

THE POISSON GEOMETRY OF THE CONJUGATION QUOTIENT MAP FOR SIMPLE ALGEBRAIC GROUPS AND DEFORMED POISSON W-ALGEBRAS

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ABSTRACT. We define Poisson structures on certain transversal slices to conjugacy classes in complex simple algebraic groups. These structures are analogous to the Poisson structures introduced in papers [5, 13] on the Slodowy slices in complex simple Lie algebras. The quantum deformations of these Poisson structures are known as W-algebras of finite type. As an application of our definition we obtain some new Poisson structures on the coordinate rings of simple Kleinian singularities.

INTRODUCTION

In 1970 Brieskorn conjectured that simple singularities can be obtained as intersections of the nilpotent cone of a complex simple Lie algebra \mathfrak{g} with transversal slices to adjoint orbits of subregular nilpotent elements in \mathfrak{g} (see [2]). This conjecture was proved by Slodowy in book [23] where for each nilpotent element e a suitable transversal slice $s(e)$ to the adjoint orbit of e was constructed.

The simple singularities appear as some singularities of the fibers of the adjoint quotient map $\delta_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{h}/W$ generated by the inclusion $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$, where \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} and W is the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$. The fibers of the adjoint quotient map are unions of adjoint orbits in \mathfrak{g} . Each fiber of $\delta_{\mathfrak{g}}$ contains a single orbit which consists of regular elements. The singularities of the fibers correspond to irregular elements. For irregular e the restriction of the adjoint quotient map $\delta_{\mathfrak{g}}$ to the slice $s(e)$ has some singular fibers, and $s(e)$ can be regarded as a deformation of these singularities.

In papers [5, 13] using Hamiltonian reduction it was shown that the slices $s(e)$ can be naturally equipped with Poisson structures and that these Poisson structures can be quantized. This yields noncommutative deformations of the singularities of the fibers of the adjoint quotient map. These noncommutative deformations called W-algebras of finite type play an important role in the classification of the generalized Gelfand–Graev representations of the Lie algebra \mathfrak{g} [9, 13].

In this paper we are going to outline a similar construction for algebraic groups. Let G be a complex simple algebraic group with Lie algebra \mathfrak{g} . In case of algebraic groups, instead of the adjoint quotient map, one should consider the conjugation quotient map $\delta_G : G \rightarrow T/W$ generated by the inclusion $\mathbb{C}[T]^W \simeq \mathbb{C}[G]^G \hookrightarrow \mathbb{C}[G]$, where T is the maximal torus of G corresponding to the Cartan subalgebra \mathfrak{h} and W is the Weyl group of the pair (G, T) . Some fibers of this map are singular and one can study these singularities by restricting δ_G to certain transversal slices

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to conjugacy classes in G . These slices were defined in paper [20], and they are associated to pairs (\mathfrak{p}, s) , where \mathfrak{p} is a parabolic subalgebra in \mathfrak{g} and s is an element of the Weyl group W .

To describe the Poisson structure on the slice $s(e)$ by means of Hamiltonian reduction we first recall that the space \mathfrak{g}^* dual to \mathfrak{g} carries the standard Kirillov-Kostant Poisson bracket. The coadjoint action of the adjoint group G' of \mathfrak{g} on \mathfrak{g}^* is Hamiltonian, i.e. this action preserves the Poisson bracket on \mathfrak{g}^* . One can restrict this action to a unipotent subgroup $N \subset G'$ associated to the nilpotent element e as follows.

Let (e, h, f) be an \mathfrak{sl}_2 -triple associated to e , i.e. elements $f, h \in \mathfrak{g}$ obey the following commutation relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Under the action of $\text{ad } h$ we have a decomposition

$$(1) \quad \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \quad \text{where } \mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}.$$

Denote by χ the element of \mathfrak{g}^* which corresponds to e under the isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ induced by the Killing form. The skew-symmetric bilinear form ω on $\mathfrak{g}(-1)$ defined by $\omega(x, y) = \chi([x, y])$ is nondegenerate. Fix an isotropic Lagrangian subspace l of $\mathfrak{g}(-1)$ with respect to ω . Let

$$\mathfrak{n} = l \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i),$$

and N the Lie subgroup of G' which corresponds to the Lie subalgebra $\mathfrak{n} \subset \mathfrak{g}$.

The restriction of the coadjoint action of G' to N is still Hamiltonian, and hence the quotient \mathfrak{g}^*/N naturally acquires a Poisson structure. We denote by $\mu : \mathfrak{g}^* \rightarrow \mathfrak{n}^*$ the moment map for this action. Note that the quotient \mathfrak{g}^*/N is not smooth, and we understand the Poisson structure on \mathfrak{g}^*/N in the sense that the set $C^\infty(\mathfrak{g}^*)^N$ of N -invariant functions on \mathfrak{g}^* is a Poisson subalgebra in $C^\infty(\mathfrak{g}^*)$.

In papers [8, 11] it was proved that the reduced Poisson manifold $\mu^{-1}(\chi|_{\mathfrak{n}})/N$ corresponding to the value $\chi|_{\mathfrak{n}}$ of the moment map can be identified with $s(e)$ under the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ induced by the Killing form. This result is known as the Kostant cross-section theorem. By this theorem the slice $s(e)$ naturally acquires a Poisson structure, and $s(e)$ becomes a Poisson submanifold of the quotient \mathfrak{g}^*/N . As a variety the slice $s(e)$ can be identified with $e + \mathfrak{z}(f)$, where $\mathfrak{z}(f)$ is the centralizer of f in \mathfrak{g} .

Now we briefly describe the main construction of this paper. As we already mentioned above algebraic group counterparts of the Slodowy slices $s(e)$ were introduced in paper [20]. We shall equip these transversal slices to conjugacy classes in the group G with some Poisson structures. These slices are of the form $N_s Z s^{-1}$ where $s \in G$ is a representative of an element of the Weyl group W , $N_s = \{n \in N \mid sns^{-1} \in \overline{P}\}$, N is the unipotent radical of a parabolic subgroup P of G with Levi factor L , \overline{P} is the opposite parabolic subgroup, and $Z = \{z \in L \mid szs^{-1} = z\}$ is the centralizer of s in L . If the action of s on N by conjugations has no fixed points then $N_s Z s^{-1}$ is indeed a transversal slice to conjugacy classes in G (for a precise statement see [20] or Proposition 1 below). Moreover, the quotient $N Z s^{-1} N / N$ with respect to the action of N on $N Z s^{-1} N$ by conjugations is isomorphic to $N_s Z s^{-1}$ (see [20] or Proposition 2 below), and hence $N_s Z s^{-1}$ is a subvariety of G/N .

Now in order to equip the slice $N_s Z s^{-1}$ with a Poisson structure we recall that the action of G on itself by conjugations can be naturally put into the context of

Poisson geometry. Namely, equip the group G with the structure of a quasitriangular Poisson-Lie group, i.e. with a Poisson bracket on G such that the product map $G \times G \rightarrow G$ is a Poisson mapping. (A recollection of the results on Poisson-Lie groups and Poisson geometry used in this paper can be found in Section 2). If we denote by $(\mathfrak{g}, \mathfrak{g}^*)$ the tangent Lie bialgebra of G then one can define the dual Poisson-Lie group G^* with the tangent Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$. As a manifold the dual Poisson-Lie group G^* is isomorphic to a dense open subset in G , and there is a Poisson group action, i.e. a Poisson map $G \times G^* \rightarrow G^*$, called the dressing action. If we realize G^* as the dense open subset in G then the dressing action is induced by the action of G on itself by conjugations. Denote by G_* the group G equipped with the Poisson structure induced from G^* . In Lemma 12 and Theorem 13 it is proved that under some compatibility condition for N , s and the Poisson-Lie group structure on G the subgroup $N \subset G$ is admissible, in the sense that G_*/N is naturally equipped with the Poisson structure induced from G_* , and that the quotient $N_s Z s^{-1} \simeq N Z s^{-1} N / N$ is a Poisson submanifold in G_*/N . Here as in case of Lie algebras the quotient G_*/N is singular, and the Poisson structure on G_*/N should be understood in the sense that the set $C^\infty(G_*)^N$ of N -invariant functions on G_* is a Poisson subalgebra in $C^\infty(G_*)$.

The definition of the Poisson structure on the slice $N_s Z s^{-1}$ is similar to the definition of the Poisson structure on the Slodowy slice $s(e)$ discussed above, the variety $N Z s^{-1} N$ being a counterpart of the level surface $\mu^{-1}(\chi|_{\mathfrak{n}})$ of the moment map in case of Lie algebras. We call the Poisson algebra of regular functions on the slice $N_s Z s^{-1}$ a deformed Poisson W-algebra. The deformed Poisson W-algebras can be naturally quantized in the framework of quantum group theory. This will be explained in a subsequent paper.

In case when s is a representative of a Coxeter element in W and P is a Borel subgroup of G the slice $N_s s^{-1}$ was introduced by R. Steinberg in [24] (in this case the subgroup Z is trivial) and the Poisson structure on the slice $N_s s^{-1}$ was defined in [19]. Actually this Poisson structure is trivial, and the quantization of the Poisson algebra of regular functions on $N_s s^{-1}$ is isomorphic to the center of the quantized algebra of regular functions on the Poisson-Lie group G^* (see [21]). This fact is of primary importance for classification of the Whittaker representations of the quantum group $U_q(\mathfrak{g})$ [22].

Note that the G -invariant functions on G_* lie in the center of the Poisson algebra of functions on G_* [18]. Therefore the intersections of the slices $N_s Z s^{-1}$ with the fibers of the conjugation quotient map $\delta_G : G \rightarrow T/W$ are also equipped with Poisson structures. In particular, if such a fiber is singular our construction yields a Poisson structure related to the corresponding singularity. In Section 4 we study in detail the case when s is subregular in G and \mathfrak{p} is a parabolic subalgebra in \mathfrak{g} . It is shown in [20] that in this case the intersections of the slice $N_s Z s^{-1}$ with the fibers of the conjugation quotient map may only have simple isolated singularities, and we show that using Theorem 13 one can equip the slice $N_s Z s^{-1}$ and the coordinate rings of the corresponding simple singularities with Poisson structures.

In the simplest case when $\mathfrak{g} = \mathfrak{sl}_3$, s is a representative of the longest element of the Weyl group and \mathfrak{p} is a Borel subalgebra one can calculate the corresponding Poisson structures explicitly. We show that in this case the singular fiber $\delta_G^{-1}(1)$ of the map $\delta_G : N_s Z s^{-1} \rightarrow T/W$ has simple A_2 -type singularity, and the corresponding Poisson structure on the singular fiber is proportional to that obtained in [13]

for the singular fiber $\delta_{\mathfrak{g}}^{-1}(0)$ of the adjoint quotient map $\delta_{\mathfrak{g}} : s(e) \rightarrow \mathfrak{h}/W$ in case of a subregular nilpotent element $e \in \mathfrak{g}$.

