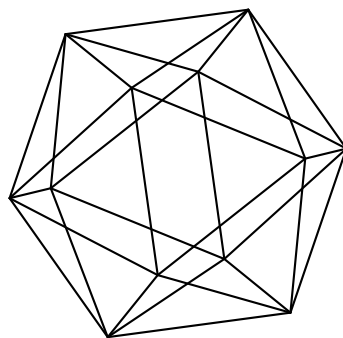# Max-Planck-Institut für Mathematik Bonn

Complexity of solving tropical linear systems

by

Dima Grigoriev

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Dima Grigoriev

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

CNRS, Mathématiques
Université de Lille
59655 Villeneuve d'Ascq
France

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Dima Grigoriev

CNRS, Mathématiques, Université de Lille

Villeneuve d'Ascq, 59655, France

`Dmitry.Grigoryev@math.univ-lille1.fr`

http://en.wikipedia.org/wiki/Dima_Grigoriev

Abstract

An algorithm is designed solving a tropical linear system with the complexity polynomial in the size of the system.

Introduction

We say that a *tropical linear system*

$$\min_{1 \le j \le n} \{a_{i,j} + x_j\},\ 1 \le i \le m \tag{1}$$

(or $(m \times n)$-matrix $A = (a_{i,j})$) has a *tropical solution* $x = (x_1 \ldots, x_n)$ if for every row $1 \le i \le m$ there are two columns $1 \le k < l \le n$ such that

$$a_{i,k} + x_k = a_{i,l} + x_l = \min_{1 \le j \le n} \{a_{i,j} + x_j\}$$

(see e. g. [1], [7]). Our purpose is to design an algorithm to solve (1).

In Section 1 we assume that coefficients $a_{i,j} \in \mathbb{Z}$ (we call it the case of *finite coefficients*) and that $0 \le a_{i,j} \le M$ for all i, j. We describe an algorithm which yields a solution $x \in \mathbb{Z}^n$ of (1) or detects its insolvability within the complexity polynomial in M, n, m. The algorithm runs by induction on m and starting with any solution of the first $m-1$ equations of (1) the algorithm modifies it in a solution of (1) or detects the insolvability of (1). One can view the algorithm as a tropical analogue of the Gram-Schmidt process with respect to the *tropical norm* introduced in Section 1.

In Section 2 we study the case of the (*extended*) coefficients $a_{i,j} \in \mathbb{Z}_\infty := \mathbb{Z} \cup \infty$ and look for a solution $x \in \mathbb{Z}_\infty^n$ of (1) with not all its coefficients x_j, $1 \le j \le n$ equal ∞. We assume that $0 \le a_{i,j} \le M$ for all finite coefficients. We describe an algorithm which solves (1) also within the complexity polynomial in M, n, m. Reordering the columns and rows of $(m \times n)$-matrix $A := (a_{i,j})$ the algorithm brings it to a block form $(A_{p,q})$, $1 \le p, q \le t$ such that each of the first $t-1$ diagonal blocks $A_{i,i}$, $1 \le i \le t-1$ has no (tropical) solution,

and all upper-triangular blocks $A_{p,q}$, $1 \leq p < q \leq t$ have all entries equal ∞. It would be interesting to solve (1) within the complexity polynomial in $\log M$, n, m.

In Section 3 we study tropical *non-homogeneous* linear systems

$$\min_{1 \leq j \leq n} \{a_{i,j} + x_j, \, a_i\}, \, 1 \leq i \leq m \tag{2}$$

and describe an algorithm for their solving relying on the algorithm from Section 2 with a similar complexity bound.

In Section 4 as a consequence of the algorithm from Section 2 we give a characterization of solvability of (1) in terms of the *tropical* and *Kapranov ranks* of matrix A (their definitions are reminded in Section 4) and generalize this characterization to the extended real coefficients from $\mathbb{R}_\infty$. Note that for finite coefficients from $\mathbb{R}$ this follows from [3].

In Section 5 we describe an algorithm which tests whether (1) has a unique (in the *tropical projective space*) solution also within the complexity polynomial in M, n, m (answering a question posed to the author by Thorsten Theobald). It would be interesting to calculate the dimension of the set of tropical solutions of (1) within a similar complexity. The latter problem is apparently difficult since in [1] an example of a linear polynomial ideal is exhibited with an exponential lower bound on the size of its tropical bases.

Note that in [2] it was shown that one can test the tropical singularity of a square matrix in polynomial time. It is known that calculations of the tropical rank [5] and of the Kapranov rank [6] are both NP-hard in general. Moreover, it is proved in [6] that the problem of solving systems of polynomial equations over a given infinite field is reducible to the problem of testing whether the Kapranov rank of a matrix over this field equals 3. In [7] it was established that solving tropical polynomial systems (already of degrees 2) is NP-complete.

We mention that even in the classical algebra there are known two different notions of a rank of an $(m \times n)$-matrix A over a commutative integral domain K [4]. Define $Rk(A)$ to be the minimal r such that $A = X_1 \cdot Y_1 + \cdots + X_r \cdot Y_r$ for suitable $(m \times 1)$-matrices $X_1, \ldots, X_r$ and $(1 \times n)$-matrices $Y_1, \ldots, Y_r$ over K. Obviously $Rk(A)$ is greater or equal to the usual rank $rk(A)$ and can be greater than the latter up to a factor 2 over polynomial rings K.

