Regular holonomic $\mathcal{D}$-modules on skew symmetric matrices

Philibert Nang

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Abstract

We give a classification of regular holonomic $\mathcal{D}$-modules on skew-symmetric matrices attached to the linear action of the general linear group GL_N.

1 Introduction

The framework of the general theory of $\mathcal{D}$-modules was built up by M. Sato, T. Kawai and M. Kashiwara. It is a very powerful point of view, bringing ideas from algebra and algebraic geometry to the analysis of systems of differential equations. The interest in the theory of $\mathcal{D}$-modules is due to its applications in various parts of mathematics such as the representation theory, the singularity theory, the cohomology of singular spaces etc. Perhaps the first systematic use of $\mathcal{D}$-modules appeared in [13]. Since then there have appeared several articles by Kashiwara and others. We should also mention the contribution of B. Malgrange. Furthermore Z. Mebkhout used the theory of $\mathcal{D}$-modules to study the topology of singular varieties. Last but not least we mention the work of A. Beilinson and J. Bernstein regarding the algebraic aspect of the theory. Among the $\mathcal{D}$-modules we single out a class of objects of utmost importance: the regular holonomic $\mathcal{D}$-modules. One of the main problem in the $\mathcal{D}$-modules theory consists in the classification of these objects. Let us point out that several authors have taken an interest in it, notably L. Boutet de Monvel [1], P. Deligne, R. MacPherson and K. Vilonen [9] etc. One knows by the Riemann-Hilbert correspondence (see.[5] or [10]) that there is a general equivalence between the category consisting of regular holonomic $\mathcal{D}_V$-modules with characteristic variety Σ and the category consisting of perverse sheaves on V (where V denotes a complex manifold) with microsupport Σ. This gives a classification of regular holonomic $\mathcal{D}$-modules theoretically, but in practice the classification of perverse sheaves is not always much simpler. A more accessible problem is as follows: given a complex manifold V on which a Lie group acts linearly with finitely many orbits $(V_j)_{j\in J}$; the problem is to classify regular holonomic $\mathcal{D}_V$-modules whose characteristic variety is contained in the union of conormal bundles ($\Sigma := \bigcup\limits_{j\in J} \overline{T^*_{V_j}V}$) to these orbits. Closely equivalent: those that admit

a good filtration stable by infinitesimal generators of the group. These modules form a full category denoted $\mathrm{Mod}_{\Sigma}^{\mathrm{rh}}(\mathcal{D})$. In this paper we consider the action of $GL_N(\mathbb{C})$ on skew-symmetric tensors. This last induces a linear action on $V := \Lambda^2(\mathbb{C}^N)$, which we will think of as the space of skew-symmetric matrices. There are $\left(\left[\frac{N}{2}\right]+1\right)$ orbits V_{2k} $\left(0 \leq k \leq \left[\frac{N}{2}\right]\right)$($\left[\frac{N}{2}\right]$ is the integer part of $\frac{N}{2}$): the set of rank $2k$ matrices in V. This study is done here for N even which is the most interesting case. Note that here there is a natural algebra associated to this situation: the (graded) algebra $\mathcal{A}$ of (polynomial coefficients) differential operators acting on polynomials of the pfaffian, which is a quotient of the algebra $\overline{\mathcal{A}}$ of $SL_N(\mathbb{C})$-invariant differential operators on V (see. section 3).
The main result of this paper is the theorem 18 saying that there is an equivalence of categories between the category $\mathrm{Mod}_{\Sigma}^{\mathrm{rh}}(\mathcal{D})$ consisting of regular holonomic $\mathcal{D}$-modules as above and the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ consisting of graded $\mathcal{A}$-modules of finite type for the Euler vector field on V. The algebra $\mathcal{A}$ is described simply by generators and relations (see Proposition 7) thanks to a beautiful skew-Capelli identity construted by R. Howe and T. Umeda (see. [3, p. 592, Corollary (11.3.19)]). This also leads to the description of the latter category as an "elementary" category consisting of finite dimensional vector spaces and linear maps between them satisfying certain relations (Quiver category) on which one can see what are the simple or indecomposable objects (see. section 6). The following example is provided to illustrate the theoretical results:

Example 1 *Denote $pf(X)$ the pfaffian of $X \in V$, $pf(D)$ its dual and θ the Euler vector field on V.The $\mathcal{D}_V$-module $\mathcal{O}_V$ is an object in $\mathrm{Mod}_{\Sigma}^{\mathrm{rh}}(\mathcal{D})$. It is generated by an element $e_0 = 1_V$ such that $\theta e_0 = 0$ and $pf(D)e_0 = 0$. This yields a graded $\mathcal{A}$-module of finite type in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ with a basis (e_q) where $q = mk$ $(k \in \mathbb{N})$ such that $pf(D)e_0 = 0$ and satisfying the folowing system:*

$$S_0 = \begin{cases} \theta e_q = q e_q \ (q = mk,\ k \in \mathbb{N}) \\ pf(X)\, e_q = e_{q+m}, \\ pf(D)\, e_q = \prod_{t=0}^{m-1} \left(\frac{q}{m} + 2t\right) e_{q-m} \end{cases}. \tag{1}$$

Throught the paper we assume that the reader has some familiarity with the language of $\mathcal{D}$-modules. He may consult the very nice book [6] that provides a good account of an introduction to the general theory of $\mathcal{D}$-modules. Finally we should recall that in precedent papers the author has obtained similar results for $\mathcal{D}$-modules on $\mathbb{C}^n$associated to the action of the orthogonal group (see. [11]), and on $M_n(\mathbb{C})$ the space of complex square matrices associated to the action of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ (see. [12]).

2 Definitions and preliminary results

We have the following proposition.

Proposition 2 *The orbits V_{2k} $\left(0 \leq k \leq \left[\frac{N}{2}\right]\right)$ are simply connected.*

Proof. Recall that the fundamental group of a homogeneous space G/H, where G is a connected group, is the component group of H. If the skew - symmetric matrices are thought as the skew bilinear forms, then a typical point is determined by
a) the radical of the form, which is a subspace,
and
b) a non degenerate skew-symmetric form on the quotient. The stabilizer of this consists of all transformations which preserves the radical, and which act as an isometry of the quotient. This group is the product of a connected unipotent group with the GL of the radical and the symplectic group. Here the coefficient field is $\mathbb{C}$, then this group is connected, so the orbits should be simply connected.
■

Denote θ the Euler vector field on $\Lambda^2\mathbb{C}^N$. Let $\mathcal{M}$ be a $\mathcal{D}$-module.

Definition 3 *A section s in $\mathcal{M}$ is homogeneous if* $\dim_{\mathbb{C}} \mathbb{C}[\theta]\, s < \infty$. *$s$ is homogeneous of degree $\lambda \in \mathbb{C}$, if there exists $j \in \mathbb{N}$ such that $(\theta - \lambda)^j s = 0$.*

We recall the following useful theorem (see [11, Theorem 1.3.]):

Theorem 4 *Let $\mathcal{M}$ be a coherent $\mathcal{D}_{\Lambda^2\mathbb{C}^N}$-module with a good filtration $(\mathcal{M}_k)_{k\in\mathbb{Z}}$ stable by θ. Then*
i) $\mathcal{M}$ is generated by finitely many homogeneous global sections,
ii) For any $k \in \mathbb{N}$, $\lambda \in \mathbb{C}$, the vector space $\Gamma\left(\Lambda^2\mathbb{C}^N, \mathcal{M}_k\right) \bigcap \left[\bigcup_{p\in\mathbb{N}} \ker(\theta-\lambda)^p\right]$ of homogeneous global sections of $\mathcal{M}_k$ of degree λ is finite dimensional.

Denote by G is the quotient of $GL_N(\mathbb{C})$ by the kernel of its action on skew symmetric matrices. Let $\widetilde{G} := SL_N(\mathbb{C})$ be its universal covering.