In general the Poisson structures on the slices $N_s Z s^{-1}$ are difficult to describe explicitly. In Section 5 we consider the case when s is the reflection with respect to a long root β and $\mathfrak{p} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ where the components $\mathfrak{g}(i)$ are defined by formula (1) where e is a root vector corresponding to β . In this case the Poisson structure on the slice $N_s Z s^{-1}$ can be described explicitly. An analogous Poisson structure on the slice $s(e)$ for a long root vector e in \mathfrak{g} was considered in [14]. The quantization of this Poisson structure is related to the Joseph ideal of the universal enveloping algebra $U(\mathfrak{g})$.

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1. TRANSVERSAL SLICES TO CONJUGACY CLASSES IN ALGEBRAIC GROUPS

In this section we recall the definition of the algebraic group counterparts of the Slodowy slices and an analogue of the Kostant cross–section theorem for them.

Let G be a complex semisimple algebraic group, \mathfrak{g} its Lie algebra. Let P be a parabolic subgroup of G , L its Levi factor, and N the unipotent radical of P . Denote by \mathfrak{p} , \mathfrak{l} and \mathfrak{n} the Lie algebras of P , L and N , respectively. Let \overline{P} be the opposite parabolic subgroup and \overline{N} the unipotent radical of \overline{P} . Denote by $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{n}}$ the Lie algebras of \overline{P} and \overline{N} , respectively.

Let $s \in W$ be an element of the Weyl group W of G . Fix a representative of s in G . We denote this representative by the same letter, $s \in G$. The element $s \in G$ naturally acts on G by conjugations. Let Z be the set of s -fixed points in L ,

$$Z = \{z \in L \mid s z s^{-1} = z\},$$

and

$$N_s = \{n \in N \mid s n s^{-1} \in \overline{P}\}.$$

Clearly, Z and N_s are subgroups in G , and Z normalizes both N and N_s . Denote by \mathfrak{n}_s and \mathfrak{z} the Lie algebras of N_s and Z , respectively.

Now consider the subvariety $N_s Z s^{-1} \subset G$.

Proposition 1. ([20], **Proposition 2.1**) *Assume that for each $x \in \mathfrak{n}$ and $y \in \overline{\mathfrak{n}}$, $x, y \neq 0$ there exist $k, k' \in \mathbb{N}$ such that $\text{Ad}(s^k)(x) \notin \mathfrak{p}$ and $\text{Ad}(s^{k'})(y) \notin \overline{\mathfrak{p}}$. Then the conjugation map $G \times N_s Z s^{-1} \rightarrow G$ is smooth, and the variety $N_s Z s^{-1} \subset G$ is a transversal slice to the set of conjugacy classes in G .*

The variety $N_s Z s^{-1}$ is called the transversal slice in G associated to the pair (\mathfrak{p}, s) . The following statement is an analogue of the Kostant cross–section theorem for the slice $N_s Z s^{-1} \subset G$.

Proposition 2. ([20], **Proposition 2.2**) *Assume that for each $x \in \mathfrak{n}$, $x \neq 0$ there exists $k \in \mathbb{N}$ such that $\text{Ad}(s^k)(x) \notin \mathfrak{n}$. Then the conjugation map*

$$(2) \quad \alpha : N \times N_s Z s^{-1} \rightarrow N Z s^{-1} N$$

is an isomorphism of varieties.

The variety $N Z s^{-1} N$ is isomorphic to $N_s Z s^{-1} N$ and hence the image of map (2) is isomorphic to $N_s Z s^{-1} N \simeq N Z s^{-1} N$ as a variety.

Now, following [20], we recall that to each Weyl group element one can naturally associate some parabolic subalgebras which satisfy the conditions of Propositions 1 and 2. First recall that in the classification theory of conjugacy classes in Weyl group W of complex simple Lie algebra \mathfrak{g} the so-called primitive (or semi-Coxeter in another terminology) elements play a primary role. The primitive elements $w \in W$ are characterized by the property $\det(1 - w) = \det a$, where a is the Cartan matrix of \mathfrak{g} . According to the results of [3] each element s of the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ is a primitive element in the Weyl group W' of a regular semisimple Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of the form

$$\mathfrak{g}' = \mathfrak{h}' + \sum_{\alpha \in \Delta'} X_\alpha,$$

where Δ' is a root subsystem of the root system Δ of \mathfrak{g} , X_α is the root subspace of \mathfrak{g} corresponding to root α , and \mathfrak{h}' is a Lie subalgebra of \mathfrak{h} .

Proposition 3. ([20], **Proposition 2.4**) *Let s be an element of the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$. Denote by the same letter a representative of s in G . Assume that s is primitive in the Weyl group W' of a pair $(\mathfrak{g}', \mathfrak{h}')$, where \mathfrak{g}' is a regular subalgebra of \mathfrak{g} and $\mathfrak{h}' \subset \mathfrak{h}$ is the Cartan subalgebra of \mathfrak{g}' . Let \mathfrak{p} be the parabolic subalgebra in \mathfrak{g} defined by $\mathfrak{p} = \bigoplus_{n \geq 0} \mathfrak{g}_n$, where $\mathfrak{g}_n = \{x \in \mathfrak{g} \mid [h, x] = nx\}$ for some $h \in \mathfrak{h}'$. Let \mathfrak{l} and \mathfrak{n} be the Levi factor and the nilradical of \mathfrak{p} . Denote by P , L and N the Lie subgroups of G corresponding to Lie subalgebras \mathfrak{p} , \mathfrak{l} and \mathfrak{n} , respectively. Let*

$$Z = \{z \in L \mid szs^{-1} = z\}$$

be the centralizer of s in L , and

$$N_s = \{n \in N \mid sns^{-1} \in \overline{P}\},$$

where \overline{P} is the parabolic subalgebra opposite to P . Then the conjugation map $G \times N_s Z s^{-1} \rightarrow G$ is smooth, the variety $N_s Z s^{-1} \subset G$ is a transversal slice to the set of conjugacy classes in G , and the conjugation map

$$(3) \quad N \times N_s Z s^{-1} \rightarrow N Z s^{-1} N$$

is an isomorphism of varieties.

2. POISSON-LIE GROUPS AND POISSON REDUCTION

In this section we recall some results related to Poisson-Lie groups and Poisson geometry (see [4, 6, 15, 18]). These results will be used in the next section to equip the slices $N_s Z s^{-1}$ defined in Section 1 with Poisson structures.

Let G be a finite-dimensional Lie group equipped with a Poisson bracket, \mathfrak{g} its Lie algebra. G is called a Poisson-Lie group if the multiplication $G \times G \rightarrow G$ is a Poisson map. A Poisson bracket satisfying this axiom is degenerate and, in particular, is identically zero at the unit element of the group. Linearizing this bracket at the unit element defines the structure of a Lie algebra in the space $T_e^* G \simeq \mathfrak{g}^*$. The pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called the tangent bialgebra of G .

Lie brackets in \mathfrak{g} and \mathfrak{g}^* satisfy the following compatibility condition:

Let $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ be the dual of the commutator map $[\cdot, \cdot]_* : \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^*$. Then δ is a 1-cocycle on \mathfrak{g} (with respect to the adjoint action of \mathfrak{g} on $\mathfrak{g} \wedge \mathfrak{g}$).

Let c_{ij}^k and f_c^{ab} be the structure constants of \mathfrak{g} and \mathfrak{g}^* , respectively, with respect to the dual bases $\{e_i\}$ and $\{e^i\}$ in \mathfrak{g} and \mathfrak{g}^* . The compatibility condition means that

$$c_{ab}^s f_s^{ik} - c_{as}^i f_b^{sk} + c_{as}^k f_b^{si} - c_{bs}^k f_a^{si} + c_{bs}^i f_a^{sk} = 0.$$

This condition is symmetric with respect to exchange of c and f . Thus if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, then $(\mathfrak{g}^*, \mathfrak{g})$ is also a Lie bialgebra.

The following proposition shows that the category of finite-dimensional Lie bialgebras is isomorphic to the category of finite-dimensional connected simply connected Poisson–Lie groups.

Proposition 4. ([4], **Theorem 1.3.2**) *If G is a connected simply connected finite-dimensional Lie group, every bialgebra structure on \mathfrak{g} is the tangent bialgebra of a unique Poisson structure on G which makes G into a Poisson–Lie group.*

Let G be a finite-dimensional Poisson–Lie group, $(\mathfrak{g}, \mathfrak{g}^*)$ the tangent bialgebra of G . The connected simply connected finite-dimensional Poisson–Lie group corresponding to the Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$ is called the dual Poisson–Lie group and denoted by G^* .

In this paper we shall need a special class of factorizable Lie bialgebras. This class is slightly smaller than the class of quasitriangular Lie bialgebras. A Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is called a *factorizable* if the following conditions are satisfied (see [6, 15]):

(1) \mathfrak{g} is equipped with a non-degenerate invariant scalar product $\langle \cdot, \cdot \rangle$.

We shall always identify \mathfrak{g}^* and \mathfrak{g} by means of this scalar product.

(2) The dual Lie bracket on $\mathfrak{g}^* \simeq \mathfrak{g}$ is given by

$$(4) \quad [X, Y]_* = \frac{1}{2} ([rX, Y] + [X, rY]), \quad X, Y \in \mathfrak{g},$$

where $r \in \text{End } \mathfrak{g}$ is a skew symmetric, with respect to the non-degenerate invariant scalar product, linear operator.

(3) r satisfies the modified classical Yang–Baxter equation:

$$(5) \quad [rX, rY] - r([rX, Y] + [X, rY]) = -[X, Y], \quad X, Y \in \mathfrak{g}.$$

The skew-symmetric operator r on \mathfrak{g} satisfying the modified classical Yang–Baxter equation is called the classical r -matrix (or simply r -matrix).

