Orthogonal curvilinear coordinate systems corresponding to singular spectral curves *

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1 Introduction

In this paper we study the limiting case of the Krichever construction of orthogonal curvilinear coordinate systems when the spectral curve becomes singular.

Theory of orthogonal curvilinear coordinate systems was very popular among differential geometers in the 19th century and in the first half of the 20th century (Dupin, Gauss, Lame, Bianchi, Darboux) and the classification problem was basically solved at the beginning of the 20th century (see the book by Darboux [2] which summarizes this stage of the development of this theory). These coordinate systems are interesting due to search of systems solved by the separation of variables method and to modern problems of theory of hydrodynamical type systems and topological field theory (Dubrovin, Novikov, Tsarev, Krichever, see references in [3, 8]).

In the problem of explicit constructing such systems a breakthrough was achieved by Zakharov [8] who by using the dressing method first applied the methods of integrable systems to this problem. Onto the finite-gap integration method this approach was extended by Krichever [3]. Therewith the initial data for a construction of such a system consist of a Riemann surface, i.e., the spectral curve, which in [3] is assumed to be nonsingular and some other additional quantities related to it. We briefly recall the constructions by Zakharov and Krichever in §2.

In the case when the spectral curve becomes singular and reducible such that all its irreducible components are smooth rational complex curves the procedure of constructing orthogonal curvilinear coordinate systems is crucially simplified and reduces to simple computations with elementary functions (see §3). Therewith we show how such well-known coordinate systems as the polar coordinates on the plane, the cylindrical coordinates in the three-space, and the spherical

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coordinates in $\mathbb{R}^n$ with $n \geq 3$ fit in this scheme (we expose these constructions together with constructions of some other coordinate systems in §4).

We remark that the inverse problems with such spectral curves were already studied in relations to their applications (see, for instance, the paper [6] on surface theory and the paper [7] on the Hitchin system). Although this case is very special this paper also shows that explicit solutions corresponding to it are important for applications.

2 Methods of constructing orthogonal curvilinear coordinates

2.1 Preliminary facts

A curvilinear coordinate system $u = (u^1, \dots, u^n)$ in the Euclidean n-space $\mathbb{R}^n$ is called n-orthogonal if the metric in these coordinates takes the form

$$ds^2 = H_1^2(du^1)^2 + \dots + H_n^2(du^n)^2.$$

Therewith the functions $H_j = H_j(u)$ are called the Lame coefficients and the condition that the curvature tensor vanishes takes the form

$$\frac{\partial^2 H_i}{\partial u^j \partial u^k} = \frac{1}{H_j}\frac{\partial H_j}{\partial u^k}\frac{\partial H_i}{\partial u^j} + \frac{1}{H_k}\frac{\partial H_k}{\partial u^j}\frac{\partial H_i}{\partial u^k}, \tag{1}$$

$$\frac{\partial}{\partial u^j}\left(\frac{1}{H_j}\frac{\partial H_i}{\partial u^j}\right) + \frac{\partial}{\partial u^i}\left(\frac{1}{H_i}\frac{\partial H_j}{\partial u^i}\right) + \sum_{k \neq i \neq j} \frac{1}{H_k^2}\frac{\partial H_i}{\partial u^k}\frac{\partial H_j}{\partial u^k} = 0. \tag{2}$$

There are $\frac{n(n-1)(n-2)}{2}$ and $\frac{n(n-1)}{2}$ equations in the systems (1) and (2) respectfully, the equations (1) are equivalent to the condition that $R_{ijik} = 0, j \neq k$, and the equations (2) are equivalent to $R_{ijij} = 0$. Other components of the curvature tensor R_{ijkl} always vanish for a diagonal metric. Hence the system of equations (1)–(2) for the Lame coefficients is strongly overdetermined. A general solution to this system is parameterized by $\frac{n(n-1)}{2}$ functions of two variables.

The order of the system (1)–(2) is minimized by introducing the rotation coefficients

$$\beta_{ij} = \frac{\partial_{u^i} H_j}{H_i}. \tag{3}$$

Then the equations (1) and (2) take the form

$$\partial_{u^k}\beta_{ij} = \beta_{ik}\beta_{kj}, \tag{4}$$

$$\partial_{u^i}\beta_{ij} + \partial_{u^j}\beta_{ij} + \sum_{k \neq i,l} \beta_{ki}\beta_{kj} = 0. \tag{5}$$

Given a solution β_{ij} to these equations, the Lame coefficients are found from (3) as a solution to the Cauchy problem

$$H_i(0, \dots, 0, u^i, 0, \dots, 0) = h_i(u^i).$$

Therewith such a solution depends on the initial data for this problem, i.e., on n functions h_i of one variable.We remark that the compatibility of (3) is equivalent to (4).

The immersion problem, i.e., the determination of the Euclidean coordinates $x = (x^1, \ldots, x^n)$ as the functions of $u = (u^1, \ldots, u^n)$, reduces to solving the overdetermined system of second order linear equations

$$\frac{\partial^2 x^k}{\partial u^i \partial u^j} = \sum_l \Gamma^l_{ij} \partial_{u^l} x^k, \tag{6}$$

and in our case the Christoffel symbols have the form

$$\Gamma^k_{ij} = 0,\ i \neq j \neq k; \quad \Gamma^k_{kj} = \frac{\partial_{u^j} H_k}{H_k}; \quad \Gamma^k_{ii} = -\frac{H_i \partial_{u^k} H_i}{(H_k)^2},\ k \neq i.$$

By (1) and (2), the system of equations (6) is compatible and determines an n-orthogonal curvilinear coordinate system up to motions of $\mathbb{R}^n$.

2.2 Zakharov's method [8]

The abstract n waves problem has the form

$$\sum_{i,j,k} \varepsilon_{ijk} (I_j \partial_{u^i} Q I_k - I_i Q I_j Q I_k) = 0, \tag{7}$$

where $Q(u)$ is the unknown $(n \times n)$-matrix function, the matrices $I_j = I_j(u^j)$ are pairwise commuting, $\varepsilon_{ijk} = 1$ for $i > j > k$ and changes the sign after an odd permutation of indices.

