our consideration of Au_p.

Let us now analyze the effects of the various operators starting with G. We have $G = \sum_{j=0}^{d} G_j\partial_r^j$ with G_j of type zero. The normal derivatives do not affect the form of (1), so assume that $G = G_0$. For each fixed p, Gu_p belongs to $\mathcal{K}_R^{\infty,\gamma-\mu}(X^\wedge)$ for a certain finite asymptotic type R independent of p. Suppose p varies over a bounded open set K. Since u_p depends holomorphically on p, $M_{t\to z}Gu_p$ will be a holomorphic function of both, z and p, as p varies over K and z varies over a set of the kind

$$R_{\alpha,\beta} = \{z \in \mathbf{C} : \alpha < \operatorname{Re} z < \beta, \operatorname{Im} z > k_0\}, \tag{2}$$

where k_0 is a constant depending on K, α, and $\beta, \alpha < \beta$ arbitrary.

Now consider $\omega_j t^j \operatorname{op}_M^{\gamma_j} h_j u_p$. We first assume that $\pi_{\mathbf{C}}P_j \cap \Gamma_{\frac{1}{2}} = \emptyset$. Then we have $\omega_j t^j \operatorname{op}_M^{\gamma_j} h_j u_p = \omega_j t^j \operatorname{op}_M^0 h_j u_p + t^{j+\gamma_j}G_1 u_p$ according to 4.2.6(a), with a Green operator G_1. We may apply the above argument and see that $M_{t\to z}t^{j+\gamma_j}G_1u_p$ is a holomorphic function of z and p whenever p runs over a bounded set and z over some $R_{\alpha,\beta}$. The functions u_p all belong to $L^2(X^\wedge)$, hence also $\operatorname{op}_M^0 h_j u_p$. By 5.1.8 the Mellin transform of $(1-\omega_j)t^j \operatorname{op}_M^0 h_j u_p$ is holomorphic in $\{\operatorname{Re} z < \frac{1}{2} + j\}$. Finally

$$M_{t\to z}[t^j \operatorname{op}_M^0(h_j)\,u_p](z) = M_{t\to z}[\operatorname{op}_M^0(h_j)\,u_p](z+j) = h_j(z+j)v(z+j-p).$$

Now we fix j_0 and choose K a small neighborhood of 0. The we pick a z_0 with imaginary part so large such that all functions $h_j(z+j)$ and all functions $M_{t\to z}Gu_p$ are holomorphic near $z = z_0$ for all $p \in K$. We then integrate $M_{t\to z}Au_p$ over a small contour C around $z_0 + j_0$. By Cauchy's formula, the holomorphic contributions vanish, and (1) implies that

$$\frac{1}{2\pi i}\int_C M_{t\to z}Au_p\,dp = c\; h_{j_0}(z_0+j_0)\phi.$$

Since we may vary z_0 slightly and since we know that the h_j are meromorphic functions, we conclude that h_{j_0} and consequently all h_j are uniquely determined by Au_p.

In particular, if A is a Green operator then all h_j are zero.
Should $\pi_{\mathbb{C}} P_j \cap \Gamma_{\frac{1}{2}-j}$ be non-empty, then $\pi_{\mathbb{C}} P_j \cap \Gamma_{\frac{1}{2}-j-\epsilon} = \emptyset$ and $\pi_{\mathbb{C}} P_j \cap \Gamma_{\frac{1}{2}-\gamma_j-\epsilon} = \emptyset$ for some $\epsilon > 0$. With 4.2.3

$$\begin{aligned} t^j \mathrm{op}_M^{\gamma_j} h_j &= t^{-\epsilon}(t^j t^\epsilon \mathrm{op}_M^{\gamma_j} h_j) \\ &= t^{-\epsilon}(t^j \mathrm{op}_M^{\gamma_j} h_j(\cdot + \epsilon) t^\epsilon + G_2). \end{aligned} \tag{3}$$

Now apply the preceding argument to $t^\epsilon A$. This operator acts with respect to the weight datum $g_\epsilon = (\gamma, \gamma - \mu + \epsilon, \Theta)$, it is a Green operator if and only A is. Thus we also obtain the assertion for A. ◁

4.3.8 Theorem. *Let $A \in C^{\nu,0}_{M+G}(X^\wedge, g), g = (\gamma, \gamma - \mu, \Theta)$. Then the formal adjoint A^* of A belongs to $C^{\nu,0}_{M+G}(X^\wedge, h), h = (-\gamma + \mu, -\gamma, \Theta)$.*

Proof. Let A be as in 4.3.1. Consider the formal adjoint $A^* = A_M^* + A_G^*$. By assumption, A_G is of type zero, so it is immediate from 3.3.1(1), (2) that $A_G^* \in C_G^0(X^\wedge, g)_{Q,P}$.
By 4.1.13

$$\begin{aligned} [\omega_j t^{j-\nu} \mathrm{op}_M^{\gamma_j}(h_j)\tilde{\omega}_j]^* &= t^{-\nu} t^\nu \overline{\tilde{\omega}}_j \mathrm{op}_M^{-\gamma_j - n} h_j^{(*)} t^{j-\nu} \overline{\omega}_j \\ &= t^{-\nu} \overline{\tilde{\omega}}_j \mathrm{op}_M^{-\gamma_j - n + \nu}[T^\nu h_j^{(*)}]\, t^j \bar{\omega}_j \\ &= t^{-\nu} \overline{\tilde{\omega}}_j\, t^j \mathrm{op}_M^{-\gamma_j - n + \nu - j}[T^{\nu - j} h_j^{(*)}]\, \bar{\omega}_j \end{aligned} \tag{1}$$

according to 4.2.6(a). Since

$$(\mu - \gamma) - (\mu - \nu) - j - \frac{n}{2} = -\gamma + \nu - j - \frac{n}{2} \le -\gamma_j - n + \nu - j \le -\gamma + \mu - \frac{n}{2}$$

we obtain the assertion. ◁

4.3.9 Theorem. *Let $A \in C^{\nu,d}_{M+G}(X^\wedge, g), H \in C_G^{d'}(X^\wedge, h)_{Q,R}, K \in C_G^{d'}(X^\wedge, k)_{S,T}$, where $h = (\gamma - \mu, \delta, \Theta), k = (\delta, \gamma, \Theta)$, and Q, R, S, T are corresponding asymptotic types. Then*

$$HA \in C_G^d(X^\wedge, h_1)_{Q,\tilde{R}} \tag{1}$$

$$AK \in C_G^{d'}(X^\wedge, k_1)_{\tilde{Q},R} \tag{2}$$

with $h_1 = (\gamma, \delta, \Theta), k_1 = (\delta, \gamma - \mu, \Theta)$ and resulting asymptotic types $\tilde{Q}, \tilde{R}$.

Proof. As usual, it suffices to prove the case where the operators act on a trivial one-dimensional vector bundle over $X^\wedge$ only. Using 4.3.2 and the definition of Green operators write $A = \sum_{j=0}^d A_j \partial_r^j, H = \sum_{l=0}^{d'} H_l \partial_r^l$, where A_j and H_l are of type zero. Now

$$HA = \sum_{j=0}^{d} (\sum_{l=0}^{d'} H_l \partial_r^l A_j) \partial_r^j.$$

Let us show that each of the terms $H_l \partial_r^l A_j$ is a Green operator of type zero. In order to see this notice first that $A_{lj} := \partial_r^l A_j \in C^{\nu,0}_{M+G}(X^\wedge, g)$ as a result of the composition rules

in Boutet de Monvel's calculus (for the Mellin part of A_j) and of the representation of the Green operators of type zero as integral operators, cf. 3.3.2 (for the Green part of A_j).
Now the assertion is immediate: A_{jl} maps $\mathcal{K}^{s,\gamma}(X^\wedge)$ to $\mathcal{K}^{\infty,\gamma-\mu}(X^\wedge)$ which is in turn mapped to $\mathcal{S}_Q^\delta(X^\wedge)$ by H_l, cf. 4.3.3 and 3.3.1. The operator A_{jl} has an adjoint in $C_{M+G}^{\nu,0}(X^\wedge, g^*), g^* = (-\gamma+\mu, -\gamma, \Theta)$, for it is of type zero.
Vice versa, for any $s \geq 0, [H_l A_{jl}]^* = A_{jl}^* H_l^*; H_l^*$ maps $\mathcal{K}^{s,\delta}(X^\wedge)$ to $\mathcal{S}_R^{\mu-\gamma}(X^\wedge)$. Applying A_{jl}^*, this space is mapped into $\mathcal{S}_{\tilde{R}}^{-\gamma}(X^\wedge)$ with a resulting asymptotic type according to 4.3.3.
The proof of the statement for AK is similar. ◁

4.3.10 Theorem. *Let $A \in C_{M+G}^{\nu,d}(X^\wedge, g)$ and $B \in C_{M+G}^{\nu',d'}(X^\wedge, h)$ with $h = (\gamma+\mu', \gamma, \Theta)$ and $g = (\gamma, \gamma-\mu, \Theta)$. Then $AB \in C_{M+G}^{\nu+\nu',d'}(X^\wedge, k)$ with $k = (\gamma+\mu', \gamma-\mu, \Theta)$. The conormal symbols satisfy the relations*

$$\sigma_M^{\nu+\nu'-r}(AB) = \sum_{p+q=r} \left[T^{\nu'-q}\,\sigma_M^{\nu-p}(A)\right]\,\sigma_M^{\nu'-q}(B).$$

Proof. We already have dealt in 4.3.9 with products of elements in $C_{M+G}^{\nu,d}(X^\wedge, g)$ with Green operators from the left and the right. It is therefore sufficient to treat a product of the form

$$t^{-\nu+j}\omega\,\mathrm{op}_M^{\gamma_1}(a)\,\omega_1 t^{-\nu'+l}\mathrm{op}_M^{\gamma_2}(b)\omega_2. \tag{1}$$

Applying equality 4.2.6(1), (1) equals

$$t^{-\nu-\nu'+j+l}\omega\,\mathrm{op}_M^{\gamma_1+\nu'-l}(T^{\nu'-l}a)\,\omega_1\mathrm{op}_M^{\gamma_2}(b)\,\omega_2. \tag{2}$$

In general, we can now push the weights $\gamma_1+\nu'-l$ and γ_2 to the common possible weight $\gamma_3 = \gamma+\nu'-l-\frac{n}{2}$. Moreover, we may omit the factor ω_1 in the middle, producing an error term which is a Green operator. If we believe both statements for the moment, the fact that $\mathrm{op}_M^{\gamma_3}(T^{\nu'-l}a)\mathrm{op}_M^{\gamma_3}b = \mathrm{op}_M^{\gamma_3}[(T^{\nu'-l}a)\,b]$ gives the assertion.
The above choice of the weight is not possible in case either a or b has a singularity on $\Gamma_{\frac{n+1}{2}-\gamma-\nu+l}$. In view of 4.3.1(i) this can only happen if we have additional freedom in the choice of either γ_1 or γ_2. Then we may move the weight to a line slightly to the left or the right avoiding all singularities and still have the composition (2) well-defined as an operator from $\mathcal{K}^{s+\mu'}(X^\wedge)$ to $\mathcal{K}^{\gamma-\mu}(X^\wedge)$; the conclusion is as before.
In order to see the statement on the factor in the middle, first assume for simplicity that the types of a and b are zero. Given $u \in \mathcal{K}^{s,\gamma+\mu'}(X^\wedge), \mathrm{op}_M^{\gamma_3}(b)\,\omega_2 u \in \mathcal{H}^{\infty,\gamma+\mu'}(X^\wedge)$. Therefore $v = (1-\omega_1)\,\mathrm{op}_M^{\gamma_3}(b)\,\omega_2\,u \in \mathcal{H}^{\infty,N}(X^\wedge)$ for arbitrary $N > 0$, so that $\omega\,\mathrm{op}_M^{\gamma_3}(T^{\nu'-l}a)\,v \in [\omega_1]\mathcal{K}^{\infty,N}(X^\wedge) \hookrightarrow \mathcal{S}_O^{\gamma+\mu'}(X^\wedge)$. This mapping also is continuous as a consequence of the closed graph theorem and the fact that $\omega\,\mathrm{op}_M^{\gamma_3}(T^{\nu'-l}a)\,(1-\omega_1)\,\mathrm{op}_M^{\gamma_3}b\,\omega_2$ is continuous on $\mathcal{K}^{s,\gamma+\mu'}(X^\wedge)$. Since we had assumed the types to be zero, we may now apply the same argument to the adjoints. The general case presents no additional difficulties. ◁

For completeness we note the following result.