Definition 5 *The action of the group G (preserving the good filtration) on a $\mathcal{D}_{\Lambda^2\mathbb{C}^N}$-module $\mathcal{M}$ is given by an isomorphism $u : p_1^+(\mathcal{M}) \xrightarrow{\sim} p_2^+(\mathcal{M})$ where $p_1 : G \times \Lambda^2\mathbb{C}^N \longrightarrow \Lambda^2\mathbb{C}^N$ is the projection on $\Lambda^2\mathbb{C}^N$, and $p_2 : G \times \Lambda^2\mathbb{C}^N \longrightarrow \Lambda^2\mathbb{C}^N$ defines the action of G on $\Lambda^2\mathbb{C}^N$.*

Remark 6 *From [11, Proposition 1.6.] we see that the infinitesimal action of G on $\mathcal{M}$ lifts to an action of its universal covering $\widetilde{G} := SL_n(\mathbb{C})$ on $\mathcal{M}$.*

3 Invariant differential operators for skew-symmetric matrices

In this section we describe $SL_N(\mathbb{C})$-invariant (polynomial coefficients) differential operators. Let us consider the action of $GL_N(\mathbb{C})$ on skew-symmetric tensors. This action $(\lambda(g) : X \to gXg^t)$ $(g \in GL_N(\mathbb{C}))$ naturally takes place in $\Lambda^2\mathbb{C}^N$ (which we will think of as the space of skew-symmetric matrices) and in particular on the algebra $\mathcal{P}\left(\Lambda^2\mathbb{C}^N\right)$ of polynomials on $\Lambda^2\mathbb{C}^N$. Thus we have a natural homomorphism (also denoted λ) from the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_N)$ to the algebra $\mathcal{PD}\left(\Lambda^2\mathbb{C}^N\right)$ of (polynomial coefficients) differential operators. This homomorphism λ maps the center $\mathcal{Z}(\mathcal{U}(\mathfrak{gl}_N))$ to the algebra $\mathcal{PD}\left(\Lambda^2\mathbb{C}^N\right)^{GL_N}$ of GL_N-invariant differential operators with polynomial coefficients, and this restriction to the center is known to be surjective (see. [3]). Thus it is natural to consider the concrete correspondence of GL_N-invariant differential operators and the central elements of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_N)$ in more details (the concrete Capelli problem see. [3, (10.4)]). Here the central elements of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_N)$ are the skew Capelli elements which were introduced in [3, p. 592, Remark (a)] to construct a skew Capelli identity for multiplicity free action of GL_N on skew symmetric matrices (see [3, p. 592, Corollary (11.3.19)]). Note that the explicit description of these skew Capelli elements was given by [8, p. 457, Theorem 3.2] in terms of the traces of powers of a matrix defined by the standard infinitesimal generators of GL_N . Now we know from [3, p. 589, (11.3.4)] (see. also [4, p. 741]) that the canonical generators of the algebra $\mathcal{PD}\left(\Lambda^2\mathbb{C}^N\right)^{GL_N}$ of GL_N-invariant differential operators on $\Lambda^2\mathbb{C}^N$ are the following skew Capelli operators defined with the Pfaffian:

$$\Gamma_k^\Lambda := \sum_{|I|=2k} pf(X_I)\, pf(D_I) \tag{2}$$

where Γ_k^Λ has degree k as differential operator $(1 \le k \le \left[\frac{N}{2}\right])$. Here, X_I and D_I indicate the submatrices $X_I = (x_{ij})_{i,j\in I}$ and $D_I = (\partial_{ij})_{i,j\in I}$ for $I \subseteq \{1, 2, \cdots, N\}$, respectively. These GL_N-invariant operators are commutative, and isomorphic to the center of the universal enveloping algebra of $\mathfrak{gl}_N(\mathbb{C})$. This is discussed in [3, p.581 (10.3), (10.4), p.589-593, and p.612 Table(15.1)].

3.1 SL_N-Invariant differential operators

Now denote $\overline{\mathcal{A}} := \mathcal{PD}\left(\Lambda^2\mathbb{C}^N\right)^{SL_N}$ the algebra of SL_N-invariant differential operators with polynomial coefficients on $\Lambda^2\mathbb{C}^N$. This is sligtly larger than that of GL_N-invariant operators described from the $\Gamma_k^\Lambda := \sum\limits_{|I|=2k} pf(X_I)\, pf(D_I)$. Then we deduce generators for $\overline{\mathcal{A}}$ by adding operators $pf(X)$ and $pf(D)$. Note that here $\Gamma_1^\Lambda =: \theta$ is the Euler vector field on $\Lambda^2\mathbb{C}^N$. Denote $\mathcal{J} \subset \overline{\mathcal{A}}$ the ideal of SL_N-invariant operators annihilating SL_N-invariant polynomials $\mathbb{C}[pf(X)]$.

Proposition 7 *The algebra $\overline{\mathcal{A}}$ of SL_N-invariant differential operators on $\Lambda^2\mathbb{C}^N$ is generated over $\mathbb{C}$ by $pf(X)$, $\theta := \Gamma_1^\Lambda$, Γ_2^Λ, $\cdots$, $\Gamma_{\frac{N}{2}-1}^\Lambda$, $pf(D)$ subject to the following relations modulo $\mathcal{J}$*

$$(a) \qquad [\theta, pf(X)] = \tfrac{N}{2} pf(X), \qquad [\theta, pf(D)] = -\tfrac{N}{2} pf(D)$$

$$(b) \qquad \left[\Gamma_k^\Lambda, \Gamma_l^\Lambda\right] = 0 \text{ for } k, l = 1, \cdots, \tfrac{N}{2} - 1$$

$$(c) \qquad pf(X)\, pf(D) = \prod_{t=0}^{\frac{N}{2}-1} \left(\frac{\theta}{\left(\frac{N}{2}\right)} + 2t \right)$$

$$(d)\qquad pf(D)\,pf(X)=\prod_{t=0}^{\frac{N}{2}-1}\left(\frac{\theta}{\left(\frac{N}{2}\right)}+2t+1\right)$$

$$(e)\qquad [pf(D),pf(X)]=\prod_{t=0}^{\frac{N}{2}-1}\left(\frac{\theta}{\left(\frac{N}{2}\right)}+2t+1\right)-\prod_{t=0}^{\frac{N}{2}-1}\left(\frac{\theta}{\left(\frac{N}{2}\right)}+2t\right)$$

$$(f)\qquad \Gamma_k^{\Lambda}=\alpha_k\prod_{t=0}^{k-1}\left(\frac{\theta}{\left(\frac{N}{2}\right)}+2t\right)$$

$$(g)\qquad \left[\Gamma_k^{\Lambda},pf(X)\right]=\alpha_k pf(X)\left\{\prod_{t=0}^{k-1}\left(\frac{\theta}{\left(\frac{N}{2}\right)}+2t+1\right)-\prod_{t=0}^{k-1}\left(\frac{\theta}{\left(\frac{N}{2}\right)}+2t\right)\right\}$$

$$(h)\qquad \left[\Gamma_k^{\Lambda},pf(D)\right]=\alpha_k\left\{\prod_{t=0}^{k-1}\left(\frac{\theta}{\left(\frac{N}{2}\right)}+2t-1\right)-\prod_{t=0}^{k-1}\left(\frac{\theta}{\left(\frac{N}{2}\right)}+2t\right)\right\}pf(D)$$

where $\alpha_k:=\frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{N}{2}+1-k\right)\Gamma(k+1)}$.

Before going to proof of Proposition 7 we recall the explicit eigenvalues of the Capelli operators Γ_k^{Λ}. From [8, p. 463, Formula (3.10)] we deduce $(1\le k\le \frac{N}{2})$

$$\Gamma_k^{\Lambda}pf(X)^s=\frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{N}{2}+1-k\right)\Gamma(k+1)}\prod_{t=0}^{k-1}(s+2t)\,pf(X)^s \tag{3}$$

where $\Gamma(z)$ is the gamma function. As an application $(k=\frac{N}{2},\Gamma_{\frac{N}{2}}^{\Lambda}=pf(X)\,pf(D))$ of this formula we note the following evaluation of the (simplest) Cayley type formula (b-function) attached to $\Lambda^2\mathbb{C}^N$ (see. [8, p. 463, Corollary 3.13]):

$$pf(D)\,pf(X)^s=\prod_{t=0}^{\frac{N}{2}-1}(s+2t)\,pf(X)^{s-1}. \tag{4}$$

Proof. of Proposition 7. Consider the following non commutative algebra

$$\overline{\mathcal{B}}:=\mathbb{C}\left\langle pf(X),\theta,\Gamma_2^{\Lambda},\cdots,\Gamma_{\frac{N}{2}-1}^{\Lambda},pf(D)\right\rangle\subset\overline{\mathcal{A}}. \tag{5}$$

