On the bottom of the spectrum of regular graphs

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Abstract. We establish a formula for the smallest value λ_0 of the spectrum of the Laplacian on a regular graph in terms of the critical exponent of the action of the fundamental group of the graph on its universal cover. This formula is a discrete analog of a formula due to Sullivan for the Laplacian on manifolds with constant negative curvature. We also obtain a characterization of λ_0-recurrent regular graphs.

§1.—Introduction and statement of the results

Let $k \geq 3$ be an integer. Consider a $k-$*regular* graph G, that is, a connected simplicial complex of dimension 1 where every vertex belongs to exactly k edges, and let S denote the set of vertices of G. Let $\ell^2(S)$ be the Hilbert space of square-summable functions on S and P the linear self-adjoint operator on $\ell^2(S)$ which associates to each element f in that space the element Pf defined by:

$$Pf(x) = \frac{1}{k} \sum_y f(y),$$

where the sum is taken over all the vertices y which are connected to x by an edge. The *Laplacian* of the graph G is the operator Δ defined by $\Delta f = f - Pf$.

It is well-known that the norm of P is equal to its spectral radius $r(P)$, that the spectrum of P is a compact subset of the segment $[-r(P), r(P)]$ and that $r(P)$ belongs to the spectrum of P. It follows that $\lambda_0(G) = 1 - r(P)$ is the smallest element of the spectrum of the Laplacian Δ. We call $\lambda_O(G)$ the *bottom of the spectrum of* G. It is clear

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that we have $0 \leq \lambda_0(G) \leq 1$. (For general properties on the Laplacian on graphs, the reader can consult for example [DK] and [MW].)

There is a canonical length metric on G for which the length of each edge is equal to 1. The universal covering space X of G is a k−homogeneous tree. It has an induced length metric, and the fundamentral group Γ of G acts propertly and isometrically on X. Let x and y be two arbitrary points of X. Then the *critical exponent* $\delta(G)$ of Γ is defined by the formula

$$\delta(G) = \limsup_{R\to\infty} \frac{1}{R} \log(\mathrm{card}\{\gamma \in \Gamma | \mathrm{dist}(x, \gamma y) \leq R\}).$$

The *Poincaré series* associated to G

$$\eta_s(x,y) = \sum_{\gamma\in\Gamma} \exp\big(- s\,\mathrm{dist}(x, \gamma y)\big)$$

is therefore divergent for $s < \delta(G)$ and convergent for $s > \delta(G)$. It can easily be shown that $\delta(G)$ does not depend on the chosen points x and y, and that we always have $0 \leq \delta(G) \leq \log(k-1)$.

We prove the following theorem, which is an analog, in the setting of graphs, of Sullivan's generalization of a formula of Elstrodt-Patterson for manifolds of constant negative curvature (see [Sul], Theorem 2.17):

Theorem 1.— *The bottom of the spectrum $\lambda_0(G)$ of a k−regular graph G is given by the following formula:*

$$\lambda_0(G) = \begin{cases} 1 - 2\dfrac{\sqrt{k-1}}{k} & if \quad \delta(G) \leq \dfrac{1}{2}\log(k-1) \\ \dfrac{1}{k}(1 - e^{-\delta(G)})(k - 1 - e^{\delta(G)}) & if \quad \delta(G) \geq \dfrac{1}{2}\log(k-1), \end{cases}$$

where $\delta(G)$ denotes the critical exponent of the action of the group $\Gamma = \pi_1(G)$ on the universal cover of G.

The proof of Theorem 1 is given in §3 below.

Example.—Let G' be any compact graph which is k'−regular, with $k' \leq k$, and define the graph G by adding at each vertex of G' a connected piece of a homogeneous tree of degre k so that the resulting graph is k−regular. Then $\delta(G) = \delta(G') = \log(k'-1)$. Applying Theorem 1, we obtain

$$\lambda_0(G) = \begin{cases} 1 - 2\dfrac{\sqrt{k-1}}{k} & if \quad (k'-1)^2 \leq k-1 \\ \dfrac{(k-k')(k'-2)}{k(k'-1)} & if \quad (k'-1)^2 \geq k-1. \end{cases}$$

In Figure 1, we give an example of this construction with $k' = 3$ and $k = 4$, and therefore $\lambda_0(G) = \frac{1}{8}$. In this figure, G' is the 1-skeleton of a tetrahedron (in bold lines) and G is the full graph drawn.