4.3.11 Lemma. *Let A be as in 4.3.1 and assume additionally that $\mu = 0, 0 \neq -\nu \in \mathbf{N}$. Then*

$$A : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge)$$

is compact for every $s > -\frac{1}{2}$.

Proof. This is immediate from the continuity properties in 4.3.3 and the fact that the image of the Mellin part of A is in fact contained in $\mathcal{K}^{\infty,\gamma+\nu}(X^\wedge)$ which is compact in $\mathcal{K}^{s,\gamma}(X^\wedge)$. ◁

4.3.12 Definition. Let $A \in C^{0,d}_{M+G}(X^\wedge, g), \gamma \in \mathbf{R}, d \in \mathbf{N}, g = (\gamma, \gamma, \Theta), \Theta = (-k, 0]$. We shall say that the operator $I + A$ is elliptic, if

$$I + \sigma^0_M(A)(z) : H^s(X, V_1) \oplus H^s(Y, V_3) \to H^s(X, V_1) \oplus H^s(Y, V_3)$$

is invertible for all $s > d - \frac{1}{2}$ and all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$.

As a preparation for the proof of Theorem 4.3.15, below, we need the following teo lemmata.

4.3.13 Lemma. *Let P be a Mellin asymptotic type, $d \in \mathbf{N}$, and $h \in M_P^{-\infty,d}(X)$. Then $I + h(z) \in \mathcal{B}^{-\infty,d}(X)$ is an invertible operator on $H^s(X, V_1) \oplus H^s(Y, V_3), s > d - \frac{1}{2}$, for all but countably many $z \in \mathbf{C}$. Moreover, there is a Mellin asymptotic type Q and an $f \in M_Q^{-\infty,d}(X)$ such that*

$$[I + h(z)]^{-1} = I + f(z).$$

Proof. For each $z \in \mathbf{C}\backslash\pi_{\mathbf{C}}P, h(z)$ is a regularizing operator of type d in Boutet de Monvel's calculus. Therefore $\{I+h(z) : z \in \mathbf{C}\backslash\pi_{\mathbf{C}}P\}$ is a holomorphic family of Fredholm operators on $H^s(X, V_1) \oplus H^s(Y, V_3)$. For each $\beta \in \mathbf{R}$ we know that $h(\beta + i\tau) \in \mathcal{B}^{-\infty,d}(X; \mathbf{R}_\tau)$; the corresponding estimates are uniform for β in compact intervals. Consequently,

$$\|h(\beta + i\tau)\| = O(\langle\tau\rangle^{-1})$$

on each such strip so that $I + h(z)$ will be invertible for large imaginary parts (here the norm is taken in $\mathcal{L}(H^s(X, V_1) \oplus H^s(Y, V_3))$, and the estimate is in fact much better).
Therefore, a well-known theorem from operator theory, cf. [27], Section 2.2.5, asserts that $[I+h(z)]^{-1}$ exists on $\mathbf{C}\backslash\pi_{\mathbf{C}}P$ except for a discrete set which can have accumulation points only in the singularity set of P. Moreover, the above theorem states that all singularities of $[I + h(z)]^{-1}$ are poles and that the coefficients in the Laurent expansion in these poles are finite rank operators.
Let us show that the singularities of $[I + h(z)]^{-1}$ have no accumulation point. For each pole, say z_0, of h, we can write $I + h(z) = I + h_0(z) + \sum_{j=1}^M F_j(z - z_0)^{-j}$ with suitable $M \in \mathbf{N}, h_0 \in M_P^{-\infty,d}(X)$ holomorphic near z_0, and finite rank operators F_j. Note that $h_0 \not\equiv -I$. Writing $h_s(z) = \sum_{j=1}^M F_j(z - z_0)^{-j}$ we have

$$I + h(z) = (I + h_0(z))\,(I + (I + h_0(z))^{-1}h_s(z)).$$

Since h_0 is holomorphic near z_0, and $I + h_0(z)$ is Fredholm and invertible for large imaginary parts, the above-mentioned theorem asserts that the inverse to the first factor on the right hand side exists in a small neighborhood of z_0 except possibly in z_0. For the second we notice that $(I + h_0(z))^{-1}h_s(z)$ can be written in the form $H(z) + \sum_{k=1}^{N} A_k(z - z_0)^{-k}$ with suitable H, holomorphic near z_0, and finite rank operators $A_1, \ldots, A_N$. Moreover, the operator $(I + h_0(z))^{-1}h_s(z)$ will vanish on the intersection $\mathcal{N}_0$ of the kernels of the operators $F_j, j = 1, \ldots, M$, which is finite-codimensional. Denoting by $\mathcal{N}_1$ the intersection of the kernels of the operators $A_1, \ldots, A_N$, we have $H(z)u = 0$ for all u in the finite-codimensional space $\mathcal{N}_0 \cap \mathcal{N}_1$. Lemma 4.3.14, below, therefore implies that the second factor also is invertible near z_0 although possibly not for $z = z_0$.

Let us now have a closer look at the inverse to $I + h(z)$. From Theorem 2.3.8 we know that, for fixed $z, [I + h(z)]^{-1} = I + f(z)$ with $f(z) \in \mathcal{B}^{-\infty,d}(X)$. By the above considerations, the singularities of f are all poles, they have no accumulation points in finite strips $\{c_1 \le \mathrm{Re}\,(z) \le c_2\}$, and the coefficients in the Laurent expansion are regularizing finite rank operators in Boutet de Monvel's calculus.

It remains to check the condition on the decay of f outside the poles. Write $\|\cdot\|$ for an arbitrary semi-norm in $\mathcal{B}^{-\infty,d}(X)$. Fix a strip $\{c_1 \le \mathrm{Re}\,(z) \le c_2\}$. Instead of multiplying by an excision function of the poles we simply consider f outside the compact set $K = \{z : c_1 \le \mathrm{Re}\, z \le c_2, -R \le \mathrm{Im}\, z \le R\}$, where $R > 0$ is so large that K not only contains all poles of f in the strip, but also $\|h(z)\| < \frac{1}{2}$ outside K. Then $[I + h(z)]^{-1} = O(1)$ outside K. We conclude from the identity $(1 - x)^{-1} = 1 + x(1 - x)^{-1}$ and the estimates on h that for arbitrary $N > 0$ we have $\|f(z)\| = O(\langle \mathrm{Im}\, z\rangle^{-N})$. Hence f is a Mellin symbol with asymptotics. ◁

In the proof we have used the following lemma.

4.3.14 Lemma. *Let $U \subset \mathbb{C}$ be an open neighborhood of 0, E a Banach space, $N \in \mathbb{N}$, and $A_1, \ldots, A_N \in \mathcal{L}(E)$ operators of finite rank. Let H be a holomorphic function on U with values in $\mathcal{L}(E)$ such that $H(z)u = 0$ for all u in a finite-codimensional subspace K_0 of E. Then there is a $\delta > 0$ such that the meromorphic $\mathcal{L}(E)$-valued function*

$$F(z) = I + H(z) + \sum_{k=1}^{N} A_k z^{-k}$$

is invertible for all $0 < |z| < \delta$.

Proof. The intersection K_1 of the kernels of $A_1, \ldots, A_N$ is a closed subspace of E with finite codimension, and so is $K = K_0 \cap K_1$. Hence we may choose a finite-dimensional subspace C of E such that $E = K \oplus C$ is topologically direct. With respect to this decomposition, F has the matrix

$$F(z) = \begin{bmatrix} I & * \\ 0 & * \end{bmatrix}$$

for, if $u \in K$, then $F(z)u = u$. In particular, $F(z) : \mathcal{L}(E) \to \mathcal{L}(E)$ is invertible if and only if $P_C F(z) : C \to C$ is invertible; here P_C is the projection onto C along K. Now

$$P_C F(z) = I_C + P_C H(z) P_C + \sum P_C A_{N-k} P_C z^{-k}$$

can be regarded as a meromorphic matrix-valued function. Applying Cramer's rule, it is invertible in a neighborhood of $z = 0$, except possibly the point 0 itself. $\lhd$

4.3.15 Theorem. *Let $A \in C^{0,d}_{M+G}(X^\wedge, g)$ be as in 4.3.12, and suppose $I + A$ is elliptic. Then*

$$I + A : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2}}(Y^\wedge, V_3) \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2}}(Y^\wedge, V_3) \end{array}$$

is a Fredholm operator for all $s \geq d$.

Proof. Write $h = \sigma^0_M(A)$; denote by P the Mellin asymptotic type of h. By Lemma 4.3.13 $[I + h(z)]^{-1}$ exists outside a discrete set in $\mathbf{C}$ and equals $I + h_1(z)$ for some $h_1 \in M_Q^{-\infty,d}(X)$ and a suitable Mellin asymptotic type Q. By assumption $I + h(z)$ is invertible for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$, so h_1 has no singularity along this line.
We construct a Fredholm inverse $I + B$ to $I + A$ by letting $B = \mathrm{op}^\gamma_M(h_1)$. Clearly, $I + B \in C^{0,d}_{M+G}(X^\wedge, g)$. By Theorem 4.3.10, the identity $(I + h(z))(I + h_1(z)) = I$ implies that $\sigma^0_M((I + A)(I + B) - I) = 0 = \sigma^0_M((I + B)(I + A) - I)$. Therefore $(I + A)(I + B) - I$ and $(I + B)(I + A) - I$ belong to $C^{-1,d}_{M+G}(X^\wedge, g)$, thus are compact operators by 4.3.11. $\lhd$

Theorem 4.3.17, below, gives a slightly weaker converse of Theorem 4.3.15. The following lemma shows that the weights γ can be shifted to arbitrary values by 'weight conjugation'.

4.3.16 Lemma. Weight Conjugation. *Let $g = (\gamma, \gamma - \mu, \Theta)$ and $k = k(t) \in C^\infty(\mathbf{R}_+)$ be a strictly positive function equal to t for small t and equal to 1 for large t. Then for arbitrary $\rho \in \mathbf{R}$*

$$\begin{bmatrix} k^\rho & 0 \\ 0 & k^\rho \end{bmatrix} C^{\nu,d}_{M+G}(X^\wedge, g) \begin{bmatrix} k^{-\rho} & 0 \\ 0 & k^{-\rho} \end{bmatrix} = C^{\nu,d}_{M+G}(X^\wedge, g_\rho)$$

with $g_\rho = (\gamma + \rho, \gamma + \rho - \mu, \Theta)$.