We take for I_j the diagonal matrices with the diagonal $(0, \ldots, 0, 1, 0, \ldots, 0)$ (i.e., the unit is on the j-th place). The equations (7) take the form

$$\partial_{u^j} Q_{ik} = Q_{ij} Q_{jk},$$

i.e., coincide with (4). Let us consider an auxiliary function $\widetilde{Q} = \tilde{Q}(u, s)$, with $s = u^{n+1}$ an additional variable and I_{n+1} the unit matrix, satisfying the equations

$$\partial_{u^j} [I_i, \widetilde{Q}] - \partial_{u^i} [I_j, \widetilde{Q}] + I_i \partial_s \widetilde{Q} I_j - I_j \partial_s \widetilde{Q} I_i - [[I_i, \widetilde{Q}], [I_j, \widetilde{Q}]] = 0, \tag{8}$$

where $i, j = 1, \ldots, n+1$. It is shown that if $\widetilde{Q}$ satisfies (8) then for a fixed value of s the matrix function $\widetilde{Q}$ is a solution to the n waves problem. The system (8) admits the Lax representation

$$[L_i, L_j] = 0, \quad L_j = \partial_{u^j} + I_j \partial_s + [I_j, \widetilde{Q}].$$

The dressing method consist in the following procedure. Let us consider the integral equation of the Marchenko type:

$$K(s, s', u) = F(s, s', u) + \int_s^\infty K(s, q, u) F(q, s', u) dq, \tag{9}$$

where $F(s,s',u)$ is a matrix function such that it satisfies the equation

$$\partial_{u^i} F + I_i \partial_s F + \partial_{s'} F I_i = 0 \tag{10}$$

and (9) has a unique solution. Then the function

$$\tilde{Q}(s,u) = K(s,s,u) \tag{11}$$

satisfies (8) and therefore for any fixed value of s the function $Q(u) = \widetilde{Q}(s,u)$ satisfies (7). Moreover if the differential reduction

$$\partial_{s'} F_{ij}(s,s',u) + \partial_s F_{ji}(s',s,u) = 0, \tag{12}$$

holds (here F_{ij} are the entries of the matrix F) then $\widetilde{Q}$ also satisfies (5).

The system consisting of (10) and (12) admits the following solution [8]:

- *Let $\Phi_{ij}(x,y), i < j$, be arbitrary $\frac{n(n-1)}{2}$ functions of two variables and $\Phi_{ii}(x,y)$ be n arbitrary skew-symmetric functions:*

  $$\Phi_{ii}(x,y) = -\Phi_{ii}(y,x).$$

  We put

  $$F_{ij} = \partial_s \Phi_{ij}(s-u^i, s'-u^j),\ i<j,$$
  $$F_{ji} = \partial_s \Phi_{ij}(s'-u^i, s-u^j),$$
  $$F_{ii} = \partial_s \Phi_{ii}(s-u^i, s'-u^i).$$

  The matrix function $F = (F_{ij})$ satisfies (10) and (12) and therefore a solution K to (9) with such a matrix F gives for any fixed value of s the rotation coefficients of an orthogonal coordinate system: $Q_{ij}(u) = K_{ij}(s,s,u)$, i.e. a solution to the system (4)–(5).

Remark 1. Since we have $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$ functional parameters, i.e., $\Phi_{ij}, i \le j$, and a general solution depends on $\frac{n(n-1)}{2}$ functional parameters, this method gives equivalence classes of dressings as it is explained in [8].

2.3 Krichever's method [3]

Let Γ be a smooth connected complex algebraic curve. We take three divisors on Γ:

$$P = P_1 + \ldots + P_n, \quad D = \gamma_1 + \ldots + \gamma_{g+l-1}, \quad R = R_1 + \ldots + R_l,$$

where g is the genus of Γ, $P_i, \gamma_j, R_k \in \Gamma$. We denote by k_i^{-1} some local parameter on Γ near P_i, $i = 1,\ldots,n$.

The Baker–Akhiezer function corresponding to the data

$$S = \{P, D, R\}$$

is a function $\psi(u^1,\dots,u^n,z),\ z\in\Gamma$, meeting the following conditions:

1) $\psi\exp(-u^ik_i)$ is analytic near P_i, $i=1,\dots,n$;

2) ψ is meromorphic on $\Gamma\backslash\{\cup P_i\}$ with poles at γ_j, $j=1,\dots,g+l-1$;

3) $\psi(u,R_k)=1$, $k=1,\dots,l$.

For a generic divisor D such a function exists and unique. Moreover it is expressed in terms of the theta function of Γ [3].

If the curve Γ is not connected it is assumed that the restriction of the Baker–Akhiezer function onto every connected component meets the conditions above.

We take an additional divisor $Q=Q_1+\ldots+Q_n$ on Γ such that $Q_i\in\Gamma\setminus\{P\cup D\cup R\}, i=1,\dots,n$, and denote by x^j the following function

$$x^j(u^1,\dots,u^n)=\psi(u^1,\dots,u^n,Q_j),\ j=1,\dots,n.$$

There is the following Krichever theorem [3]:

- *Let Γ admit a holomorphic involution $\sigma:\Gamma\to\Gamma$ such that*

  1) this involution has exactly $2m, m\le n$, fixed points which are just the points $P_1,\dots,P_n$ from P and $2m-n$ points from Q;

  2) $\sigma(Q)=Q$, i.e, non-fixed points from Q are interchanged by the involution:

  $$\sigma(Q_k)=Q_{\sigma(k)},\quad k=1,\dots,n;$$

  3) $\sigma(k_i^{-1})=-k_i^{-1}$ near P_i, $i=1,\dots,n$;

  4) there exists a meromorphic differential Ω on Γ such that its divisors of zeroes and poles are of the form

  $$(\Omega)_0=D+\sigma D+P,\quad (\Omega)_\infty=R+\sigma R+Q.$$

  It is assumed that $\Gamma_0=\Gamma/\sigma$ is a smooth algebraic curve.

  Then, as it is easy to show, Ω is a pull-back of some meromorphic differential Ω_0 on Γ_0 and the following equalities hold:

  $$\sum_{k,l}\eta_{kl}\partial_{u^i}x^k\partial_{u^j}x^l=\varepsilon_i^2h_i^2\delta_{ij}$$

  where

  $$h_i=\lim_{P\to P_i}\left(\psi e^{-u^ik_i}\right),\quad \eta_{kl}=\delta_{k,\sigma(l)}\mathrm{res}_{Q_k}\Omega_0,$$ [1]

  and

  $$\Omega_0=\frac{1}{2}\left(\varepsilon_i^2\lambda_i+O(\lambda_i)\right)d\lambda_i,\ \lambda_i=k_i^{-2},\ \text{ at } P_i,\ i=1,\dots,n.$$

[1] Given a nonsingular fixed point Q_i of σ, there is a parameter k near it such that $\sigma(k)=-k, k(Q_i)=0$. Therefore $\lambda=k^2$ is a local parameter near Q_i on Γ/σ and we have $\Omega=\left(\frac{a}{k}+\ldots\right)dk, \Omega_0=\frac{1}{2}\left(\frac{a}{\lambda}+\ldots\right)d\lambda$, which implies $\mathrm{res}_{Q_i}\Omega_0=\frac{1}{2}\mathrm{res}_{Q_i}\Omega$.