Define operators $r_{\pm} \in \text{End } \mathfrak{g}$ by

$$r_{\pm} = \frac{1}{2} (r \pm id).$$

We shall need some properties of the operators r_{\pm} . Denote by \mathfrak{b}_{\pm} and \mathfrak{n}_{\mp} the image and the kernel of the operator r_{\pm} :

$$(6) \quad \mathfrak{b}_{\pm} = \text{Im } r_{\pm}, \quad \mathfrak{n}_{\mp} = \text{Ker } r_{\pm}.$$

Proposition 5. ([1, 17]) *Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a factorizable Lie bialgebra. Then*

(i) $\mathfrak{b}_{\pm} \subset \mathfrak{g}$ is a Lie subalgebra, the subspace \mathfrak{n}_{\pm} is a Lie ideal in \mathfrak{b}_{\pm} , $\mathfrak{b}_{\pm}^{\perp} = \mathfrak{n}_{\pm}$.

(ii) \mathfrak{n}_{\pm} is an ideal in \mathfrak{g}^* .

(iii) \mathfrak{b}_{\pm} is a Lie subalgebra in \mathfrak{g}^* . Moreover $\mathfrak{b}_{\pm} = \mathfrak{g}^*/\mathfrak{n}_{\mp}$.

(iv) $(\mathfrak{b}_{\pm}, \mathfrak{b}_{\pm}^*)$ is a subbialgebra of $(\mathfrak{g}, \mathfrak{g}^*)$ and $(\mathfrak{b}_{\pm}, \mathfrak{b}_{\pm}^*) \simeq (\mathfrak{b}_{\pm}, \mathfrak{b}_{\mp})$.

The classical Yang–Baxter equation implies that r_{\pm} , regarded as a mapping from \mathfrak{g}^* into \mathfrak{g} , is a Lie algebra homomorphism. Moreover, $r_+^* = -r_-$, and $r_+ - r_- = id$.

The skew-symmetric classical r -matrices associated to complex simple Lie algebras were classified in [1]. A closely related class of graded r -matrices was described in [17]. We shall need this classification in the next section.

Proposition 6. ([1, 17]) *Let \mathfrak{g} be a complex simple Lie algebra and r a classical r -matrix on \mathfrak{g} , i.e. r is a solution to the classical Yang–Baxter equation and r is skew-symmetric with respect to the Killing form on \mathfrak{g} . Then the corresponding Lie subalgebras $\mathfrak{b}_\pm \subset \mathfrak{g}$ are contained in some parabolic subalgebras $\mathfrak{p}_\pm \subset \mathfrak{g}$, $\mathfrak{b}_\pm \subset \mathfrak{p}_\pm$, and the Lie subalgebras $\mathfrak{n}_\pm \subset \mathfrak{b}_\pm$ contain the nilradicals \mathfrak{k}_\pm of \mathfrak{p}_\pm , respectively, $\mathfrak{k}_\pm \subset \mathfrak{n}_\pm$. Moreover, if we denote by \mathfrak{b} the Borel subalgebra contained in \mathfrak{p}_+ then \mathfrak{p}_- contains the opposite Borel subalgebra $\bar{\mathfrak{b}}$.*

Fix a system Γ of simple positive roots in \mathfrak{g} associated to the Borel subalgebra \mathfrak{b} and denote by \mathfrak{h} the corresponding Cartan subalgebra contained in \mathfrak{b} . Denote also by $\Gamma_{1,2} \subset \Gamma$ the subsystems of positive roots corresponding to the parabolic subalgebras \mathfrak{p}_\pm so that

$$\mathfrak{p}_+ = \mathfrak{b} + \sum_{\alpha \in \widehat{\Gamma}_1} \mathfrak{g}_{-\alpha}, \quad \mathfrak{p}_- = \bar{\mathfrak{b}} + \sum_{\alpha \in \widehat{\Gamma}_2} \mathfrak{g}_\alpha,$$

where $\widehat{\Gamma}_{1,2}$ are the sets of positive roots which are linear combinations of elements of $\Gamma_{1,2}$. Let \mathfrak{m}_\pm be the Levi factor of \mathfrak{p}_\pm , and \mathfrak{l}_\pm the semisimple part of \mathfrak{m}_\pm .

Then there is a one-to-one mapping $\tau : \Gamma_1 \rightarrow \Gamma_2$ such that

- τ is isometric with respect to the canonical scalar product $\langle \cdot, \cdot \rangle$ on Γ induced by the Killing form,

$$(7) \quad \langle \tau(\alpha), \tau(\beta) \rangle = \langle \alpha, \beta \rangle;$$

- for any $\alpha \in \Gamma_1$ there exists a natural number k such that

$$(8) \quad \tau^k(\alpha) \notin \Gamma_1,$$

and any element $X \in \mathfrak{g}$ can be uniquely represented in the form

$$(9) \quad X = X_+ + X_- + (1 - \theta)X_0 + X_{\mathfrak{h}},$$

where $X_\pm \in \mathfrak{k}_\pm$, $X_0 \in \mathfrak{m}_+ \ominus \mathfrak{h}$, $X_{\mathfrak{h}} \in \mathfrak{h}$, $\mathfrak{m}_+ \ominus \mathfrak{h}$ is the orthogonal complement of \mathfrak{h} in \mathfrak{m}_+ with respect to the Killing form, and $\theta : \mathfrak{l}_+ \rightarrow \mathfrak{l}_-$ is the Lie algebra homomorphism induced by τ , i.e.

$$\theta(X_\alpha) = X_{\tau(\alpha)}, \quad \theta(X_{-\alpha}) = X_{-\tau(\alpha)}, \quad \theta(H_\alpha) = H_{\tau(\alpha)}, \quad \alpha \in \Gamma_1$$

for the systems of Weyl generators $X_{\pm\alpha}$, H_α , $\alpha \in \Gamma_1$ and $X_{\pm\beta}$, H_β , $\beta \in \Gamma_2$ of \mathfrak{l}_\pm , respectively.

Moreover, the restriction r_0 of the operator r to the Cartan subalgebra \mathfrak{h} leaves \mathfrak{h} invariant, $r_0 : \mathfrak{h} \rightarrow \mathfrak{h}$, and we have

$$(10) \quad rX = X_+ - X_- + (1 + \theta)X_0 + r_0X_{\mathfrak{h}}.$$

The operator r_0 satisfies the properties

$$(11) \quad (r_0 - id)|_{\mathfrak{h} \cap \mathfrak{l}_+} = (r_0 + id)\theta|_{\mathfrak{h} \cap \mathfrak{l}_+}, \quad r_0^* = -r_0,$$

and the data $(\Gamma_1, \Gamma_2, \tau, r_0)$ completely determine the r -matrix r up to an automorphism of the Lie algebra \mathfrak{g} , i.e. every skew-symmetric solution to the Yang–Baxter equation for the Lie algebra \mathfrak{g} can be transformed by a suitable automorphism to the operator defined by formulas (9) and (10) for some data $(\Gamma_1, \Gamma_2, \tau, r_0)$ satisfying conditions (7), (8) and (11).

If the tangent Lie bialgebra of a Poisson–Lie group G is factorizable then one can describe the dual group G^* in terms of G as follows. Put $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}$ (direct sum of two copies). The mapping

$$(12) \quad \mathfrak{g}^* \rightarrow \mathfrak{d} : X \mapsto (X_+, X_-), \quad X_\pm = r_\pm X$$

is a Lie algebra embedding. Thus we may identify \mathfrak{g}^* with a Lie subalgebra in \mathfrak{d} .

Naturally, embedding (12) extends to an embedding

$$G^* \rightarrow G \times G, \quad L \mapsto (L_+, L_-).$$

We shall identify G^* with the corresponding subgroup in $G \times G$.

Now we explicitly describe Poisson structures on the Poisson–Lie group G and on its dual group G^* .

For every group A with Lie algebra \mathfrak{a} we define left and right gradients $\nabla\varphi, \nabla'\varphi \in \mathfrak{a}^*$ of a function $\varphi \in C^\infty(A)$ by the formulae

$$(13) \quad \begin{aligned} \xi(\nabla\varphi(x)) &= \left(\frac{d}{dt} \right)_{t=0} \varphi(e^{t\xi}x), \\ \xi(\nabla'\varphi(x)) &= \left(\frac{d}{dt} \right)_{t=0} \varphi(xe^{t\xi}), \quad \xi \in \mathfrak{a}. \end{aligned}$$

The canonical Poisson bracket on Poisson–Lie group G with factorizable tangent bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ has the form:

$$(14) \quad \{\varphi, \psi\} = \frac{1}{2} \langle r\nabla\varphi, \nabla\psi \rangle - \frac{1}{2} \langle r\nabla'\varphi, \nabla'\psi \rangle,$$

where r is the corresponding r -matrix.

The canonical Poisson bracket on the dual Poisson–Lie group G^* can be described in terms of the original group G and the classical r -matrix r .

Proposition 7. *Denote by G_* the group G equipped with the following Poisson bracket*

$$(15) \quad \{\varphi, \psi\}_* = \langle r\nabla\varphi, \nabla\psi \rangle + \langle r\nabla'\varphi, \nabla'\psi \rangle - 2 \langle r_+ \nabla'\varphi, \nabla\psi \rangle - 2 \langle r_- \nabla\varphi, \nabla'\psi \rangle,$$

where all the gradients are taken with respect to the original group structure on G .

Then the map $q : G^* \rightarrow G_*$ defined by

$$(16) \quad q(L_+, L_-) = L_+ L_-^{-1}$$

is a Poisson mapping and the image of q is a dense open subset in G_* .

Now we recall some facts on Poisson reduction and Poisson group actions. A Poisson group action of a Poisson–Lie group A on a Poisson manifold M is a group action $A \times M \rightarrow M$ which is also a Poisson map (as usual, we suppose that $A \times M$ is equipped with the product Poisson structure). In [16] it is proved that if the space M/A is a smooth manifold, there exists a unique Poisson structure on M/A such that the canonical projection $M \rightarrow M/A$ is a Poisson map.

The main example of Poisson group actions is the so-called dressing action. The dressing action can be described as follows (see [12, 16]).

Proposition 8. *Let G be a Poisson–Lie group with factorizable tangent Lie bialgebra, G^* the dual group. Then there exists a unique left Poisson group action*

$$G \times G^* \rightarrow G^*, \quad (g, (L_+, L_-)) \mapsto g \circ (L_+, L_-).$$

Moreover, if $q : G^* \rightarrow G_*$ is the map defined by formula (16) then

$$q(g \circ (L_+, L_-)) = g L_- L_+^{-1} g^{-1},$$

i.e. the conjugation map $G \times G_* \rightarrow G_*$ is a Poisson group action of the Poisson–Lie group G on the Poisson manifold G_* .