We show that $\overline{\mathcal{A}}=\overline{\mathcal{B}}$. Let V^* be the dual of $V:=\Lambda^2\mathbb{C}^N$ and (X,ξ) be matrices in $V\times V^*$. We first show that $\mathrm{gr}\overline{\mathcal{A}}$ is free abelian, generated by $pf(X)$, $\gamma_1,\cdots,\gamma_{\frac{N}{2}-1}$, $pf(\xi)$ the symbols of $pf(X)$, θ, $\Gamma_2^{\Lambda},\cdots,\Gamma_{\frac{N}{2}-1}^{\Lambda}$, $pf(D)$ respectively. Put $\mu_j=dz_{2j-1}dz_{2j}$ for $j=1,\cdots,\frac{N}{2}$ and consider matrices $X^t:=t\mu_1+\sum_{j=2}^{\frac{N}{2}}\mu_j\in V$ and $\xi^{[t]}:=\xi^{t_1,\cdots t_{\frac{N}{2}}}=\frac{t_1}{t}\mu_1+\sum_{j=2}^{\frac{N}{2}}t_j\mu_j\in V^\star$, $t\neq 0$. The invariant polynomials $\gamma_k\left(X^t,\xi^{[t]}\right)$ are exactly the elementary symmetric polynomials $s_k(t_j):=\sum_{i_1<\cdots<i_k}t_{i_1}\cdots t_{i_k}$, for $k,j=1,\cdots,\frac{N}{2}$(This can be seen from [8, Equations (2.2), (2.4) and (2.6), p.452, 453.]). Let $f=f(X,\xi)$ be a polynomial in (X,ξ), there exists a polynomial q such that

$$f\left(X^t,\xi^{[t]}\right)=q\left(t,t_1,\cdots,t_{\frac{N}{2}}\right) \tag{6}$$

is a polynomial in the variables $t, t_1, \cdots, t_{\frac{N}{2}}$. Note that if the variables t_j are permuted one remains in the same orbit (of $Sp \subset SL$). So if f is an invariant polynomial, then $f\left(X^t, \xi^{[t]}\right)$ is a polynomial of t and of the $s_k(t_j)$ elementary symmetric polynomials for $k, j = 1, \cdots, \frac{N}{2}$:

$$f\left(X^t, \xi^{[t]}\right) = \widetilde{q}\left(t,\, s_1(t_j), \cdots, s_{\frac{N}{2}}(t_j)\right). \tag{7}$$

Then the difference

$$f(X, \xi) - \widetilde{q}\left(pf(X),\, \gamma_1, \cdots, \gamma_{\frac{N}{2}-1},\, pf(\xi)\right) \tag{8}$$

is a polynomial in (X, ξ) vanishing on the set ($(\frac{N}{2}+1)$-affine space) of $\left(X^t, \xi^{[t]}\right)$. Denote $\mathcal{R} := \bigsqcup_t \widetilde{G} \cdot \left(X^t, \xi^{[t]}\right)$ the union of orbits of points $\left(X^t, \xi^{[t]}\right)$ in one of affine spaces of $V \times V^*$. Assume f is invariant, then $f - \widetilde{q}$ is invariant and vanishes on $\mathcal{R}$. To see that $f - \widetilde{q}$ vanishes everywhere (i.e. $\mathrm{gr}\overline{\mathcal{A}} = \mathrm{gr}\overline{\mathcal{B}}$) it remains to show that $\mathcal{R}$ is open in $V \times V^*$. For this purpose consider $X : V \to V^*$, $\xi : V^* \to V$ skew symmetric: generically X (resp. ξ) is invertible then there exits $g \in SL$ such that ${}^t g X g^{-1} = t\mu_1 + \sum_{j=2}^{\frac{N}{2}} \mu_j$ (here ${}^t g$ denote the transpose of g) (see.[2, Abstract p.119 and Theorem 4 p. 125]). Moreover the matrice $X\xi$ is diagonalizable with distinct eigenvalues, so there exists h a symplectic transformation (preserving X) such that $g\xi({}^t h)^{-1} = \frac{t_1}{t}\mu_1 + \sum_{j=2}^{\frac{N}{2}} t_j \mu_j$. This means exactly that the orbit of $\left(X^t, \xi^{[t]}\right)$ contains a Zariski open dense set. So $\mathcal{R}$ is open in $V \times V^*$, and $f - \widetilde{q}$ vanishes everywhere.

$$\text{Thus} \qquad \mathrm{gr}\overline{\mathcal{A}} = \mathrm{gr}\overline{\mathcal{B}}. \tag{9}$$

Now, if P is an invariant operator of degree $m \geq 0$ ($P \in \overline{\mathcal{A}}$), its symbol $\sigma(P)$ is also invariant and from (9) we get $\sigma(P) = \sigma(P)\left(pf(X), \gamma_1, \cdots, \gamma_{\frac{N}{2}-1}, pf(\xi)\right)$. Then P can be written as the sum of an operator $Q \in \overline{\mathcal{B}}$ (a polynomial in the Γ_k^Λ's, $pf(X)$, $pf(D)$) and $R \in \overline{\mathcal{A}}$ an invariant operator of degree at most $m-1$:

$$P = Q\left(pf(X), \theta, \Gamma_2^\Lambda, \cdots, \Gamma_{\frac{N}{2}-1}^\Lambda, pf(D)\right) + R, \text{ with } \deg(R) \leq m-1. \tag{10}$$

By recurrence on the degree of the operator we see that P is a polynomial in the Γ_k^Λ's, $pf(X)$, $pf(D)$ that is $P \in \overline{\mathcal{B}}$. Therefore

$$\overline{\mathcal{A}} = \overline{\mathcal{B}}. \tag{11}$$

The remaining part of the proof is devoted to desmontrating the relations (a), (b), (c), (d), (e), (f), (g). The algebra $\overline{\mathcal{A}}$ acts on $\mathbb{C}[pf(X)]$ the ring of

$\widetilde{G}$-invariant polynomials. Now (a) is obvious since the pfaffian is homogeneous of degree $\frac{N}{2}$. (b) holds since the GL_N-invariant operators commute (see. [3, P. 581, (10.3) (The abstract Capelli problem) and p. 612, Table (15.1) : (line 3, column 3).]). To verify (c), (d) and (e), (f), (g) we use the relations (4), (3). ■

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Now put $\mathcal{A} := \overline{\mathcal{A}}/\mathcal{J}$ the quotient algebra of $\overline{\mathcal{A}}$ by the ideal $\mathcal{J}$.

Corollary 8 *$\mathcal{A}$ is generated by $pf\,(X)$, θ, $pf\,(D)$ satisfying the relations*

$$[\theta, pf\,(X)] = \tfrac{N}{2} pf\,(X)\,, \quad [\theta, pf\,(D)] = -\tfrac{N}{2} pf\,(D)\,,$$
$$pf\,(X)\, pf\,(D) = \prod_{t=0}^{\frac{N}{2}-1} \left(\frac{\theta}{\left(\frac{N}{2}\right)} + 2t \right).$$

Proof. Let P be in $\overline{\mathcal{A}}$, we decompose it into homogeneous components ($P = \sum_{j\in\mathbb{Z}} P_j$) P_j of degree $j\frac{N}{2}$ (ie. $[\theta, P_j] = j\frac{N}{2}P_j$) so that if $j = 0$ then $P_0 = \varphi\,(\theta)$ is a polynomial in θ. Indeed, P_0 acts on $\mathbb{C}\,[pf(X)]$ then $P_0 \in \mathbb{C}\left[pf(X), \frac{\partial}{\partial(pf(X))}\right]$ with $pf(X)\frac{\partial}{\partial(P(X))} = \frac{1}{\frac{N}{2}}\theta$. If $j > 0$ then $pf(D)^j P_j = \psi\,(\theta)$ is a polynomial in θ because $pf(D)^j P_j$ is homogeneous of degree 0. Likewise if $j < 0$ then $pf(X)^{-j} P_j = \phi\,(\theta)$ is a polynomial in θ. Thus for any P_j (SL_N-invariant) homogeneous of degree $j\frac{N}{2}$, its class modulo $\mathcal{J}$ is of the form

$$P_j \bmod \mathcal{J} = \begin{cases} pf(X)^j \phi_j\,(\theta) & \text{if } j \geq 0 \\ pf(D)^{-j} \psi_j\,(\theta) & \text{if } j \leq 0 \end{cases} \tag{12}$$

where $\phi_j\,(\theta)$, $\psi_j\,(\theta)$ are (polynomials) homogeneous of degree 0.