1 Solving tropical linear systems with finite coefficients

In this section we study the case of finite coefficients $a_{i,j} \in \mathbb{Z}$ of system (1) and assume that $0 \leq a_{i,j} \leq M$, $1 \leq i \leq m$, $1 \leq j \leq n$. Then w.l.o.g. one can look for a solution $x = (x_1, \ldots, x_n)$ with coefficients $x_j \geq 0$, $1 \leq j \leq n$ being also integers. We introduce the notation of the *tropical norm* of a vector $||x|| = \sum_{1 \leq j \leq n} x_j - n \cdot \min_{1 \leq j \leq n} \{x_j\}$.

Theorem 1.1 *There is an algorithm which for an input (1) either finds its solution or detects its insolvability within complexity $O(M \cdot \log M \cdot n^2 \cdot m^2)$.*

Lemma 1.2 *If (1) has a solution $(x_1, \ldots, x_n)$ then (1) has a solution $(x'_1, \ldots, x'_n)$ satisfying $0 \leq x'_j \leq M$, $x'_j \leq x_j$, $1 \leq j \leq n$.*

Proof of lemma. One can suppose w.l.o.g. that $\min_{1 \leq j \leq n}\{x_j\} = 0$. Therefore for each row i it is fulfilled $\min_{1 \leq j \leq n}\{a_{i,j} + x_j\} \leq M$. Hence if column j_0 satisfies the property $a_{i,j_0} + x_{j_0} = \min_{1 \leq j \leq n}\{a_{i,j} + x_j\}$ for suitable row i (we call such column j_0 *active*) then $x'_{j_0} := x_{j_0} \leq M$. For any non-active j_0 one can put $x'_{j_0} := \min\{x_{j_0}, M\}$. ∎

Proof of Theorem 1.1 we carry out by induction on m. Assume by inductive hypothesis that the algorithm has already produced a current solution x for $((m-1) \times n)$-submatrix A' of matrix A excluding the first row of A such that $0 \leq x_j \leq M$, $1 \leq j \leq n$. Reordering the columns we suppose that $a_{1,1} + x_1 = \min_{1 \leq j \leq n}\{a_{1,j} + x_j\}$. The algorithm modifies vector x (keeping the property of being a solution of A') until the modified vector becomes a solution also for the first row or detects that A has no solutions. One can assume that $a_{1,1} + x_1 < a_{1,j} + x_j$, $j \geq 2$, otherwise the algorithm terminates the inductive step.

We construct by recursion a subset J of columns. At the beginning $J = \{1\}$. For a current $J = \{1, \ldots, k\}$ for each $1 \leq i \leq k$ we have

$$a_{i,i} + x_i = \min_{1 \leq j \leq n}\{a_{i,j} + x_j\} < a_{i,j_1} + x_{j_1}, \; j_1 > i \qquad (3)$$

Suppose that there exists a row $i = i_{k+1}$ for which there is a unique $j_0 \notin J$ such that $a_{i,j_0} + x_{j_0} = \min_{1 \leq j \leq n}\{a_{i,j} + x_j\}$. Clearly, $i > k$ due to (3). Transpose column j_0 with $k+1$ and row i with $k+1$, respectively. Put current $J := \{1, \ldots, k+1\}$. Then (3) is fulfilled for new J.

Now assume that the algorithm fails to augment J. Observe that J does not depend on the order of choosing rows $i = i_{k+1}$ in the above construction.

First suppose that $J = \{1, \ldots, n\}$. In this case $(n \times n)$-submatrix of A induced by its first n rows is tropically nonsingular, consequently (1) has no solution and the algorithm halts.

Now let $k = |J| < n$. If $k = 1$ we add to x_1 number $\min_{2 \leq j \leq n}\{a_{1,j} + x_j\} - (a_{1,1} + x_1) \geq 1$ and obtain a solution of (1). Thereupon apply Lemma 1.2 to the obtained solution, thereby the algorithm terminates the inductive step.

Thus, from now on we assume that $k > 1$. We call row i *attracted* if for every j_0 such that $a_{i,j_0} + x_{j_0} = \min_{1 \leq j \leq n}\{a_{i,j} + x_j\}$, we have $j_0 \in J$. Obviously, the first row is attracted. Reordering the rows one may suppose that exactly first l rows are attracted. Note that for any row $i > l$ there are at least two different columns $j_1, j_2 \notin J$ such that $a_{i,j_1} + x_{j_1} = a_{i,j_2} + x_{j_2} = \min_{1 \leq j \leq n}\{a_{i,j} + x_j\}$.

For $1 \leq i \leq l$ denote

$$a_i := \min_{k < j \leq n}\{a_{i,j} + x_j\} - \min_{1 \leq j \leq n}\{a_{i,j} + x_j\} \geq 1$$

and $a := \min_{1 \leq i \leq l}\{a_i\}$. The algorithm modifies vector $(x_1, \ldots, x_n)$ in such a way that

$$y_j := x_j + a, \; 1 \leq j \leq k; \quad y_j := x_j, \; j > k.$$

Then vector $y := (y_1, \ldots, y_n)$ is still a solution of A' and $a_{1,1} + y_1 = \min_{1 \leq j \leq n}\{a_{1,j} + y_j\}$. Moreover, the tropical norm

$$||(a_{1,j} + y_1, \ldots, a_{1,n} + y_n)|| = ||(a_{1,j} + x_1, \ldots, a_{1,n} + x_n)|| - a \cdot (n - k)$$

has dropped.

Thereupon the algorithm applies Lemma 1.2 to vector $(y_1, \ldots, y_n)$. Observe that this does not change the tropical norm since each of the first k columns is active (taking into account that $k > 1$) and hence every $y_j \leq M$, $j > k$ and thereby y_j does not change in the course of application of Lemma 1.2.