Proof. In view of the fact that for arbitrary s, γ, ρ

$$\begin{array}{rcl} k^\rho : \mathcal{K}^{s,\gamma}(X^\wedge) & \rightarrow & \mathcal{K}^{s,\gamma+\rho}(X^\wedge) \\ \text{and} \quad k^\rho : \mathcal{K}^{s,\gamma}(Y^\wedge) & \rightarrow & \mathcal{K}^{s,\gamma+\rho}(Y^\wedge) \end{array}$$

is an isomorphism, the assertion is obvious for Green operators.
For an operator of the form $\omega(t)\, t^{-\nu+j}\, \mathrm{op}^{\gamma_j}_M h_j\, \tilde{\omega}(t)$ we may assume that the support of ω and $\tilde{\omega}$ is so small that, there, k^ρ and t^ρ coincide. Then we obtain the assertion from the identity $t^\rho \mathrm{op}^{\gamma_j}_M h_j t^{-\rho} = \mathrm{op}^{\gamma_j+\rho}_M(T^\rho h_j)$. $\lhd$

4.3.17 Theorem. *Let $A \in C^{0,0}_{M+G}(X^\wedge, g)$, and suppose that*

$$I + A : \mathcal{K}^{0,\gamma}(X^\wedge, V_1) \oplus \mathcal{K}^{0,\gamma-\frac{1}{2}}(Y^\wedge, V_3) \rightarrow \mathcal{K}^{0,\gamma}(X^\wedge, V_1) \oplus \mathcal{K}^{0,\gamma-\frac{1}{2}}(Y^\wedge, V_3)$$

is a Fredholm operator. Then $I + A$ is elliptic.

Proof. In view of Lemma 4.3.14 we may assume that $\gamma = \frac{n}{2}$. Then $\mathcal{K}^{0,\gamma}(X^\wedge, V_1) \oplus \mathcal{K}^{0,\gamma-\frac{1}{2}}(Y^\wedge, V_3) = L^2(X, V_1) \oplus L^2(Y, V_3)$. In fact the proof, below, shows that the case of vector bundles and the presence of Y only causes notational difficulties. Let us therefore prove the statement for the case of a trivial one-dimensional vector bundle over X only. The Fredholm property implies an *a priori* estimate: There is a compact operator K and a $C > 0$ such that

$$\|u\|_{L^2(X^\wedge)} \leq C\,\|(I+A)u\|_{L^2(X^\wedge)} + \|Ku\|_{L^2(X^\wedge)}.$$

We will now assume that there is a $z_0 \in \Gamma_{\frac{n+1}{2}-\gamma} = \Gamma_{\frac{1}{2}}$ such that

$$I + \sigma_M^0(A)(z_0) : H^0(X) \to H^0(X)$$

is *not* invertible and show that this leads to a contradiction to the above *a priori* estimate by constructing a sequence of functions $\{\phi_m\}$ in $\mathcal{K}^{0,\frac{n}{2}}(X^\wedge)$ with norm equal to 1, ϕ_m converging weakly to zero, and $(I+A)\phi_m \to 0$.
To this end identify first $H^0(X)$ and $L^2(X)$, write $a = \sigma_M^0(A), h = I + a$. If $h(z_0)$ is not invertible, then there is a sequence $\{u_m\} \subseteq C_0^\infty(X)$ with $h(z_0)u_m \to 0$ in $L^2(X)$ while $\|u_m\|_{L^2(X)} = 1, m = 1, 2, \ldots$.
Now let $0 \neq f \in C_0^\infty(\mathbf{R}_+), \beta, \beta_0, \delta \in \mathbf{R}, c \in \mathbf{R}_+$. Let $g_c(t) = t^{(c-1)\delta - i\beta_0} f(t^c)$. Then $t^\delta g_c(t) = t^{c\delta - i\beta_0} f(t^c)$, hence

$$\begin{aligned} M(g_c)(\delta + i\beta) &= M(t^\delta g_c)(i\beta) = M([t^c]^{\delta - i\beta_0/c} f(t^c))(i\beta) \\ &= c^{-1} M(t^{\delta - i\beta_0/c} f)(i\beta/c) = c^{-1}(Mf)(\delta + i\frac{\beta - \beta_0}{c}). \end{aligned}$$

Let $z_0 = \frac{1}{2} + i\beta_0$, and replace δ by $\frac{1}{2}$. Then the above identities yield

$$(Mg_c)(\frac{1}{2} + i\beta) = c^{-1}(Mf)(\frac{1}{2} + i\frac{\beta - \beta_0}{c}),$$

hence

$$\begin{aligned} &\int_{\Gamma_{\frac{1}{2}}} |Mg_c(z)|^2\, |dz| = \int_{\mathbf{R}} |Mg_c(\frac{1}{2} + i\beta)|^2\, d\beta \\ &= c^{-2} \int_{\mathbf{R}} |Mf(\frac{1}{2} + i\frac{\beta - \beta_0}{c})|^2\, d\beta = c^{-1} \int_{\mathbf{R}} |Mf(\frac{1}{2} + iu)|^2\, du. \end{aligned}$$

In other words: For all $c > 0$ and all $m \in \mathbf{N}$, the norm of $c^{\frac{1}{2}} g_c u_m$ in $\mathcal{K}^{0,\frac{n}{2}}(X^\wedge) = L^2(X^\wedge)$ is constant. Moreover, as $c \to 0$ the functions $c^{\frac{1}{2}} g_c u_m$ weakly tend to zero. In order to see this, choose arbitrary functions $\phi \in C_0^\infty(\mathbf{R}_+), v \in C^\infty(X)$. Then

$$\begin{aligned} &|(c^{\frac{1}{2}} g_c u_m, \phi v)_{\mathcal{K}^{0,\frac{n}{2}}(X^\wedge)}| \\ =\ & \int_{\Gamma_{\frac{1}{2}}} (c^{\frac{1}{2}} Mg_c(z) u_m, M\phi v)_{L^2(X)}\, |dz| \\ \leq\ & \int_{\mathbf{R}} |c^{-\frac{1}{2}} Mf(\frac{1}{2} + i\frac{\beta - \beta_0}{c}) M\phi(\frac{1}{2} + i\beta)|\, d\beta\ \ \|u_m\|_{L^2(X)} \|v\|_{L^2(X)}, \\ \leq\ & \text{const}\ c^{\frac{1}{2}} \int_{\mathbf{R}} |Mf(\frac{1}{2} + is) M\phi(\frac{1}{2} + i(sc + \beta_0))|\, ds\ \ \|u_m\|_{L^2(X)} \|v\|_{L^2(X)}, \\ \leq\ & \text{const}\ c^{\frac{1}{2}} \int_{\mathbf{R}} |Mf(\frac{1}{2} + is)| ds\ \|M\phi\|_{\sup} \|u_m\|_{L^2(X)} \|v\|_{L^2(X)} \to 0 \end{aligned}$$

as $c \to 0$, since $Mf \in L^1(\mathbf{R}_+)$.
Finally let us show that $(I+A)(c^{\frac{1}{2}} g_c(t) u_m(x))$ tends to zero in $\mathcal{K}^{0,\frac{n}{2}}(X^\wedge)$ as $m \to \infty$ and $c \to 0$.

$$
\begin{aligned}
& \|\mathrm{op}_M^0 h(c^{\frac{1}{2}} g_c u_m)\|^2_{\mathcal{K}^{0,\frac{n}{2}}(X^\wedge)} \\
= & \int_{\Gamma_{\frac{1}{2}}} \|h(z) c^{\frac{1}{2}} M g_c(z) u_m\|^2_{L^2(X)} \, |dz| \\
= & \int_{\mathbf{R}} \|h(\frac{1}{2} + i\beta)\, c^{-\frac{1}{2}} M f(\frac{1}{2} + i\,\frac{\beta-\beta_0}{c}) u_m(x)\|^2_{L^2(X)} \, d\beta \\
= & \int_{\mathbf{R}} \|h(\frac{1}{2} + i\beta_0 + ics)\, Mf(\frac{1}{2} + is)\, u_m(x)\|^2_{L^2(X)} \, ds.
\end{aligned}
$$

The last expression tends to zero as $c \to 0$ and $m \to \infty$ by Lebesgue's theorem on dominated convergence, since
(i) the integrand can be estimated by

$$
|Mf(\frac{1}{2} + is)|^2 \sup_{\substack{0<c\le 1 \\ m\in\mathbf{N}, s\in\mathbf{R}}} \|h(\frac{1}{2} - \gamma + i\beta_0 + ics) u_m\|^2_{L^2(X)}
$$

noting that the first factor is an L^1-function of s, and since
(ii) for each s

$$
\|h(\frac{1}{2} + i\beta_0 + ics) u_m\|^2_{L^2(X)} \to 0 \quad \text{as} \quad m \to \infty \quad \text{and} \quad c \to 0. \tag{1}
$$

Therefore, the sequence $\phi_m(x,t) = m^{-\frac{1}{2}}\, g_{1/m}(t)\, u_m(x), m = 1, 2, \ldots$, will lead to a contradiction to the *a priori* estimate and thus prove our assertion. ◁

4.3.18 Theorem. *Let $A \in C^{0,d}_{M+G}(X^\wedge, g)$ be as in 4.3.12 and let $I+A$ be elliptic. Suppose that for some $s > d - \frac{1}{2}$*

$$
I + A : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2}}(Y^\wedge, V_3) \end{array} \to \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma-\frac{1}{2}}(Y^\wedge, V_3) \end{array}
$$

is invertible. Then there is an $A' \in C^{0,d}_{M+G}(X^\wedge, g)$ such that

$$
(I+A)^{-1} = I + A'.
$$

Proof. Write $\mathcal{K} = \mathcal{K}^{s,\gamma}(X^\wedge, V_1) \oplus \mathcal{K}^{s,\gamma-\frac{1}{2}}(Y^\wedge, V_3)$. Since $I+A$ is elliptic, the construction in Theorem 4.3.15 immediately gives an operator B_1 such that $(I+B_1)(I+A) = I - C$ on $\mathcal{K}$ with $C \in C^{-1,d}_{M+G}(X^\wedge, g)$. Hence for $B = (I + \sum_{j=1}^{k-1} C^j)(I+B_1) - I$

$$
(I + \sum_{j=1}^{k-1} C^j)(I+B_1)(I+A) = I + G,
$$

where, by 4.3.4, $G \in C^{-k,d}_{M+G}(X^\wedge, g)$ is in fact a Green operator in $C^d_G(X^\wedge, g)_{Q,R}$ with suitable asymptotic types Q and R; in particular, it is compact on $\mathcal{K}$.
By a classical result in operator theory, cf. [32], there is an $r \in \mathbf{N}$ such that

$$\begin{aligned} \mathcal{N} &= \mathcal{N}((I+G)^r) = \mathcal{N}\left((I+G)^{r+1}\right) \\ \mathcal{R} &= \mathcal{R}((I+G)^r) = \mathcal{R}\left((I+G)^{r+1}\right) \\ \mathcal{K} &= \mathcal{N} \oplus \mathcal{R} & (1) \\ I+G &: \mathcal{R} \to \mathcal{R} \quad \text{is bijective.} & (2) \end{aligned}$$

The kernel $\mathcal{N}$ of $(I+G)^r$ is a finite-dimensional subspace of $\mathcal{S}_Q(X^\wedge)$. Let $\{\phi_1, \ldots, \phi_m\}$ be an orthonormal basis. Define the operator P on $\mathcal{K}$ by $Pf = \sum_{j=1}^m (f, \phi_j)\phi_j$. Then P is the orthogonal projection onto $\mathcal{N}$. Moreover, it is a Green operator of type zero since it has an integral kernel in $\mathcal{S}_Q(X^\wedge) \otimes \mathcal{S}_{\overline{Q}}(X^\wedge)$; here $\overline{Q}$ is the conjugate asymptotic type, cf. 3.3.2.
Let $f_j = (I+A)\phi_j, j = 1, \ldots, m$. Since the ϕ_j are linearly independent and since $I + A$ is invertible, the f_j will be linearly independent functions in $\mathcal{S}_{\tilde{Q}}(X^\wedge)$ for a suitable asymptotic type $\tilde{Q}$. Now define the operator $F : \mathcal{K} \to \mathcal{K}$ by

$$F(f_j) = \phi_j \quad \text{on span}\{f_1, \ldots, f_m\}, \quad = 0 \quad \text{on span}\{f_1, \ldots, f_m\}^\perp.$$

Then F is a Green operator of type zero, since it has an integral kernel in $\mathcal{S}_Q(X^\wedge) \otimes \mathcal{S}_{\overline{\tilde{Q}}}(X^\wedge)$, and it is easily checked that it is a relative inverse to $(I+A)P$:

(i) $F(I+A)PF = F$, since both sides map each f_j to ϕ_j and functions in the complement of their span to zero, and

(ii) $(I+A)PF(I+A)P = (I+A)P$, for all ϕ_j are mapped to f_j, while the functions orthogonal to span$\{\phi_1, \ldots, \phi_m\}$ are mapped to zero.