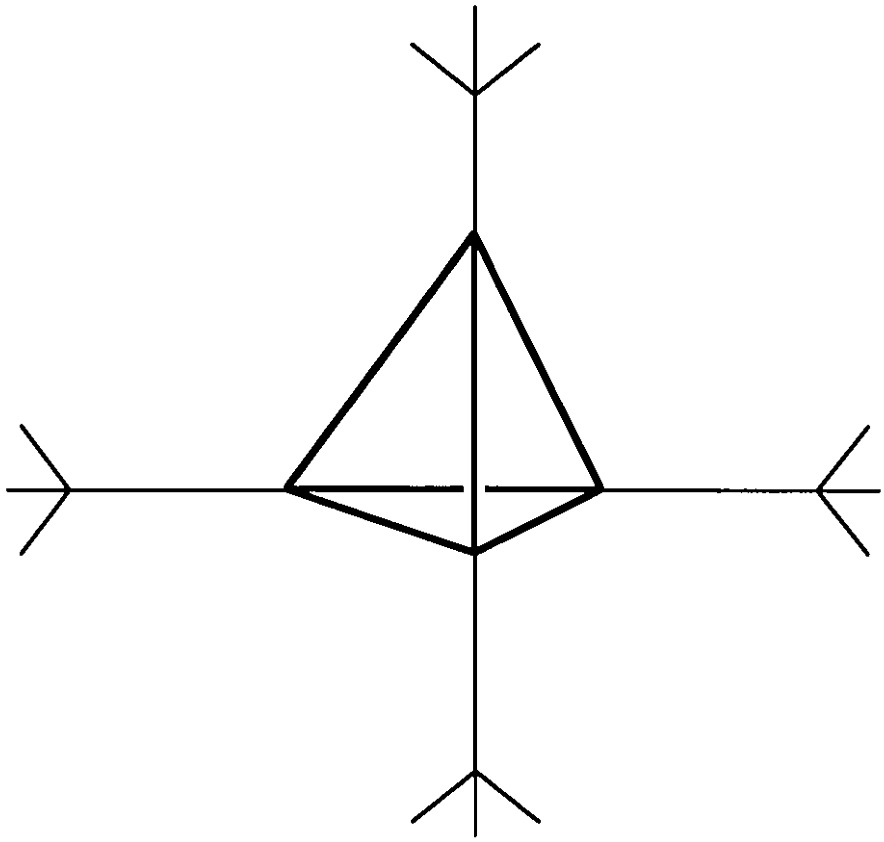

Figure 1

The graph G is called λ_0*−recurrent* if its λ−Green kernel $N_\lambda^G(x,y)$ is divergent at $\lambda = \lambda_0(G)$ for some (or equivalently for every) $x, y \in S$. (Note that in [MW], p. 215, λ_0−recurrent graphs are called $r(P)$−recurrent graphs). We shall also prove the following

Theorem 2.— *The k−regular graph G is λ_0−recurrent if and only if the following two conditions hold: $\delta(G) \geq \frac{1}{2}\log(k-1)$ and the Poincaré series η_s of G is divergent at $s = \delta(G)$.*

§2.— Green kernels

Let λ be a real number < 1. Recall that the λ−Green kernel of the graph G is the map $N_\lambda^G : S \times S \to [0, \infty]$ defined by

$$N_\lambda^G(x,y) = \sum_{n=0}^{\infty} (1-\lambda)^{-n-1} p^{(n)}(x,y)$$

where $\big(p^{(n)}(x,y)\big)$ is the n−th power of the matrix $\big(p(x,y)\big)$, $(x,y) \in S \times S$, defined by $p(x,y) = \frac{1}{k}$ if x and y are connected by an edge and $p(x,y) = 0$ otherwise. In probabilistic terms, $p^{(n)}(x,y)$ is the probability, starting at x, to reach the point y in n steps by means of the symmetric random walk on S, that is, the random walk for which the probability for going from one vertex to a neighboring vertex is $\frac{1}{k}$.

We have the following characterization of λ_0 (see for example [MW], Theorem 4.4):

Proposition 1.— *Let x and y be two arbitrary vertices of G. For every $\lambda < \lambda_0(G)$, we have $N_\lambda^G(x,y) < \infty$. For every λ such that $\lambda_0(G) < \lambda < 1$, we have $N_\lambda^G(x,y) = \infty$.*

Finally, let us recall that there is an explicit formula for the λ−Green kernel of a homogeneous tree, which is due to Kesten (see for example [MW] or [CP] §5):

Proposition 2.— *Let X be a k−homogeneous tree, and let x and y be arbitrary vertices of X. Then*

(i) if $1 - 2\frac{\sqrt{k-1}}{k} < \lambda < 1$, then $N_\lambda^X(x,y) = \infty$;