Thus, we have described a single iteration of the algorithm. The nest iteration starts with the modified vector y replacing x. The complexity of the execution of the iteration can be bounded by $O(\log M \cdot m \cdot n)$. The total number of iterations does not exceed tropical norm $||(a_{1,j} + x_1, \ldots, a_{1,n} + x_n)|| \leq 2 \cdot M \cdot (n - 1)$. Since the described induction (considering each time one more row of matrix A) requires m steps, we conclude the complexity bound in Theorem 1.1. ∎

2 Solving tropical linear systems with coefficients extended by infinity

From now on we assume that entries of (1) $a_{i,j} \in \mathbb{Z}_\infty$ and $0 \leq a_{i,j} \leq M$ when $a_{i,j} \in \mathbb{Z}$. We are looking for solutions $(x_1, \ldots, x_n)$ over $\mathbb{Z}_\infty$ with not all the coordinates equal ∞.

Theorem 2.1 *There is an algorithm which for a tropical linear system (1) over $\mathbb{Z}_\infty$ either finds a solution or detects its insolvability within complexity $O(M \cdot \log(M \cdot n) \cdot n^4 \cdot m^2)$.*

Lemma 2.2 *If $(x_1, \ldots, x_n) \in (\mathbb{R}_\infty)^n$ is a solution of (1) then there exists a solution $(x_1', \ldots, x_n')$ of (1) such that for any $1 \leq j, j_1 \leq n$ it holds:*
- *$x_j' = \infty$ iff $x_j = \infty$;*
- *$0 \leq x_j' \leq \min\{x_j, (M + 1) \cdot n\}$, provided that $x_j \neq \infty$;*
- *$x_j - x_{j_1} > M$ iff $x_j' - x_{j_1}' > M$.*

Proof of Lemma 2.2. In the course of the proof we will modify vector $(x_1, \ldots, x_n)$ keeping for it the same notation. One can assume w.l.o.g. that $0 = x_1 = \min_{1 \leq j \leq n}\{x_j\}$. Consider a graph whose vertices are finite coordinates x_j, and a pair of coordinates x_p, x_q is connected by an edge if for some row i we have $a_{i,p} + x_p = a_{i,q} + x_q = \min_{1 \leq j \leq n}\{a_{i,j} + x_j\} \neq \infty$.

Consider a connected component of the graph which contains x_1. Let the component contain p vertices, and after their reordering one can assume that it consists of $x_1, \ldots, x_p$, hence $x_j \leq M \cdot (p - 1)$, $1 \leq j \leq p$. After reordering the coordinates one can assume that $x_{p+1} = \min_{j>p}\{x_j\}$. If $x_{p+1} = \infty$ the Lemma is proved. Otherwise, if $x_{p+1} \geq M \cdot p + 1$ then replace x_j with $x_j - x_{p+1} + (M \cdot p + 1)$ for all $j > p$. Take a connected component of the graph which contains x_{p+1}. Let it consist of q vertices $x_{p+1}, \ldots, x_{p+q}$. As above we

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conclude that $x_{p+j} \leq M \cdot (p + q - 1) + 1$, $1 \leq j \leq q$. Continuing in this way we complete the proof of Lemma. ∎

The proof of Theorem 2.1 we carry out by induction on m and first formulate the inductive hypothesis. Suppose that $((m-1) \times n)$-submatrix A' of A (after reordering the rows and columns) has a block structure $(A_{p,q})$ where $A_{p,q}$ is of size $u_p \times v_q$, $1 \leq p, q \leq t-1$, while the lowest blocks $\overline{A_{t,q}}$ are of sizes $\overline{u_t} \times v_q$, $1 \leq q < t$, the rightmost blocks $\overline{A_{p,t}}$ are of sizes $u_p \times \overline{v_t}$, $1 \leq p < t$, finally the diagonal block $\overline{A_{t,t}}$ is of size $\overline{u_t} \times \overline{v_t}$ where $\overline{u_t} = m - 1 - u_1 - \cdots - u_{t-1}$, $\overline{v_t} = n - v_1 - \cdots - v_{t-1}$.

Also a vector $(y_1, \ldots, y_n) \in \mathbb{Z}^n$, $0 \leq y_j \leq (M+1) \cdot n, 1 \leq j \leq n$ is yielded. For each diagonal block $A_{p,p} = (a_{u+i,v+j})$, $1 \leq p < t$, $u := u_1 + \cdots u_{p-1}$, $1 \leq i \leq u_p$, $v := v_1 + \cdots v_{p-1}$, $1 \leq j \leq v_p$ (except for the lowest diagonal block $\overline{A_{t,t}}$) we have $a_{u+i,v+i} + y_{v+i} = \min_{1 \leq j \leq v_p} \{a_{u+i,v+j} + y_{v+j}\}$ for $1 \leq i \leq v_p$ and $a_{u+i,v+i} + y_{v+i} < a_{u+i,v+j}$, $i < j \leq v_p$. Therefore, in particular $u_p \geq v_p$, $p < t$. It is not excluded that $\overline{u_t} = 0$, while the case $\overline{v_t} = 0$ would mean that the algorithm under description terminates with the output that system (1) has no solutions (cf. below Lemma 2.5).

