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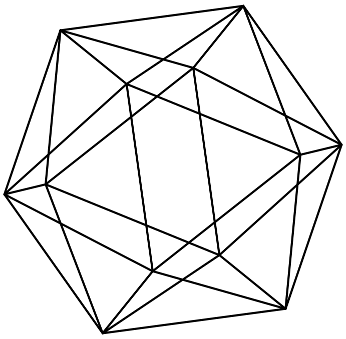
by

Wojciech Kucharz
Krzysztof Kurdyka



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Wojciech Kucharz
Krzysztof Kurdyka

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University
ul. Łojasiewicza 6
30-348 Kraków
Poland

Laboratoire de Mathématiques
UMR 5175 du CNRS
Université de Savoie
Campus Scientifique
73376 Le Bourget-du-Lac Cedex
France

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Abstract. We prove that if a linear equation, whose coefficients are continuous rational functions on a nonsingular real algebraic surface, has a continuous solution, then it also has a continuous rational solution. This is known to fail in higher dimensions.

Key words. Linear equation, continuous rational solution, real algebraic variety.

Mathematics subject classification (2010). 14P25, 14E05, 26C15, 13A15.

1 Introduction

Fefferman and Kollár [5] study the following problem. Given continuous functions $f_1, \dots, f_r$ on $\mathbb{R}^n$, which continuous functions φ can be written in the form

$$\varphi = \varphi_1 f_1 + \cdots + \varphi_r f_r,$$

where the φ_i are continuous functions on $\mathbb{R}^n$? Moreover, if φ and the f_i have some regularity properties, can one choose the φ_i to have the same (or weaker) regularity properties? In other words, the questions are about solutions of linear equations of the form

$$f_1 y_1 + \cdots + f_r y_r = \varphi.$$

The problem is hard even if φ and the f_i are polynomial functions. In [5], two different ways to solve the problem are presented: the Glaeser–Michael method and the algebraic geometry approach. Each of them consists of a rather complex procedure and it does not seem possible to give a concise answer in general.

In this note we settle the problem in a simple manner, assuming that $n = 2$ and the f_i are continuous rational functions. Actually, our results are more general and settle the corresponding problem for functions defined on any nonsingular real algebraic surface.

A complex version of the problem under consideration was studied by Brenner [3], Epstein and Hochster [4], and Kollár [9].

Convention 1.1. By a function we always mean a real-valued function.

Notation 1.2. If $f_1, \dots, f_r$ are functions defined on some set S, then

$$Z(f_1, \dots, f_r) \coloneqq \{x \in S \mid f_1(x) = 0, \dots, f_r(x) = 0\}.$$

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We now recall the *pointwise test* (or PT for short) introduced in [5, p. 235].

Definition 1.3. Let Ω be a metric space and let $f_1, \dots, f_r$ be continuous functions on Ω. We say that a continuous function φ on Ω satisfies the PT for the f_i if for every point $p \in \Omega$, the following two equivalent conditions hold:

(a) The function φ can be written as

$$\varphi = \psi_1^{(p)} f_1 + \cdots + \psi_r^{(p)} f_r,$$

where the $\psi_i^{(p)}$ are functions on Ω that are continuous at p.

(b) The function φ can be written as

$$\varphi = \varphi^{(p)} + c_1^{(p)} f_1 + \cdots + c_r^{(p)} f_r,$$

where $c_i^{(p)} \in \mathbb{R}$ and the functions $A_i^{(p)}$ defined by

$$A_i^{(p)} = \frac{\varphi^{(p)} f_i}{f_1^2 + \cdots + f_r^2} \quad \text{on } \Omega \setminus Z^{(p)} \quad \text{and} \quad A_i^{(p)} = 0 \quad \text{on } Z^{(p)},$$

with $Z^{(p)} := Z(f_1, \dots, f_r) \cup \{p\}$, are continuous at p.