(ii) if $\lambda \leq 1 - 2\frac{\sqrt{k-1}}{k}$, then $N_\lambda^X(x,y) = C\exp\big(-D\,dist(x,y)\big)$, where $C = C(k,\lambda) > 0$ is a constant and where $D = D(k,\lambda)$ is the largest of the two solutions of the following equation:

$$\lambda = \frac{1}{k}(1 - e^{-D})(k - 1 - e^{D}).$$

§3.—Proof of Theorems 1 and 2

Proof of Theorem 1: Let x and y be two vertices of the graph G, and let u and v be two vertices of the universal covering X which project respectively to x and y. To simplify the notations, we let

$$\lambda_m = 1 - 2\frac{\sqrt{k-1}}{k}.$$

We have, for every $\lambda < 1$,

$$N_\lambda^G(x,y) = \sum_{\gamma\in\Gamma} N_\lambda^X(u, \gamma v).$$

Therefore, by Proposition 2, $N_\lambda^G(x,y) = \infty$ for $\lambda_m < \lambda < 1$. Using Proposition 1, we deduce

$$\lambda_0(G) \leq \lambda_m \tag{1}$$

For $\lambda < \lambda_m$, using the notations of Proposition 2, we obtain

$$N_\lambda^G(x,y) = C\sum_{\gamma\in\Gamma} \exp\big(-D\,\mathrm{dist}(u,\gamma v)\big)$$

$$= C\eta_D(u,v).$$

Therefore, for $\lambda < \lambda_m$, we have

$$N_\lambda^G(x,y) = \infty \text{ if } D < \delta(G) \tag{2}$$

$$N_\lambda^G(x,y) < \infty \text{ if } D > \delta(G). \tag{3}$$

Now the function $D = D(\lambda)$ is decreasing for $\lambda < \lambda_m$ and we have

$$D(\lambda_m) = \frac{1}{2}\log(k-1).$$

Using (1), (2) and (3), Theorem 1 follows from Proposition 1. ∎

Proof of Theorem 2: We keep the same notations as in the proof of Theorem 1. We have

$$N_{\lambda_0(G)}^G(x,y) = C\eta_{D_O}(u,v)$$

where $D_0 = D\big(\lambda_0(G)\big)$ is again the largest root of the equation given in Proposition 2 (*ii*) for $\lambda = \lambda_0(G)$.

We have $D_0 \geq \frac{1}{2}\log(k-1)$. Therefore, if $\delta(G) < \frac{1}{2}\log(k-1)$, then the graph is not λ_0-recurrent. If $\delta(G) \geq \frac{1}{2}\log(k-1)$, then $\delta(G) = D_0$ by Theorem 1. The proof of Theorem 2 follows. ∎

We conclude with a few remarks:

1.— We have $\lambda_0(G) = 0$ if and only if the graph G is amenable (*cf.* [MW], Corollary 5.6). Therefore, Theorem 1 shows in particular that the amenability of G is equivalent to the condition $\delta(G) = \log(k-1)$. In the particular case where the graph G is the Cayley graph of a finitely generatoed group, with respect to one of its finite generating systems, we recover the cogrowth theorem of Grigorchuck-Cohen (see [Gri], [Coh], [Szw] and [Nor]). We are grateful to A. Valette who, after a preprint version of this note was circulated, has pointed to us the paper [Nor] where the amenability of a regular graph G is shown to be equivalent to the fact that $\delta(G) = \log(k-1)$, using considerations which are in the same spirit as the ones of the present paper.

2.— To use a celebrated formula of M. Kac's, Theorem 1 shows that we can "hear" the critical exponent $\delta(G)$ of a regular graph G in the case where $\delta(G) \geq \frac{1}{2}\log(k-1)$.

3.— Let ∂X be the boundary at infinity of the tree X, and let $\Lambda \subset \partial X$ be the limit set of $\Gamma = \pi_1(G)$, that is, the set of accumulation points in ∂X of the Γ−orbit of an arbitrary point of X. Let $\dim(\Lambda)$ denote the Hausdorff dimension of Λ. This dimension is taken relatively to the visual metrics on ∂X. (Let us recall that the *visual metric* on ∂X, seen from the point $x \in X$, is the metric d_x defined by $d_x(\xi, \xi') = \exp(-L)$, where L is the length of the common path of the two geodesic rays starting at x and converging respectively to the points ξ and ξ' of ∂X.). If the graph G has compact core, then $\delta(G) = \dim(\Lambda)$ (see [Coo]). (Recall that the graph G is said to have compact core if the group Γ is convex cocompact, or equivalently if G contains a compact subgraph G_0 such that each component of $G \setminus G_0$ is simply connected.) Therefore, by Theorem 1, we can hear $\dim(\Lambda)$ if G has compact core and $\dim(\Lambda) \geq \frac{1}{2}\log(k-1)$. Note also that graphs with compact core have a divergent Poincaré series at $s = \lambda_0(G)$ (cf. [Coo]). Therefore, Theorem 2 shows that the k−regular graph with compact core is λ_0−recurrent if and only if $\dim(\Lambda) \geq \frac{1}{2}\log(k-1)$. Note finally that all the graphs in the family of examples described in §1 above have compact core. In particular, the graph of Figure 1 is λ_0−recurrent.

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