Every entry of each upper-triangular block $A_{p,q}$, $p < q$ (as well as of $\overline{A_{p,t}}$, $p < t$) equals ∞. Moreover, the vector from $\mathbb{Z}_\infty^n$ whose coordinates in the first $t-1$ blocks equal ∞ and in the last t-th block coincide with y_j for $v_1 + \cdots v_{t-1} < j \leq n$, is a tropical solution of matrix A'.

For the sake of simplifying notations denote matrix $B' = \overline{A_{t,t}}$ of size $(r - 1 := \overline{u_t}) \times (s := \overline{v_t})$. One can assume that B' has no rows consisting fully of ∞ entries, otherwise the corresponding row of matrix A' one can join to the previous $(t-1)$-th block.

We assume that matrix A' is obtained from A by deleting its $(m - r + 1)$-th row. By $B = (b_{i,j})$, $1 \leq i \leq r$, $1 \leq j \leq s$ denote $(r \times s)$-submatrix of A located in its lower right corner. Deleting the first row from B we obtain B'. Also one can suppose w.l.o.g. that $b_{1,1} + y_{n-s+1} = \min_{1 \leq j \leq s} \{b_{1,j} + y_{n-s+j}\}$.

The algorithm will modify vector $(y_{n-s+1}, \ldots, y_n)$ (keeping for a current vector the same notation) while preserving the property that $(y_{n-s+1}, \ldots, y_n)$ is a (tropical) solution of B'. One can assume w.l.o.g. that $b_{1,1} + y_{n-s+1} < \min_{2 \leq j \leq s} \{b_{1,j} + y_{n-s+j}\}$, since otherwise the vector from $\mathbb{Z}_\infty^n$ with all coordinates in the first $t-1$ blocks equal ∞ and coinciding with vector $(y_{n-s+1}, \ldots, y_n)$ in the last t-th block provides a solution of A which would terminate the inductive step. In this case in the block structure $u_1, \ldots, u_{t-1}, v_1, \ldots, v_{t-1}, \overline{v_t}$ do not change, while $\overline{u_t}$ increases by one. Applying Lemma 2.2 one can assume w.l.o.g. that $0 \leq y_j \leq (M+1) \cdot n$, $n - s + 1 \leq j \leq n$.

As above in the proof of Theorem 1.1 the algorithm constructs recursively a set $J \subset \{n - s + 1, \ldots, n\}$ of columns of matrix B, while modifying vector $(y_{n-s+1}, \ldots, y_n)$, and we describe a single iteration of this modification. As in the proof of Theorem 1.1 $(n - s + 1) \in J$. Again as above introduce the set of attracted rows. For every attracted row i denote

$$b_i := \min_{j \notin J} \{b_{i,j-n+s} + y_j\} - \min_{j \in J} \{b_{i,j-n+s} + y_j\} \geq 1.$$

Then $b := \min\{b_i\} \geq 1$ where min is taken over all the attracted rows. Thus, the algorithm modifies vector $(y_{n-s+1}, \ldots, y_n)$ adding b to every y_j for $j \in J$. Thereupon the algorithm

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applies Lemma 2.2 to vector $(y_{n-s+1}, \ldots, y_n)$ which satisfies B'. Hence one can assume w.l.o.g. that $0 \leq y_j \leq (M+1) \cdot s$, $n - s + 1 \leq j \leq n$.

The algorithm introduces the following *directed* graph G with s vertices $\{n - s + 1, \ldots, n\}$. There is an edge in G from j_1 to j_2 if $y_{j_1} - y_{j_2} \leq M$. Observe that an application of Lemma 2.2 to vector $(y_{n-s+1}, \ldots, y_n)$ does not change graph G. Denote by $S \subset \{n - s + 1, \ldots, n\}$ the set of all the vertices which can be reached in G starting with vertex $n - s + 1$. In the course of executing the algorithm, while modifying J, G, S we keep for them the same notations.

Lemma 2.3 *After any subsequent iteration of the algorithm set J remains to be a subset of S. Set S after an iteration becomes a subset of S before the iteration.*

Proof of Lemma 2.3. At the current iteration inclusion $J \subset S$ holds by virtue of construction of J since for any row i and any pair of columns $j \in S$, $l \notin S$ we have $b_{i,l-n+s} + y_l \neq b_{i,j-n+s} + y_j$, unless $b_{i,l-n+s} = b_{i,j-n+s} = \infty$. Therefore, after the modification of vector $(y_{n-s+1}, \ldots, y_n)$ its coordinates y_j for $j \in J$ increase, while the other coordinates do not change. Consequently, the modified S is a subset of the previous S. ∎

Lemma 2.4 *For any attracted row i and any $l \notin S$ we have $b_{i,l-n+s} = \infty$.*

Proof of Lemma 2.4. If $b_{i,l-n+s} < \infty$ then $b_{i,l-n+s} + y_l < b_{i,j-n+s} + y_j$ for any $j \in S$ which contradicts to that row i is attracted and that $J \subset S$ (due to Lemma 2.3). ∎

Now assume that $J = S$. Denote by $v_t := \#J$ and by u_t the number of attracted rows of B. Reorder the rows and the columns of B (and respectively, of A) in such a way that the set of the first v_t columns of B coincides with J, and the set of the first u_t rows of B coincides with the set of attracted rows of B. Moreover, one can suppose that for any $1 \leq i \leq v_t$ we have

$$b_{i,i} + y_{n-s+i} = \min_{1 \leq j \leq s} \{b_{i,j} + y_{n-s+j}\} < \min_{i < l \leq s} \{b_{i,l} + y_{n-s+l}\}. \tag{4}$$