Remark 2. This theorem stays true if instead of 1) we assume that the functions $\psi \exp(-f^i(u^i)k_i)$ are analytic near P_i where f^i are some functions of one variable, $i = 1, \dots, n$. Therefore we do not differ orthogonal coordinate systems which are obtained by coordinate changes of the form

$$u^i \to f^i(u^i).$$

Remark 3. Krichever's theorem gives a construction of these coordinates by using the formalism of Baker–Akhiezer functions. In fact, it is clear from the proof that the uniqueness of the Baker–Akhiezer function stays valid if we replace the condition $\psi(u, R_k) = 1, i = 1, \dots, l$, by

$$\psi(u, R_k) = d_k, \quad k = 1, \dots, l, \tag{13}$$

where all constants d_k do not vanish. From that we deduce that we even can assume only that not all constants d_k vanish:

$$|d_1|^2 + \ldots + |d_l|^2 \neq 0, \tag{14}$$

and under this assumption the main results of [3] still hold.

For distinguishing the cases when this Theorem give positive-definite metrics it needs to impose some other conditions on the spectral data [3]:

- *If there is an antiholomorphic involution $\tau : \Gamma \to \Gamma$ such that all fixed points of σ are fixed by τ and*

  $$\tau^*(\Omega) = \overline{\Omega}$$

  (for that it is enough to assume that $\tau(k_i^{-1}) = \overline{k_i^{-1}}$ at $P_i, i = 1, \dots, n$, and τ maps divisors Q, R, and D into themselves: $\tau(Q) = Q, \tau(R) = R, \tau(D) = D$, however these divisors do not necessarily consist of fixed points of τ), then the coefficients $H_i(u)$ are real valued for $u^1, \dots, u^n \in \mathbb{R}$.

- *$u^1, \dots, u^n$ are n-orthogonal coordinates in the flat n-space with the metric $\eta_{kl} dx^k dx^l$.* [2]

- *Provided that all points from Q are fixed by the involution σ and*

  $$\mathrm{res}_{Q_1} \Omega_0 = \ldots = \mathrm{res}_{Q_n} \Omega_0 = \eta_0^2 > 0, \tag{15}$$

  the functions $x^1(u^1, \dots, u^n), \dots, x^n(u^1, \dots, x^n)$ solve the immersion problem for n-orthogonal coordinates $u^1, \dots, u^n$ and $ds^2 = H_1^2 (du^1)^2 + \ldots + H_n^2 (du^n)^2$ with

  $$H_i = \frac{\varepsilon_i h_i}{\eta_0}, \quad i = 1, \dots, n.$$

The analog of Krichever's construction for discrete orthogonal systems was developed in [1].

Krichever's results allow us to assume that n-orthogonal coordinate systems which are expressed in terms of elementary functions correspond to limiting cases when the spectral curve is singular.

[2] It is easy to see that if there are points Q_k and $Q_l, l = \sigma(k)$, which are interchanged by the involution then the metric η is indefinite.

3 Coordinate systems corresponding to singular spectral curves

Let Γ be an algebraic complex curve with singularities. Then there exists a morphism of a nonsingular algebraic curve $\Gamma_{\rm nm}$:

$$\pi : \Gamma_{\rm nm} \to \Gamma,$$

such that

1) there is a finite set S of points from $\Gamma_{\rm nm}$ with the equivalence relation $\sim$ on this set such that π maps S exactly into the set $\mathrm{Sing} = \mathrm{Sing}\,\Gamma$ formed by all singular points of Γ, and therewith the preimage of every point from Sing consists in a class of equivalent points;

2) the mapping $\pi : \Gamma_{\rm nm} \setminus S \to \Gamma \setminus \mathrm{Sing}$ is a smooth one-to-one projection;

3) any regular mapping $F : X \to \Gamma$ of a nonsingular algebraic variety X with an everywhere dense image $F(X) \subset \Gamma$ descends through $\Gamma_{\rm nm}$, i.e., $F = \pi G$ for some regular mapping $G : X \to \Gamma_{\rm nm}$.

A mapping π meeting these properties is called the normalization of Γ and is unique. The genus of $\Gamma_{\rm nm}$ is called the geometric genus of Γ and is denoted by $p_g(\Gamma)$.

However another genus comes into the Riemann–Roch formula, i.e., the arithmetic genus $p_a(\Gamma)$ which is a sum of the geometric genus and some positive-valued contribution of singularities (the points from Sing). For a nonsingular curve we have $p_a = p_g$.

For example, let us consider the case of multiple points when on $\Gamma_{\rm nm}$ we choose s families $D_1, \dots, D_s$ consisting of $r_1, \dots, r_s$ points all of which are pairwise different. Let Γ is obtained by gluing together points from each family. Then

$$p_a(\Gamma) = p_g(\Gamma) + \sum_{i=1}^{s} (r_i - 1).$$

A meromorphic 1-form ω on $\Gamma_{\rm nm}$ defines a regular differential on Γ if for any point $P \in \mathrm{Sing}$ we have

$$\sum_{\pi^{-1}(P)} \mathrm{res}\ (f\omega) = 0$$

for any meromorphic function f, on $\Gamma_{\rm nm}$, which descends to a function on Γ, i.e., takes the same value at points from each divisor D_i, and does not have poles in $\pi^{-1}(P)$. Regular differentials may have poles in the preimages of singular sets. It is easy to notice that the dimension of the space of regular differentials equals $p_a(\Gamma)$.

In the general case all these notions are exposed in [4] (for using it in the finite gap integration we gave some short expositions in [5, 6]). We only remind the Riemann–Roch theorem for singular curves.