Note that conditions (a) and (b) are indeed equivalent. If (a) holds, then so does (b) with

$$\varphi^{(p)} = (\psi_1^{(p)} - \psi_1^{(p)}(p)) f_1 + \cdots + (\psi_r^{(p)} - \psi_r^{(p)}(p)) f_r \quad \text{and} \quad c_i^{(p)} = A_i^{(p)}.$$

Conversely, (b) implies (a) with $\psi_i^{(p)} = c_i^{(p)} + A_i^{(p)}$.

Clearly, the PT is a basic necessary condition for existence of continuous functions $\varphi_1, \dots, \varphi_r$ on Ω satisfying $\varphi = \varphi_1 f_1 + \cdots + \varphi_r f_r$.

For background on real algebraic geometry the reader may consult [2]. By a *real algebraic variety* we mean a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of regular functions (such an object is called an affine real algebraic variety in [2]). Recall that any quasi-projective real algebraic variety is a real algebraic variety in the sense just defined, cf. [2, Prop. 3.2.10, Thm. 3.4.4]. Each real algebraic variety carries also the Euclidean topology, which is determined by the usual metric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties will refer to the Euclidean topology.

We say that a function f defined on a real algebraic variety X is *continuous rational* if it is continuous on X and regular on some Zariski open dense subset of X. We denote by $P(f)$ the smallest Zariski closed subset of X such that f is regular on $X \setminus P(f)$. The continuous rational functions form a subring of the ring of all continuous functions on X. Any regular function on X is continuous rational. The converse does not hold in general, even if X is nonsingular.

Example 1.4. The function f on $\mathbb{R}^2$, defined by

$$f(x, y) = \frac{x^3}{x^2 + y^2} \quad \text{for } (x, y) \neq (0, 0) \quad \text{and} \quad f(0, 0) = 0,$$

is continuous rational but it is not regular; in fact, $P(f) = \{(0, 0)\}$.

Recently, continuous rational functions have attracted a lot of attention, cf. [1, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. On nonsingular varieties they coincide with regulous functions introduced by Fichou, Huisman, Mangolte and Monnier [6].

Our first result, to be proved in Section 2, is the following.

Theorem 1.5. *Let X be a nonsingular real algebraic surface and let $f_1, \ldots, f_r$ be continuous rational functions on X. For a continuous function φ on X, the following conditions are equivalent:*

(a) *The function φ can be written in the form*

$$\varphi = \varphi_1 f_1 + \cdots + \varphi_r f_r,$$

where the φ_i are continuous functions on X.

(b) *The function φ satisfies the PT for the f_i.*

An example of Hochster [5, p. 236], which involves simple polynomial functions on $\mathbb{R}^3$, shows that Theorem 1.5 cannot be extended to varieties of higher dimension.

In Section 3 we prove the following.

Theorem 1.6. *Let X be a nonsingular real algebraic surface and let $f_1, \ldots, f_r$ be continuous rational functions on X. For a continuous rational function φ on X, the following conditions are equivalent:*

(a) *The function φ can be written in the form*

$$\varphi = \varphi_1 f_1 + \cdots + \varphi_r f_r,$$

where the φ_i are continuous rational functions on X.

(b) *The function φ satisfies the PT for the f_i.*

Furthermore, if (b) *holds, then the φ_i in* (a) *can be chosen so that $P(\varphi_i)$ is a finite set contained in $Z(f_1, \ldots, f_r) \cup P(f_1) \cup \ldots \cup P(f_r) \cup P(\varphi)$.*

As a straightforward consequence we get

Corollary 1.7. *Let X be a nonsingular real algebraic surface and let $f_1, \ldots, f_r$ be continuous rational functions on X. For a continuous rational function φ on X, the following conditions are equivalent:*

(a) *The function φ can be written in the form*

$$\varphi = \varphi_1 f_1 + \cdots + \varphi_r f_r,$$

where the φ_i are continuous rational functions on X.

(b) *The function φ can be written in the form*

$$\varphi = \psi_1 f_1 + \cdots + \psi_r f_r,$$

where the ψ_i are continuous functions on X.