Now put $\mathcal{K} := \overline{\mathcal{A}} \left(pf\,(X)\, pf\,(D) - \prod_{t=0}^{\frac{N}{2}-1} \left(\frac{\theta}{\left(\frac{N}{2}\right)} + 2t \right) \right) \overline{\mathcal{A}} \subset \mathcal{J}$ we show that $\mathcal{J} = \mathcal{K}$. Let P be in $\mathcal{J}$. Since P annihilates $pf(X)^m$, $m \geq 0$, then its homogeneous components $P_j = pf(X)^j Q\,(\theta) \bmod \mathcal{K}$ if $j \geq 0$ (resp.$pf(D)^{-j} Q\,(\theta) \bmod \mathcal{K}$ if $j \leq 0$) also annihilate $pf(X)^m$. This means that the polynomial in m vanishes $Q\left(\frac{N}{2}m\right) = 0$ for $m > j$. Then we may deduced that the polynomial in $\lambda \in \mathbb{C}$, $Q\left(\frac{N}{2}\lambda\right) = 0$. Therefore $Q = 0 \bmod \mathcal{K}$ and $\mathcal{J} = \mathcal{K}$. ■

4 Invariant sections of $\mathcal{D}$-modules on skew symmetric matrices

This section is devoted to the main general argument of the paper. The idea is to show that the $\mathcal{D}_{\Lambda^2\mathbb{C}^N}$-modules studied here are generated by their invariant global sections under the action of $SL_N\,(\mathbb{C})$. The general structure of the demonstration uses classical methods and the fact that such $\mathcal{D}_{\Lambda^2\mathbb{C}^N}$-modules are essentially inverse images of modules over $\mathbb{C}$ by the pfaffian map.

Theorem 9 *A module $\mathcal{M}$ in $\mathrm{Mod}_{\Sigma}^{\mathrm{rh}}(\mathcal{D})$ is generated by its SL_N-invariant global sections.*

Before going to the proof of the theorem, we need the following results.

4.1 $\mathcal{D}$-modules with support in $\overline{V}_{N-2}$

From now on $N = 2m$. Denote $\overline{V}_{2k} := \left\{X \in \Lambda^2\mathbb{C}^{2m} \ / \ \mathrm{rank}\,(X) \leq 2k\right\}$ the set of skew-symmetric matrices of rank $2k$ or less for $0 \leq k \leq m$. Then $\overline{V}_{2m-2} = \left\{X \in \Lambda^2\mathbb{C}^{2m} \ / \ pf\,(X) = 0\right\}$is the hypersurface defined by the pfaffian map $pf : \Lambda^2\mathbb{C}^{2m} \longrightarrow \mathbb{C}, X \longrightarrow t$. Here we study $\mathcal{D}$-modules with support on $\overline{V}_{2k}$. These modules will be used in the sequel to prove the central theorem 9.

4.1.1 Meromorphic sections

Let $\mathcal{N}$ be a $\mathcal{D}_{\mathbb{C}}$-module and $pf^+\,(\mathcal{N})$ its inverse image by the pfaffian map. Note that the transfer module $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m} \xrightarrow{pf} \mathbb{C}}$ is flat over $pf^{-1}\,(\mathcal{D}_{\mathbb{C}})$ thanks to the flatness of the pfaffian map. Thus the inverse image pf^+ is an exact functor. If $\mathcal{N}$ is a regular holonomic $\mathcal{D}_{\mathbb{C}}$-module with singularity at $t = 0$ (t is a coordinate on $\mathbb{C}$) then its inverse image $pf^+\,(\mathcal{N})$ decomposes at least as $\mathcal{N}$. If the operator of multiplication by t is invertible on the $\mathcal{D}_{\mathbb{C}}$-module $\mathcal{N}$ then the multiplication by the pfaffian pf is invertible on $pf^+\,(\mathcal{N})$. In particular in this case, any meromorphic section defined on $\Lambda^2\mathbb{C}^{2m}\backslash\overline{V}_{2m-2}$ extends to the whole $\Lambda^2\mathbb{C}^{2m}$. Precisely, if $\mathcal{M}$ is a $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$-module, we set $\overline{\mathcal{M}} := \Gamma_{\left[\Lambda^2\mathbb{C}^{2m}|\overline{V}_{2m-2}\right]}(\mathcal{M}) = \varinjlim \mathcal{H}om_{\mathcal{O}_{\Lambda^2\mathbb{C}^{2m}}}\left(\mathcal{I}^k, \mathcal{M}\right)$ (where $\mathcal{I}$ is the defining ideal of $\overline{V}_{2m-2}$) the algebraic module of meromorphic sections of $\mathcal{M}$ with pole in the pfaffian hypersurface $\overline{V}_{2m-2}$(see. [6]). We have a canonical morphism $\mathcal{M} \longrightarrow \overline{\mathcal{M}}$.

Proposition 10 *Let $\mathcal{N}$ be a holonomic $\mathcal{D}_{\mathbb{C}}$-module with regular singularity at $t = 0$. Assume that the multiplication by t defines an automorphism of $\mathcal{N}$ then*

(i) the multiplication by the pfaffian pf defines an automorphism of $pf^+\,(\mathcal{N})$,

(ii) we have the isomorphism

$$pf^+\,(\mathcal{N}) \xrightarrow{\sim} \overline{pf^+\,(\mathcal{N})}. \tag{13}$$

Proof. (i) follows from [7, Lemma 1.2, p.166] (see. [7, Definition 1.1, (1.3) p.164-165]) and [7, Remark 1.1., (1.4) p.165]. Next, recall [6] that we have an exact sequence $0 \rightarrow \Gamma_{\left[\overline{V}_{2m-2}\right]}(\mathcal{M}) \rightarrow \mathcal{M} \rightarrow \overline{\mathcal{M}}$ where $\Gamma_{\left[\overline{V}_{2m-2}\right]}(\mathcal{M}) = \varinjlim \mathcal{H}om_{\mathcal{O}_{\Lambda^2\mathbb{C}^{2m}}}\left(\mathcal{O}_{\Lambda^2\mathbb{C}^{2m}}/\mathcal{I}^m, \mathcal{M}\right)$ is the subsheaf of $\mathcal{M}$ of sections annihilated by some power of $\mathcal{I}$. Since pf gives a bijection on $pf^+\,(\mathcal{N})$, [7, Remark 1.1., (1.4) and (1.3) p.165] asserts that $\mathcal{H}^k_{\left[\overline{V}_{2m-2}\right]}\left(pf^+\,(\mathcal{N})\right) = 0 \ ^\forall k$. Then from the previous exact sequence, we get $pf^+\,(\mathcal{N}) \simeq \overline{pf^+\,(\mathcal{N})}$. ∎

4.1.2 Study of $pf^{+}\left(\mathcal{O}\left(\frac{1}{t}\right)\right)$

Here we describe the subquotient modules of $F := \mathcal{O}_{\Lambda^2\mathbb{C}^{2m}}\left(\frac{1}{pf(X)}\right)$. Actually F is generated by its SL_N-invariant homogeneous sections $e_s = pf\,(X)^s$ where $s \le 0$. From formulas (4) and (3) we get the relations $(1 \le k \le \frac{N}{2} = m)$

$$\begin{aligned} pf\,(X)\,.e_s &= e_{s+1}, \qquad \theta e_s = \frac{N}{2} s e_s && (14)\\ pf\,(D)\,e_s &= \prod_{t=0}^{\frac{N}{2}-1} (s+2t)\, e_{s-1} && (15)\\ \Gamma_k^{\Lambda} e_s &= \frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{N}{2}+1-k\right)\Gamma\left(k+1\right)} \prod_{t=0}^{k-1} (s+2t)\, e_s && (16) \end{aligned}$$

where $\Gamma\,(z)$ is the gamma function.