Then the algorithm constructs a modified block decomposition of A being a refinement of the block decomposition from the inductive hypothesis: the last $\overline{u_t}$ rows (respectively, the last $\overline{v_t}$ columns) of A are partitioned into the first u_t rows and the remaining $\overline{u_{t+1}} := \overline{u_t} - u_t$ rows (respectively, into the first v_t columns and the remaining $\overline{v_{t+1}} := \overline{v_t} - v_t$ columns). Thus, as blocks of A we obtain the new ones $A_{t,q}$, $q \leq t$; $A_{p,t}$, $p \leq t$; $\overline{A_{t+1,q}}$, $q \leq t + 1$; $\overline{A_{p,t+1}}$, $p \leq t + 1$. The diagonal block $A_{t,t}$ satisfies the inductive hypothesis by its construction, see (4), and each entry of $\overline{A_{t,t+1}}$ equals ∞ due to Lemma 2.4. This completes the inductive step for m rows (i. e. for matrix A).

The algorithm terminates when it is impossible to continue its work. It can happen when either all the rows of (1) or all its columns are exhausted. First consider the case when all the rows of (1) are exhausted (i. e. A contains all the rows of (1)), but not all the columns are exhausted. Then two possibilities can occur. Either a (modified) vector $(y_{n-s+1}, \ldots, y_n)$ is a solution of matrix B, then the algorithm terminates before completing

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a block decomposition of matrix A (at the inductive step) and outputs a solution of (1) (see above). Or the inductive step is completed with all the rows of B being attracted (since all the rows of (1) are exhausted) and with $J = S \neq \{n - s + 1, \ldots, n\}$ (since not all the columns of (1) are exhausted). In the latter case $\overline{u_t} = u_t$, in other words blocks $\overline{A_{t+1,q}}$ are void, block $\overline{A_{t,t+1}}$ is not empty with each its entry equal ∞. Then (1) has a solution whose coordinates at first t blocks equal ∞ and at $(t + 1)$-th block equal, say, 0 (or other arbitrary integers).

Secondly, consider the case when all the columns of (1) are exhausted, i. e. $J = \{n - s + 1, \ldots, n\}$. Then observe that $\overline{u_t} = u_t$, $\overline{v_t} = v_t$, thus blocks $\overline{A_{t+1,q}}$; $\overline{A_{p,t+1}}$, $1 \leq p, q \leq t+1$ being void. Consider $(n \times n)$-submatrix $\tilde{C} = (\tilde{c}_{i,j})$ of A consisting of its first v_p rows from each block of decomposition of A, $1 \leq p \leq t$. Denote by $C = (c_{i,j})$ matrix such that $c_{i,j} := \tilde{c}_{i,j} + y_j$, $1 \leq i, j \leq n$. Evidently, the tropical linear systems with matrices $\tilde{C}$ and C have solutions simultaneously.

Lemma 2.5 *Let $(n \times n)$-matrix C be decomposed into blocks $C_{p,q}$ of sizes $v_p \times v_q$, $1 \leq p, q \leq t$; $n = v_1 + \cdots + v_t$. Moreover, for each diagonal block $C_{p,p} = (c_{\overline{v}+i,\overline{v}+j})$, $1 \leq i, j \leq v_p$, $1 \leq p \leq t$ where $\overline{v} = v_1 + \cdots v_{p-1}$ we have*

$$c_{\overline{v}+i,\overline{v}+i} = \min_{1 \leq j \leq v_p} \{c_{\overline{v}+i,\overline{v}+j}\} < \min_{i < l \leq v_p} \{c_{\overline{v}+i,\overline{v}+l}\}$$

for every $1 \leq i \leq v_p$. In addition, any entry of an upper-triangular block $C_{p,q}$, $p < q$ equals ∞. Then a tropical linear system with matrix C has no solution over $\mathbb{Z}_\infty$.

Proof of Lemma 2.5. Suppose that vector $(z_1, \ldots, z_n)$ is a tropical solution of matrix C. Let p be the first block $(z_{\overline{v}+1}, \ldots, z_{\overline{v}+v_p})$ of $(z_1, \ldots, z_n)$ which contains a finite coordinate. Then $(z_{\overline{v}+1}, \ldots, z_{\overline{v}+v_p})$ is a tropical solution of matrix $C_{p,p}$. Take a unique $1 \leq j_0 \leq v_p$ such that

$$z_{\overline{v}+j_0} = \min_{1 \leq j \leq v_p} \{z_{\overline{v}+j}\} < \min_{1 \leq j \leq j_0} \{z_{\overline{v}+j}\}.$$

Then we conclude that $(z_{\overline{v}+1}, \ldots, z_{\overline{v}+v_p})$ is not a tropical solution of j_0-th row of matrix $C_{p,p}$ because $c_{\overline{v}+j_0,\overline{v}+j_0} + z_{\overline{v}+j_0} < \min_{1 \leq j \leq v_p, j \neq j_0} \{c_{\overline{v}+j_0,\overline{v}+j} + z_{\overline{v}+j}\}$. The achieved contradiction proves the Lemma. ∎

Lemma 2.5 implies the correctness of the described algorithm: it outputs a solution of (1) over $\mathbb{Z}_\infty$ iff (1) is solvable.