Corollary 1.7 cannot be extended to varieties of higher dimension. A relevant example, involving polynomial functions on $\mathbb{R}^3$, is provided by Kollár and Nowak [10, Example 6]. Furthermore, the argument used in [10, Example 6] shows that Corollary 1.7 does not hold for the singular real algebraic surface $S \subset \mathbb{R}^3$ that appears there.

We conclude this section with an example.

Example 1.8. Consider the functions $f_1(x,y) = x^3$, $f_2(x,y) = y^3$, $\varphi(x,y) = x^2y^2$ on $\mathbb{R}^2$. We have

$$\varphi = \varphi_1 f_1 + \varphi_2 f_2,$$

where φ_1, φ_2 are continuous rational functions on $\mathbb{R}^2$ defined by

$$\begin{aligned} \varphi_1(x,y) &= \frac{x^5y^2}{x^6+y^6} \quad \text{for } (x,y) \neq (0,0), \quad \varphi_1(0,0) = 0, \\ \varphi_2(x,y) &= \frac{x^2y^5}{x^6+y^6} \quad \text{for } (x,y) \neq (0,0), \quad \varphi_2(0,0) = 0. \end{aligned}$$

However, φ cannot be written as a linear combination of f_1 and f_2 with coefficients that are regular (or $\mathcal{C}^\infty$) functions on $\mathbb{R}^2$, as can be seen by comparing the Taylor's expansions at $(0,0)$.

2 Continuous solutions

We begin with some preliminary results.

Lemma 2.1. *Let Ω be a metric space and let $f_1, \ldots, f_r, \varphi$ be continuous functions on Ω such that the set $Z(f_1, \ldots, f_r)$ is nowhere dense in Ω and φ satisfies the PT for the f_i. Assume that $f_i = gg_i$, where g and the g_i are continuous functions on Ω. Then there exists a unique continuous function ψ on Ω such that $\varphi = g\psi$. Furthermore, ψ satisfies the PT for the g_i.*

Proof. Note that the set $Z(g)$ is nowhere dense in Ω. To prove existence of ψ (uniqueness is then automatic) it suffices to show that for every point $p \in \Omega$ the limit

$$\lim_{x \to p} \frac{\varphi(x)}{g(x)}, \quad \text{where } x \in \Omega \setminus Z(g),$$

exists. This readily follows since φ can be written as

$$\varphi = \psi_1^{(p)} f_1 + \cdots + \psi_r^{(p)} f_r = g(\psi^{(p)} g_1 + \cdots + \psi_r^{(p)} g_r),$$

where the $\psi_i^{(p)}$ are functions on Ω that are continuous at p.

It remains to prove that ψ satisfies the PT for the g_i. We set $Z^{(p)} := Z(f_1, \ldots, f_r) \cup \{p\}$ and write φ in the form

$$\varphi = \varphi^{(p)} + c_1^{(p)} f_1 + \cdots + c_r^{(p)} f_r = \varphi^{(p)} + g(c_1^{(p)} g_1 + \cdots + c_r^{(p)} g_r),$$

where $c_i^{(p)} \in \mathbb{R}$ and the functions $A_i^{(p)}$, defined by

$$A_i^{(p)} = \frac{\varphi^{(p)} f_i}{f_1^2 + \cdots + f_r^2} \quad \text{on } \Omega \setminus Z^{(p)} \quad \text{and} \quad A_i^{(p)} = 0 \quad \text{on } Z^{(p)},$$

are continuous at p. Defining $\psi^{(p)}$ by

$$\psi = \psi^{(p)} + c_1^{(p)} g_1 + \cdots + c_r^{(p)} g_r,$$

we get $\varphi^{(p)} = g\psi^{(p)}$. Consequently,

$$\frac{\varphi^{(p)} f_i}{f_1^2 + \cdots + f_r^2} = \frac{\psi^{(p)} g_i}{g_1^2 + \cdots + g_r^2} \quad \text{on } \Omega \setminus Z(f_1, \ldots, f_r).$$

Since the set $Z(f_1, \ldots f_r)$ is nowhere dense in Ω, it follows that ψ satisfies the PT for the g_i. $\square$