Let $L(D)$ be the space of meromorphic functions on Γ with poles at the points from $D = \sum n_P P$ of order less or equal that n_P, and let $\Omega(D)$ be the space of regular differentials on Γ which has at every point $P \in \mathrm{Sing}$ a zero of order at least n_P. The Riemann–Roch theorem reads

$$\dim L(D) - \dim \Omega(D) = \deg D + 1 - p_a(\Gamma).$$

For generic divisor D with $\deg D > g$ we have $\dim \Omega(D) = 0$ and the Riemann–Roch theorem takes the form

$$\dim L(D) = \deg D + 1 - p_a(\Gamma).$$

Theorem 1 *Krichever's theorem (see §2.3) holds for singular algebraic curves provided that g is replaced by $p_a(\Gamma)$, i.e., by the arithmetic genus of Γ, and the assumption that Γ/σ is a nonsingular curve is replaced by the condition that $P_1, \dots, P_n$ and the poles of Ω are nonsingular points.*

Moreover we may assume that ψ meets the conditions (13) and (14) instead of $\psi(u, R_k) = 1, k = 1, \dots, l$.

Proof of this theorem is basically the same as the original Krichever proof in [3]. The uniqueness of ψ is established by using the general theory of Baker–Akhiezer functions. In the cases studied in §§ 3.1, 3.2 and 4 such a uniqueness is trivial since we are working with rational curves. The identity

$$\sum_{k,l} \eta_{kl} \partial_{u^i} x^k \partial_{u^j} x^l - \varepsilon_i^2 h_i^2 \delta_{ij} = 0$$

is equivalent to the identity

$$\sum \mathrm{res}\, (\partial_{u^i} \psi(u, X) \partial_{u^j} \psi(u, \sigma(X)) \Omega) = 0$$

and is obtained from it by explicit calculations of the residues.

Remark 4 (main). In the case when Γ_{nm} is a union of smooth rational curves, i.e., copies of $\mathbb{C}P^1$, the procedure of constructing Baker–Akhiezer functions and orthogonal coordinates is very simple: it reduces to simple computations with elementary functions and does not go far than solving systems of linear equations. However singular curves of algebraic genus g are obtained via degeneration from smooth curves of the same genus. Therewith qualitative properties of solutions, which correspond to these curves, of nonlinear equations are inherited, i.e., such solutions are rather complicated.

We restrict ourselves by the most simple case when Γ is a reducible curve consisting of components $\Gamma_1, \dots, \Gamma_s$ isomorphic to $\mathbb{C}P^1$. These components may intersect each other at some points.

A regular differential Ω on Γ is defined by some differentials $\Omega_1, \dots, \Omega_s$ on $\Gamma_1, \dots, \Gamma_s$ which may have poles at the intersections of components and moreover if P is such an intersection point of the components $\Gamma_{i_1}, \dots, \Gamma_{i_r}$ then

$$\sum_{k=1}^{r} \mathrm{res}_P\ \Omega_{i_k} = 0.$$

The arithmetic genus g_a is the dimension of the space of holomorphic regular differentials, i.e., differentials such that Ω_j may have poles only at intersections of different components.

For different combinatorial configurations of rational components and intersection points Theorem 1 is written in absolutely elementary form and the construction of orthogonal coordinates reduces to solving some systems of linear equations. It is simpler to demonstrate that by some examples which we expose below. For simplicity we assume that on every component there is defined some complex parameter.

3.1 2-orthogonal coordinate systems

Example 1. Let Γ consists of two copies of $\mathbb{C}P^1$, i.e., of Γ_1 and Γ_2, which intersect each other at two points:

$$a \sim b, \quad (-a) \sim (-b), \quad \{a, -a\} \subset \Gamma_1, \{b, -b\} \subset \Gamma_2$$

(see Fig. 1). We have $p_a(\Gamma) = 1$.

The Baker–Akhiezer function takes the form

$$\psi_1(u^1, u^2, z_1) = e^{u^1 z_1} \left(f_0(u^1, u^2) + \frac{f_1(u^1, u^2)}{z_1 - \alpha_1} + \ldots + \frac{f_k(u^1, u^2)}{z_1 - \alpha_{s_1}} \right), \ z_1 \in \Gamma_1,$$

$$\psi_2(u^1, u^2, z_2) = e^{u^2 z_2} \left(g_0(u^1, u^2) + \frac{g_1(u^1, u^2)}{z_2 - \beta_1} + \ldots + \frac{g_n(u^1, u^2)}{z_2 - \beta_{s_2}} \right), \ z_2 \in \Gamma_2.$$

$$\psi_1(a) = \psi_2(b), \quad \psi_1(-a) = \psi_2(-b). \tag{16}$$

It has two essential singularities at the points $P_1 = \infty \in \Gamma_1$ and $P_2 = \infty \in \Gamma_2$.

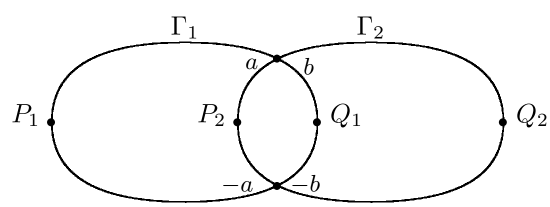

Fig. 1.

The general normalization condition takes the form

$$\psi_1(R_{1,i}) = d_{1,i}, \quad \psi_2(R_{2,j}) = d_{2,j} \tag{17}$$

where $R_{1,i} \in \Gamma_1, i = 1, \ldots, l_1$, and $R_{2,j} \in \Gamma_2, j = 1, \ldots, l_2$. We also have

$$l = l_1 + l_2 = s_1 + s_2.$$

Let

$$\Omega_1 = \frac{(z_1^2-\alpha_1^2)\dots(z_1^2-\alpha_{l_1}^2)}{z_1(z_1^2-a^2)(z_1^2-R_{1,1}^2)\dots(z_1^2-R_{1,l_1}^2)}dz_1,$$

$$\Omega_2 = \frac{(z_2^2-\beta_1^2)\dots(z_2^2-\beta_{l_2}^2)}{z_2(z_2^2-b^2)(z_2^2-R_{2,1}^2)\dots(z_2^2-R_{2,l_2}^2)}dz_2.$$

We put

$$Q_1 = 0 \in \Gamma_1, \quad Q_2 = 0 \in \Gamma_2.$$

If the following equalities hold

$$\mathrm{res}_a\Omega_1 = -\mathrm{res}_b\Omega_2, \quad \mathrm{res}_{-a}\Omega_1 = -\mathrm{res}_{-b}\Omega_2, \quad \mathrm{res}_{Q_1}\Omega_1 = \mathrm{res}_{Q_2}\Omega_2,$$

then the differential Ω defined by Ω_1 and Ω_2 is regular, the condition (15) is satisfied and, by Theorem 1, the coordinates u^1 and u^2 such that

$$x^1(u) = \psi_1(u,0), \quad x^2(u) = \psi_2(u,0)$$

are orthogonal.