The notion of Poisson group actions may be generalized as follows. Let $A \times M \rightarrow M$ be a Poisson group action of a Poisson–Lie group A on a Poisson manifold M . A subgroup $K \subset A$ is called *admissible* if the set $C^\infty(M)^K$ of K -invariants is a Poisson subalgebra in $C^\infty(M)$. If space M/K is a smooth manifold, we may identify the algebras $C^\infty(M/K)$ and $C^\infty(M)^K$. Hence there exists a Poisson structure on M/K such that the canonical projection $M \rightarrow M/K$ is a Poisson map. The space M/K is called the *reduced Poisson manifold*. In order to shorten the notation we shall say that M/K inherits a Poisson structure from M even in case when the quotient M/K is not smooth. In this case the Poisson structure on M/K should be understood in the sense that the set $C^\infty(M)^K$ is a Poisson subalgebra in $C^\infty(M)$.

The following proposition proved in [16] gives a sufficient criterion for K to be an admissible subgroup of a Poisson–Lie group A .

Proposition 9. *Let $(\mathfrak{a}, \mathfrak{a}^*)$ be the tangent Lie bialgebra of A . A connected Lie subgroup $K \subset A$ with Lie algebra $\mathfrak{k} \subset \mathfrak{a}$ is admissible if $\mathfrak{k}^\perp \subset \mathfrak{a}^*$ is a Lie subalgebra.*

In particular, A itself is admissible. (Note that $K \subset A$ is a Poisson–Lie subgroup if and only if $\mathfrak{k}^\perp \subset \mathfrak{a}^*$ is an *ideal*; in that case the tangent Lie bialgebra of K is $(\mathfrak{k}, \mathfrak{a}^*/\mathfrak{k}^\perp)$.)

Even if M is symplectic the reduced Poisson bracket on M/K is usually degenerate. The difficult part of reduction is the description of the symplectic leaves in M/K . In case of Hamiltonian group actions the appropriate technique is provided by the use of the moment map. Although a similar notion of the *nonabelian moment map* in the context of Poisson group theory is also available [12], it is less convenient. The following simple assertion may be extracted from [12] and will serve as a substitute.

Let M be a Poisson manifold, $\pi : M \rightarrow B$ a Poisson submersion. Hamiltonian vector fields $\xi_\varphi, \varphi \in \pi^*C^\infty(B)$, generate an integrable distribution \mathfrak{H}_π in TM .

Proposition 10. *Let $V \subset M$ be a submanifold. Then $W = \pi(V) \subset B$ is a Poisson submanifold if and only if V is an integral manifold of \mathfrak{H}_π .*

Assume that this condition is satisfied. Then the Poisson bracket on W can be described as follows. Let $N_V \subset T^*M|_V$ be the conormal bundle of V . Clearly, $T^*V \simeq T^*M|_V/N_V$.

Proposition 11. *Let $\varphi, \psi \in C^\infty(W)$; put $\varphi^* = \pi^*\varphi|_V, \psi^* = \pi^*\psi|_V$. Let $\overline{d\varphi}, \overline{d\psi} \in T^*M|_V$ be any representatives of $d\varphi^*, d\psi^* \in T^*V$. Denote by $P_M \in \wedge^2 T^*M$ the Poisson tensor associated to the Poisson bracket on M . Then*

$$(17) \quad \pi^* \{ \varphi, \psi \} |_V = \langle P_M, \overline{d\varphi} \wedge \overline{d\psi} \rangle ;$$

in particular, the r.h.s. of equation (17) does not depend on the choice of $\overline{d\varphi}, \overline{d\psi}$.

3. POISSON REDUCTION AND DEFORMED W ALGEBRAS

Let G be a complex simple algebraic group, \mathfrak{g} its Lie algebra. In this section we equip slices defined in Section 1 with Poisson structures. We keep the notation introduced in Sections 1 and 2.

Let r be a classical r-matrix on \mathfrak{g} . Assume that r is skew symmetric with respect to the Killing form on \mathfrak{g} and equip the group G with the standard Lie–Poisson bracket (14) associated with this r-matrix. By Proposition 8 the action by conjugations of the Poisson–Lie group G on the Poisson manifold G_* is Poisson if G_* carries

Poisson bracket (15) associated with the same r -matrix. We would like to show that the unipotent radical K_+ of the parabolic subgroup $P_+ \subset G$ with Lie algebra \mathfrak{p}_+ introduced in Proposition 6 contains some unipotent admissible subgroups $N \subset G$, and hence the conjugation action of G on G_* can be restricted to subgroups N in such a way that the quotients G_*/N carry natural Poisson structures. Then we show that under some compatibility conditions the slices $N_s Z s^{-1}$ introduced in Proposition 1 are Poisson submanifolds of the quotients G_*/N .

Lemma 12. *Let G be a complex simple algebraic group, \mathfrak{g} its Lie algebra and r a classical r -matrix on \mathfrak{g} which is skew-symmetric with respect to the Killing form. Let \mathfrak{k}_+ be the nilradical of the parabolic subalgebra $\mathfrak{p}_+ \subset \mathfrak{g}$ introduced in Proposition 6 for r . Let $\mathfrak{n} \subset \mathfrak{k}_+$ be the nilradical of a parabolic subalgebra \mathfrak{p} containing the same Borel subalgebra as \mathfrak{p}_+ . Let N be the Lie subgroup of G corresponding to the Lie algebra \mathfrak{n} . Then $N \subset G$ is an admissible subgroup in the Poisson-Lie group G equipped with the standard Poisson bracket (14).*

Proof. By Proposition 9 it suffices to show that $\mathfrak{p} = \mathfrak{n}^\perp \subset \mathfrak{g}^*$ is a Lie subalgebra. Recall that according to formula (12) \mathfrak{g}^* can be identified with the Lie subalgebra $r_+ \mathfrak{g} + r_- \mathfrak{g}$ of $\mathfrak{g} + \mathfrak{g}$. Using formulas (9) and (10) one can describe the Lie algebra $\mathfrak{g}^* \simeq r_+ \mathfrak{g} + r_- \mathfrak{g}$ as follows

$$(18) \quad \mathfrak{g}^* \simeq \left\{ (X_+ + X_0 + \frac{1}{2}(r_0 + id)X_{\mathfrak{h}}, -X_- + \theta X_0 + \frac{1}{2}(r_0 - id)X_{\mathfrak{h}}), \right. \\ \left. X_{\pm} \in \mathfrak{k}_{\pm}, X_0 \in \mathfrak{m}_+ \ominus \mathfrak{h}, X_{\mathfrak{h}} \in \mathfrak{h} \right\}.$$

In the last formula we use the notation introduced in Proposition 6, and $\mathfrak{m}_+ \ominus \mathfrak{h}$ is the orthogonal complement of \mathfrak{h} in \mathfrak{m}_+ with respect to the Killing form.

Using description (18) of the Lie algebra \mathfrak{g}^* and recalling that \mathfrak{k}_{\pm} , \mathfrak{m}_{\pm} , \mathfrak{p}_{\pm} , \mathfrak{n} and \mathfrak{p} are direct sums of root spaces with respect to the adjoint action of the Cartan subalgebra \mathfrak{h} one can identify $\mathfrak{p} = \mathfrak{n}^\perp \subset \mathfrak{g}^*$ with the following Lie subalgebra in $\mathfrak{g} + \mathfrak{g}$

$$(19) \quad \mathfrak{p} = \mathfrak{n}^\perp \simeq \left\{ (Z_+ + Z_0 + \frac{1}{2}(r_0 + id)Z_{\mathfrak{h}}, -Z_- + \theta Z_0 + \frac{1}{2}(r_0 - id)Z_{\mathfrak{h}}), \right. \\ \left. Z_+ \in \mathfrak{k}_+, Z_- \in \mathfrak{k}_- \cap \mathfrak{p}, Z_0 \in V, Z_{\mathfrak{h}} \in \mathfrak{h} \right\},$$

where

$$V = \{Z_0 \in \mathfrak{m}_+ \ominus \mathfrak{h} : \theta Z_0 \in \mathfrak{m}_- \cap \mathfrak{p}\}.$$

Now using the Yang–Baxter equation for r and the fact that \mathfrak{k}_{\pm} are ideals in \mathfrak{p}_{\pm} and \mathfrak{p} is a Lie subalgebra in \mathfrak{g} one checks straightforwardly that the subset in \mathfrak{g}^* defined by the r.h.s. of formula (19) is a Lie subalgebra of $r_+ \mathfrak{g} + r_- \mathfrak{g} \simeq \mathfrak{g}^*$. \square

Now we restrict the action of G on G_* by conjugations to the subgroup $N \subset G$ introduced in the previous lemma. By Lemma 12 and the remark before Proposition 9 the space G_*/N inherits a reduced Poisson structure from G_* .

Now assume that the conditions of Proposition 2 are satisfied for N and for some $s \in W$, where W is the Weyl group of G . Then the quotient $N Z s^{-1} N / N \simeq N_s Z s^{-1}$ is a subspace of the quotient G_*/N which carries the reduced Poisson bracket. We would like to investigate under which conditions $N Z s^{-1} N / N \simeq N_s Z s^{-1} \subset G_*/N$ is a Poisson submanifold in G_*/N .

Theorem 13. *Let G be a complex simple algebraic group, \mathfrak{g} its Lie algebra and r an r -matrix on \mathfrak{g} which is skew-symmetric with respect to the Killing form. Let*

\mathfrak{k}_+ be the nilradical of the parabolic subalgebra $\mathfrak{p}_+ \subset \mathfrak{g}$ introduced in Proposition 6 for r . Let $\mathfrak{n} \subset \mathfrak{k}_+$ be the nilradical of a parabolic subalgebra \mathfrak{p} containing the same Borel subalgebra as $\mathfrak{p}_+ \subset \mathfrak{g}$. Let N be the Lie subgroup of G corresponding to the Lie algebra \mathfrak{n} . Assume that the conditions of Proposition 2 are satisfied for N and for some representative s of an element $s \in W$, where W is the Weyl group of G . Then N is an admissible subgroup of G , and $N_s Z s^{-1}$ is a Poisson submanifold of G_*/N if and only if the following conditions are satisfied

$$(20) \quad \text{Ad}s^{-1}(\mathfrak{k}_- \cap \mathfrak{p}) \subset \mathfrak{n} + \mathfrak{z}, \quad \text{Ad}s^{-1}\theta|_V = id \pmod{\mathfrak{n} + \mathfrak{z}}, \quad [\mathfrak{z}, V] \subset \mathfrak{n} + \mathfrak{z}, \\ \text{Ad}s^{-1}r_-|_{\mathfrak{h}} - r_+|_{\mathfrak{h}} = 0 \pmod{\mathfrak{n} + \mathfrak{z}}, \quad [\mathfrak{z}, \mathfrak{b}_- \cap \mathfrak{h}] \subset \mathfrak{n} + \mathfrak{z}.$$

Proof. By Proposition 10 it is sufficient to check that the Hamiltonian vector fields generated by N -invariant functions on G_* are tangent to $NZs^{-1}N$ iff conditions (20) are satisfied.