Lemma 2.2. *Let X be an irreducible nonsingular real algebraic variety and let $f_1, \ldots, f_r$ be continuous rational functions on X, not all identically equal to 0. Then the Zariski closure of $Z(f_1, \ldots, f_r)$ is Zariski nowhere dense in X. In particular, $Z(f_1, \ldots, f_r)$ is Euclidean nowhere dense in X.*

Proof. Setting $f = f_1^2 + \cdots + f_r^2$, we get $Z(f) = Z(f_1, \ldots, f_r)$. The function f is continuous rational and satisfies

$$Z(f) \subset Z(f|_{X \setminus P(f)}) \cup P(f),$$

which implies both assertions. $\square$

Lemma 2.3. *Let X be an irreducible nonsingular real algebraic surface and let $f_1, \ldots, f_r$ be continuous rational functions on X. Then, for every point $p \in X$, there exists a Zariski open neighborhood $X^{(p)} \subset X$ of p and there exist regular functions $g_1, \ldots, g_r, g, h$ on $X^{(p)}$ such that $Z(h) \neq X^{(p)}$, $Z(g_1, \ldots, g_r) \subset \{p\}$, and $hf_i = gg_i$ on $X^{(p)}$ for $i = 1, \ldots, r$.*

Proof. We can find regular functions $\lambda_1, \ldots, \lambda_r, \mu$ on X such that $Z(\mu) \neq X$ and $f_i = \lambda_i / \mu$ on $X \setminus Z(\mu)$ for $i = 1, \ldots, r$. Since X is nonsingular, the local ring of X at each point $p \in X$ is a unique factorization domain. Consequently, there exists a Zariski open neighborhood $X^{(p)} \subset X$ of p and there exist regular functions $g_1, \ldots, g_r, g$ on $X^{(p)}$ such that $\lambda_i = gg_i$ on $X^{(p)}$ and $Z(g_1, \ldots, g_r) \subset \{p\}$. To complete the proof it suffices to set $h := \mu|_{X^{(p)}}$. $\square$

Lemma 2.4. *Let X be an irreducible nonsingular real algebraic surface and let $f_1, \ldots, f_r$ be continuous rational functions on X, not all identically equal to 0. Let φ be a continuous function on X that satisfies the PT for the f_i. Then, for every point $p \in X$, there exists a Zariski open neighborhood $X^{(p)} \subset X$ of p and there exist continuous functions $\alpha_1^{(p)}, \ldots, \alpha_r^{(p)}$ on $X^{(p)}$ and real numbers $c_1^{(p)}, \ldots, c_r^{(p)}$ such that*

$$\varphi = \alpha_1^{(p)} f_1 + \cdots + \alpha_r^{(p)} f_r \quad \text{on } X^{(p)}, \quad \text{and}$$

$$\alpha_i^{(p)} = c_i^{(p)} + \frac{(\varphi - (c_1^{(p)} f_1 + \cdots + c_r^{(p)} f_r)) f_i}{f_1^2 + \ldots + f_r^2} \quad \text{on } X^{(p)} \setminus Z(f_1, \ldots, f_r)$$

for $i = 1, \ldots, r$.

Proof. By Lemma 2.3, we can find a Zariski open neighborhood $X^{(p)} \subset X$ of p and regular functions $g_1, \ldots, g_r, g, h$ on $X^{(p)}$ such that

$$Z(g_1, \ldots, g_r) \subset \{p\}, \tag{1}$$

$$hf_i = gg_i \quad \text{on } X^{(p)} \quad \text{for } i = 1, \ldots, r, \tag{2}$$

and $Z(h) \neq X^{(p)}$. Since φ satisfies the PT for the f_i, it follows that $h\varphi|_{X^{(p)}}$ satisfies the PT for the $hf_i|_{X^{(p)}} = gg_i$. According to Lemma 2.2, the set

$$Z(hf_1|_{X^{(p)}}, \ldots, hf_r|_{X^{(p)}}) = Z(gg_1, \ldots, gg_r)$$

is nowhere dense in $X^{(p)}$. Hence, in view of Lemma 2.1, there exists a unique continuous function ψ on $X^{(p)}$ such that