Let us consider the simplest case $l_1 = 0$ and $l_2 = 1$.

We have

$$\psi_1 = e^{u^1 z_1} f_0(u^1,u^2), \quad \psi_2 = e^{u^2 z_2}\left(g_0(u^1,u^2) + \frac{g_1(u^1,u^2)}{z_2 - c}\right).$$

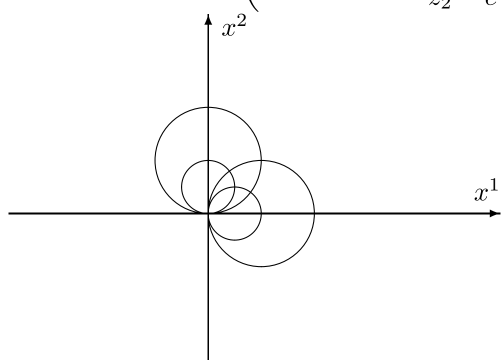

Fig. 2.

The gluing conditions at the intersection points and the normalization condition are

$$\psi_1(a) = \psi_2(b), \quad \psi_1(-a) = \psi_2(-b), \quad \psi_2(r) = 1, \; r = R \in \Gamma_2,$$

which implies

$$\psi_1 = e^{u^1 z_1}\left(\frac{2b(c-r)e^{au^1+(b-r)u^2}}{(b+c)(b-r)e^{2bu^2} - (b+r)(b-c)e^{2au^1}}\right),$$

$$\psi_2 = e^{u^2 z_2}\left(\frac{e^{-ru^2}((b-c)e^{2au^1}+(b+c)e^{2bu^2})(c-r)}{(b+c)(b-r)e^{2bu^2}-(b-c)(b+r)e^{2au^1}}+\right.$$
$$\left.\frac{1}{z_2-c}\frac{(b^2-c^2)(r-c)e^{-ru^2}(e^{2au^1}-e^{2bu^2})}{(b+c)(r-b)e^{2bu^2}+(b-c)(b+r)e^{2au^1}}\right).$$

The differential Ω is defined by the differentials

$$\Omega_1 = -\frac{1}{z_1(z_1^2-a^2)}dz_1,\ \Omega_2 = -\frac{(z_2^2-c^2)}{z_2(z_2^2-b^2)(z_2^2-r^2)}dz_2.$$

We have the following regularity condition for Ω:

$$\mathrm{res}_a\Omega_1 = \mathrm{res}_{-a}\Omega_1 = -\frac{1}{2a^2} = -\mathrm{res}_b\Omega_2 = -\mathrm{res}_{-b}\Omega_2 = \frac{(b^2-c^2)}{2b^2(b^2-r^2)},$$

and the condition (15) takes the form

$$\mathrm{res}_{Q_1}\Omega_1 = \frac{1}{a^2} = \mathrm{res}_{Q_2}\Omega_2 = \frac{c^2}{r^2b^2}$$

which implies

$$a = \frac{br}{c},\ r = \frac{b}{\sqrt{2+\frac{b^2}{c^2}}}.$$

After the substitution $u^1 = \log y^1,\ u^2 = \log y^2$ the formulas for the coordinates are written as

$$x^1 = \frac{-2b(r-c)}{(c-b)(b+r)}(y^2)^{-r}\left(\frac{\frac{(y^2)^b}{(y^1)^a}}{1+\frac{(b+c)(b-r)}{(c-b)(b+r)}\frac{(y^2)^{2b}}{(y^1)^{2a}}}\right),$$

$$x^2 = \frac{b(c-r)}{c(b+r)}(y^2)^{-r}\left(\frac{1+\frac{(b+c)}{(c-b)}\frac{(y^2)^{2b}}{(y^1)^{2a}}}{1+\frac{(b+c)(b-r)}{(c-b)(b+r)}\frac{(y^2)^{2b}}{(y^1)^{2a}}}\right)$$

and by straightforward computations we obtain

$$(x^1)^2+\left(x^2-(y^2)^{-r}\frac{b(c-r)}{c(b^2-r^2)}\right)^2 = (y^2)^{-2r}\frac{b^2(c-r)^2}{c^2(b^2-r^2)^2}.$$

Therefore the coordinate lines $y^2 = \mathrm{const}$, i.e., $u^2 = \mathrm{const}$, are the circles centered on the x^2 axis. For $b = \pm 1$ these circles touch the x^1 axis and another family of coordinate lines consist of circles centered at the x^1 axis and touching the x^2 axis (see Fig. 2).

Example 2. Let Γ be the same as in Example 1 however all essential singularities lie in one copy of $\mathbb{C}P^1$ and the divisor Q lies in another copy (see Fig. 3)

$$P_1 = \infty,\ P_2 = 0 \in \Gamma_1,\quad Q_1 = \infty,\ Q_2 = 0 \in \Gamma_2.$$

We define the Baker–Akhiezer function as follows:

$$\psi_1(u,z_1)=e^{u^1 z_1+\frac{u^2}{z_1}}\left(f_0(u)+\frac{f_1(u)}{z_1-\alpha_1}+\ldots+\frac{f_k(u)}{z_1-\alpha_{s_1}}\right),\ z_1\in\Gamma_1;$$

$$\psi_2(u,z_2)=\left(g_0(u)+\frac{g_1(u)}{z_2-\beta_1}+\ldots+\frac{g_n(u)}{z_2-\beta_{s_2}}\right),\ z_2\in\Gamma_2.$$

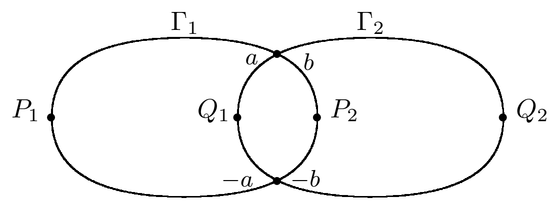

Fig. 3.