Using F we construct an inverse to $I+A$. Let $L = \left([(I+B)(I+A)]^{r-1}(I+B) + F\right)(I+A)$. Then L is of the form $I + G'$ with a Green operator of type d. In particular, it is a Fredholm operator of index zero. Let us show that it is invertible by showing that its kernel is trivial: Let $h \in \mathcal{K}$ with $Lh = 0$. Then $0 = Lh = [(I+B)(I+A)]^r h + F(I+A)h$. Since the first summand belongs to $\mathcal{R}$ while the second belongs to $\mathcal{N}$, (1) implies that both are zero. Now write $h = h_\mathcal{N} + h_\mathcal{R}$ with $h_\mathcal{N} \in \mathcal{N}, h_\mathcal{R} \in \mathcal{R}$. Then $0 = [(I+B)(I+A)]^r(h_\mathcal{N} + h_\mathcal{R}) = [(I+B)(I+A)]^r h_\mathcal{R}$, by definition of $\mathcal{N}$. On the other hand (2) now implies that $h_\mathcal{R} = 0$. Using (ii), we conclude that

$$0 = F(I+A)h_\mathcal{N} = PF(I+A)Ph_\mathcal{N} = (I+A)^{-1}(I+A)PF(I+A)Ph_\mathcal{N} = Ph_\mathcal{N} = h_\mathcal{N}.$$

Thus $h = 0$, and L is invertible. Finally we can apply 3.3.10 and obtain an inverse to L of the form $I + G''$, thus an inverse to $I + A$ in $C^{0,d}_{M+G}(X^\wedge, g)$. ◁

4.3.19 Corollary. Let $g = (\gamma, \gamma, \Theta), A \in C^{0,0}_{M+G}(X^\wedge, g)\}$, and suppose that $I + A$ is invertible on $\mathcal{K}^{0,\gamma}(X, V_1) \oplus \mathcal{K}^{0,\gamma-\frac{1}{2}}(Y, V_3)$. By combining 4.3.17 and 4.3.18 we see that there is an inverse of the form $I + B, B \in C^{0,0}_{M+G}(X^\wedge, g)$.

5 Appendix

5.1 The Mellin Transform

For the sake of completeness we shall collect in this section a few simple facts about the Mellin transform. The proofs will be omitted. Most statements are elementary, for details cf. [27].

5.1.1 Definition. The Mellin transform is defined for $f \in C_0^\infty(\mathbf{R}_+)$ by

$$(Mf)(z) = \int_{\mathbf{R}_+} t^{z-1} f(t)dt, \qquad z \in \mathbf{C}.$$

In order to indicate that the argument of f is t while that of Mf is z, we will occasionally write $M_{t\to z}(f(t))(z)$.

5.1.2 Lemma. *We have the following elementary properties. Let $f \in C_0^\infty(\mathbf{R}_+)$.*

(a) *If $g(t) = t^\beta f(t), \beta \in \mathbf{C}$ then $M(g)(z) = (Mf)(z+\beta)$.*

(b) *If $g(t) = (-t\partial_t)f(t)$, then $M(g)(z) = z\,(Mf)(z)$.*

(c) *If $g(t) = \ln t f(t)$, then $M(g)(z) = \frac{d}{dz}(Mf)(z)$.*

(d) *If $g(t) = f(t^\rho), \rho \in \mathbf{C}$, then $(Mg)(z) = \rho^{-1}(Mf)(\rho^{-1}z)$.*

(e) *If $g(t) = f(ct), c > 0$, then $(Mg)(z) = c^{-z}(Mf)(z)$.*

5.1.3 Lemma. *For $f \in C_0^\infty(\mathbf{R}_+), Mf$ is a holomorphic function on $\mathbf{C}$. Moreover, it is rapidly decreasing on each of the lines*

$$\Gamma_\beta = \{z \in \mathbf{C} : \operatorname{Re} z = \beta\}.$$

Moreover, Mf satisfies the corresponding estimates uniformly for β in compact intervals.

5.1.4 Theorem. *The Mellin transform extends to an isomorphism*

$$M : L^2(\mathbf{R}_+) \to L^2(\Gamma_{\frac{1}{2}});$$

in fact we have "Parseval's identity"

$$\int_0^\infty f(t)\overline{g(t)}\,dt = \frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{2}}} Mf(z)\overline{Mg(z)}\,dz.$$

5.1.5 Definition. The weighted Mellin transform

$$M_\beta : t^\beta L^2(\mathbf{R}_+) \to L^2(\Gamma_{\frac{1}{2}-\beta}) \tag{1}$$

is defined by

$$(M_\beta f)(z) = M(t^{-\beta} f)(z+\beta).$$

By 5.1.2(a), $M_\beta f = Mf|_{\Gamma_{\frac{1}{2}-\beta}}$ for $f \in C_0^\infty(\mathbf{R}_+)$; by 5.1.4, (1) is an isomorphism.

The following lemma is easily deduced from Lemma 5.1.2.

5.1.6 Lemma. *Let ω be a cut-off function near zero, $p \in \mathbf{C}, \operatorname{Re} p < \frac{1}{2}$, and $k \in \mathbf{N}$. Then*

(a) $M\omega(z) = z^{-1} M(-t\partial_t\omega)(z)$.

Note: *Since $-t\partial_t\omega \in C_0^\infty(\mathbf{R}_+)$, its Mellin transform is rapidly decreasing on each line Γ_β, uniformly for β in compact intervals.*
In particular, if χ is a smooth function on $\mathbf{C}$ which vanishes near zero and is equal to 1 near infinity, then $\chi M\omega$ is rapidly decreasing on each line Γ_β, uniformly for β in compact intervals.

(b)

$$\begin{aligned} M_{t\to z}(t^{-p}\ln^k t\,\omega(t))(z) &= \frac{d^k}{dz^k}(M\omega)(z-p) \\ &= \frac{d^k}{dz^k}(-z^{-1}\phi(z))(z-p) =: \psi(z) \end{aligned}$$

where $\phi = M(t\partial_t\omega)$. In particular, ψ is a meromorphic function in $\mathbf{C}$ with a single pole of order $k+1$ in p. If χ is a smooth function on $\mathbf{C}$ which vanishes near p and is equal to 1 near infinity, then $\chi\psi$ is rapidly decreasing on each line Γ_β, uniformly for β in compact intervals.

(c) *Let $\omega_\sigma(t) = \omega(\sigma t), \sigma > 0$. By* 5.1.2(e)

$$M\omega_\sigma(z) = \sigma^{-z}(M\omega)(z).$$

In particular, $M\omega_\sigma(z) \to 0$ as $\sigma \to \infty$ on $\{\operatorname{Re} z > 0\}$ and $u_\sigma(z) = M_{t\to z}(t^{-p}\ln^k t\,\omega_\sigma(t)) \to 0$ as $\sigma \to \infty$ on $\{\operatorname{Re} z > \operatorname{Re} p\}$.
If χ is as in (b), *then χu_σ tends to 0 on $\{\operatorname{Re} z > \operatorname{Re} p\}$ as σ tends to ∞, uniformly on all lines Γ_β, for β in compact subintervals of $\mathbf{R}$.*

(d) *If instead of $\operatorname{Re} p < \frac{1}{2}$ we have $\operatorname{Re} p > \frac{1}{2}$ and if we define*

$$v(z) = M_{t\to z}(t^{-p}\ln^k t\,\omega(t^{-1})),$$

then v also is meromorphic on $\mathbf{C}$ with a single pole of order $k+1$ in p. If χ is as in (b), *then χv is rapidly decreasing on each line Γ_β, uniformly for β in compact intervals.*

The following Paley-Wiener type results can be found in Jeanquartier's paper [13].

5.1.7 Theorem. *Let F be an entire function.*
(a) *F is the Mellin transform of a distribution supported in the interval $[a^{-1}, a], a \geq 1$, if and only if it satisfies the inequality*

$$|F(z)| \leq C\,\langle z\rangle^m\; a^{|Re(z)|}, \qquad z \in \mathbf{C} \tag{1}$$

for some $m \in \mathbf{N}$ and $C > 0$.
(b) *F is the Mellin transform of a C^∞ function supported in $[a^{-1}, a], a \geq 1$, if and only if for every $m \in \mathbf{N}$ there is a $C_m > 0$ such that*

$$|F(z)| \leq C_m\,\langle z\rangle^{-m}\; a^{|Re(z)|}, \qquad z \in \mathbf{C}. \tag{2}$$

5.1.8 Lemma. *Let $\omega \in C_0^\infty(\overline{\mathbf{R}}_+), \omega \equiv 1$ near zero, $f \in L^2(\overline{\mathbf{R}}_+)$. Then $M_{t\to z}[(1-\omega)f] \in \mathcal{A}(\mathrm{Re}\; z < \frac{1}{2})$ and $M_{t\to z}[(1-\omega)f]|_{\Gamma_\beta} \in L^2(\Gamma_\beta)$ for all $\beta \leq \frac{1}{2}$, uniformly for β in compact intervals.*

One also has an inversion theorem.

5.1.9 Theorem. *Let $\beta \in \mathbf{R}$. The inverse of the weighted Mellin transform in* 5.1.5(1)

$$M_\beta : t^\beta L^2(\mathbf{R}_+) \to L^2(\Gamma_{\frac{1}{2}-\beta})$$

is given by

$$M_\beta^{-1} = \frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{2}-\beta}} t^{-z} h(z)\, dz.$$

We finally give a detailed proof for the statement on the adjoint of Mellin operators in 4.1.13:

5.1.10 Lemma. *Let $a \in M_P^{\mu,0}(X), \mu \in \mathbf{Z}, \gamma \in \mathbf{R}$, let P be a Mellin asymptotic type with $\pi_{\mathbf{C}}P \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$, and let V, W be vector bundles over X. Then the operator*

$$A = \mathrm{op}_M^\gamma a : \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V) \to \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, W)$$

has a formal adjoint A^* with respect to the sesquilinear pairings between

$$\mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V) \quad \text{and} \quad \mathcal{H}_0^{-s,-\gamma-\frac{n}{2}}(X^\wedge, V)$$

on one hand and

$$\mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, W) \quad \text{and} \quad \mathcal{H}_{\{0\}}^{-s+\mu,-\gamma-\frac{n}{2}}(X^\wedge, W)$$

on the other hand. Here, the index $\{0\}$ means that we use the $\mathcal{H}_0$–spaces for $s - \mu \geq 0$ and the usual $\mathcal{H}$–spaces otherwise, cf. 3.3.1. We have

$$A^* = \mathrm{op}_M^{-\gamma-n} a^{(*)} \quad \text{with} \quad a^{(*)} = a(n+1-\bar{z})^*;$$

the last asterisk indicates the formal L^2-adjoint of the operator $a(z) : C^\infty(\overline{X}, V) \to C^\infty(\overline{X}, W)$.