$$h\varphi|_{X^{(p)}} = g\psi. \tag{3}$$

Furthermore, ψ satisfies the PT for the g_i. Consequently, taking (1) into account, we can write ψ in the form

$$\psi = \psi^{(p)} + c_1^{(p)} g_1 + \cdots + c_r^{(p)} g_r, \tag{4}$$

where $c_i^{(p)} \in \mathbb{R}$ and the functions $B_i^{(p)}$ on $X^{(p)}$, defined by

$$B_i^{(p)} = \frac{\psi^{(p)} g_i}{g_1^2 + \cdots + g_r^2} \quad \text{on } X^{(p)} \setminus \{p\} \quad \text{and} \quad B_i^{(p)}(p) = 0, \tag{5}$$

are continuous at p. It follows that the $B_i^{(p)}$ are continuous on $X^{(p)}$.

Defining $\varphi^{(p)}$ by

$$\varphi = \varphi^{(p)} + c_1^{(p)} f_1 + \cdots + c_r^{(p)} f_r \tag{6}$$

and making use of (2)–(6), we get

$$B_i^{(p)} = \frac{\varphi^{(p)} f_i}{f_1^2 + \cdots + f_r^2} \quad \text{on } X^{(p)} \setminus (\{p\} \cup Z(g)).$$

By continuity,

$$B_i^{(p)} = \frac{\varphi^{(p)} f_i}{f_1^2 + \cdots + f_r^2} \quad \text{on } X^{(p)} \setminus Z(f_1, \ldots, f_r). \tag{7}$$

The functions $\alpha_i^{(p)} \coloneqq c_i^{(p)} + B_i^{(p)}$ are continuous on $X^{(p)}$ and in view of (6), (7) they satisfy

$$\varphi = \alpha_1^{(p)} f_1 + \cdots + \alpha_r^{(p)} f_r \quad \text{on } X^{(p)} \setminus Z(f_1, \ldots, f_r).$$

By continuity, the last equality holds on $X^{(p)}$. The proof is complete. □

Proof of Theorem 1.5. By Lemma 2.4, a partition of unity argument completes the proof. □

Lemma 2.4 contains more information than we needed for the proof of Theorem 1.5. However, the full statement will be used to prove Theorem 1.6 in Section 3.

3 Continuous rational solutions

We will frequently use, not necessarily explicitly referring to it, the following fact: If X is a nonsingular real algebraic variety, $X^0 \subset X$ a Zariski open subset, and $U \subset X$ a Euclidean open subset, then $X^0 \cap U$ is Euclidean dense in U.

Lemma 3.1. *Let X be a nonsingular real algebraic variety, $\psi\colon X \to \mathbb{R}$ a regular function, and $f\colon X \setminus Z(\psi) \to \mathbb{R}$ a continuous rational function. Then there exists an integer $N_0 > 0$ such that for every integer $N \geq N_0$, the function $\psi^N f$, extended by 0 on $Z(\psi)$, is continuous rational on X.*

Proof. According to a variant of the Łojasiewicz inequality [2, Prop. 2.6.4], it suffices to prove that f is a semialgebraic function. This is straightforward since the graph of the function f restricted to $(X \setminus Z(\psi)) \setminus P(f)$ is a semialgebraic subset of $(X \setminus Z(\psi)) \times \mathbb{R}$, whose closure is equal to the graph of f. □

Lemma 3.2. *Let X be a nonsingular real algebraic variety and let $\{X^1, \ldots, X^m\}$ be a Zariski open cover of X. Let $f_1, \ldots, f_r, \varphi$ be continuous rational functions on X such that for $j = 1, \ldots, m$ the restriction $\varphi|_{X^j}$ can be written in the form*

$$\varphi|_{X^j} = \sum_{i=1}^{r} \varphi_{ij} f_i|_{X^j},$$

where the φ_{ij} are continuous rational functions on X^j. Then φ can be written in the form

$$\varphi = \sum_{i=1}^{r} \varphi_i f_i,$$

where the φ_i are continuous rational functions on X with

$$P(\varphi_i) \subset \bigcup_{j=1}^{m} (P(\varphi_{ij}) \cup (X \setminus X^j)).$$