The gluing and normalization conditions have the same forms (16) and (17). Let

$$\Omega_1=\frac{z_1(z_1^2-\alpha_1^2)\ldots(z_1^2-\alpha_{l_1-1}^2)}{(z_1^2-a^2)(z_1^2-R_{1,1}^2)\ldots(z_1^2-R_{1,l_1}^2)}dz_1,$$

$$\Omega_2=\frac{(z_2^2-\beta_1^2)\ldots(z_2^2-\beta_{l_2+1}^2)}{z_2(z_2^2-b^2)(z_2^2-R_{2,1}^2)\ldots(z_2^2-R_{2,l_2}^2)}dz_2.$$

By Theorem 1, if

$$\mathrm{res}_a\Omega_1=-\mathrm{res}_b\Omega_2,\ \mathrm{res}_{-a}\Omega_1=-\mathrm{res}_{-b}\Omega_2,\ \mathrm{res}_{Q_1}\Omega_2=\mathrm{res}_{Q_2}\Omega_2,$$

then we have

$$\partial_{u^1}x^1\partial_{u^2}x^1+\partial_{u^1}x^2\partial_{u^2}x^2=0.$$

Let us consider the simplest case: $l_1=s_2=1, l_2=s_1=0$, $r=R\in\Gamma_1$, $d_{1,1}=1$. We have

$$\psi_1=e^{u^1 z_1+\frac{u^2}{z_1}}f(u),\quad \psi_2=\left(g_0(u)+\frac{g_1(u)}{z_2-c}\right),$$

$$\Omega_1=\frac{z_1}{(z_1^2-a^2)(z_1^2-r^2)}dz_1,\quad \Omega_2=-\frac{(z_2^2-c^2)}{z_2(z_2^2-b^2)}dz_2.$$

We have

$$\mathrm{res}_a\Omega_1=\mathrm{res}_{-a}\Omega_1=\frac{1}{2(a^2-r^2)},\quad \mathrm{res}_b\Omega_2=\mathrm{res}_{-b}\Omega_2=\frac{(b^2-c^2)}{2b^2},$$

$$\mathrm{res}_{Q_1}\Omega_2 = 1, \quad \mathrm{res}_{Q_2}\Omega_2 = -\frac{b^2}{c^2},$$

and the regularity condition for Ω and (15) are satisfied exactly when

$$b = \pm ic, \quad a^2 - r^2 = -\frac{1}{2}.$$

For a particular solution $b = i,\ c = -1,\ a = \frac{i}{2},\ r = \frac{1}{2}$, the immersion formulas take the form

$$x^1 = e^{-\frac{u^1}{2}-2u^2}\left(\cos\left(\frac{u^1}{2} - 2u^2\right) + \sin\left(\frac{u^1}{2} - 2u^2\right)\right),$$

$$x^2 = e^{-\frac{u^1}{2}-2u^2}\left(\cos\left(\frac{u^1}{2} - 2u^2\right) - \sin\left(\frac{u^1}{2} - 2u^2\right)\right).$$

By the substitution

$$y^1 = \frac{u^1}{2},\ y^2 = 2u^2$$

we obtain

$$x^1 = e^{-y^1-y^2}(\cos(y^1 - y^2) + \sin(y^1 - y^2)),$$
$$x^2 = e^{-y^1-y^2}(\cos(y^1 - y^2) - \sin(y^1 - y^2)).$$

Therewith the "lines" $y^1 + y^2 = \mathrm{const}$ correspond to circles centered at the origin $x = 0$, and the "lines" $y^1 - y^2 = \mathrm{const}$ define in the x-space rays drawing from the origin.

3.2 3-orthogonal coordinate systems

EXAMPLE 3. Let Γ consist of three components Γ_1, Γ_2 and Γ_3 which are copies of $\mathbb{C}P^1$ and have four intersection points as it is shown on Fig. 4:

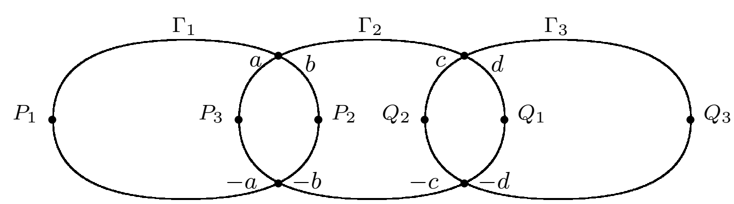

Fig. 4

$$\pm a \sim \pm b, \quad \pm c \sim \pm d, \quad \pm a \in \Gamma_1,\ \pm b, \pm c \in \Gamma_2, \quad \pm d \in \Gamma_3.$$

Let us put

$$P_1 = \infty \in \Gamma_1, \quad P_2 = 0 \in \Gamma_1, \quad P_3 = \infty \in \Gamma_2,$$

$$Q_1 = 0 \in \Gamma_2, \quad Q_2 = \infty \in \Gamma_3, \quad Q_3 = 0 \in \Gamma_3,$$
$$l = 1, \quad r = R \in \Gamma_1, \quad \psi_1(r) = 1.$$

We have

$$\psi_1(u, z_1) = e^{u^1 z_1 + \frac{u^2}{z_1}} f(u), \quad z_1 \in \Gamma_1,$$
$$\psi_2(u, z_2) = e^{u^3 z_2} \left(g_0(u) + \frac{g_1(u)}{z_2 - \beta} \right), \quad z_2 \in \Gamma_2,$$
$$\psi_3(u, z_3) = h_0(u) + \frac{h_1(u)}{z_3 - \gamma}, \quad z_3 \in \Gamma_3.$$

with

$$\psi_1(\pm a) = \psi_2(\pm b), \ \psi_2(\pm c) = \psi_3(\pm d), \quad \psi_1(r) = 1.$$

Let us take Ω defined by the differentials

$$\Omega_1 = \frac{z_1 dz_1}{(z_1^2 - a^2)(r^2 - z_1^2)}, \quad \Omega_2 = \frac{(\beta^2 - z_2^2) dz_2}{z_2(z_2^2 - b^2)(z_2^2 - c^2)}, \quad \Omega_3 = \frac{(\gamma^2 - z_3^2) dz_3}{z_3(z_3^2 - d^2)}.$$

The regularity condition for Ω and (15) are satisfied for

$$a^2 = -\frac{1}{12}, \ b = -\frac{1}{3}, \ c = d = i, \ \beta = bc, \ \gamma = -1$$

in which case we have

$$x^1 = \sqrt{2} e^{-\frac{u^1}{2} - 2u^2} \cos\left(\frac{1}{12}(3\pi + 2\sqrt{3}(u^1 - 2(6u^2 + u^3))) \right),$$
$$x^2 = \sqrt{2} e^{-\frac{u^1}{2} - 2u^2} \left(\cos\left(\frac{u^1 - 2(6u^2 + u^3)}{2\sqrt{3}} \right) \sin\left(\frac{\pi}{4} + u^3 \right) + \right.$$
$$\left. \sin\left(\frac{u^1 - 2(6u^2 + u^3)}{2\sqrt{3}} \right) \cos\left(\frac{\pi}{12} + u^3 \right) \right),$$
$$x^3 = \sqrt{2} e^{-\frac{u^1}{2} - 2u^2} \left(\cos\left(\frac{u^1 - 2(6u^2 + u^3)}{2\sqrt{3}} \right) \cos\left(\frac{\pi}{4} + u^3 \right) - \right.$$
$$\left. \sin\left(\frac{u^1 - 2(6u^2 + u^3)}{2\sqrt{3}} \right) \sin\left(\frac{\pi}{12} + u^3 \right) \right).$$