Proof. By definition $\mathrm{op}_M^\gamma a = t^\gamma M^{-1}T^{-\gamma}aT^\gamma M t^{-\gamma}$. So let $u \in C_0^\infty(\overline{X}^\wedge, V), v \in C_0^\infty(\overline{X}^\wedge, W)$. In the following computation, we shall freely use Parseval's identity for Mellin transforms, cf. 5.1.4, and employ the notation $\rho = n + \gamma$.

$$
\begin{aligned}
& (\mathrm{op}_M^\gamma(a)u, v)_{\mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge,W),\mathcal{H}_{\{0\}}^{-s+\mu,-\gamma-\frac{n}{2}}(X^\wedge,W)} \\
= & \int_{\Gamma_{\frac{n+1}{2}}} (M_{t\to z}\mathrm{op}_M^\gamma(a)\,u, M_{t\to z}v)_{L^2(X,W)}\,dz \\
= & \int_{\Gamma_{\frac{1}{2}}} (M(t^{\frac{n}{2}}\mathrm{op}_M^\gamma(a)\,u), M(t^{\frac{n}{2}}v))_{L^2(X,W)}\,dz \\
= & \int_0^\infty (t^{\frac{n}{2}+\gamma}M^{-1}T^{-\gamma}a\,T^\gamma M(t^{-\gamma}u), t^{\frac{n}{2}}v)_{L^2(X,W)}\,dt \\
= & \int_0^\infty (M^{-1}T^{-\gamma}a\,T^\gamma M(t^{-\gamma}u), t^{n+\gamma}v)_{L^2(X,W)}\,dt \\
= & \int_{\Gamma_{\frac{1}{2}}} (T^{-\gamma}M^{-1}T^{-\gamma}a\,T^\gamma M(t^{-\gamma}u), M(t^{n+\gamma}v))_{L^2(X,W)}\,dz \\
= & \int_{\Gamma_{\frac{1}{2}-\gamma}} (a(z)\,T^\gamma M(t^{-\gamma}u), T^{-\gamma}M(t^{n+\gamma}v))_{L^2(X,W)}\,dz \\
= & \int_{\Gamma_{\frac{1}{2}-\gamma}} (T^\gamma M(t^{-\gamma}u), a^*(z)\,T^{-\gamma}M(t^{n+\gamma}v))_{L^2(X,V)}\,dz \\
= & \int_{\Gamma_{\frac{1}{2}}} (M(t^{-\gamma}u), T^\gamma a^* T^{-\gamma}M(t^{n+\gamma}v))_{L^2(X,V)}\,dz \\
= & \int_0^\infty (t^{-\gamma}u, M^{-1}T^\gamma a^* T^{-\gamma}M(t^{n+\gamma}v))_{L^2(X,V)}\,dz \\
= & \int_0^\infty (u, t^{-\gamma}M^{-1}T^\gamma a^* T^{-\gamma}M(t^{n+\gamma}v))_{L^2(X,V)}\,dt \\
= & \int_0^\infty (u, t^{n-\rho}M^{-1}T^{\rho-n}a^* T^{-\rho+n}M(t^\rho v))_{L^2(X,V)}\,dt \\
= & \int_0^\infty (t^{\frac{n}{2}}u, t^{\frac{n}{2}-\rho}M^{-1}T^\rho a^{(*)}\,T^{-\rho}M(t^\rho v))_{L^2(X,V)}\,dt \\
= & \int_{\Gamma_{\frac{n+1}{2}}} (M_{t\to z}u, M_{t\to z}\mathrm{op}_M^{-\rho}(a^{(*)})\,v)_{L^2(X,V)}\,dz \\
= & (u, \mathrm{op}_M^{-\rho}(a^{(*)})\,v)_{\mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge,V),\mathcal{H}_0^{-s,-\gamma-\frac{n}{2}}(X^\wedge,V)}
\end{aligned}
$$

This was our assertion. In the third identity from the bottom we used that $T^{\rho-n}a^*T^{-\rho+n} = T^\rho a^*(\cdot - n)T^{-\rho}$ and that, on $\Gamma_{\frac{1}{2}}$, we have $z - n = 1 - n - \bar{z}$. Notice that for the above consideration we did not need the holomorphy of a. Whenever it holds it will allow further conclusions on the adjoint. For the computation we only need that a be defined on $\Gamma_{\frac{1}{2}-\gamma}$ and that all integrals make sense. ◁

5.2 The Left-Over Term in the Composition of Pseudodifferential Operators

In this section we shall give the proof of the statement in Theorem 2.2.5. The presence of the parameter λ does not require major deviations from the classical route. We will, however, emphasize the new representation on the singular Green operators. Recall the notation $\mathrm{op}^+_{x_n}p = [\mathrm{op}_{x_n}p]_+ = \mathrm{r}^+\mathrm{op}_{x_n}p\,\mathrm{e}^+$.

5.2.1 Proposition. *Let $\Omega' \subseteq \mathbf{R}^{n-1}$ be open, $\Omega = \Omega' \times \mathbf{R}$. Moreover, let $\mu, \nu \in \mathbf{Z}, p \in S^{\mu}_{1,0,tr}(\Omega, \mathbf{R}^n; \mathbf{R}^l), q \in S^{\nu}_{1,0,tr}(\Omega, \mathbf{R}^n; \mathbf{R}^l)$. Suppose that either $p(x,\xi,\lambda)$ or $q(x,\xi,\lambda)$ vanishes for x_n outside a compact set. Then*

$$L(p,q) = \mathrm{op}^+_{x_n}p \circ_n \mathrm{op}^+_{x_n}q - [\mathrm{op}_{x_n}p \circ_n \mathrm{op}_{x_n}q]_+ \tag{1}$$

induces a parameter-dependent singular Green operator of order $\mu+\nu$ and type $\max\{\nu, 0\}$.

In order to save notation but also to be slightly more general, let us assume that $\Omega = \mathbf{R}^n$ and consider symbols in the uniform symbol classes. For convenience, the proof is broken up into a series of steps called lemmata. We will, however, keep assumptions and notation fixed in this subsection.

5.2.2 Lemma. *We may assume that $p(x', x_n, \xi)$ and $q(x', x_n, \xi)$ vanish for $|x_n| > \epsilon$, where $\epsilon > 0$ is arbitrary.*

Proof. A priori, the condition that one of the symbols vanishes for large x_n ensures that the second term at the right hand side of 5.2.1(1) is well-defined. If q vanishes for x_n close to zero, then $L(p,q) = 0$. On the other hand, let p vanish for x_n near zero, and let $\phi \in C_0^\infty(\mathbf{R})$ be supported in a sufficiently small neighborhood of zero. Then

$$\mathrm{op}\,p = (\mathrm{op}\,p)\phi + (\mathrm{op}\,p)(1-\phi) = \mathrm{op}\,r + (\mathrm{op}\,p)(1-\phi)$$

with a regularizing symbol r – simply compute the asymptotic expansion. Therefore

$$\mathrm{op}^+_{x_n}p \circ_n \mathrm{op}^+_{x_n}q = [\mathrm{op}_{x_n}p(1-\phi) \circ_n \mathrm{op}_{x_n}q]_+ + \mathrm{op}^+_{x_n}r'$$

with regularizing r'. Since $\mathrm{op}^+_{x_n}r'$ also induces a regularizing singular Green operator, we have the desired result. ◁

We will therefore assume that p and q vanish for $|x_n| > 1$.

5.2.3 Lemma. *We may write $p = p_d + p_0, q = q_d + q_0$, where p_d, q_d are polynomials in (ξ, λ), and where for all $k \in \mathbf{N}$*

$$\begin{aligned} \partial^k_{x_n}p_0(x', 0, \xi', \langle\xi', \lambda\rangle\,\xi_n, \lambda) &\in S^{\mu}\hat{\otimes}_\pi H_0 \\ \partial^k_{x_n}q_0(x', 0, \xi', \langle\xi', \lambda\rangle\,\xi_n, \lambda) &\in S^{\nu}\hat{\otimes}_\pi H_0. \end{aligned}$$

Proof. We have a decomposition induced by the transmission property and the fact that $H_d, d \in \mathbf{N}$ is the direct sum of H_0 and the space of all polynomials of degree less than d, cf. 2.1.3,2.1.5. Differentiating the decomposition with respect to the variables (ξ', λ) shows that the part which is a polynomial in ξ_n also is a polynomial with respect to ξ' and λ. ◁

Let us first study the behavior of the polynomial parts of p and q.

5.2.4 Lemma. *In the notation of* 5.2.3,

$$L(p_d, q) = 0. \tag{1}$$

If we write $q_d = \sum_{j=0}^{\nu} s_j(x', \xi', \lambda)\xi_n^j$ *with polynomials* $s_j \in S^{\nu-j}(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \times \mathbf{R}^l)$, *then*

$$L(p, q_d) = \sum_{j=0}^{\nu-1} k_j \gamma_j, \tag{2}$$

where, as usual, $\gamma_j(f) = \lim_{t\to 0^+} \partial_{x_n}^j f(x', t)$ *and*

$$k_j(x', \xi', D_n, \lambda) = -i \sum_{m=j+1}^{\nu} \mathrm{r}^+ p(x, \xi', D_n, \lambda) s_m(x', \xi', \lambda) D_n^{m-1-j}(u \otimes \delta). \tag{3}$$

The k_j *are parameter-dependent potential symbols of order* $\mu + \nu - j - \frac{1}{2}$; *the* γ_j *are trace symbols of order* $j + \frac{1}{2}$ *and type* $j + 1$.