Proof. We may assume that X is irreducible and the X^j are all nonempty. Then each X^j is Euclidean dense in X. We choose a regular function ψ_j on X with $Z(\psi_j) = X \setminus X^j$. By Lemma 3.1, there exists a positive integer N such that the φ_{ij} can be written as

$$\varphi_{ij} = \frac{a_{ij}}{\psi_j^N} \quad \text{on } X^j, \tag{1}$$

where the a_{ij} are continuous rational functions on X. It follows that

$$\psi_j^N \varphi = \sum_{i=1}^{r} a_{ij} f_i \tag{2}$$

on X^j. By continuity, (2) holds on X. Multiplying both sides of (2) by ψ_j^N and summing over j, we get

$$b\varphi = \sum_{i=1}^{r} b_i f_i, \tag{3}$$

where

$$b = \sum_{j=1}^{m} \psi_j^{2N} \quad \text{and} \quad b_i = \sum_{j=1}^{m} a_{ij} \psi_j^N.$$

The function b is regular with $Z(b) = \varnothing$, which implies that $\varphi_i := b_i/b$ is a continuous rational function on X. In view of (3) we have

$$\varphi = \sum_{i=1}^{r} \varphi_i f_i.$$

By construction,

$$P(\varphi_i) \subset \bigcup_{j=1}^{m} P(a_{ij}),$$

while (1) implies

$$P(a_{ij}) \subset P(\varphi_{ij}) \cup (X \setminus X^j).$$

The proof is complete. $\square$

We will make use of rational maps and rational functions understood in the standard way. A *rational map* $F\colon X \dashrightarrow Y$, between real algebraic varieties X and Y, is the equivalence class of regular maps with values in Y, defined on Zariski open dense subsets of X; two such regular maps $f_1\colon X^1 \to Y$ and $f_2\colon X^2 \to Y$ are declared to be equivalent if $f_1|_{X^0} = f_2|_{X^0}$ for some Zariski open dense subset $X^0 \subset X^1 \cap X^2$. We denote by $\operatorname{dom}(F)$ the union of all the domains of regular maps representing F. Thus F determines a regular map $F\colon \operatorname{dom}(F) \to Y$. The polar set $\operatorname{pole}(F) := X \setminus \operatorname{dom}(F)$ is Zariski nowhere dense in X. If $Y = \mathbb{R}$, then F is called a *rational function* on X. The rational functions on X form a ring (a field, if X is irreducible), denoted $\mathbb{R}(X)$.

Definition 3.3. A rational function R on a real algebraic variety X is said to be *locally bounded* if for every point $p \in X$, one can find a Zariski open dense subset $X_p \subset X$, a Euclidean open neighborhood $U_p \subset X$ of p, and a real number $M_p > 0$ such that

$$|R(x)| \leq M_p \quad \text{for all } x \in U_p \cap \operatorname{dom}(R) \cap X_p.$$

It readily follows that the set of all locally bounded rational functions on X forms a subring of $\mathbb{R}(X)$.

Actually, Definition 3.3 would not be affected if we substituted for each X_p the set X^{ns} of all nonsingular points of X. Definition 3.3 imposes no restriction if the point p is not in the Euclidean closure of X^{ns}.

Example 3.4. Consider the Whitney umbrella $W := (X^2 = y^2 z) \subset \mathbb{R}^3$. The set of singular points of W is the z-axis. The rational function $1/(z+1)$ is locally bounded on W, but it is not locally bounded on the z-axis.

We will consider locally bounded rational functions only on nonsingular varieties. A typical example is the following.

Example 3.5. The rational function $xy/(x^2+y^2)$ on $\mathbb{R}^2$ is locally bounded (even bounded), but it cannot be extended to a continuous function on $\mathbb{R}^2$.

Lemma 3.6. *Let X be a nonsingular real algebraic variety and let R be a locally bounded rational function on X. Then the polar set* $\operatorname{pole}(R)$ *is of codimension at least* 2.