4.1.3 Relations

Note that the $\mathcal{D}_{\Lambda^2\mathbb{C}^N}$-module F has $\frac{N}{2}+1$ submodules denoted by F_s, generated respectively by $e_{-s} = pf\,(X)^{-s}$ $(s = 0, 1, \cdots, \frac{N}{2})$ in $\mathcal{O}_{\Lambda^2\mathbb{C}^N}\left(\frac{1}{pf}\right)$:

$$F_0 := \mathcal{O}_{\Lambda^2\mathbb{C}^N} \subset F_1 := \mathcal{D}_{\Lambda^2\mathbb{C}^N} pf\,(X)^{-1} \subset \cdots \subset F_{\frac{N}{2}} := \mathcal{D}_{\Lambda^2\mathbb{C}^N} pf\,(X)^{-\frac{N}{2}}. \quad (17)$$

Consider the following quotient modules of $\mathcal{O}_{\Lambda^2\mathbb{C}^N}\left(\frac{1}{pf(X)}\right)$ by the $F_{\frac{N}{2}-s-1}$ which will be used in the sequel:

$$R_s := \mathcal{O}_{\Lambda^2\mathbb{C}^N}\left(\frac{1}{pf}\right)/F_{\frac{N}{2}-s-1} = \mathcal{O}_{\Lambda^2\mathbb{C}^N}\left(\frac{1}{pf}\right)/\mathcal{D}pf^{-\frac{N}{2}+s+1}. \qquad (18)$$

The R_s are generated by $e_{-\frac{N}{2}+s} \bmod e_{-\frac{N}{2}+s+1}$, $e_{-\frac{N}{2}+s-1} \bmod e_{-\frac{N}{2}+s+1}$, $\cdots$, $e_{-\frac{N}{2}} \bmod e_{-\frac{N}{2}+s+1}$ homogeneous sections of degree $\frac{N}{2}\left(-\frac{N}{2}+s\right)$, $\frac{N}{2}\left(-\frac{N}{2}+s-1\right)$, $\cdots$, $-\frac{N}{4}^2$ respectively. Put $\widetilde{e}_{-\frac{N}{2}+s-k} := e_{-\frac{N}{2}+s-k} \bmod e_{-\frac{N}{2}+s+1}$ for $k = 0,\cdots,s$:

$$R_s := \left\{ \begin{array}{c} \text{generators } \widetilde{e}_{-\frac{N}{2}+s},\ \widetilde{e}_{-\frac{N}{2}+s-1}, \cdots,\ \widetilde{e}_{-\frac{N}{2}}, \\ pf\,(X)\,\widetilde{e}_{-\frac{N}{2}+s} = 0 \\ \theta \widetilde{e}_{-\frac{N}{2}+s} = \frac{N}{2}\left(-\frac{N}{2}+s\right)\widetilde{e}_{-\frac{N}{2}+s} \\ pf\,(X_I)\,\widetilde{e}_{-\frac{N}{2}+s} = 0,\ |I| = s+1 \text{ for } 0 \le s \le \frac{N}{2}-1 \\ pf\,(D_I)\,\widetilde{e}_{-\frac{N}{2}+s} = 0,\ |I| = s \text{ for } 0 \le s \le \frac{N}{2}. \end{array} \right. \qquad (19)$$

thanks to the relations (14), (15), (16). Thus $\mathrm{Char}\,(R_s) := \overline{T^*_{V_{2s}} X}$.

Lemma 11 *The $R_s := \mathcal{O}\left(\frac{1}{pf}\right)/F_{\frac{N}{2}-s-1}$ are modules with support on $\overline{V}_{2s}$.*

4.1.4 Extension

In this subsection we show that any section s of the $\mathcal{D}_X$-module R_s in the complementary of $\overline{V}_{2s-2}$ extends to the whole $\Lambda^2\mathbb{C}^N$.

Proposition 12 *A section $s \in \Gamma\left(\Lambda^2\mathbb{C}^{2m}\backslash\overline{V}_{2s-2}, R_s\right)$ of the $\mathcal{D}_{\Lambda^2\mathbb{C}^N}$-module R_s in the complementary of $\overline{V}_{2s-2}$ extends to the whole $\Lambda^2\mathbb{C}^{2m}$ ($s = 1, \cdots, m-1$).*

Proof. First, note that the hypersurface $\overline{V}_{2m-2}$ is smooth out of $\overline{V}_{2m-4}$ and it is a normal variety along V_{2m-4} (smooth). Likewise the variety $\overline{V}_{2s}$ is smooth out of $\overline{V}_{2s-2}$ and normal along V_{2s-2} for $s = 1, \cdots, m-1$. Next, the $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$-module R_s is the union of modules $\mathcal{O}_{\Lambda^2\mathbb{C}^N}\widetilde{e}_{-\frac{N}{2}+s-j}$ $(0 \leq j \leq s)$ such that the associated graded modules $\mathrm{gr}(R_s)$ is the sum of modules $\mathcal{O}_{T^*_{V_{2s}}\Lambda^2\mathbb{C}^N}\widetilde{e}_{-\frac{N}{2}+s-j}$ $(0 \leq j \leq s)$. In this case the property of extension here is true for functions because $\overline{V}_{2s}$ is normal along $\overline{V}_{2s-2}$ $(s = 1, \cdots, m-1)$. ■

4.2 Inverse image by the pfaffian map

Now let $i : \mathbb{C} \longrightarrow \Lambda^2\mathbb{C}^{2m}$, $t \mapsto \omega_t := tdz_1dz_2 + dz_3dz_4 + \cdots + dz_{2m-1}dz_{2m}$ (with $\frac{\omega_t^m}{n!} = tdz_1dz_2\cdots dz_{2m}$ and $pf\left(\omega_t\right) = t$) be a section of the pfaffian map $pf : \Lambda^2\mathbb{C}^{2m} \longrightarrow \mathbb{C}$. Denote by $D = \omega_{\mathbb{C}} = \mathbb{C}dz_1dz_2 + dz_3dz_4 + \cdots + dz_{2m-1}dz_{2m}$ its image. We need the following lemma:

Lemma 13 *The line D is non characteristic for $\mathcal{M}$ in $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D})$ i.e. $\overline{T^*_D\Lambda^2\mathbb{C}^N} \bigcap char(\mathcal{M}) \subset T^*_{\Lambda^2(\mathbb{C}^N)}\Lambda^2\left(\mathbb{C}^N\right)$. In other words $char(\mathcal{M}) \bigcap T^*\Lambda^2\left(\mathbb{C}^N\right)_{|D} \longrightarrow T^*D$ is a proper morphism.*

Proof. Note that $V_{2k} \cap D = \begin{cases} \{\omega_0\} \text{ if } k = m-1 \\ \emptyset \text{ if } k = 0, \cdots m-2 \\ D\backslash\{\omega_0\} \text{ if } k = m \end{cases}$. We show that $\overline{T^*_D\Lambda^2\mathbb{C}^N} \bigcap \overline{T^*_{V_{2m-2}}\Lambda^2\mathbb{C}^N}$ is contained in the zero section $T^*_{\Lambda^2\mathbb{C}^N}\Lambda^2\mathbb{C}^N$. It suffices to check at the point ω_0 $(t = 0)$ which is the only point of the line D above which the characteristic variety $\mathrm{char}(\mathcal{M})$ has a non zero covector $\xi_0 \neq 0$. Note that this covector ξ_0 is parallel to dpf (the conormal bundle to pfaffian variety) and on the line D we have $dpf\left(X\right) = dt \neq 0$, that is, $\xi_0 \notin \overline{T^*_D\Lambda^2\mathbb{C}^N}$. ■

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Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_{\Lambda^2(\mathbb{C}^N)}$-module along $\mathrm{char}(\mathcal{M})$. Since the line D is non characteristic for $\mathcal{M}$ then $\mathcal{M}$ is canonically isomorphic to the inverse image $pf^+i^+\left(\mathcal{M}\right)$ in the neighborhood of D. The sheaf $\mathcal{H}\mathrm{om}_{\mathcal{D}_X}\left(\mathcal{M}, pf^+i^+\left(\mathcal{M}\right)\right)$ is constructible (see. [6]) and it is also locally constant on the fibers of the pfaffian map. Let u be its canonical section defined in the neighborhood of D (corresponding with the isomorphism $\mathcal{M} \xrightarrow{\sim} pf^+i^+\left(\mathcal{M}\right)$ which induces the identity on D). Now, thanks to the simply connectedness of the fibers of the pfaffian map (see. Proposition 2) we have the following proposition:

Proposition 14 *The canonical isomorphism $u : \mathcal{M} \xrightarrow{\sim} pf^+i^+\left(\mathcal{M}\right)$ defined in the neighborhood of D such that $i^+.u = Id_{|D}$, extends to $\Lambda^2\mathbb{C}^{2m}\backslash\overline{V}_{2m-4}$.*

Proof. The local canonical section $u : \mathcal{M}_{|_D} \xrightarrow{\sim} pf^+ i^+ (\mathcal{M})_{|_D}$ is defined out of $V_{2m-4} \cup \cdots \cup V_2 \cup V_0$(the singular part of the pfaffian variety $\overline{V}_{2m-2} := pf^{-1}\{0\}$). Then we focus our attention on the orbits $V_{2m} = \Lambda^2\mathbb{C}^{2m} \backslash pf^{-1}\{0\}$ and $V_{2m-2} = pf^{-1}\{0\} \setminus (V_{2m-4} \cup \cdots \cup V_2 \cup V_0)$. These orbits are simply connected (see. proposition 2):
(i) $\pi_1(V_{2m}) = \{1\}$
(ii) $\pi_1(X_{2m-2}) = \{1\}$.