Let $\varphi \in C^\infty(G_*)^N$. Then $\varphi(vg) = \varphi(gv)$ for all $v \in N$, $g \in G_*$, and hence $Z = \nabla\varphi - \nabla'\varphi \in \mathfrak{p}$. Since $\nabla'\varphi(g) = \text{Ad } g^{-1}(\nabla\varphi(g))$ we can rewrite the Poisson bracket (15) on G_* in the following form:

$$\{\varphi, \psi\}_\tau(g) = 2 \langle r_+ Z - \text{Ad } g(r_- Z), \nabla\psi(g) \rangle.$$

Thus using the right trivialization of TG the Hamiltonian field generated by φ can be written in the following form:

$$(21) \quad \xi_\varphi(g) = r_+ Z - \text{Ad } g(r_- Z).$$

According to formulas (12) and (19) we also have

$$(22) \quad r_+ Z = Z_+ + Z_0 + \frac{1}{2}(r_0 + id)Z_{\mathfrak{h}}, \quad r_- Z = -Z_- + \theta Z_0 + \frac{1}{2}(r_0 - id)Z_{\mathfrak{h}},$$

where the components

$$Z_+ \in \mathfrak{k}_+, \quad Z_- \in \mathfrak{k}_- \cap \mathfrak{p}, \quad Z_0 \in V, \quad Z_{\mathfrak{h}} \in \mathfrak{h}$$

are uniquely defined by decomposition (9).

Now assume that $g \in NZs^{-1}N$, $g = ncs^{-1}n'$, $n, n' \in N$, $c \in Z$. Then from (21) and (23) we deduce that

$$(23) \quad \xi_\varphi(ncs^{-1}n') = Z_+ + Z_0 + r_+ Z_{\mathfrak{h}} - \text{Ad}(ncs^{-1}n')(-Z_- + \theta Z_0 + r_- Z_{\mathfrak{h}}).$$

On the other hand, in the right trivialization of TG the tangent space $T_{ncs^{-1}n'}NZs^{-1}N$ is identified with $\mathfrak{n} + \mathfrak{z} + \text{Ad}(ncs^{-1}n')\mathfrak{n}$.

Now using the fact that $\mathfrak{z} \subset \mathfrak{l}$, where \mathfrak{l} is the Levi factor of \mathfrak{p} , and \mathfrak{n} is an ideal in \mathfrak{p} one checks straightforwardly that the vector field (23) is tangent to $T_{ncs^{-1}n'}NZs^{-1}N$ at each point $ncs^{-1}n'$, $n, n' \in N$, $c \in Z$ if and only if conditions (20) are satisfied. \square

Note that for any $X \in \mathfrak{g}$ and any regular function φ on G the functions $\langle \nabla\varphi, X \rangle$ and $\langle \nabla'\varphi, X \rangle$ are regular. Therefore the space of regular functions on G is closed with respect to Poisson bracket (15). Now assume that conditions (20) and the conditions of Proposition 2 are satisfied for the unipotent radical N of a parabolic subgroup P of G , for some r -matrix r and some representative s of Weyl group element s . Since by Proposition 2 the projection $NZs^{-1}N \rightarrow N_s Z s^{-1}$ induced by the map $G_* \rightarrow G_*/N$ is a morphism of varieties, Proposition 11 implies that the algebra of regular functions on $N_s Z s^{-1}$ is closed under the reduced Poisson bracket defined on $N_s Z s^{-1}$ in the previous theorem. We call this Poisson algebra

the deformed Poisson W -algebra associated to the triple (r, s, \mathfrak{p}) and denote it by $W_{(r,s,\mathfrak{p})}(\mathfrak{g})$.

Now to each Weyl group element we associate a natural transversal slice which can be equipped with a Poisson structure. First, following Proposition 3, we construct a parabolic subalgebra \mathfrak{p} of \mathfrak{g} associated to element $s \in W$ in such a way that the semisimple part of the Levi factor \mathfrak{l} of \mathfrak{p} is contained in the centralizer of s in \mathfrak{g} .

Assume as in the end of Section 1 that s is primitive in the Weyl group W' of a pair $(\mathfrak{g}', \mathfrak{h}')$, where \mathfrak{g}' is a regular subalgebra of \mathfrak{g} and $\mathfrak{h}' \subset \mathfrak{h}$ is the Cartan subalgebra of \mathfrak{g}' . In our construction we shall use some generic elements $h \in \mathfrak{h}'$ that we are going to define now. Let Δ^\perp be the subset of the root system Δ which consists of the roots orthogonal to \mathfrak{h}' ,

$$\Delta^\perp = \{\alpha \in \Delta : \alpha \perp \mathfrak{h}'\}.$$

Obviously, Δ^\perp is a root subsystem of Δ . Since s is contained in the Weyl subgroup W' , and elements of W' act trivially on the orthocomplement of \mathfrak{h}' in \mathfrak{h} , Δ^\perp consists of the roots which are fixed by the action of s . Denote by $\tilde{\Delta}$ the complementary subset of Δ^\perp in Δ , $\tilde{\Delta} = \Delta \setminus \Delta^\perp$ and by $\tilde{\Delta}'$ the orthogonal projection of $\tilde{\Delta}$ onto \mathfrak{h}' . The set $\tilde{\Delta}'$ is a finite subset of the real form $\mathfrak{h}'_{\mathbb{R}}$ of \mathfrak{h}' , the real span of simple coroots in \mathfrak{h}' . Let Π be the union of hyperplanes in $\mathfrak{h}'_{\mathbb{R}}$ which are orthogonal to the elements of $\tilde{\Delta}'$,

$$\Pi = \cup_{\alpha \in \tilde{\Delta}'} V_\alpha, V_\alpha = \{x \in \mathfrak{h}'_{\mathbb{R}} : \alpha(x) = 0\}.$$

Let h be an arbitrary element of $\mathfrak{h}'_{\mathbb{R}}$ which belongs to the complement of Π in $\mathfrak{h}'_{\mathbb{R}}$, $h \in \mathfrak{h}'_{\mathbb{R}} \setminus \Pi$. Denote by \mathfrak{p} the parabolic subalgebra of \mathfrak{g} associated to h , $\mathfrak{p} = \bigoplus_{n \geq 0} \mathfrak{g}_n$, where $\mathfrak{g}_n = \{x \in \mathfrak{g} \mid [h, x] = nx\}$. By the choice of h the semisimple part \mathfrak{m} of the Levi factor \mathfrak{l} of \mathfrak{p} is the semisimple subalgebra of \mathfrak{g} with the root system Δ^\perp . Therefore \mathfrak{m} is fixed by the action of $\text{Ad}s$, where we denote a representative of the Weyl group element s in G by the same letter. Thus \mathfrak{m} is contained in the set of fixpoints of the operator $\text{Ad}s$. In fact in this case $\mathfrak{z} = \mathfrak{m} \oplus \mathfrak{h}_z$ and $\mathfrak{z} \cap \mathfrak{h} = \mathfrak{h}'^\perp$, where \mathfrak{h}_z is a Lie subalgebra in the center of \mathfrak{l} and \mathfrak{h}'^\perp is the orthogonal complement of \mathfrak{h}' in \mathfrak{h} with respect to the Killing form.

Moreover, if we denote by N and Z the subgroups of G corresponding to the Lie subalgebras \mathfrak{n} and \mathfrak{z} and by N_s the subgroup of N defined by $N_s = \{n \in N \mid sns^{-1} \in \overline{N}\}$, where \overline{N} is the opposite unipotent subgroup of G then, according to Proposition 3, the corresponding variety $N_s Z s^{-1} \subset G$ is a transversal slice to the set of conjugacy classes in G , and the conjugation map

$$(24) \quad N \times N_s Z s^{-1} \rightarrow N Z s^{-1} N$$

is an isomorphism of varieties.

Note that according to the definition of N_s the dimension of the slice $N_s Z s^{-1}$ is equal to $l(s) + F + \dim \mathfrak{h}'^\perp$, where $l(s) = \dim N_s$ is the length of s in W with respect to the system of simple positive roots associated to the Borel subalgebra contained in \mathfrak{p} , i.e. the number of the corresponding simple reflections entering a reduced decomposition of s in W , and F is the number of fixed points of s in the root system Δ .

Now using theorem 13 we equip the variety $N_s Z s^{-1} \subset G$ with a Poisson structure. Let \mathfrak{b} be the Borel subalgebra of \mathfrak{g} contained in \mathfrak{p} and \mathfrak{k} the nilradical of \mathfrak{b} . Denote by $\overline{\mathfrak{b}}$ and $\overline{\mathfrak{k}}$ the opposite Borel and nilpotent subalgebras of \mathfrak{g} . One can

verify straightforwardly that the r matrix for which the conditions of Theorem 13 are satisfied with s and N fixed above is

$$(25) \quad r = P_{\mathfrak{k}} - P_{\bar{\mathfrak{k}}} + \frac{1+s}{1-s} P_{\mathfrak{h}'},$$

where $P_{\mathfrak{k}}, P_{\bar{\mathfrak{k}}}$ and $P_{\mathfrak{h}'}$ are the orthogonal projection operators, with respect to the Killing form, onto $\mathfrak{k}, \bar{\mathfrak{k}}$ and \mathfrak{h}' , respectively. In case of r -matrix (25) the subalgebras \mathfrak{p}_{\pm} and $\bar{\mathfrak{k}}_{\pm}$ introduced in Proposition 6 are $\mathfrak{p}_{+} = \mathfrak{b}$, $\mathfrak{p}_{-} = \bar{\mathfrak{b}}$, $\bar{\mathfrak{k}}_{+} = \mathfrak{k}$, $\bar{\mathfrak{k}}_{-} = \bar{\mathfrak{k}}$, the subsets $\Gamma_{1,2} \subset \Gamma$ are trivial, so that $\mathfrak{m}_{\pm} = \mathfrak{h}$, and $r_0 = \frac{1+s}{1-s} P_{\mathfrak{h}'}$.

Note that in case of r -matrix (25) the only nontrivial condition arising in Theorem 13 is

$$\text{Ad}s^{-1}r_{-}|_{\mathfrak{h}} - r_{+}|_{\mathfrak{h}} = 0 \pmod{\mathfrak{n} + \mathfrak{z}}.$$

This condition fixes r_0 completely.