Proof. Using the inclusion $\mathbb{R} \subset \mathbb{P}^1(\mathbb{R})$, we obtain a rational map $R^*\colon X \dashrightarrow \mathbb{P}^1(\mathbb{R})$ determined by R. The polar set $\operatorname{pole}(R^*)$ is of codimension at least 2 [8, p. 129, Thm. 2.17]. Since R is locally bounded, we have $\operatorname{pole}(R) = \operatorname{pole}(R^*)$, which completes the proof. $\square$

Remark 3.7. Let X be a nonsingular real algebraic variety. Any continuous rational function f on X determines a rational function $\tilde{f}$ on X, which is represented by the regular function $f|_{X \setminus P(f)}$. Clearly, $P(f) = \operatorname{pole}(\tilde{f})$. Furthermore, if g is a continuous rational function on X, not identically equal to 0 on any irreducible component of X, then the quotient $\tilde{f}/\tilde{g}$ is a well defined rational function on X (see Lemma 2.2). To simplify notation, we will prefer to say "the rational function f" or "the rational function f/g" instead of writing $\tilde{f}$ or $\tilde{f}/\tilde{g}$, respectively. For the rational function f/g, we have

$$\operatorname{pole}(f/g) \subset P(f) \cup P(g) \cup Z(g).$$

Lemma 3.8. *Let X be an irreducible nonsingular real algebraic variety and let $\varphi, f_1, \ldots, f_r$ be continuous rational functions on X, where the f_i are not all identically equal to 0. For $i = 1, \ldots, r$ and $\mathbf{c} = (c_1, \ldots, c_r) \in \mathbb{R}^r$, let*

$$R_{\mathbf{c}i} := \frac{(\varphi - (c_1 f_1 + \cdots + c_r f_r)) f_i}{f_1^2 + \cdots + f_r^2}.$$

If φ satisfies the PT for $f_1, \ldots, f_r$, then each rational function $R_{\mathbf{c}i}$ is locally bounded on X.

Proof. Let $Z := Z(f_1, \ldots, f_r)$ and let $S \subset X$ be an arbitrary subset. The $R_{\mathbf{c}i}$ are well defined functions on $X \setminus Z$. Setting $R_i = R_{\mathbf{0}i}$, where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^r$, we get

$$R_{\mathbf{c}i} = R_i - \frac{c_1 f_i f_1 + \cdots + c_r f_i f_r}{f_1^2 + \cdots + f_r^2}.$$

Since

$$\left| \frac{f_i f_j}{f_1^2 + \cdots + f_r^2} \right| \leq \frac{1}{2} \quad \text{on } X \setminus Z,$$

it follows that $R_{\mathbf{c}i}$ is bounded on $S \cap (X \setminus Z)$ if and only if R_i is such.

Suppose that φ satisfies the PT for $f_1, \ldots, f_r$, fix a point $p \in X$, and set $Z^{(p)} = Z \cup \{p\}$. The function φ can be written in the form

$$\varphi = \varphi^{(p)} + c_1^{(p)} f_1 + \cdots + c_r^{(p)} f_r,$$

where $c_i^{(p)} \in \mathbb{R}$ and the functions $A_i^{(p)}$ on X, defined by

$$A_i^{(p)} = \frac{\varphi^{(p)} f_i}{f_1^2 + \cdots + f_r^2} \quad \text{on } X \setminus Z^{(p)} \quad \text{and} \quad A_i^{(p)} = 0 \quad \text{on } Z^{(p)},$$

are continuous at p. In particular, the $A_i^{(p)}$ are bounded on some Euclidean open neighborhood $U_p \subset X$ of p. Consequently, the functions $R_{\mathbf{c}^{(p)}i}$, where $\mathbf{c}^{(p)} = (c_1^{(p)}, \ldots, c_r^{(p)})$, are bounded on $U_p \cap (X \setminus Z)$, which in turn implies that the R_i are bounded on $U_p \cap (X \setminus Z)$. This conclusion remains valid if Z is replaced by its Zariski closure V in X. The proof is complete since the subset $X \setminus V \subset X$ is Zariski open dense by Lemma 2.2. □