Note that the fundamental group $\pi_1(V_{2m})$ (resp. $\pi_1(V_{2m-2})$) acts on the constructible sheaf $\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, pf^+ i^+ \mathcal{M})$. This sheaf is trivial on $V_{2m} = \bigcup_{t \neq 0} pf^{-1}(t)$ and on $V_{2m-2} = pf^{-1}(0) \setminus (V_{2m-4} \cup \cdots \cup V_2 \cup V_0)$. Then the local section $u : \mathcal{M}_{|_D} \xrightarrow{\sim} pf^+ i^+ (\mathcal{M})_{|_D}$extends globally to the union $V_{2m} \cup V_{2m-2} = \Lambda^2\mathbb{C}^{2m} \setminus (V_{2m-4} \cup \cdots \cup V_2 \cup V_0)$. ■

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Denote $\mathcal{N} := i^+\mathcal{M}$ the restriction of the $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$-module $\mathcal{M}$ to the transversal line D. From Proposition 14, we deduce that the $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$-module $\mathcal{M}$ is isomorphic to $pf^+\mathcal{N}$ on $\Lambda^2\mathbb{C}^{2m} \backslash \overline{V}_{2m-2}$:

$$\mathcal{M}_{|_{\Lambda^2\mathbb{C}^{2m} \backslash \overline{V}_{2m-2}}} \simeq pf^+\mathcal{N}_{|\Lambda^2\mathbb{C}^{2m} \backslash \overline{V}_{2m-2}} \tag{20}$$

Recall that $\overline{\mathcal{M}}$ (see section 4.1.1) stands for the $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$-module of meromorphic sections of $\mathcal{M}$ defined on $\Lambda^2\left(\mathbb{C}^{2m}\right) \backslash \overline{V}_{2m-2}$. According to an argument of Kashiwara, since $\mathcal{M}$ and $pf^+\mathcal{N}$ are regular holonomic and isomorphic out of $\Lambda^2\left(\mathbb{C}^{2m}\right) \backslash \overline{V}_{2m-2}$, then their corresponding meromorphic modules are also isomorphic that is

$$\overline{\mathcal{M}} \simeq \overline{pf^+\mathcal{N}}\ . \tag{21}$$

Now let us consider the following morphism (see section 4.1.1)

$$\mathcal{M} \longrightarrow \overline{\mathcal{M}} \left(\simeq \overline{pf^+\mathcal{N}} \right). \tag{22}$$

By using the basic fact that $\overline{pf^+\mathcal{N}} \simeq pf^+\mathcal{N}$ (see relation (13) of Proposition 10) and the morphism (22), we deduce that there exists a morphism

$$v : \mathcal{M} \longrightarrow pf^+\mathcal{N} \tag{23}$$

which is an isomorphism out of the pfaffian hypersurface $\overline{V}_{2m-2}$.

Lemma 15 *The image $v(\mathcal{M}) \subset pf^+\mathcal{N}$ is a $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$-module generated by its SL_N-invariant homogeneous global sections.*

Now, we are in position to prove the theorem 9.

4.3 Proof of Theorem 9

Recall $F := \mathcal{O}_{\Lambda^2\mathbb{C}^{2m}}\left(\frac{1}{pf}\right)$ and $R_s := \mathcal{O}\left(\frac{1}{pf}\right)/\mathcal{D}pf^{-\frac{N}{2}+s+1}$ (see. section 4.1.3). The $\mathcal{D}$-module F is generated by its SL_N-invariant homogeneous sections $e_s =$

pf^s where $s \leq 0$ subject to the relations (14),(15 (16). In particular, F has $\left(\frac{N}{2}+1\right)$ subquotient modules denoted by R_s, generated by $\widetilde{e}_{-\frac{N}{2}+s}, \widetilde{e}_{-\frac{N}{2}+s-1}, \cdots,$ $\widetilde{e}_{-\frac{N}{2}}$ $(0 \leq s \leq \frac{N}{2} = m)$, and supported by $\overline{V}_{2s}$ (see. Lemma 11) .

Let us denote by $\widetilde{\mathcal{M}} \subset \mathcal{M}$ the submodule generated over $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$ by SL_N-invariant homogeneous global sections (that is the module $\widetilde{\mathcal{M}} := \mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}\{u \in \Gamma\left(\Lambda^2\mathbb{C}^{2m}, \mathcal{M}\right)^{SL_N}, \dim_{\mathbb{C}} \mathbb{C}\left[\theta\right] u < \infty\}$). We will see successively that the quotient module $\mathcal{M}\diagup\widetilde{\mathcal{M}}$ is supported by $\overline{V}_{2s}$ $(0 \leq s \leq m-1)$, and the monodromy is trivial since $\overline{V}_{2s}\backslash\overline{V}_{2s-2}$ is simply connected.

To begin with, $\mathcal{M}/\widetilde{\mathcal{M}}$ is supported by $\overline{V}_{2m-2}$: indeed $\mathcal{M}$ is isomorphic in $\Lambda^2\mathbb{C}^{2m}\backslash\overline{V}_{2m-2}$ to a $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$-module $pf^+\left(\mathcal{N}\right)$ (see. formula (20) and Proposition 14). We may assume that the operator of multiplication by t is invertible on $\mathcal{N}$ such that there exists a morphism $v : \mathcal{M} \longrightarrow pf^+\left(\mathcal{N}\right)$ which is an isomorphism out of $\overline{V}_{2m-2}$ (see. (23)). The image $v\left(\mathcal{M}\right)$ is a submodule of $pf^+\left(\mathcal{N}\right)$ so it is generated by its SL_N-invariant homogeneous global sections (see. Lemma 15). If σ is a SL_N-invariant homogeneous global section of a quotient of $\mathcal{M}$ then σ lifts to an invariant homogeneous global section $\widetilde{\sigma}$ of $\mathcal{M}$ $(\widetilde{\sigma} \in \Gamma\left(X, \mathcal{M}\right)^{SL_N}$, $\dim_{\mathbb{C}} \mathbb{C}\left[\theta\right] \widetilde{\sigma} < \infty)$. This means that $\mathcal{M}/\widetilde{\mathcal{M}}$ is supported by $\overline{V}_{2m-2}$. Next, if $\mathcal{M}$ is supported by $\overline{V}_{2m-2}$, it is isomorphic out of $\overline{V}_{2m-4}$ to a direct sum of copies of R_{m-1}, then there is a morphism $\mathcal{M} \longrightarrow R^q_{m-1}$ whose sections extend (see. Proposition 12) such that $\mathcal{M}/\widetilde{\mathcal{M}}$ is supported by $\overline{V}_{2m-2}$ because the submodules of R_{m-1} are also generated by their invariant homogeneous sections. In the same way by induction on k, if $\mathcal{M}$ is with support on $\overline{V}_{2s}$ $(0 \leq S \leq m-2)$ then there is a morphism $\mathcal{M} \longrightarrow R^q_s$ which is an isomorphism out of $\overline{V}_{2s-2}$, such that $\mathcal{M}/\widetilde{\mathcal{M}}$ is with support on $\overline{V}_{2s-2}$ because the submodules of R_s are also generated by their invariant homogeneous sections. Finally, if $\mathcal{M}$ is supported by V_0 (the Dirac module with support at the origin) then the result is obvious.This ends the proof of theorem 9.