Now Theorem 13 implies that the the slice $N_s Z s^{-1}$ inherits a Poisson structure from the Poisson manifold G_* associated to r -matrix (25). We denote by $W_s(\mathfrak{g})$ the corresponding deformed Poisson W -algebra $W_{(r,s,p)}(\mathfrak{g})$. We also call $W_s(\mathfrak{g})$ the W -algebra associated to the Weyl group element s .

4. THE ALGEBRA $W_s(\mathfrak{g})$ IN CASE OF SUBREGULAR s AND SIMPLE SINGULARITIES

Let G be a simple complex algebraic group. Recall that simple singularities correspond to some irregular points of the fibers of the conjugation quotient map $\delta_G : G \rightarrow T/W$ generated by the inclusion $\mathbb{C}[T]^W \simeq \mathbb{C}[G]^G \hookrightarrow \mathbb{C}[G]$, where T is a maximal torus of G and W is the Weyl group of the pair (G, T) . More precisely, if one restricts the conjugation quotient map to a transversal slice S to the set of conjugacy classes in G of dimension $\text{rank } G + 2$, $\delta_G|_S : S \rightarrow T/W$, then any fiber $\delta_G|_S^{-1}(t), t \in T/W$ may only have isolated singularities which are rational double points. In [20] we introduced suitable transversal slices of dimension $\text{rank } G + 2$ in G and constructed simple singularities in terms of these slices. These slices are of the type considered in Proposition 1. In this section using Theorem 13 we equip these slices with Poisson structures. We start with the geometric construction of the slices of dimension $\text{rank } G + 2$ suggested in [20].

Proposition 14. ([20], **Proposition 4.3**) *Let G be a complex simple algebraic group with Lie algebra \mathfrak{g} , \mathfrak{b} a Borel subalgebra of \mathfrak{g} containing Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$, T is the maximal torus of the Borel subgroup $B \subset G$ corresponding to the Borel subalgebra \mathfrak{b} , N is the unipotent radical of B , \bar{B} is the opposite Borel subgroup in G . Let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$. Denote by $\Gamma = \{\alpha_1, \dots, \alpha_r\}$, $r = \text{rank } \mathfrak{g}$ the corresponding system of simple positive roots of \mathfrak{g} and by Δ the root system of \mathfrak{g} . For each root $\alpha \in \Delta$ we denote denote by $s_{\alpha} \in W$ the corresponding reflection. Let s the element of the Weyl group of \mathfrak{g} defined as follows (below we use the convention of [7] for the numbering of simple roots)*

- for \mathfrak{g} of type A_{n-1} , $\mathfrak{g} = \mathfrak{sl}_n$,

$$s = s_{\alpha_1} \dots s_{\alpha_{l-1}} s_{\alpha_l + \alpha_{l+1}} s_{\alpha_{l+2}} \dots s_{\alpha_{n-1}}, \quad (l = 1, \dots, n-2),$$

- for \mathfrak{g} of type B_n , $\mathfrak{g} = \mathfrak{so}_{2n+1}$,

$$s = s_{\alpha_1} \dots s_{\alpha_{n-2}} s_{\alpha_{n-1} + \alpha_n} s_{\alpha_n},$$

- for \mathfrak{g} of type C_n , $\mathfrak{g} = \mathfrak{sp}_{2n}$,

$$s = s_{\alpha_1} \dots s_{\alpha_{n-2}} s_{2\alpha_{n-1} + \alpha_n} s_{\alpha_n},$$

- for \mathfrak{g} of type D_n , $\mathfrak{g} = \mathfrak{so}_{2n}$,

$$s = s_{\alpha_1} \cdots s_{\alpha_{n-4}} s_{\alpha_{n-3} + \alpha_{n-2}} s_{\alpha_{n-2} + \alpha_{n-1}} s_{\alpha_{n-1}} s_{\alpha_n},$$

- for \mathfrak{g} of type E_6 ,

$$s = s_{\alpha_1} s_{\alpha_2 + \alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_3 + \alpha_6} s_{\alpha_6},$$

- for \mathfrak{g} of type E_7 ,

$$s = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3 + \alpha_4} s_{\alpha_5} s_{\alpha_6} s_{\alpha_7} s_{\alpha_4 + \alpha_7},$$

- for \mathfrak{g} of type E_8 ,

$$s = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4 + \alpha_5} s_{\alpha_5 + \alpha_8} s_{\alpha_6} s_{\alpha_7} s_{\alpha_8},$$

- for \mathfrak{g} of type F_4 ,

$$s = s_{\alpha_1} s_{\alpha_2} s_{\alpha_2 + 2\alpha_3} s_{\alpha_3 + \alpha_4},$$

- for \mathfrak{g} of type G_2 ,

$$s = s_{3\alpha_1 + \alpha_2} s_{\alpha_2}.$$

Denote a representative for s in G by the same letter, $s \in G$.

Then s is subregular in G and the conditions of Propositions 1 and 2 are satisfied for the pair (\mathfrak{b}, s) , and hence the variety $N_s Z s^{-1}$, where $Z = \{z \in T \mid z s z^{-1} = z\}$, $N_s = \{n \in N \mid n s n^{-1} \in \overline{B}\}$, is a transversal slice to the set of conjugacy classes in G and the map $N \times N_s Z s^{-1} \rightarrow N Z s^{-1} N$ is an isomorphism of varieties. The slice $N_s Z s^{-1}$ has dimension $r + 2$, $r = \text{rank } \mathfrak{g}$.

Now we recall the construction of simple singularities using the previous proposition.

Proposition 15. ([20], Proposition 4.4) *Let \mathfrak{g} be a complex simple Lie algebra, G a complex simple algebraic group with Lie algebra \mathfrak{g} , \mathfrak{b} a Borel subalgebra of \mathfrak{g} , T is the maximal torus of the Borel subgroup $B \subset G$ corresponding to the Borel subalgebra \mathfrak{b} , N is the unipotent radical of B , \overline{B} is the opposite Borel subgroup in G , and W the Weyl group of the pair (G, T) . Let s be the element of the Weyl group of \mathfrak{g} defined in Proposition 14. Denote a representative of s in G by the same letter. Let $Z = \{z \in T \mid z s z^{-1} = z\}$ and $N_s = \{n \in N \mid n s n^{-1} \in \overline{B}\}$.*

Then the fibers of the restriction of the adjoint quotient map to the transversal slice $N_s Z s^{-1}$ to the conjugacy classes in G , $\delta_G : N_s Z s^{-1} \rightarrow T/W$, are normal surfaces with isolated singularities. A point $x \in N_s Z s^{-1}$ is an isolated singularity of such a fiber iff x is subregular in G , and $N_s Z s^{-1}$ can be regarded as a deformation of this singularity.

Moreover, if $t \in T$, and $x \in N_s Z s^{-1}$ is a singular point of the fiber $\delta_G^{-1}(t)$, then x is a rational double point of type ${}_h \Delta_i$ for a suitable $i \in \{1, \dots, m\}$, where Δ_i are the components in the decomposition of the Dynkin diagram $\Delta(t)$ of the centralizer $Z_G(t)$ of t in G , $\Delta(t) = \Delta_1 \cup \dots \cup \Delta_m$. If Δ_i is of type A , D or E then ${}_h \Delta_i = \Delta_i$; otherwise ${}_h \Delta_i$ is the homogeneous diagram of type A , D or E associated to Δ_i by the rule ${}_h B_n = A_{2n-1}$, ${}_h C_n = D_{n+1}$, ${}_h F_4 = E_6$, ${}_h G_2 = D_4$.

Now we equip the slices defined in Proposition 14 with Poisson structures. Let $\mathfrak{p} = \mathfrak{b}$ be a Borel subalgebra of \mathfrak{g} containing Cartan subalgebra \mathfrak{h} and s a representative of the element of the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ defined in Proposition 14. Then the nilradical $\mathfrak{n} = \mathfrak{k}$ of \mathfrak{b} is a maximal nilpotent subalgebra of \mathfrak{g} . Denote by N the Lie subgroup of G corresponding to \mathfrak{n} , by $\overline{\mathfrak{b}}$ and $\overline{\mathfrak{k}}$ the opposite Borel and

nilpotent subalgebras of \mathfrak{g} and by \mathfrak{z}^\perp the orthogonal complement to \mathfrak{z} in \mathfrak{h} with respect to the Killing form. Then one can easily verify that the r matrix for which the conditions of Theorem 13 are satisfied with s and N fixed above is

$$(26) \quad r = P_{\mathfrak{k}} - P_{\overline{\mathfrak{k}}} + \frac{1+s}{1-s} P_{\mathfrak{z}^\perp},$$

where $P_{\mathfrak{k}}, P_{\overline{\mathfrak{k}}}$ and $P_{\mathfrak{z}^\perp}$ are the orthogonal projection operators, with respect to the Killing form, onto $\mathfrak{k}, \overline{\mathfrak{k}}$ and \mathfrak{z}^\perp , respectively. Remark also that $\mathfrak{z} = 0$ in all cases except for $\mathfrak{g} = \mathfrak{sl}_n$ when $\mathfrak{z} = \mathbb{C}(\omega_l - \omega_{l+1})$, where ω_l and ω_{l+1} are the fundamental weights dual to α_l and α_{l+1} , respectively (here we keep the notation of Proposition 14). Therefore $\mathfrak{z}^\perp = \mathfrak{h}$ in all cases except for $\mathfrak{g} = \mathfrak{sl}_n$.

Now Theorem 13 implies that the slice $N_s Z s^{-1}$ inherits a Poisson structure from the Poisson manifold G_* associated to r -matrix (26). Note that by Proposition 15 the slice $N_s Z s^{-1}$ is a deformation of a simple singularity. Therefore the corresponding algebra $W_{(r,s,p)}(\mathfrak{g}) = W_s(\mathfrak{g})$ can be regarded as a Poisson deformation of the simple singularity. Note that the regular functions which are invariant with respect to the action of G on G_* by conjugations lie in the center of the Poisson algebra of regular functions on G_* (see [18]). The restrictions of these functions to the slice $N_s Z s^{-1}$ lie in the center of the Poisson algebra $W_s(\mathfrak{g})$. Therefore the fibers of the map $\delta_G : N_s Z s^{-1} \rightarrow T/W$ also inherit Poisson structures from $N_s Z s^{-1}$. In particular, the Poisson structures on the singular fibers generate Poisson brackets on the coordinate rings of Kleinian singularities.

In general the Poisson structure on $N_s Z s^{-1}$ and the Poisson bracket on the corresponding algebra $W_s(\mathfrak{g})$ are difficult to calculate. We shall consider the simplest nontrivial example when one can do that explicitly.