Now we estimate the complexity of the algorithm. We recall that in the course of an iteration modifying vector $(y_{n-s+1}, \ldots, y_n)$ the modified set S becomes a subset of the previous set S (see Lemma 2.3). First we bound from above the number of iterations while S does not change. Observe that the integer $N := (s - 1) \cdot y_{n-s+1} - y_{n-s+2} - \cdots - y_n$ increases after every iteration because the algorithm adds an integer $b \geq 1$ to each y_j for $j \in J \subset S$ (due to Lemma 2.3), while $n - s + 1 \in J$, in addition $J \neq S$ (otherwise, the algorithm completes the inductive step). At the beginning of the inductive step $N \geq -(s - 1) \cdot (M + 1) \cdot n$ (cf. Lemma 2.2). If N becomes larger than $M \cdot s^2$ then S should change (since not all the vertices of S become reachable in graph G). Therefore, after at

most of $O(M \cdot s \cdot n)$ iterations set S changes. Again due to Lemma 2.3 S can be modified at most s times. Thus, the whole number of iterations in the inductive step is less than $O(M \cdot s^2 \cdot n) \leq O(M \cdot n^3)$.

The complexity of executing a single iteration is bounded by $\log(M \cdot n) \cdot m \cdot n$ (cf. Lemma 2.2). The number of inductive steps (augmenting the set of rows of (1) under consideration) does not exceed m. Summarizing, this provides the complexity bound $O(M \cdot \log(M \cdot n) \cdot n^4 \cdot m^2)$ of the algorithm and completes the proof of Theorem 2.1. ■

3   Solving tropical non-homogeneous linear systems

Treating (1) as a tropical homogeneous linear system one can consider its non-homogeneous counterpart (2). Denote by $\hat{A}$ the matrix of size $m \times (n + 1)$ obtained from $A = (a_{i,j})$ by joining as the last $(n + 1)$-th column $(a_1, \ldots, a_m)^T$. Then (2) has a tropical solution over $\mathbb{Z}_\infty$ iff the homogeneous linear system with the matrix $\hat{A}$ has a tropical solution $(x_1, \ldots, x_n, x_{n+1})$ such that $x_{n+1} \neq \infty$. We describe an algorithm which can test the existence of such a solution.

The algorithm from Theorem 2.1 brings matrix $\hat{A}$ (after handling all its m rows) to the block form $(A_{p,q})$ with block sizes $u_1, \ldots, u_t;\ v_1, \ldots, v_t$ (possibly $u_t = 0$). We assume that the homogeneous system with the matrix $\hat{A}$ has a tropical solution (which is detected by the algorithm from Theorem 2.1), otherwise (2) has no tropical solution.

The proof of Lemma 2.5 entails that any solution of the homogeneous system with the matrix $\hat{A}$ has coordinates equal ∞ in the first $t - 1$ blocks of sizes $v_1, \ldots, v_{t-1}$ (we remind that the algorithm from Theorem 2.1 reorders the columns and rows of $\hat{A}$). On the other hand, there is a solution with all finite coordinates in the last t-th block of size v_t. Thus, the criterion of solvability of (2) is that the last $(n + 1)$-th column of $\hat{A}$ belongs to the last t-th block.

Assume that the entries $a_{i,j},\ a_i$ satisfy the same bounds as $a_{i,j}$ from (1). Making use of Theorem 2.1, we get

Corollary 3.1 *There is an algorithm which for an input (2) either finds its solution over $\mathbb{Z}_\infty$ or detects its insolvability within complexity $O(M \cdot \log(M \cdot n) \cdot n^4 \cdot m^2)$.*

4   Solvability of tropical linear systems via tropical and Kapranov ranks

As a direct consequence of Theorem 2.1 we get a criterion of solvability of a tropical linear system (1) over $\mathbb{Z}_\infty$ in terms of its tropical and Kapranov ranks [3].

Similar to matrices over $\mathbb{Z}$ we call $(n \times n)$-matrix $A = (a_{i,j})$ *tropically non-singular* if there exists a unique assignment $\{a_{i,\pi(i)}\}_{1 \leq i \leq n}$ for a permutation $\pi \in Sym(n)$ with a minimal sum $\sum_{1 \leq i \leq n} a_{i,\pi(i)}$ (in this case the latter sum is obviously finite). Then as

usually, the *tropical rank* of an $(m \times n)$-matrix is defined as the maximal size of tropically non-singular submatrices.

For an $(m \times n)$-matrix $A = (a_{i,j})$ its *lifting* is defined as an $(m \times n)$-matrix $F = (f_{i,j})$ over the field of Puiseux series $K = \mathbb{C}((t^{1/\infty}))$ such that $\mathrm{ord}(f_{i,j}) = a_{i,j}$ or $f_{i,j} = 0$ when $a_{i,j} = \infty$. Then the Kapranov rank of A is said to be less or equal to r if there exists a lifting F of A with the rank (over K) at most r.

Corollary 4.1 *The following three statements are equivalent:*
i) A tropical linear system (1) with $(m \times n)$-matrix A has a solution over $\mathbb{Z}_\infty$;
ii) The tropical rank of A is less than n;
iii) The Kapranov rank of A is less than n.

Proof of Corollary 4.1. The implication iii) $\Rightarrow$ ii) is evident (cf. e. g. [3]). In [3] it is also shown the equivalence of ii) and iii) for matrices over $\mathbb{R}$ (so, with finite coefficients).

The implication ii) $\Rightarrow$ i) follows from Theorem 2.1. Indeed, if (1) has no solutions, the algorithm designed in the proof of Theorem 2.1 terminates by exhausting the columns of (1). Hence there is an $(n \times n)$-submatrix $\tilde{C} = (\tilde{c}_{i,j})$ of A such that $(n \times n)$-matrix $C = (c_{i,j})$ for which $c_{i,j} = \tilde{c}_{i,j} + y_j$ for an appropriate vector $(y_1, \ldots, y_n) \in \mathbb{Z}^n$ fulfils the properties of Lemma 2.5. Clearly, matrix C has a unique minimal assignment located on its diagonal and thereby is tropically non-singular, the same holds for $\tilde{C}$ as well.