Proof of Theorem 1.6. It suffices to prove that (b) implies (a) together with the extra conditions stipulated on the φ_i. Let us suppose that (b) holds. We may assume that X is irreducible and the f_i are not all identically equal to 0. According to Lemma 2.4, for each point $p \in X$, we can find a Zariski open neighborhood $X^{(p)} \subset X$ of p and continuous functions $\alpha_1^{(p)}, \ldots, \alpha_r^{(p)}$ on $X^{(p)}$ such that

$$\varphi = \alpha_1^{(p)} f_1 + \cdots + \alpha_r^{(p)} \quad \text{on } X^{(p)}$$

and

$$\alpha_i^{(p)} = c_i^{(p)} + R_i^{(p)} \quad \text{on } X^{(p)} \setminus Z,$$

where $Z = Z(f_1, \ldots, f_r)$, $c_i^{(p)} \in \mathbb{R}$, and

$$R_i^{(p)} = \frac{(\varphi - (c_1^{(p)} f_1 + \cdots + c_r^{(p)} f_r)) f_i}{f_1^2 + \cdots + f_r^2} \quad \text{on } X^{(p)} \setminus Z.$$

We regard the $R_i^{(p)}$ as rational functions on X and set

$$X_0^{(p)} := \operatorname{dom}(R_1^{(p)}) \cap \ldots \cap \operatorname{dom}(R_r^{(p)}).$$

Clearly,

$$X \setminus X_0^{(p)} \subset Z.$$

Furthermore, according to Lemmas 3.6 and 3.8, $X \setminus X_0^{(p)}$ is a finite set. Since the set $X^{(p)} \setminus Z$ is Euclidean dense in X (see Lemma 2.2), it follows that

$$\alpha_i^{(p)} = R_i^{(p)} \quad \text{on } X^{(p)} \cap X_0^{(p)}.$$

Consequently, we obtain a well defined continuous rational function $\beta_i^{(p)}$ on $X_1^{(p)} := X^{(p)} \cup X_0^{(p)}$ by setting

$$\beta_i^{(p)} = \alpha_i^{(p)} \quad \text{on } X^{(p)} \quad \text{and} \quad \beta_i^{(p)} = c_i^{(p)} + R_i^{(p)} \quad \text{on } X_0^{(p)}.$$

By construction,

$$\varphi = \beta_1^{(p)} f_1 + \cdots + \beta_r^{(p)} f_r \quad \text{on } X_1^{(p)}.$$

Now it is easy to complete the proof. We choose a finite collection of points $p_1, \ldots, p_m$ in X so that the sets $X^j := X_1^{(p_j)}$ form a cover of X. Setting $\varphi_{ij} := \beta_i^{(p_j)}$, we get

$$\varphi|_{X^j} = \sum_{i=1}^{r} \varphi_{ij}|_{X^j}, \quad P(\varphi_{ij}) \subset (P(\varphi) \cup Z \cup P(f_1) \cup \ldots \cup P(f_r)) \cap (X \setminus X_0^{(p_j)}).$$

By Lemma 3.2, there exist continuous rational functions $\varphi_1, \ldots, \varphi_r$ on X such that

$$\varphi = \varphi_1 f_1 + \cdots + \varphi_r f_r$$

and

$$P(\varphi_i) \subset \bigcup_{j=1}^{r} (P(\varphi_{ij}) \cup (X \setminus X^j)).$$

The functions φ_i satisfy all the requirements. □

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Wojciech Kucharz
Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University
Łojasiewicza 6
30-348 Kraków
Poland
E-mail address: `Wojciech.Kucharz@im.uj.edu.pl`

Krzysztof Kurdyka
Laboratoire de Mathématiques
UMR 5175 du CNRS
Université de Savoie
Campus Scientifique
73 376 Le Bourget-du-Lac Cedex
France
E-mail address: `kurdyka@univ-savoie.fr`