Proof. Identities (1), (2), and (3) are straightforward. They follow from the identity $\partial_{x_n} \mathrm{e}^+ f = \mathrm{e}^+ \partial_{x_n} f + \gamma_0 f\, \delta$, valid for $f \in C^1(\mathbf{R})$, with Dirac's delta function at the origin, cf. [9], (2.6.18, 19, 20). By 2.2.13 we also have the statement concerning the γ_j. Moreover, it is clear that $s_m \in S_{1,0}^{\nu-m}(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathbf{C}, \mathbf{C})$ and that $p(x, \xi, \lambda)\xi_n^{m-j-1} \in S_{1,0,tr}^{\mu+m-j-1}(\mathbf{R}^n, \mathbf{R}^n \times \mathbf{R}^l)$. In order to prove the result it is therefore sufficient to show the following: If $a = a(x, \xi, \lambda)$ belongs to $S^\mu(\mathbf{R}^n, \mathbf{R}^n \times \mathbf{R}^l)$, then the operator-valued symbol $k(x', \xi', D_n, \lambda)$ defined by

$$k(x', \xi', D_n, \lambda)u = a(x, \xi', D_n, \lambda)(u \otimes \delta), \qquad u \in \mathcal{S}(\mathbf{R}^{n-1}), \tag{4}$$

belongs to $S^{\mu+\frac{1}{2}}(\mathbf{R}^{n-1}, \mathbf{R}^{n-1}; \mathbf{C}, \mathcal{S}(\mathbf{R}_+))$. To this end we will estimate the norm of

$$\|\kappa_{\langle\xi',\lambda\rangle^{-1}} D_{\xi'}^\alpha D_{x'}^\beta k(x', \xi', D_n, \lambda)\|_{\mathcal{L}(\mathbf{C}, H^{s,t}(\mathbf{R}_+))}$$

for arbitrary $s, t \geq 0$. For fixed x', ξ', λ, this is a multiplication operator with a rapidly decreasing function $\phi_{\alpha,\beta}(x', \xi', x_n, \lambda)$. So let us show that for all $k, m \in \mathbf{N}$,

$$x_n^k D_{x_n}^m \phi_{\alpha,\beta}(x', \xi', x_n, \lambda) = \mathrm{O}(\langle\xi', \lambda\rangle^{\mu-|\alpha|+\frac{1}{2}}). \tag{5}$$

The function $\phi_{\alpha,\beta}$ is given by

$$\phi_{\alpha,\beta}(x', \xi', x_n, \lambda) = \text{const } \langle\xi', \lambda\rangle^{-\frac{1}{2}} \int e^{i\langle\xi',\lambda\rangle^{-1} x_n \xi_n} D_{\xi'}^\alpha D_{x'}^\beta a(x', \langle\xi', \lambda\rangle^{-1} x_n, \xi', \lambda)\, d\xi.$$

We may assume that $\alpha = \beta = 0$, for $D_{\xi'}^\alpha D_{x'}^\beta a$ is of order $\mu - |\alpha|$, and ignore the constant. So consider

$$\begin{aligned} & x_n^k D_{x_n}^m \langle\xi', \lambda\rangle^{\frac{1}{2}} \phi_{0,0}(x', \xi', x_n, \lambda) \\ = & \sum_{m_1+m_2=m} c_{m_1 m_2} \int e^{i\langle\xi',\lambda\rangle^{-1} x_n \xi_n} x_n^k \langle\xi', \lambda\rangle^{-m} \xi_n^{m_1} [D_{x_n}^{m_2} a](x', \langle\xi', \lambda\rangle^{-1} x_n, \xi, \lambda) d\xi_n \\ = & \sum_{m_1+m_2=m} c_{m_1 m_2} \int e^{i\langle\xi',\lambda\rangle^{-1} x_n \xi_n} \langle\xi', \lambda\rangle^{k-m} \xi_n^{m_1} [D_{x_n}^{m_2} (-D_{\xi_n})^k a](x', \langle\xi', \lambda\rangle^{-1} x_n, \xi, \lambda) d\xi_n \end{aligned} \tag{6}$$

after integration by parts. Now $\xi_n^{m_1} D_{x_n}^{m_2}(-D_{\xi_n})^k a$ also has the transmission property and is of order $\leq \mu + m - k$. In view of the factor $\langle \xi', \lambda \rangle^{k-m}$ it is no restriction to assume $k = m = 0$. A Taylor expansion gives

$$a(x,\xi,\lambda) = \sum_{k=0}^{M-1} \frac{x_n^k}{k!}(\partial_{x_n}^k a)(x',0,\xi,\lambda) + x_n^M a_M(x,\xi,\lambda).$$

Notice that a_M belongs to $S^{\mu}_{1,0,tr}(\mathbf{R}^n, \mathbf{R}^n; \mathbf{R}^l)$. Plugging this into (6), we obtain two types of expressions, the first ones corresponding to the terms in the summation, the second ones to the remainder. For the first ones note that

$$\begin{aligned} & \int e^{i\langle \xi',\lambda\rangle^{-1} x_n \xi_n} \frac{x_n^k}{k!}(\partial_{x_n}^k a)(x',0,\xi,\lambda)d\xi_n \\ = & \frac{\langle \xi',\lambda\rangle^k}{k!} \int e^{i\langle \xi',\lambda\rangle^{-1} x_n \xi_n}[\partial_{x_n}^k(-D_{\xi_n})^k a](x',0,\xi,\lambda)d\xi_n \\ = & \frac{\langle \xi',\lambda\rangle^{k+1}}{k!} \int e^{ix_n u_n}[\partial_{x_n}^k(-D_{\xi_n})^k a](x',0,\xi',\langle \xi',\lambda\rangle u_n,\lambda)du_n. \end{aligned} \tag{7}$$

Since a satisfies the transmission condition, $[\partial_{x_n}^k(-D_{\xi_n})^k a](x',0,\xi',\langle \xi',\lambda\rangle u_n,\lambda) \in S^{\mu-k}_{1,0}\hat{\otimes}_\pi H$. Correspondingly, expression (7) is $O(\langle \xi',\lambda\rangle^{\mu+1})$. In view of the factor $\langle \xi',\lambda\rangle^{\frac{1}{2}}$ in (6), this is exactly what we want.

For the analysis of the terms associated with the remainder, we perform the corresponding transformations. We choose M so large that $\mu - M < -1$, then the expression corresponding to (7) is

$$\langle \xi',\lambda\rangle^{M+1} \int e^{ix_n u_n}((-D_{\xi_n})^M a_M)(x',\langle \xi',\lambda\rangle^{-1} x_n,\xi',\langle \xi',\lambda\rangle u_n,\lambda)du_n,$$

which also is $O(\langle \xi',\lambda\rangle^{\mu+1})$. ◁

5.2.5 Definition. Let J be the reflection operator on functions in $\mathbf{R}^n : Ju(x',x_n) = u(x',-x_n)$.

5.2.6 Lemma. *We may write*

$$L(p,q) = g^+(p)g^-(q), \tag{1}$$

with

$$\begin{aligned} g^+(p)(x',\xi',D_n,\lambda) &= \mathrm{r}^+ p(x',\xi',D_n,\lambda)\mathrm{e}^- J : \mathcal{S}(\mathbf{R}_+) \to \mathcal{S}(\mathbf{R}_+); & (2)\\ g^-(q)(x',\xi',D_n,\lambda) &= J\mathrm{r}^- q(x',\xi',D_n,\lambda)\mathrm{e}^+ : \mathcal{S}(\mathbf{R}_+) \to \mathcal{S}(\mathbf{R}_+). & (3) \end{aligned}$$

Moreover, for $0 \neq w \in \mathbf{R}$ let

$$\begin{aligned} \tilde{p}(x,\xi',w,\lambda) &= (2\pi)^{-\frac{1}{2}} \int e^{iw\xi_n} p(x,\xi,\lambda)d\xi_n, & (4)\\ \tilde{q}(x,\xi',w,\lambda) &= (2\pi)^{-\frac{1}{2}} \int e^{iw\xi_n} q(x,\xi,\lambda)d\xi_n & (5) \end{aligned}$$

denote the inverse Fourier transforms of p and q, respectively, with respect to ξ_n. Integration by parts shows that both integrals make sense as oscillatory integrals. In this notation, $g^+(p)(x',\xi',D_n,\lambda)$ and $g^-(q)(x',\xi',D_n,\lambda)$ are the integral operators on $\mathcal{S}(\mathbf{R}_+)$ with the distributional kernels

$$\begin{aligned}\tilde{g}^+(p)(x',\xi',x_n,y_n,\lambda) &= (2\pi)^{-\frac{1}{2}}\,\tilde{p}(x,\xi',w,\lambda)|_{w=x_n+y_n}, &(6)\\ \tilde{g}^-(q)(x',\xi',x_n,y_n,\lambda) &= (2\pi)^{-\frac{1}{2}}\,\tilde{q}(x,\xi',w,\lambda)|_{w=-x_n-y_n}. &(7)\end{aligned}$$

Proof. Identity (1) is immediate since, on $\mathcal{S}(\mathbf{R}_+)$,

$$[\mathrm{r}^+\mathrm{op}_{x_n}p\,\mathrm{e}^+][\mathrm{r}^+\mathrm{op}_{x_n}q\,\mathrm{e}^+] - \mathrm{r}^+\mathrm{op}_{x_n}p\circ_n \mathrm{op}_{x_n}q\mathrm{e}^+ = \mathrm{r}^+\mathrm{op}_{x_n}p\,(\mathrm{e}^+\mathrm{r}^+-1)\,\mathrm{op}_{x_n}q\,\mathrm{e}^+,$$

and since $\mathrm{e}^+\mathrm{r}^+-1=\mathrm{e}^-\mathrm{r}^-$, ignoring the value in zero. Identities (6) and (7) are immediate from the usual formula for the integral kernel of a pseudodifferential operator in connection with (2) and (3). ◁

We shall need the following lemma.

5.2.7 Lemma. *Let $r\in S^{\mu}_{1,0,tr}(\mathbf{R}^n,\mathbf{R}^n;\mathbf{R}^l)$ and let $(x',\xi',\lambda)\in\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}\times\mathbf{R}^l$ be fixed.*
(a) *For $x_n,y_n>0$ define*

$$\tilde{g}(x',\xi',x_n,y_n,\lambda)=\left(\mathcal{F}^{-1}_{\xi_n\to w}r\right)(x',0,\xi',w,\lambda)|_{w=x_n+y_n}. \qquad (1)$$

Then the operator-valued symbol $g(x',\xi',D_n,\lambda)$ defined by

$$g(x',\xi',D_n,\lambda)f(x_n)=\int_0^\infty \tilde{g}(x',\xi',x_n,y_n,\lambda)f(y_n)dy_n$$

for $f\in\mathcal{S}(\mathbf{R}_+)$ belongs to $S^{\mu}(\mathbf{R}^{n-1},\mathbf{R}^{n-1}\times\mathbf{R}^l;\mathcal{S}'(\mathbf{R}_+),\mathcal{S}(\mathbf{R}_+))$, i.e. defines a singular Green symbol of order μ and type zero.
(b) *Similarly, letting*

$$\tilde{h}(x',\xi',x_n,y_n,\lambda)=\left(\mathcal{F}^{-1}_{\xi_n\to w}r\right)(x',0,\xi',w,\lambda)|_{w=-x_n-y_n} \qquad (2)$$

we obtain an operator-valued symbol h in $S^{\mu}(\mathbf{R}^{n-1},\mathbf{R}^{n-1}\times\mathbf{R}^l;\mathcal{S}'(\mathbf{R}_+),\mathcal{S}(\mathbf{R}_+))$.
Note: *Since r satisfies the transmission condition we know that $r(x',0,\xi',\langle\xi',\lambda\rangle\,\xi_n,\lambda)\in S^{\mu}_{1,0}\hat{\otimes}_\pi H$. Since g is defined from the values of the inverse Fourier transform on $\mathbf{R}_+$, it only depends on the part of $r(x',0,\xi',\langle\xi',\lambda\rangle\,\xi_n,\lambda)$ in H^+, while h only depends on the part in H_0^-.*

Proof. (a) We have to show that for all $s,t\in\mathbf{N}$ and all multi-indices α,β,γ

$$\|\kappa_{\langle\xi',\lambda\rangle^{-1}}D^\alpha_{\xi'}D^\beta_{x'}D^\gamma_\lambda g(x',\xi',D_n,\lambda)\kappa_{\langle\xi',\lambda\rangle}\|_{\mathcal{L}(H_0^{-s,-t}(\mathbf{R}_+),H^{s,t}(\mathbf{R}_+))}=O(\langle\xi',\lambda\rangle^{\mu-|\alpha|-|\gamma|}). \qquad (3)$$

It is obviously sufficient to prove the case $|\alpha|=|\beta|=|\gamma|=0$, otherwise we might consider $D^\alpha_{\xi'}D^\beta_{x'}D^\gamma_\lambda r$. A calculation shows that $\kappa_{\langle\xi',\lambda\rangle^{-1}}g(x',\xi',D_n,\lambda)\kappa_{\langle\xi',\lambda\rangle}$ is the integral operator with the kernel $\langle\xi',\lambda\rangle^{-1}\,\tilde{g}(x',\xi',\langle\xi',\lambda\rangle^{-1}x_n,\langle\xi',\lambda\rangle^{-1}y_n,\lambda)$.