To establish the remaining implication i) $\Rightarrow$ iii) consider a solution $(x_1, \ldots, x_n) \in (\mathbb{Z}_\infty)^n$ of A. We take a vector $z := (z_1, \ldots, z_n) \in K^n$ such that $z_j = t^{x_j}$ or $z_j = 0$ when $x_j = \infty$. Our purpose is to produce an $(m \times n)$-matrix $F = (f_{i,j})$ over K such that $F \cdot z = 0$ and $\mathrm{ord}(f_{i,j}) = a_{i,j}$ or $f_{i,j} = 0$ when $a_{i,j} = \infty$ (i. e. F will be a lifting of A).

Fix a row i for the time being. If $\min_{1 \leq j \leq n}\{a_{i,j} + x_j\} = \infty$ we have $f_{i,j} \cdot z_j = 0$, $1 \leq j \leq n$. Now let $a_{i,l} + x_l = \min_{1 \leq j \leq n}\{a_{i,j} + x_j\} < \infty$ for all $l \in L$ for a certain subset $L \subset \{1, \ldots, n\}$ with at least two elements. We look for $f_{i,j} = \sum_{k \geq a_{i,j}} g_{j,k} \cdot t^k$ as polynomials with indeterminate coefficients $g_{j,k} \in \mathbb{Z}$. Fix in an arbitrary way all $f_{i,j} := t^{a_{i,j}}$ (when $a_{i,j} < \infty$) for all j except a single $l_0 \in L$. Expanding equality $\sum_{1 \leq j \leq n} f_{i,j} \cdot z_j = 0$ in the powers of t we obtain in the unique way a polynomial $f_{i,l_0} = -(\#L - 1) \cdot t^{a_{i,l_0}} + \cdots \in \mathbb{Z}[t]$ with $\mathrm{ord}(f_{i,l_0}) = a_{i,l_0}$. Since the rank of F (being a lifting of A) is less than n, we establish iii). $\blacksquare$

Remark 4.2 *For (extended) rational coefficients $a_{i,j} \in \mathbb{Q}_\infty$ Theorem 2.1 and Corollary 4.1 hold literally.*

Remark 4.3 *For (extended) real coefficients $a_{i,j} \in \mathbb{R}_\infty$ statements i) and ii) of Corollary 4.1 are equivalent. Indeed, for the implication ii) $\Rightarrow$ i) one can in the proof of Theorem 2.1 replace the induction with the transfinite induction, while modifying vector $(y_{n-s+1}, \ldots, y_n)$ and proving existence of a solution of (1) (again matrix C from Lemma 2.5 is tropically non-singular).*

To prove the inverse implication i) $\Rightarrow$ ii) assume that $(x_1, \ldots, x_n) \in (\mathbb{R}_\infty)^n$ is a solution of a tropical square linear system (1), i. e. $m = n$, and that A has a unique minimal

assignment. Reordering the rows and the columns of A one can suppose w.l.o.g. that $x_j = \infty$ iff $j > k$ and in addition that the unique minimal assignment is located on the diagonal of A. Then vector $(x_1, \ldots, x_k) \in \mathbb{R}^k$ is a solution of $(k \times k)$-submatrix $A_k = (a_{i,j})$, $1 \le i, j \le k$ of A in its upper left corner. Consider a directed graph H with k vertices $x_1, \ldots, x_k$. For a pair of vertices x_i, x_j, $i \ne j$ there is an edge (x_i, x_j) in H if $a_{i,j} + x_j = \min_{1 \le l \le k} \{a_{i,l} + x_l\}$. Since $(x_1, \ldots, x_k)$ is a solution of A_k, for any $1 \le i \le k$ there is $1 \le j \le k$ such that H contains edge (x_i, x_j). Therefore, there exists a simple cycle $x_{i_1}, x_{i_2}, \ldots, x_{i_s}$ in H. Then the assignment of A obtained from the diagonal one by means of replacing $a_{i_1,i_1}, a_{i_2,i_2}, \ldots, a_{i_s,i_s}$ with $a_{i_1,i_2}, a_{i_2,i_3}, \ldots, a_{i_s,i_1}$, has the same sum as the diagonal one. This contradiction with the tropical singularity of A proves ii).

It is an interesting question whether the complexity of the algorithm designed in Theorem 2.1 depends polynomially on $\log M$ (rather than M)? Or is there another way to test solvability of (1) within the complexity polynomial in $\log M$, n, m? A similar question was posed in [1], [7]. On the other hand, we note that one can detect solvability of (1) by verifying the tropical singularity of all $(n \times n)$-submatrices of A (see Corollary 4.1), thus within the complexity polynomial in $\log M$, $\binom{m}{n}$, cf. [2].

5 Testing uniqueness of a solution of a tropical linear system

Let $y = (y_1, \ldots, y_n) \in \mathbb{Z}_\infty^n$ be a solution of (1) (being yielded, say, by the algorithm designed in Theorem 2.1). One can suppose w.l.o.g. that $0 \le y_j \le (M+1) \cdot n + 1$, $1 \le j \le n$, cf. Lemma 2.2. Our purpose is to test whether y is unique (in the tropical projective space [3]) solution of (1). We refer to two vectors as different if they are different in the tropical projective space. The set $S_\infty(y) \subset \{1, \ldots, n\}$ of all $1 \le l \le n$ such that $y_l = \infty$ we call the *infinity support* of y.