The first nontrivial example of Poisson deformed W -algebras appears in case when $\mathfrak{g} = \mathfrak{sl}_3$. We are going to describe the slice $N_s Z s^{-1}$ and the algebra $W_s(\mathfrak{sl}_3)$ explicitly. We use the usual matrix realization of the Lie algebra \mathfrak{sl}_3 by complex 3×3 traceless matrices,

$$\mathfrak{sl}_3 = \{X \in Mat_3(\mathbb{C}), \text{tr } X = 0\},$$

and take \mathfrak{h} to be the subalgebra of traceless diagonal matrices, and \mathfrak{n} the subalgebra of lower triangular matrices.

If s is the longest element in the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ then one can choose a representative $s \in SL(3)$ of this element as follows:

$$(27) \quad s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The Lie group Z of the \mathfrak{h} -component \mathfrak{z} of the centralizer of s consists of the matrices of the following form

$$(28) \quad Z = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-2} & 0 \\ 0 & 0 & t \end{pmatrix}, t \in \mathbb{C}^* \right\},$$

and the Lie group N_s coincides with N in this case, $N = N_s = \{n \in N \mid sns^{-1} \in \overline{P}\}$. The transversal slice $N_s Z s^{-1}$ can be described explicitly using (27) and (28),

$$(29) \quad N_s Z s^{-1} = \left\{ S = \begin{pmatrix} 0 & 0 & t \\ 0 & -t^{-2} & \alpha \\ t & \beta & \gamma \end{pmatrix}, t \in \mathbb{C}^*, \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

The algebra of regular functions on the slice $N_s Z s^{-1}$ is generated by four functions $\varphi_\alpha, \varphi_\beta, \varphi_\gamma, \varphi_t$,

$$\varphi_\alpha(S) = \alpha, \varphi_\beta(S) = \beta, \varphi_\gamma(S) = \gamma, \varphi_t(S) = t, S \in N_s Z s^{-1}.$$

In order to describe the Poisson algebra $W_s(\mathfrak{sl}_3)$ we have to calculate the Poisson brackets of these functions. According to Proposition 11 we have to find N -invariant extensions of $\varphi_\alpha, \varphi_\beta, \varphi_\gamma, \varphi_t$ and then calculate their Poisson brackets in the Poisson algebra of regular functions on G_* .

Introduce functions $\bar{\varphi}_\alpha, \bar{\varphi}_\beta, \bar{\varphi}_\gamma, \bar{\varphi}_t$ on G_* as follows

$$(30) \quad \begin{aligned} \bar{\varphi}_\alpha(g) &= g_{13}^2(g_{11}g_{23} - g_{13}g_{21}) + g_{23}, \\ \bar{\varphi}_\beta(g) &= g_{32} - g_{12}g_{13}^{-3}(1 + g_{13}^2g_{33}), \\ \bar{\varphi}_\gamma(g) &= g_{11} + g_{22} + g_{33} + g_{13}^{-2}, \\ \bar{\varphi}_t(g) &= g_{13}, \end{aligned}$$

where

$$(31) \quad g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \in SL(3).$$

One can check straightforwardly that the functions $\bar{\varphi}_\alpha, \bar{\varphi}_\beta, \bar{\varphi}_\gamma, \bar{\varphi}_t$ are invariant with respect to the action of N on G_* by conjugations and that their restrictions to $N_s Z s^{-1}$ coincide with $\varphi_\alpha, \varphi_\beta, \varphi_\gamma, \varphi_t$, respectively.

Now according to Proposition 11 the Poisson brackets of the functions $\varphi_\alpha, \varphi_\beta, \varphi_\gamma, \varphi_t$ in the Poisson algebra $W_{(r,s,p)}(\mathfrak{sl}_3)$ are equal to the restrictions to $N_s Z s^{-1}$ of the Poisson brackets of $\bar{\varphi}_\alpha, \bar{\varphi}_\beta, \bar{\varphi}_\gamma, \bar{\varphi}_t$ regarded as functions on the Poisson manifold G_* . These Poisson brackets can be calculated using Poisson brackets of functions φ_{ij} , $\varphi_{ij}(g) = g_{ij}$, where $g \in SL(3)$ is given by formula (31).

From formula (15) and from the definition (25) of the r -matrix we have

$$(32) \quad \begin{aligned} \{\varphi_{ij}, \varphi_{km}\} &= \varphi_{im}\varphi_{kj}(\varepsilon_{ik} + \varepsilon_{mj}) + 2\delta_{im} \sum_{l>i} \varphi_{kl}\varphi_{lj} - \\ &\quad - 2\delta_{jk} \sum_{l>j} \varphi_{il}\varphi_{lm} + \varphi_{ij}\varphi_{km}(\delta_{im} - \delta_{jk}), \\ \varepsilon_{ij} &= \begin{cases} 1 & \text{if } i > j \\ 0 & \text{if } i = j \\ -1 & \text{if } i < j \end{cases}. \end{aligned}$$

Now Proposition 11, formulas (32) and (30) imply that

$$(33) \quad \begin{aligned} \{\varphi_\alpha, \varphi_\beta\} &= 2(\varphi_t^2 - \varphi_t^{-4} - \varphi_t^{-2}\varphi_\gamma), \\ \{\varphi_t, \varphi_\alpha\} &= \varphi_t\varphi_\alpha, \\ \{\varphi_t, \varphi_\beta\} &= -\varphi_t\varphi_\beta, \\ \{\varphi_t, \varphi_\gamma\} &= 0, \\ \{\varphi_\gamma, f\} &= \{\varphi_t^{-2}, f\} \end{aligned}$$

for any regular function $f = f(\alpha, \beta, \gamma, t)$. Poisson brackets (33) completely determine the Poisson structure of the Poisson algebra $W_s(\mathfrak{sl}_3)$.

Now we describe the associated Poisson structures on the fibers of the conjugation quotient map $\delta_G : N_s Z s^{-1} \rightarrow T/W$, where T is the maximal torus in G with

Lie algebra \mathfrak{h} . The fibers of the adjoint quotient map are the intersections of the level surfaces of the regular class functions on G with the slice $N_s Z s^{-1}$.

Recall that the regular functions which are invariant with respect to the action of G on G_* by conjugations lie in the center of the Poisson algebra of regular functions on G_* (see [18]). The restrictions of these functions to the slice $N_s Z s^{-1}$ lie in the center of the Poisson algebra $W_s(\mathfrak{sl}_3)$.

In case of $\mathfrak{g} = \mathfrak{sl}_3$ there are two algebraically independent G -invariant functions on G_* , $\overline{\varphi}_1(g) = \text{tr}g$ and $\overline{\varphi}_2(g) = \text{tr}g^2$. The restrictions φ_1 and φ_2 of these functions to $N_s Z s^{-1}$ can be expressed in terms of $\varphi_\alpha, \varphi_\beta, \varphi_\gamma, \varphi_t$ as follows

$$(34) \quad \begin{aligned} \varphi_1 &= \varphi_\gamma - \varphi_t^{-2}, \\ \varphi_2 &= 2\varphi_t^2 + \varphi_t^{-4} + 2\varphi_\alpha\varphi_\beta + \varphi_\gamma^2. \end{aligned}$$

If we denote by Z_{c_1, c_2} the Poisson ideal in $W_s(\mathfrak{sl}_3)$ generated by the elements $\varphi_1 - c_1$ and $\varphi_2 - c_2$ then the quotient $W_s(\mathfrak{sl}_3)/Z_{c_1, c_2}$ is a Poisson algebra which can be regarded as Poisson algebra of functions on the fiber of the conjugation quotient map $\delta_G : N_s Z s^{-1} \rightarrow T/W$ defined by the equations $\varphi_1(S) = c_1$, $\varphi_2(S) = c_2$, $S \in N_s Z s^{-1}$.

In particular if $c_1 = c_2 = 3$ the corresponding fiber is singular and lies in the unipotent variety of G . If we introduce new variables $x = t^2\alpha$, $y = t^2\beta$, $z = t^2 + 1$ then the elements of the singular fiber satisfy the following equations in $N_s Z s^{-1}$

$$(35) \quad \begin{aligned} \gamma &= \frac{1}{z-1} + 3, z \neq 1, \\ z^3 + xy &= 0. \end{aligned}$$

The first equation in (35) allows to eliminate γ , and the second equation defines A_2 -type simple singularity according to the A-D-E classification.

If we introduce the functions $\varphi_x, \varphi_y, \varphi_z \in W_s(\mathfrak{sl}_3)/Z_{3,3}$ on the singular fiber of the conjugation quotient map by

$$\varphi_x(x, y, z) = x, \varphi_y(x, y, z) = y, \varphi_z(x, y, z) = z$$

then their Poisson brackets in the Poisson algebra $W_s(\mathfrak{sl}_3)/Z_{3,3}$ take the form

$$(36) \quad \begin{aligned} \{\varphi_x, \varphi_y\} &= 6(\varphi_z - 1)\varphi_z^2, \\ \{\varphi_z, \varphi_x\} &= 2(\varphi_z - 1)\varphi_x, \\ \{\varphi_z, \varphi_y\} &= -2(\varphi_z - 1)\varphi_y. \end{aligned}$$

(37)

Poisson structure (36) on the singular fiber of the conjugation quotient map is proportional to the Poisson structure on the singular fiber of the adjoint quotient map $\delta_{\mathfrak{g}} : s(e) \rightarrow \mathfrak{h}/W$ derived in [13] for the Slodowy slice $s(e)$ at a subregular element $e \in \mathfrak{sl}_3$, the coefficient proportionality being $\varphi_z - 1$.

5. THE ALGEBRA $W_s(\mathfrak{g})$ IN CASE WHEN s IS THE REFLECTION WITH RESPECT TO A LONG ROOT

In this section we explicitly describe the Poisson structure of the algebra $W_s(\mathfrak{g})$ in case when s is the reflection with respect to a long root in the root system of \mathfrak{g} , $\mathfrak{p} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, the components $\mathfrak{g}(i)$ are defined by formula (1) where e is the root vector corresponding to the long root, and r is the standard r -matrix on \mathfrak{g} (see [6]).

Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} , and let Δ be the root system of \mathfrak{g} relative to \mathfrak{h} . Let $\Gamma = \{\alpha_1, \dots, \alpha_\ell\}$ be a basis of simple roots in Δ . If \mathfrak{g} is not of type A or C, there is a unique long root in Γ linked with the lowest root on the extended Dynkin diagram of \mathfrak{g} ; we call it β . For \mathfrak{g} of type A_n and C_n we set $\beta = \alpha_n$. Choose root vectors $e, f \in \mathfrak{g}$ corresponding to roots β and $-\beta$ such that $(e, [e, f], f)$ is an \mathfrak{sl}_2 -triple and put $h = [e, f]$.