5 Equivalence of categories

Recall $\overline{\mathcal{A}}$ is the algebra of SL_N-invariant differential operators, $\mathcal{J} \subset \overline{\mathcal{A}}$ the ideal annihilator of $\mathbb{C}\left[pf\left(X\right)\right]$ and $\mathcal{A}$ the quotient of $\overline{\mathcal{A}}$ by $\mathcal{J}$. Then $\mathcal{A}$ is generated by the 3 operators $pf\left(X\right)$, θ, $pf\left(D\right)$ (see. Corollary 8) such that $\left[\theta, pf\left(X\right)\right] = \frac{N}{2}pf\left(X\right)$, $\left[\theta, pf\left(D\right)\right] = -\frac{N}{2}pf\left(D\right)$ and $pf\left(X\right)pf\left(D\right) = \prod_{t=0}^{\frac{N}{2}-1}\left(\frac{\theta}{\left(\frac{N}{2}\right)} + 2t\right)$.

We denote by $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ the category of graded $\mathcal{A}$-modules T of finite type such that $\dim_{\mathbb{C}} \mathbb{C}\left[\theta\right] u < \infty$ for $^{\forall}u \in T$. In other words, $T = \underset{\lambda\in\mathbb{C}}{\oplus} T_\lambda$ is a direct sum of $\mathbb{C}$-vector spaces ($T_\lambda = \bigcup\limits_{p\in\mathbb{N}} \ker\left(\theta-\lambda\right)^p$ is finite dimensional) equipped with 3 endomorphisms $pf\left(X\right)$, θ, $pf\left(D\right)$ of degree $\frac{N}{2}$, 0, $-\frac{N}{2}$, respectively and satisfying the above relations with $(\theta-\lambda)$ being a nilpotent operator on each T_λ.

Let us recall that $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D})$ stands for the category of regular holonomic $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$-

modules whose characteristic variety is contained in Σ.
If $\mathcal{M}$ is an object in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D})$, denote by $\Psi(\mathcal{M})$ the submodule of $\Gamma\left(\Lambda^2\mathbb{C}^{2m},\mathcal{M}\right)$ consisting of SL_N-invariant homogeneous global sections u of $\mathcal{M}$ such that $\dim_{\mathbb{C}}\mathbb{C}[\theta]\,u<\infty$. Recall that (Theorem 4) $\Psi(\mathcal{M})_\lambda := [\Psi(\mathcal{M})] \bigcap \left[\bigcup_{p\in\mathbb{N}} \ker(\theta-\lambda)^p\right]$ is the $\mathbb{C}$-vector space of homogeneous global sections of degree λ of $\Psi(\mathcal{M})$ and $\Psi(\mathcal{M}) = \bigoplus_{\lambda\in\mathbb{C}} \Psi(\mathcal{M})_\lambda$. Then $\Psi(\mathcal{M})$ is an object in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$. Indeed, let $(\sigma_{1,}\cdots,\sigma_p)$ be a finite family of homogeneous invariant global sections generating the $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$-module $\mathcal{M}$ (see Theorem 9), we can see that the family $(\sigma_{1,}\cdots,\sigma_p)$ generates also $\Psi(\mathcal{M})$ as an $\mathcal{A}$-module: In fact, if $\sigma = \sum_{j=1}^{p} q_j(X,D)\,\sigma_j$ is an invariant section of $\mathcal{M}$ ($q_j\in\Gamma\left(\Lambda^2\mathbb{C}^{2m},\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}\right)$), denote by $\widetilde{q_j}$ the average of q_j over $SU_n(\mathbb{C})$ (compact maximal subgroup of SL_N), then $\widetilde{q_j}\in\overline{\mathcal{A}}$. Let $f_j = f_j(pf(X),\theta,pf(D))$ be the class of $\widetilde{q_j}$ modulo $\mathcal{J}$ that is $f_j\in\mathcal{A}$, then we also have $\sigma=\sum_{j=1}^{p}\widetilde{q_j}\sigma_j=\sum_{j=1}^{p} f_j\sigma_j$.

Conversely, if T is an object in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$, one associates to it the $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$-module

$$\Phi(T) = \mathcal{M}_0 \bigotimes_{\mathcal{A}} T \tag{24}$$

where $\mathcal{M}_0 := \mathcal{D}/\mathcal{J}$ is a $(\mathcal{D},\mathcal{A})$-module. Then $\Phi(T)$ is an object in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D})$.
Thus, we have defined two functors

$$\Psi : \mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}) \longrightarrow \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}),\ \Phi : \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}) \longrightarrow \mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}). \tag{25}$$

We need the following lemmas:

Lemma 16 *The canonical morphism*

$$T \longrightarrow \Psi(\Phi(T)),\ t \longmapsto 1\otimes t \tag{26}$$

is an isomorphism, and defines an isomorphism of functors $\mathrm{Id}_{\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})} \longrightarrow \Psi\circ\Phi$.

Proof. As above $\mathcal{M}_0 := \mathcal{D}/\mathcal{J}$. Denote by ε (the class of $1_{\mathcal{D}}$ modulo $\mathcal{J}$) the canonical generator of $\mathcal{M}_0$. Let $h\in\mathcal{D}$, denote by $\widetilde{h}\in\overline{\mathcal{A}}$ its average on $SU_N(\mathbb{C})$ and by φ the class modulo $\mathcal{J}$ that is $\varphi\in\mathcal{A}$. Since ε is SL_N-invariant, we get $\widetilde{h\varepsilon}=\widetilde{h}\varepsilon=\varepsilon\varphi$. Moreover, we have $\widetilde{h}\varphi=0$ if and only if $\widetilde{h}\in\mathcal{J}$, in other words $\varphi=0$. Therefore the average operator (over $SU_N(\mathbb{C})$) $\mathcal{D}\longrightarrow\overline{\mathcal{A}}$, $h\longmapsto\widetilde{h}$ induces a surjective morphism of $\mathcal{A}$-modules $v:\mathcal{M}_0\longrightarrow\mathcal{A}$. More generally, for any $\mathcal{A}$-module T in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ the morphism $v\otimes 1_T$ is a surjective map

$$v_T : \mathcal{M}_0 \bigotimes_{\mathcal{A}} T \longrightarrow \mathcal{A}\bigotimes_{\mathcal{A}} T = T \tag{27}$$

which is the left inverse of the morphism

$$u_T : T \longrightarrow \mathcal{M}_0 \bigotimes_{\mathcal{A}} T,\ t \longmapsto \varepsilon \otimes t \tag{28}$$

that is $(v \otimes 1_T) \circ (\varepsilon \otimes 1_T) = v(\varepsilon) = 1_T$. This means that the morphism u_T is injective. Next, the image of u_T is exactly the set of invariant sections of $\mathcal{M}_0 \bigotimes_{\mathcal{A}} T = \Phi(T)$ that is $\Psi(\Phi(T))$: indeed if $\sigma = \sum_{i=1}^{p} h_i \otimes t_i$ is an invariant section in $\mathcal{M}_0 \bigotimes_{\mathcal{A}} T$, we may replace each h_i by its average $\widetilde{h_i} \in \mathcal{A}$, then we get

$$\sigma = \sum_{i=1}^{p} \widetilde{h_i} \otimes t_i = \varepsilon \otimes \sum_{i=1}^{p} \widetilde{h_i} t_i \ \in \ \varepsilon \otimes T \tag{29}$$

that is $\sum_{i=1}^{p} \widetilde{h_i} t_i \in T$. Therefore the morphism u_T is an isomorphism from T to $\Psi(\Phi(T))$ and defines an isomorphism of functors. ■

Lemma 17 *The canonical morphism*

$$w : \Phi(\Psi(\mathcal{M})) \longrightarrow \mathcal{M} \tag{30}$$

is an isomorphism and defines an isomorphism of functors $\Phi \circ \Psi \longrightarrow \mathrm{Id}_{\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D})}$.

Proof. As in the Theorem 9 the $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$-module $\mathcal{M}$ is generated by a finite family of invariant sections $(\sigma_i)_{i=1,\cdots,p} \in \Psi(\mathcal{M})$ so that the morphism w is surjective. Anyway w is injective. Indeed let $\mathcal{Q}$ be the kernel of the morphism $w : \Phi(\Psi(\mathcal{M})) \longrightarrow \mathcal{M}$. The $\mathcal{D}_{\Lambda^2\mathbb{C}^{2m}}$-module $\mathcal{Q}$ is also generated by its invariant sections that is by $\Psi(\mathcal{Q})$. Then we get

$$\Psi(\mathcal{Q}) \subset \Psi[\Phi(\Psi(\mathcal{M}))] = \Psi(\mathcal{M}) \tag{31}$$

where we used $\Psi \circ \Phi = Id_{\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})}$ (see the previous Lemma 16). Since the morphism $\Psi(\mathcal{M}) \longrightarrow \mathcal{M}$ is injective $(\Psi(\mathcal{M}) \subset \Gamma(\Lambda^2\mathbb{C}^{2m},\ \mathcal{M}))$, we obtain$\Psi(\mathcal{Q}) = 0$. Therefore $\mathcal{Q} = 0$ (because $\Psi(\mathcal{Q})$ generates $\mathcal{Q}$). ■

.