Lemma 5.1 *Assume that there exists a solution $z = (z_1, \ldots, z_n) \in \mathbb{Z}_\infty^n$ of (1) different from y. Then there exists a solution $w = (w_1, \ldots, w_n) \in \mathbb{Z}_\infty^n$ of (1) and a pair of indices $1 \le j \ne l \le n$ with $\{j, l\} \not\subset S_\infty(y)$ and $j, l \notin S_\infty(w)$ such that*
* i) if $j, l \notin S_\infty(y)$ then $w_l \le y_l$, $w_j \le y_j$, $y_l - w_l + y_j - w_j = 1$;*
* ii) if $j \in S_\infty(y)$, $l \notin S_\infty(y)$ then $w_j - w_l = (M+1) \cdot n$.*
Moreover, $S_\infty(w) = S_\infty(y) \cap S_\infty(z)$.

    Proof. One can suppose w.l.o.g. that $0 \le z_j \le (M+1) \cdot n$, $1 \le j \le n$, cf. Lemma 2.2, and still y and z are different. If among three sets $S_\infty(y) \setminus S_\infty(z)$, $S_\infty(z) \setminus S_\infty(y)$, $\{1, \ldots, n\} \setminus (S_\infty(y) \cup S_\infty(z))$ at least two are nonempty, pick j from one of them and l from another one. Otherwise, $S_\infty(y) \setminus S_\infty(z) = S_\infty(z) \setminus S_\infty(y) = \emptyset$, in this case as j, l pick any two elements from $\{1, \ldots, n\} \setminus (S_\infty(y) \cup S_\infty(z))$ with the property that $y_l - z_l \ne y_j - z_j$ (such j, l exist since y, z are different).
    First consider the case when $j, l \notin S_\infty(y)$. Vector $z' := z + (\max\{y_l - z_l, y_j - z_j\} - 1) \cdot (1, \ldots, 1)$ is a solution of (1) (note that $\max\{y_l - z_l, y_j - z_j\} \in \mathbb{Z}$ because not both

z_j, z_l equal ∞ by virtue of the choice of j, l). Put $w := (w_1, \ldots, w_n) := \min\{y, z'\}$. Let for definiteness $y_l - z_l > y_j - z_j$. Then $w_l = y_l - 1$, $w_j = y_j$ which proves Lemma in the first case.

Secondly, assume that $j \in S_\infty(y)$, $l \notin S_\infty(y)$. Vector $y' := y + (z_j - (M+1) \cdot n - y_l) \cdot (1, \ldots, 1)$ is a solution of (1), put $w := (w_1, \ldots, w_n) := \min\{y', z\}$. Then $w_j = z_j$ and $w_l = y'_l = z_j - (M+1) \cdot n \le z_l$ due to the supposition above. ■

Now we describe an algorithm which tests whether y is a unique solution of (1). For each pair $\{j, l\} \not\subset S_\infty(y)$ the algorithm chooses integers w_j, w_l with $\min\{w_j, w_l\} = 0$ which satisfy either i) or ii) from Lemma 5.1. The algorithm extends (1) by a tropical linear equation $\min\{w_l + x_j, w_j + x_l\}$ and applies to the extended system Theorem 2.1. Lemma 5.1 implies that if y is not a unique solution of (1) then the extended system for at least one of the pairs $\{j, l\} \not\subset S_\infty(y)$ will have a solution. Conversely, if the extended system has a solution $x = (x_1, \ldots, x_n)$ then either $x_j, x_l \in \mathbb{Z}$, $x_j - x_l = w_j - w_l$ or $x_j = x_l = \infty$. In any of two latter cases vector x differs from y. Summarizing and employing Theorem 2.1 we conclude with the following

Corollary 5.2 *There is an algorithm which for a given system (1) tests whether it has a unique solution, within the complexity* $O(M \cdot \log(M \cdot n) \cdot n^7 \cdot m^2)$.

It is an open question whether one can calculate the dimension of the set of solutions of (1) (being a polyhedral complex) with the complexity which depends polynomially on n, m?

References

[1] T. Bogart, A. N. Jensen, D. Speyer, B. Sturmfels, R. R. Thomas, *Computing tropical varieties*, J. Symb. Comput. **42** (2007), 54–73.

[2] P. Butkovic, F. Hevery, *A condition for the strong regularity of matrices in the minimax algebra*, Discr. Appl. Math. **11** (1985), 209–222.

[3] M. Develin, F. Santos, B. Sturmfels, *On the rank of a tropical matrix*, In Combinatorial and computational geometry, Math. Sci. Res. Inst. Publ. **52**, Cambridge Univ. Press, Cambridge (2005), 213–242.

[4] D. Grigoriev, *Multiplicative complexity of a bilinear form over a commutative ring*, Lect. Notes Comput. Sci. **118** (1981), 281–286.

[5] K. H. Kim, F. W. Roush, *Factorization of polynomials in one variable over the tropical semiring*, arXiv:math/050116/v2

[6] K. H. Kim, F. W. Roush, *Kapranov rank vs. tropical rank*, Proc. Amer. Math. Soc. **134** (2006), 2487–2494.

[7] T. Theobald, *On the frontiers of polynomial computations in tropical geometry*, J. Symbolic Comput. **41** (2006), 1360–1375.