The space $H_0^{-s,-t}(\mathbf{R}_+)$ is the closure of $C_0^\infty(\mathbf{R}_+)$ in the norm of $H^{-s,-t}(\mathbf{R})$. We may show (3) by verifying that for all $k \le t, k' \le t', l \le s, l' \le s'$ the norm of the operator

$$x_n^k D_{x_n}^l \kappa_{\langle\xi',\lambda\rangle^{-1}} g(x',\xi',D_n,\lambda)\kappa_{\langle\xi',\lambda\rangle} x_n^{k'} D_{x_n}^{l'} \tag{4}$$

on $L^2(\mathbf{R}_+)$ is $\mathrm{O}(\langle\xi',\lambda\rangle^\mu)$. The operator in (4) has the integral kernel

$$\langle\xi',\lambda\rangle^{-1} x_n^k D_{x_n}^l y_n^{k'} (-D_{y_n})^{l'} \tilde{g}(x',\xi',\langle\xi',\lambda\rangle^{-1} x_n, \langle\xi',\lambda\rangle^{-1} y_n, \lambda). \tag{5}$$

We can estimate its operator norm by estimating the $L^2_{x_n,y_n}(\mathbf{R}^2_{++})$–norm of its kernel function. Now

$$\begin{aligned} &\|x_n^k D_{x_n}^l y_n^{k'} D_{y_n}^{l'} \tilde{g}(x',\xi',\langle\xi',\lambda\rangle^{-1} x_n, \langle\xi',\lambda\rangle^{-1} y_n,\lambda)\|_{L^2(\mathbf{R}^2_{++})} \\ \le\ &\|(x_n+y_n)^{k+k'} D_{x_n}^{l+l'}(\mathcal{F}^{-1}r)(x',0,\xi',\langle\xi',\lambda\rangle^{-1}(x_n+y_n),\lambda)\|_{L^2(\mathbf{R}^2_{++})}. \end{aligned}$$

For an integrable function f we have $\int_0^\infty\int_0^\infty f(x_n+y_n)dx_n dy_n = \int_0^\infty w f(w)dw$. The last expression above therefore equals

$$\begin{aligned} \Big(\int & w^{k+k'+1} D_w^{l+l'}(\mathcal{F}^{-1}_{\xi_n\to w} r)(x',0,\xi',\langle\xi',\lambda\rangle^{-1} w,\lambda) \\ &\cdot w^{k+k'} \overline{D_w^{l+l'}(\mathcal{F}^{-1}_{\xi_n\to w} r)}(x',0,\xi',\langle\xi',\lambda\rangle^{-1} w,\lambda)\,dw\Big)^{\frac12}. \end{aligned}$$

This we estimate by Cauchy-Schwarz' inequality

$$\begin{aligned} \le\ & \|w^{k+k'+1} D_w^{l+l'}(\mathcal{F}^{-1}_{\xi_n\to w} r)(x',0,\xi',\langle\xi',\lambda\rangle^{-1} w,\lambda)\|^{\frac12}_{L^2(\mathbf{R}_+)} \\ &\cdot \|w^{k+k'} D_w^{l+l'}(\mathcal{F}^{-1}_{\xi_n\to w} r)(x',0,\xi',\langle\xi',\lambda\rangle^{-1} w,\lambda)\|^{\frac12}_{L^2(\mathbf{R}_+)} \\ =\ & \langle\xi',\lambda\rangle^{k+k'+\frac12-l-l'} \|\mathcal{F}^{-1}_{\xi_n\to w}(D_{\xi_n} a)(x',0,\xi',\langle\xi',\lambda\rangle^{-1} w,\lambda)\|^{\frac12}_{L^2(\mathbf{R}_+)} \\ &\cdot \|\mathcal{F}^{-1}_{\xi_n\to w}(D_{\xi_n} a)(x',0,\xi',\langle\xi',\lambda\rangle^{-1} w,\lambda)\|^{\frac12}_{L^2(\mathbf{R}_+)}, \end{aligned}$$

where $a = D_{\xi_n}^{k+k'}(\xi_n^{l+l'} r)$. We note the behavior of the inverse Fourier transform under dilations with positive constants $\mathcal{F}^{-1}f(w/c) = c[\mathcal{F}^{-1}f(c\cdot)](w)$, and continue the above estimate with

$$\begin{aligned} =\ & \langle\xi',\lambda\rangle^{k+k'+l-l'+\frac32} \cdot\ \|\mathcal{F}^{-1}_{\xi_n\to w}[(D_{\xi_n} a)(x',0,\langle\xi',\lambda\rangle\xi_n,\lambda)](w)\|^{\frac12}_{L^2(\mathbf{R}_+)} \\ &\cdot \|\mathcal{F}^{-1}_{\xi_n\to w}[a(x',0,\langle\xi',\lambda\rangle\xi_n,\lambda)](w)\|^{\frac12}_{L^2(\mathbf{R}_+)}. \end{aligned} \tag{6}$$

The symbol a is of order $\mu' = \mu - k - k' + l + l'$. It also satisfies the transmission condition. So $a(x',0,\xi',\langle\xi',\lambda\rangle\xi_n,\lambda) \in S^{\mu'}_{1,0}(\mathbf{R}^{n-1},\mathbf{R}^{n-1}\times\mathbf{R}^l)\hat{\otimes}_\pi H$, and $\mathcal{F}^{-1}_{\xi_n\to w} a(x',0,\xi',\langle\xi',\lambda\rangle\xi_n,\lambda) \in S^{\mu'}_{1,0}(\mathbf{R}^{n-1},\mathbf{R}^{n-1}\times\mathbf{R}^l)\hat{\otimes}_\pi \mathcal{S}(\mathbf{R}_+)$. Therefore, the last expression is $\mathrm{O}(\langle\xi',\lambda\rangle^{\mu+1})$, which is exactly the estimate we need in view of the factor $\langle\xi',\lambda\rangle^{-1}$ in (5).
The proof of (b) is essentially the same. ◁

5.2.8 Lemma. *The operators $g^+(p)(x',\xi',D_n,\lambda)$ and $g^+(p)(x',\xi',D_n,\lambda)$ are parameter-dependent singular Green operator of orders μ and ν, respectively, and type zero. They have asymptotic expansions*

$$g^+(p)(x',\xi',D_n,\lambda) \sim (2\pi)^{-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{x_n^j}{j!} g_j^+(p)(x',\xi',D_n,\lambda) \tag{1}$$

$$g^-(q)(x',\xi',D_n,\lambda) \sim (2\pi)^{-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{x_n^j}{j!} g_j^-(q)(x',\xi',D_n,\lambda), \tag{2}$$

where $g_j^+(p)$ is the singular Green symbol obtained from $\partial_{x_n}^j p$ by the procedure in 5.2.7(a); *$g_j^-(q)$ the corresponding symbol obtained from $\partial_{x_n}^j q$ by the process in* 5.2.7(b).
Note that in view of the identity

$$\kappa_{\langle\xi',\lambda\rangle^{-1}} x_n \kappa_{\langle\xi',\lambda\rangle} f(x',x_n) = \langle\xi',\lambda\rangle^{-1} x_n f(x',x_n)$$

the multiplication operator with x_n belongs to $S^{-1}(\mathbf{R}^{n-1},\mathbf{R}^{n-1}\times\mathbf{R}^l; H^{s,t}(\mathbf{R}_+), H^{s,t-1}(\mathbf{R}_+))$ for all weighted Sobolev spaces $H^{s,t}(\mathbf{R}_+)$, so that (1) *and* (2) *indeed furnish asymptotic expansions for singular Green operators.*

Proof. Let us first consider g^+. Plugging the Taylor expansion with remainder

$$p(x,\xi,\lambda) = \sum_{j=0}^{M-1} \frac{x_n^j}{j!} \partial_{x_n}^j p(x',0,\xi,\lambda) + x_n^M p_M(x,\xi,\lambda) \tag{3}$$

into 5.2.6(6) we obtain

$$\begin{aligned} \tilde{g}^+(p)(x',\xi',x_n,y_n,\lambda) &= (2\pi)^{-\frac{1}{2}} \sum_{j=0}^{M-1} \frac{x_n^j}{j!} \partial_{x_n}^j \tilde{p}(x',0,\xi',w,\lambda)|_{w=x_n+y_n} \\ &\quad + (2\pi)^{-\frac{1}{2}} x_n^M p_M(x,\xi',w,\lambda)|_{w=x_n+y_n}. \end{aligned}$$

Again the tilde denotes the inverse Fourier transform with respect to ξ_n. This gives the beginning of the predicted asymptotic expansion. In order to justify the expansion, we will show that, given an $N \in \mathbf{N}$, the remainder induces an operator with a symbol in $S^{-N}(\mathbf{R}^{n-1},\mathbf{R}^{n-1}\times\mathbf{R}^l; H_0^{-N,-N}(\mathbf{R}_+), H^{N,N}(\mathbf{R}_+))$, provided M is large.
In order to see this we first multiply both sides of (3) by a function $0 \le \phi = \phi(x_n) \in C_0^\infty(\mathbf{R}_+)$ equal to 1 on $[-1,1]$. The left hand side will remain the same, on the right hand side, multiplication by ϕ preserves the asymptotic expansion, while the additional factor $\phi(x_n)$ with the remainder will be convenient, below.
The remainder $\phi(x_n)x_n^M\tilde{p}_M$ induces an operator-valued symbol $h_M(x',\xi',D_n,\lambda)$ by

$$h_M(x',\xi',D_n,\lambda)f(x_n) = \phi(x_n)x_n^M \int_0^\infty \tilde{p}_M(x',x_n,\xi',x_n+y_n,\lambda)f(y_n)\,dy_n.$$

Proceeding as in the proof of 5.2.7 consider the norm of $\kappa_{\langle\xi',\lambda\rangle^{-1}} D_{\xi'}^\alpha D_{x'}^\beta D_\lambda^\gamma h_M(x',\xi',D_n,\lambda)\kappa_{\langle\xi',\lambda\rangle}$ in $\mathcal{L}(H_0^{-N,-N}(\mathbf{R}_+), H^{N,N}(\mathbf{R}_+))$. Again we may assume that $|\alpha| = |\beta| = |\gamma| = 0$ and estimate instead the norm $\sup_{x_n} \|\cdot\|_{L^2_{y_n}}$ of the integral kernel of

$$x_n^k D_{x_n}^l \kappa_{\langle\xi',\lambda\rangle^{-1}} D_{\xi'}^\alpha D_{x'}^\beta D_\lambda^\gamma h_M(x',\xi',D_n,\lambda)\kappa_{\langle\xi',\lambda\rangle} x_n^{k'} D_{x_n}^{l'}$$

on $\mathcal{L}(L^2(\mathbf{R}_+))$. We have, with the obvious notation for $L^2(\mathbf{R}_+)$–spaces with respect to the corresponding variables,

$$\begin{aligned}
&\sup_{x_n} \phi(x_n) \| x_n^{k+M} D_{x_n}^l y_n^{k'} D_{y_n}^{l'} \tilde{p}_M(x', \langle \xi', \lambda \rangle^{-1} x_n, \xi', \langle \xi', \lambda \rangle^{-1} (x_n + y_n), \lambda) \|_{L^2_{y_n}} \\
\leq\ & \sup_{x_n} \phi(x_n) \| (x_n + y_n)^{k+k'+M} D_{x_n}^{l+l'} \tilde{p}(x', \langle \xi', \lambda \rangle^{-1} x_n, \xi', \langle \xi', \lambda \rangle^{-1} (x_n + y_n), \lambda) \|_{L^2_{y_n}} \\
\leq\ & \sup_{x_n} \phi(x_n) \| w^{k+k'+M} D_w^{l+l'} (\mathcal{F}^{-1}_{\xi_n \to w} p_M)(x', \langle \xi', \lambda \rangle^{-1} x_n, \xi', \langle \xi', \lambda \rangle^{-1} w, \lambda) \|_{L^2_w} \\
=\ & \sup_{x_n} \phi(x_n) \| \langle \xi', \lambda \rangle \mathcal{F}^{-1}_{\xi_n \to w} [(D_{\xi_n}^{k+k'+M} \xi_n^{l+l'} p_M)(x', \langle \xi', \lambda \rangle^{-1} x_n, \xi', \langle \xi', \lambda \rangle \xi_n, \lambda)](w) \|_{L^2_w},
\end{aligned}$$

which is $O(\langle \xi', \lambda \rangle^{\mu'})$, with $\mu' = \mu - k - k' + l + l' + 1 - M$. This gives the desired result. ◁

5.3 The Symbol of the Order Reduction is Classical

5.3.1 Lemma. *Let* $\chi \in \mathcal{S}(\mathbf{R})$. *Then*

$$\chi(\frac{\xi_n}{\langle \xi' \rangle}) \langle \xi' \rangle \in S^1_{cl}(\mathbf{R}^n).$$

As a preparation for the proof we will need the following lemmata.