It is straightforwardly checked that

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = 3e^{-u^1 - 4u^2},$$
$$x^1 - \left(\frac{1 - \sqrt{3}}{2} x^2 + \frac{1 + \sqrt{3}}{2} x^3 \right) \cos u^3 - \left(\frac{1 + \sqrt{3}}{2} x^2 + \frac{\sqrt{3} - 1}{2} x^3 \right) \sin u^3 = 0,$$
$$2(x^1)^2 - (x^2)^2 - (x^3)^2 + ((x^1)^2 + (x^2)^2 + (x^3)^2) \sin\left(\frac{u^1 - 2(6u^2 + u^3)}{\sqrt{3}} \right) = 0.$$

Therefore the "planes" $u^3 = \text{const}$ are planes passing the point $x = 0$, the "planes" $u^1 + 4u^2 = \text{const}$ are spheres centered at $x = 0$, and the "planes" $u^1 - 2(6u^2 + u^3) = \text{const}$ are cones centered at $x = 0$.

4 The classical coordinate systems

The Euclidean coordinates. Let Γ be a disjoint union of n copies $\Gamma_1, \dots, \Gamma_n$ of $\mathbb{C}P^1$. We put

$$P_j = \infty, \quad Q_j = 0, \quad R_j = -1 \in \Gamma_j, \quad \psi_j(R_j) = 1, \quad j = 1, \dots, n.$$

Then we have the differential Ω defined by the differentials

$$\Omega_j = \frac{dz_j}{z_j(z_j^2 - 1)}, \quad j = 1, \dots, n,$$

on the components of Γ. The Baker–Akhiezer function ψ is equal to

$$\psi_j = e^{u^j z_j} f_j(u^j), \quad j = 1, \dots, n,$$

and we obtain the Euclidean coordinates in $\mathbb{R}^n$ (see Remark 2):

$$x^j = e^{u^j}.$$

The polar coordinates. Let Γ consists of five irreducible components $\Gamma_1, \dots, \Gamma_5$ which intersect as it is shown on Fig. 5:

$$\{0 \in \Gamma_1\} \sim \{0 \in \Gamma_2\}, \quad \{a \in \Gamma_2\} \sim \{b_1 \in \Gamma_3\}, \quad \{-a \in \Gamma_2\} \sim \{b_2 \in \Gamma_4\},$$

$$\{c_1 \in \Gamma_3\} \sim \{d \in \Gamma_5\}, \quad \{c_2 \in \Gamma_4\} \sim \{-d \in \Gamma_5\}.$$

We define an involution σ on Γ as follows:

a) on Γ_1, Γ_2 and Γ_3 it has the form

$$\sigma(z_j) = -z_j;$$

b) Γ_3 and Γ_4 are interchanged by σ and the points $b_1, c_1, \infty \in \Gamma_3$ are mapped into the points $b_2, c_2, \infty \in \Gamma_4$ respectively. It is easy to check that

$$\sigma(z_3) = \frac{b_2 - c_2}{b_1 - c_1} z_3 + \frac{b_1 c_2 - b_2 c_1}{b_1 - c_1},$$

$$\sigma(z_4) = \frac{b_1 - c_1}{b_2 - c_2} z_4 + \frac{b_2 c_1 - b_1 c_2}{b_2 - c_2}.$$

We put

$$\beta_1 = \frac{b_2 c_1 - b_1 c_2}{b_2 - c_2}, \ \beta_2 = \frac{b_1 c_2 - b_2 c_1}{b_1 - c_1}.$$

Then $0 \in \Gamma_3$ is mapped by σ into $\beta_2 \in \Gamma_4$, and $0 \in \Gamma_4$ is mapped into $\beta_1 \in \Gamma_3$.

The divisors $P = P_1 + P_2$ and $Q = Q_1 + Q_2$ are as follows

$$P_1 = \infty \in \Gamma_1, \quad P_2 = \infty \in \Gamma_2, \quad Q_1 = 0 \in \Gamma_5, \quad Q_2 = \infty \in \Gamma_5.$$

We also take for $D = \gamma_1 + \gamma_2 + \gamma_3$ the divisor

$$\gamma_1 = 0 \in \Gamma_3, \quad \gamma_2 = 0 \in \Gamma_4, \quad \gamma_3 = \alpha \in \Gamma_5.$$

Since $\deg D = g + l - 1 = 3$ and $g = p_a(\Gamma) = 1$, we have $l = 3$. We put

$$R_1 = -1 \in \Gamma_1, \quad R_2 = \infty \in \Gamma_4, \quad R_3 = \infty \in \Gamma_5.$$

Then the Baker–Akhiezer function takes the form

$$\psi_1(u, z_1) = e^{u^1 z_1} f_1(u), \quad \psi_2(u, z_2) = e^{u^2 z_2} f_2(u),$$

$$\psi_3(u, z_3) = \frac{f_3(u)}{z_3} + \widehat{f}_3(u), \quad \psi_4(u, z_4) = \frac{f_4(u)}{z_4} + \widehat{f}_4(u),$$

$$\psi_5(u, z_5) = f_5(u) + \frac{\widehat{f}_5(u)}{z_5 - \alpha}.$$

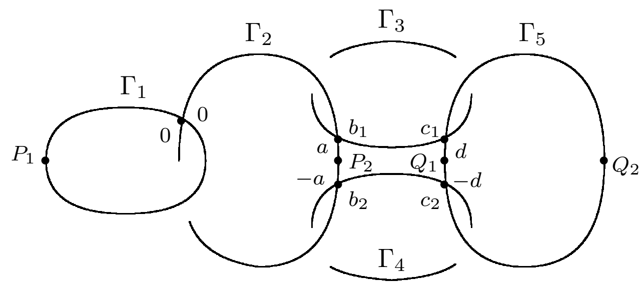

Fig. 5

Moreover we have

$$\psi_1(u, 0) = \psi_2(u, 0), \quad \psi_2(u, a) = \psi_3(u, b_1), \quad \psi_2(u, -a) = \psi_4(u, b_2),$$