The action of the inner derivation $\text{ad } h$ gives \mathfrak{g} a short \mathbb{Z} -grading

$$(38) \quad \mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2), \quad \mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$$

with $\mathfrak{g}(1) \oplus \mathfrak{g}(2)$ and $\mathfrak{g}(-1) \oplus \mathfrak{g}(-2)$ being Heisenberg Lie algebras. One knows that $\mathfrak{g}(\pm 2)$ is spanned by e and f , respectively. Denote by \mathfrak{z}_e the centralizer of e in \mathfrak{g} . The Lie algebra \mathfrak{z}_e inherits a \mathbb{Z} -grading from \mathfrak{g} , $\mathfrak{z}_e = \mathfrak{z}_e(0) \oplus \mathfrak{z}_e(1) \oplus \mathfrak{z}_e(2)$, $\mathfrak{z}_e(i) = \mathfrak{g}(i)$ for $i = 1, 2$, and the component $\mathfrak{z}_e(0)$ is the orthogonal complement to $\mathbb{C}h$ in $\mathfrak{g}(0)$ with respect to the Killing form. In particular, $\mathfrak{z}_e(0)$ is an ideal of codimension 1 in the Levi subalgebra $\mathfrak{g}(0)$.

The graded component $\mathfrak{g}(1)$ has a basis $z_1, \dots, z_s, z_{s+1}, \dots, z_{2s}$ such that the z_i 's with $1 \leq i \leq s$ (resp. $s+1 \leq i \leq 2s$) are root vectors for \mathfrak{h} corresponding to negative (resp. positive) roots, and

$$[z_i, z_j] = [z_{i+s}, z_{j+s}] = 0, \quad [z_{i+s}, z_j] = \delta_{ij}f, \quad (1 \leq i, j \leq s).$$

Fix the nondegenerate invariant form (\cdot, \cdot) on \mathfrak{g} such that $(e, f) = 1$. This enforces $(h, h) = 2$. It is well-known the restriction of (\cdot, \cdot) to \mathfrak{h} is nondegenerate and induces a W -invariant scalar product on \mathfrak{h}^* . More precisely, for all $\lambda, \mu \in \mathfrak{h}^*$ we have $(\lambda, \mu) = (t_\lambda, t_\mu)$ where $t_\lambda, t_\mu \in \mathfrak{h}$ are such that $\lambda = (t_\lambda, \cdot)$ and $\mu = (t_\mu, \cdot)$. Put $\lambda(\alpha) = 2(\lambda, \alpha)/(\alpha, \alpha)$ for all $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta$. Since (\cdot, \cdot) is a multiple of the Killing form of \mathfrak{g} , there is a constant $c \in \mathbb{C}^\times$ such that $\beta(x) = c(h, x)$ for all $x \in \mathfrak{h}$. The equality $\beta(h) = 2 = (h, h)$ now shows that $c = 1$ and $t_\beta = h$. Hence $(\gamma, \gamma) = 2$ for all long roots $\gamma \in \Delta$.

By construction the conditions of Propositions 1 and 2 are satisfied for $\mathfrak{p} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ and $s = s_\beta$, and hence the variety $N_s Z s^{-1}$ is a transversal slice to the set of conjugacy classes in G , and the map $N \times N_s Z s^{-1} \rightarrow N Z s^{-1} N$ is an isomorphism of varieties. Note that in the considered case $\mathfrak{l} = \mathfrak{g}(0)$ and $\mathfrak{z} = \mathfrak{z}_e(0)$ since $\mathfrak{z}_e(0)$ is the orthogonal complement to $\mathbb{C}h$ in $\mathfrak{g}(0)$. Moreover from the definition of grading (38) it follows that the Lie algebra \mathfrak{n}_s coincides in this case with \mathfrak{n} , and $N_s Z = Z_e$, where Z_e is the centralizer of e in G with respect to the adjoint action.

Now let \mathfrak{b} be the Borel subalgebra of \mathfrak{g} contained in \mathfrak{p} , \mathfrak{k} the nilradical of \mathfrak{b} , and $\bar{\mathfrak{k}}$ the opposite nilpotent Lie subalgebra of \mathfrak{g} . As we mentioned in the end of Section 3 the conditions of Theorem 13 are satisfied for $r = P_{\mathfrak{k}} - P_{\bar{\mathfrak{k}}} + \frac{1+s}{1-s} P_{\mathbb{C}h}$, s and \mathfrak{p} defined above, and hence the slice $N_s Z s^{-1} = Z_e s^{-1}$ inherits a Poisson structure from the Poisson manifold G_* associated to r -matrix r . Note that since $s|_{\mathbb{C}h} = -\text{id}$ the r -matrix r actually coincides with the standard r -matrix on \mathfrak{g} , $r = P_{\mathfrak{k}} - P_{\bar{\mathfrak{k}}}$ (see [6]). Using Proposition 11 we shall explicitly calculate the Poisson structure of the corresponding algebra $W_s(\mathfrak{g})$.

To each function φ on the slice $Z_e s^{-1}$ we associate its N -invariant extension φ^* to $N Z s^{-1} N$. Since for N and s fixed above the map inverse to (2) sends every element $n z s^{-1} n'$ to $n' n z s^{-1}$, $n, n' \in N$, $z \in Z$, we have

$$(39) \quad \varphi^*(n z s^{-1} n') = \varphi(n' n z s^{-1})$$

We shall also use the function φ' on Z_e associated to φ by the formula $\varphi'(nz) = \varphi(nzs^{-1})$, $n \in N$, $z \in Z$.

Let φ, ψ be two functions on the slice $Z_e s^{-1}$. In order to calculate their Poisson bracket in the algebra $W_s(\mathfrak{g})$ we have to find, according to Proposition 11, the differentials $d\varphi^*, d\psi^*$. We shall use the left trivialization of the tangent bundle TG . In this trivialization the tangent space $T_{nzs^{-1}}Z_e s^{-1}$ to the slice $Z_e s^{-1}$ at point nzs^{-1} is identified with $\mathfrak{n} + \mathfrak{z} + \text{Ad}(sz^{-1}n^{-1})\mathfrak{n} = \mathfrak{n} + \mathfrak{z} + \bar{\mathfrak{n}}$, $T_{nzs^{-1}}G = \mathfrak{n} + \mathfrak{z} + \bar{\mathfrak{n}}$.

Let e_i be a basis of \mathfrak{n} , e_i^* the dual basis of $\bar{\mathfrak{n}}$, f_i a basis of \mathfrak{z} , and f_i^* the dual basis of \mathfrak{z} . Put $g = nzs^{-1}$. Recalling definition (39) and the fact that s centralizes \mathfrak{z} we have

$$\begin{aligned}
 d\varphi^*(g) &= \frac{d}{dt}\Big|_{t=0} \left(\sum_i \varphi^*(ge^{te_i})e_i^* + \sum_i \varphi^*(ge^{tf_i})f_i^* + \sum_i \varphi^*(ge^{te_i^*})e_i \right) = \\
 &= \frac{d}{dt}\Big|_{t=0} \left(\sum_i \varphi^*(e^{te_i}g)e_i^* + \sum_i \varphi^*(nze^{tf_i}s^{-1})f_i^* + \right. \\
 (40) \quad &\quad \left. + \sum_i \varphi^*(nze^{t\text{Ads}^{-1}e_i^*s^{-1}}e_i) \right) = \\
 &= \frac{d}{dt}\Big|_{t=0} \left(\sum_i \varphi'(e^{te_i}nz)e_i^* + \sum_i \varphi'(nze^{tf_i})f_i^* + \sum_i \varphi'(nze^{t\text{Ads}^{-1}e_i^*})e_i \right) = \\
 &= P_{\bar{\mathfrak{n}}}\nabla\varphi'(nz) + \text{Ads}\nabla'\varphi'(nz),
 \end{aligned}$$

where $P_{\bar{\mathfrak{n}}}$ is the orthogonal projector onto the subspace $\bar{\mathfrak{n}}$ in \mathfrak{g} , and $\nabla\varphi', \nabla'\varphi'$ are the left (right) gradients of φ' regarded as a function on the Lie group Z_e .

Now using formula (40), Proposition 11, formula (15) for the Poisson bracket on G_* and the relation $\nabla\varphi(g) = \text{Adg}\nabla'\varphi(g)$ we obtain that

$$\begin{aligned}
 \{\varphi, \psi\}(nzs^{-1}) &= \langle r\nabla\varphi'(nz), \nabla\psi'(nz) \rangle + \langle r\nabla'\varphi'(nz), \nabla'\psi'(nz) \rangle - \\
 (41) \quad &\quad - 2\langle r_-\nabla'\varphi'(nz), \nabla\psi'(nz) \rangle - 2\langle r_+\nabla\varphi'(nz), \nabla'\psi'(nz) \rangle - \\
 &\quad - 2\langle \nabla'\varphi'(nz), \text{Ads}\nabla\psi'(nz) \rangle + 2\langle \nabla\varphi'(nz), \text{Ads}\nabla'\psi'(nz) \rangle + \\
 &\quad + \langle \text{Ad}(nz)\nabla'\varphi'(nz), \nabla\psi'(nz) \rangle - \langle \nabla\varphi'(nz), \text{Ad}(nz)\nabla'\psi'(nz) \rangle.
 \end{aligned}$$

Identifying the slice $Z_e s^{-1}$ with the Lie group Z_e we can simply write the formula for the Poisson bracket on Z_e induced by Poisson structure (41),

$$\begin{aligned}
 \{\varphi', \psi'\}(nz) &= \langle r\nabla\varphi'(nz), \nabla\psi'(nz) \rangle + \langle r\nabla'\varphi'(nz), \nabla'\psi'(nz) \rangle - \\
 &\quad - 2\langle r_-\nabla'\varphi'(nz), \nabla\psi'(nz) \rangle - 2\langle r_+\nabla\varphi'(nz), \nabla'\psi'(nz) \rangle - \\
 &\quad - 2\langle \nabla'\varphi'(nz), \text{Ads}\nabla\psi'(nz) \rangle + 2\langle \nabla\varphi'(nz), \text{Ads}\nabla'\psi'(nz) \rangle + \\
 &\quad + \langle \text{Ad}(nz)\nabla'\varphi'(nz), \nabla\psi'(nz) \rangle - \langle \nabla\varphi'(nz), \text{Ad}(nz)\nabla'\psi'(nz) \rangle.
 \end{aligned}$$

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