5.3.2 Lemma. *Let* $\phi \in C_0^\infty(\mathbf{R})$ *be a zero excision function and* $\chi \in \mathcal{S}(\mathbf{R})$. *Then for* $k \in \mathbf{Z}, \alpha$ *a multi-index,*

$$\phi(|\xi|) \chi(\frac{\xi_n}{|\xi'|}) |\xi'|^k \xi^\alpha \in S^{k+|\alpha|}_{cl}(\mathbf{R}^n)$$

Proof. Clearly, the function is well-defined and homogeneous of degree $k + |\alpha|$ for large $|\xi|$. In order to see that it belongs to $S^{k+|\alpha|}$ we only have to check that it is C^∞. Obviously, every derivative is a linear combination of functions of the same kind – except for the fact that a derivative of ϕ no longer is an excision function; it is zero near infinity, which is even better for our purposes.
So we only have to show continuity. Since ϕ vanishes near zero, the only points of interest are those of the form $\xi' = 0, \xi_n \neq 0$. If $(\xi'^{[j]}, \xi_n^{[j]})$ is a sequence with $0 \neq \xi'^{[j]} \to 0, \xi_n^{[j]} \to c \neq 0$, then $\chi(\frac{\xi_n}{|\xi'|})|\xi'|^k \to 0$, since $\chi(t)t^N \to 0$ for arbitrary N as $t \to \infty$. ◁

5.3.3 Lemma. *Let* $k \in \mathbf{Z}, \phi$ *a zero excision function, and* $\chi \in \mathcal{S}(\mathbf{R})$. *Then*

$$\chi(\frac{\xi_n}{\langle \xi' \rangle}) \langle \xi' \rangle^k - \chi(\frac{\xi_n}{|\xi'|}) |\xi'|^k \phi(\xi) = O(\langle \xi \rangle^{k-1}).$$

Proof. For small $|\xi|$ there is nothing to show. So we may assume $1 \le |\xi|$ and $\phi(\xi) = 1$. If $|\xi'|$ is small, say $|\xi'| \le \frac{1}{2}$, then necessarily $|\xi_n| \ge \frac{1}{2}$, and $|\xi_n| \ge \frac{1}{2}|\xi| \ge \frac{1}{4}\langle\xi\rangle$. For arbitrary $K \in \mathbf{N}$, and suitable constants $c, c', \dots$,

$$\chi\left(\frac{\xi_n}{\langle\xi'\rangle}\right)\langle\xi'\rangle^k \le c\left\langle\frac{\xi_n}{\langle\xi'\rangle}\right\rangle^{-K}\langle\xi'\rangle^k \le c'\langle\xi_n\rangle^{-K}\langle\xi'\rangle^k \le c''\langle\xi\rangle^{k-K},$$

and similarly

$$\chi\left(\frac{\xi_n}{|\xi'|}\right)|\xi'|^k \le c\langle\xi\rangle^{k-K}.$$

So we are left with the case where $|\xi| \ge 1, |\xi'| \ge \frac{1}{2}$; in particular $|\xi'| \sim \langle\xi'\rangle$. We shall employ the identity

$$a^k - b^k = (a-b)\sum_{j=0}^{k-1} a^j b^{k-1-j}$$

for $k \ge 0$ with $a = \langle\xi'\rangle, b = |\xi'|$. It shows that

$$\langle\xi'\rangle^k - |\xi'|^k = (\langle\xi'\rangle + |\xi'|)^{-1}\sum_{j=0}^{k-1}\langle\xi'\rangle^j|\xi'|^{k-1-j} = O(\langle\xi'\rangle^{k-2}). \tag{1}$$

For $k < 0$ we take $a = \langle\xi'\rangle^{-1}, b = |\xi'|^{-1}$ and obtain the same result, noting that

$$\frac{1}{|\xi'|} - \frac{1}{\langle\xi'\rangle} = \frac{1}{\langle\xi'\rangle|\xi'|(\langle\xi'\rangle + |\xi'|)}.$$

Now we conclude that the difference under consideration is

$$\le \left|\chi\left(\frac{\xi_n}{\langle\xi'\rangle}\right) - \chi\left(\frac{\xi_n}{|\xi'|}\right)\right|\langle\xi'\rangle^k + \left|\chi\left(\frac{\xi_n}{|\xi'|}\right)\right|\,|\langle\xi'\rangle^k - |\xi'|^k| = E_1 + E_2$$

with the obvious notation. Expression E_1 can be estimated by

$$\|\chi'\|_{\sup I(\xi)}\left|\frac{\xi_n}{\langle\xi'\rangle} - \frac{\xi_n}{|\xi'|}\right|\langle\xi'\rangle^k. \tag{2}$$

Here, $I(\xi)$ denotes the interval between $\frac{\xi_n}{\langle\xi'\rangle}$ and $\frac{\xi_n}{|\xi'|}$; the supremum over this interval is $O\left(\left\langle\frac{\xi_n}{\langle\xi'\rangle}\right\rangle^{-N}\right)$, N arbitrary.

The second factor in (2) is $|\xi_n||\xi'|^{-1}\langle\xi'\rangle^{-1}(\langle\xi'\rangle + |\xi'|)^{-1}$. The fact that $\left\langle\frac{\xi_n}{\langle\xi'\rangle}\right\rangle\langle\xi'\rangle = \langle\xi\rangle$ then shows that $E_1 = O(\langle\xi'\rangle^{k-2})$.

For E_2 we also use that $\chi(\frac{\xi_n}{|\xi'|}) = O(\left\langle\frac{\xi_n}{\langle\xi'\rangle}\right\rangle^{-N})$ in connection with (1) to obtain the estimate $E_2 = O(\langle\xi\rangle^{k-2})$. ◁

We can now prove Lemma 5.3.1. By Lemma 5.3.2, $\phi(|\xi|)\chi(\frac{\xi_n}{|\xi'|})|\xi'| \in S^1_{cl}(\mathbf{R}^n)$, moreover, we see that

$$r_1(\xi) = \chi\left(\frac{\xi_n}{\langle\xi'\rangle}\right)\langle\xi'\rangle - \phi(|\xi|)\,\chi\left(\frac{\xi_n}{|\xi'|}\right)|\xi'| \in S^0,$$

so we have the first term of the asymptotic expansion (even the first two terms).

Now write for $|\xi| \geq 1, |\xi'| \neq 0$

$$\begin{aligned} r_1(\xi) &= \left(\chi(\frac{\xi_n}{\langle\xi'\rangle}) - \chi(\frac{\xi_n}{|\xi'|})\right)\langle\xi'\rangle + \chi(\frac{\xi_n}{|\xi'|})(\langle\xi'\rangle - |\xi'|) \\ &= E_1 + E_2 \end{aligned}$$

and note that by Taylor's formula

$$\chi(t) = \sum_{j=0}^{N} \frac{1}{j!}\chi^{(j)}(t_0)(t-t_0)^j + b_N \chi^{(N+1)}(\theta)(t-t_0)^{N+1},$$

so that

$$\begin{aligned} \chi(\frac{\xi_n}{\langle\xi'\rangle}) &= \sum_{j=0}^{N} \frac{1}{j!}\chi^{(j)}(\frac{\xi_n}{|\xi'|})(\frac{\xi_n}{|\xi'|})^j \langle\xi'\rangle^{-j}(\langle\xi'\rangle + |\xi'|)^{-j} \\ &\quad + \chi^{(N+1)}(\theta)(\frac{\xi_n}{|\xi'|})^{N+1}\langle\xi'\rangle^{-N-1}(\langle\xi'\rangle + |\xi'|)^{-N-1}, \end{aligned}$$

with θ between $\frac{\xi_n}{\langle\xi'\rangle}$ and $\frac{\xi_n}{|\xi'|}$.
Similarly, we have a Taylor expansion

$$(1+t^2)^{\frac{1}{2}} = 1 + \sum_{j=1}^{N} c_j t^j + c_N(t) t^{N+1}.$$

Hence the identity $\langle\xi'\rangle = |\xi'|\left\langle\frac{1}{|\xi'|}\right\rangle$ implies that

$$\langle\xi'\rangle - |\xi'| = \sum_{j=1}^{N} c_j |\xi'|^{1-j} + c_N(|\xi'|^{-1})|\xi'|^{-1-N}.$$

Here, $c_N(|\xi'|^{-1})$ is – up to constants – a derivative of $(1+t^2)^{\frac{1}{2}}$, and these are all bounded for $N \geq 1$.
Finally a last expansion: $\frac{1}{1+\langle t\rangle} = \sum_{j=0}^{N} d_j t^j + d_N(t)t^{N+1}$. Therefore

$$\begin{aligned} (\langle\xi'\rangle + |\xi'|)^{-1} &= |\xi'|^{-1}(1 + \langle|\xi'|^{-1}\rangle)^{-1} \\ &= \sum_{j=0}^{N} d_j |\xi'|^{-1-j} + d_N(|\xi'|^{-1})|\xi'|^{-N-1}. \end{aligned}$$

Here, d_N is bounded on $\mathbf{R}$.
Using Lemma 5.3.2 this clearly yields an asymptotic expansion for E_1 and E_2.

Now for the estimates. Let us first concentrate on the estimates of the remainders with no derivatives present. With the same reasoning as in the proof of 5.3.3, we may assume that $\frac{1}{2} \leq |\xi'| \sim \langle\xi'\rangle$. We have to deal with two types of terms: those of the form

$$\chi^{(j)}(\frac{\xi_n}{|\xi'|})(\frac{\xi_n}{|\xi'|})^j |\xi'|^{-K} R_N(\xi)$$

and those of the form

$$\chi^{(j)}(\theta)(\frac{\xi_n}{|\xi'|})^j |\xi'|^{-K} \tilde{R}_N(\xi)$$

with uniformly bounded functions $R_N, \tilde{R}_N$, $K \geq 0$, and θ in the interval between $\frac{\xi_n}{\langle \xi' \rangle}$ and $\frac{\xi_n}{|\xi'|}$. Like in the proof of Lemma 5.3.3, these expressions are $O(\langle \xi \rangle^{-K})$.

Finally, we may employ the same arguments for the derivatives, because we then have to deal with expressions of essentially the same kind, cf. the considerations in the proof of Lemma 5.3.2.

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