This section ends by the following theorem established by means of the previous lemmas:

Theorem 18 *The functors* Φ *and* Ψ *induce equivalence of categories*

$$\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}) \ \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}). \tag{32}$$

6 Classification of finite type graded $\mathcal{A}$-modules

This section consists in the classification of objects in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$. A graded $\mathcal{A}$-module T in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ defines an infinite diagram consisting of finite dimensional vector spaces T_λ (with $(\theta - \lambda)$ being a nilpotent operator on each T_λ, $\lambda \in \mathbb{C}$) and linear maps between them deduced from $pf\,(X)$, θ, $pf\,(D)$:

$$\cdots \rightleftarrows T_\lambda \underset{pf(D)}{\overset{pf(X)}{\rightleftarrows}} T_{\lambda+\frac{N}{2}} \rightleftarrows \cdots \tag{33}$$

satisfying the following $(\theta - \lambda)\,T_\lambda \subset T_\lambda$, $pf\,(X)\,pf\,(D) = \prod_{t=0}^{\frac{N}{2}-1}\left(\frac{\theta}{\left(\frac{N}{2}\right)} + 2t\right)$, $pf\,(D)\,pf\,(X) = \prod_{t=0}^{\frac{N}{2}-1}\left(\frac{\theta}{\left(\frac{N}{2}\right)} + 2t + 1\right)$.

6.1 Examples

We describe graded $\mathcal{A}$-modules of finite type and the corresponding diagrams associated to $\mathcal{D}_V$-modules $\mathcal{O}_V$, $\mathcal{B}_{\{0\}|V}$, $\mathcal{O}_V\left(\frac{1}{pf(X)}\right)/\mathcal{O}_V$, which are regular holonomic with characteristic variety contained in Σ.

Example 19 *The $\mathcal{D}_V$-module $\mathcal{O}_V$ is generated by an element $e_0 = 1_V$ such that $\theta e_0 = 0$ and $pf\,(D)\,e_0 = 0$. Then its associated graded $\mathcal{A}$-module has a basis (e_q) where $q = mk$ ($k \in \mathbb{N}$) such that $pf\,(D)\,e_0 = 0$ and satisfying the system:*

$$S_0 = \begin{cases} \theta e_q = q e_q \ (q = mk,\ k \in \mathbb{N}) \\ pf\,(X)\,e_q = e_{q+m}, \\ pf\,(D)\,e_q = \prod_{t=0}^{m-1}\left(\frac{q}{m} + 2t\right) e_{q-m} \end{cases} . \tag{34}$$

Since $pf\,(D)\,e_0 = 0$ (i.e $pf\,(D)\,T_0 = 0$), the arrows at the left of T_0 in the diagram vanish.

Example 20 *The $\mathcal{D}_V$-module $\mathcal{B}_{\{0\}|V}$ is generated by an element e_{-m} such that $\theta e_{-m} = -m e_{-m}$ and $pf\,(X)\,e_{-m} = 0$. Its associated $\mathcal{A}$-module has a basis (e_q) where $q = -m - mk$ ($k \in \mathbb{N}$) such that $pf\,(X)\,e_{-m} = 0$ satisfying the system:*

$$S_1 = \begin{cases} \theta e_q = q e_q \ (q = -m - mk,\ k \in \mathbb{N}) \\ pf\,(D)\,e_q = e_{q-m}, \\ pf\,(X)\,e_q = \prod_{t=0}^{m-1}\left(\frac{q}{m} + 2t + 1\right) e_{q+m} \end{cases} \tag{35}$$

Since $pf\,(X)\,e_{-m} = 0$ (i.e $pf\,(X)\,T_{-m} = 0$), the arrows at the right of T_{-m} in the diagram vanish.

Example 21 *The $\mathcal{D}_V$-module $\mathcal{O}_V\left(\frac{1}{pf(X)}\right)/\mathcal{O}_V$ is generated by an element $e_{-m} = \frac{1}{pf(X)} \bmod \mathcal{O}_V$ such that $\theta e_{-m} = -m e_{-m}$ and $pf\,(X)\,e_{-m} = e_0$. Then its associated $\mathcal{A}$-module has a basis (e_q) where $q = -mk$ ($k \in \mathbb{N}$) satisfying a system as S_1.*

Actually diagrams studied are completely determined by a finite subset of objects and arrows. Indeed

a) For $\sigma \in \mathbb{C}/\frac{N}{2}\mathbb{Z}$, denote by $T^\sigma \subset T$ the submodule $T^\sigma = \bigoplus_{\lambda=\sigma \mathrm{mod} \frac{N}{2}\mathbb{Z}} T_\lambda$. Then T is generated by the finite direct sum of T^σ's

$$T = \bigoplus_{\sigma \,\epsilon\, \mathbb{C}/\frac{N}{2}\mathbf{Z}} T^\sigma = \bigoplus_{\sigma \,\epsilon\, \mathbb{C}/\frac{N}{2}\mathbf{Z}} \left(\bigoplus_{\lambda=\sigma \mathrm{mod} \frac{N}{2}\mathbb{Z}} T_\lambda \right). \tag{36}$$

b) If $\sigma \neq 0$ mod $\frac{N}{2}\mathbb{Z}$ $\left(\lambda = \sigma \mathrm{mod} \frac{N}{2}\mathbb{Z}\right)$, then the linear maps $pf\,(X)$ and $pf\,(D)$ are bijective. Therefore T^σ is completely determined by one element T_λ equipped with the nilpotent action of $(\theta - \lambda)$.

c) If $\sigma = 0$ mod $m\mathbb{Z}$ $(\lambda = \sigma \mathrm{mod} m\mathbb{Z})$ with $\frac{N}{2} = m$, then T^σ is completely determined by one diagrams of m elements

$$T_{-(2m-1)m} \underset{pf(D)}{\overset{pf(X)}{\rightleftarrows}} T_{-(2m-2)m} \cdots \rightleftarrows T_{-m} \underset{pf(D)}{\overset{pf(X)}{\rightleftarrows}} T_0 \tag{37}$$

In the other degrees $pf\,(X)$ or $pf\,(D)$ are bijective. Indeed, we have $T_0 \simeq pf\,(X)^k\, T_0 \simeq T_{mk}$ and $T_{-(2m-1)m} \simeq pf\,(D)^k\, T_{-(2m-1)m} \simeq T_{-(2m-1+k)m}$ $(k \in \mathbb{N})$ thanks to the relations $pf\,(X)\, pf\,(D) = \prod_{t=0}^{m-1} \left(\frac{\theta}{m} + 2t\right)$ and $pf\,(D)\, pf\,(X) = \prod_{t=0}^{m-1} \left(\frac{\theta}{m} + 2t + 1\right)$. The operator $pf\,(X)\, pf\,(D)$ (resp. $pf\,(D)\, pf\,(X)$) on T_λ has only one eigenvalue $\frac{\lambda}{m}(\frac{\lambda}{m}+2)(\frac{\lambda}{n}+4) \times \cdots \times \left(\frac{\lambda}{m} + 2m - 2\right)$ (resp. $(\frac{\lambda}{m}+1)(\frac{\lambda}{m}+3) \times \cdots \times \left(\frac{\lambda}{m} + 2m - 1\right)$) so that the equation $pf\,(X)\, pf\,(D) = \prod_{t=0}^{m-1} \left(\frac{\theta}{m} + 2t\right)$ (resp.$pf\,(D)\, pf\,(X) = \prod_{t=0}^{m-1} \left(\frac{\theta}{m} + 2t + 1\right)$) has a unique solution θ of eigenvalue λ if λ is not a critical value. Here $\lambda = 0,\ -2m,\ -4m,\ \cdots, -(2m-2)m$ or $\lambda = -m, \cdots, -(2m-1)m$ thus it is always the case.

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Philibert Nang
Max-Planck Institute for Mathematics
Vivatsgasse 7, D-53111 Bonn, Germany
E-mail adress: pnang@mpim-bonn.mpg.de