$$\psi_3(u, c_1) = \psi_5(u, d), \quad \psi_4(u, c_2) = \psi_5(u, -d).$$

We take the following normalization condition:

$$\psi_1(u, -1) = 1, \quad \psi_3(u, \infty) = 0, \ \psi_4(u, \infty) = 0.$$

This implies

$$f_1 = e^{u^1}, \quad f_2 = e^{u^1}, \quad f_3 = e^{u^1 + au^2}, \quad \widehat{f}_3 = 0, \quad f_4 = e^{u^1 - au^2}, \quad \widehat{f}_4 = 0,$$

$$f_5 = \frac{e^{u^1 - au^2}(b_1 c_2 e^{2au^2}(d - \alpha) + b_2 c_1 (d + \alpha))}{2 c_1 c_2 d},$$

$$\widehat{f}_5 = \frac{e^{u^1 - au^2}(-b_2 c_1 + e^{2au^2} b_1 c_2)(d^2 - \alpha^2)}{2 c_1 c_2 d}.$$

Is is checked by straightforward computations that for

$$a = i, \quad b_1 = \bar{b}_2 = \frac{i}{2}, \quad c_1 = \bar{c}_2 = \frac{i-1}{2}, \quad d = -i\alpha$$

the differential Ω defined by the differentials

$$\Omega_1 = -\frac{dz_1}{z_1(z_1^2-1)}, \ \Omega_2 = -\frac{dz_2}{z_2(z_2^2-a^2)}, \ \Omega_3 = -\frac{z_3(z_3-\beta_1)dz_3}{(z_3-b_1)(z_3-c_1)},$$

$$\Omega_4 = -\frac{z_4(z_4-\beta_2)dz_3}{(z_4-b_2)(z_4-c_2)}, \ \Omega_5 = -\frac{(z_5^2-\alpha^2)dz_5}{z_5(z_5^2-d^2)}$$

is regular on Γ, satisfies (15), and

$$x^1 = \psi_5(Q_1) = r\cos\varphi, \quad x^2 = \psi_5(Q_2) = r\sin\varphi$$

where $r = e^{u^1}$ and $\varphi = u^2$.

REMARK 5. We see that the values of α and d are not precisely determined and we only have the relation $\alpha = id$. Therefore as in the case of the dressing method this construction also corresponds an equivalence class of spectral data to the same metric (see Remark 1.)

The cylindrical coordinates. We take for Γ a disjoint union of the curve $\widehat{\Gamma}$ from the previous example (the polar coordinates) and a copy Γ_6 of $\mathbb{C}P^1$. All the data concerning $\widehat{\Gamma}$ are also the same as in the previous example. On Γ_6 we put $Q_3 = 0, P_3 = \infty, R_4 = -1$, and $\psi(R_4) = 1$. Then we have $\psi_6(u^3) = e^{u^3(z_6+1)}$ and

$$x^1 = \psi_5(Q_1) = r\cos\varphi, \quad x^2 = \psi_5(Q_2) = r\sin\varphi, \quad x^3 = \psi_6(Q_3) = z$$

where $r = e^{u^1}, \varphi = u^2$, and $z = u^3$.

The spherical coordinates in $\mathbb{R}^3$. The curve Γ consists of 9 irreducible components which intersect as it is shown on Fig. 6:

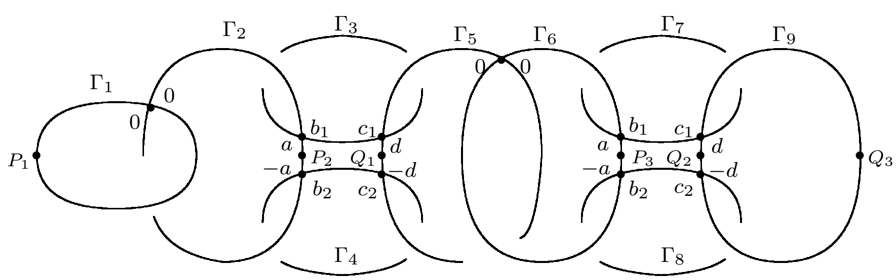

Fig. 6

$$\{0\in\Gamma_1\}\sim\{0\in\Gamma_2\},\quad \{a\in\Gamma_2\}\sim\{b_1\in\Gamma_3\},\quad \{-a\in\Gamma_2\}\sim\{b_2\in\Gamma_4\},$$
$$\{c_1\in\Gamma_3\}\sim\{d\in\Gamma_5\},\quad \{c_2\in\Gamma_4\}\sim\{-d\in\Gamma_5\},\quad \{0\in\Gamma_5\}\sim\{0\in\Gamma_6\},$$
$$\{a\in\Gamma_6\}\sim\{b_1\in\Gamma_7\},\quad \{-a\in\Gamma_6\}\sim\{b_2\in\Gamma_8\},$$
$$\{c_1\in\Gamma_7\}\sim\{d\in\Gamma_9\},\quad \{c_2\in\Gamma_8\}\sim\{-d\in\Gamma_9\}$$

where, for simplicity, we denote by the same symbol points on different components if the coordinates of these points are equal to each other (for, instance, a on Γ_2 and Γ_6).

We take

$$Q_1=\infty\in\Gamma_5,\quad Q_2=\infty\in\Gamma_9,\quad Q_3=0\in\Gamma_9,$$
$$P_1=\infty\in\Gamma_1,\quad P_2=\infty\in\Gamma_2,\quad P_3=\infty\in\Gamma_6,$$

and choose the divisor D as follows

$$\gamma_1=0\in\Gamma_3,\quad \gamma_2=0\in\Gamma_4,\quad \gamma_3=\alpha\in\Gamma_5,$$
$$\gamma_4=0\in\Gamma_7,\quad \gamma_5=0\in\Gamma_8,\quad \gamma_6=\alpha\in\Gamma_9.$$

We have $p_a(\gamma)=2$, $\deg D=6$, and therefore $l=5$. Let us put

$$R_1=-1\in\Gamma_1,\quad R_2=\infty\in\Gamma_3,\quad R_3=\infty\in\Gamma_4, R_4=\infty\in\Gamma_7,\quad R_5=\infty\in\Gamma_8.$$

The Baker–Akhiezer function is written as

$$\psi_1=e^{u^1z_1}f_1(u),\quad \psi_2=e^{u^2z_2}f_2(u),\quad \psi_3=\frac{f_3(u)}{z_3}+\widehat{f}_3(u),$$
$$\psi_4=\frac{f_4(u)}{z_4}+\widehat{f}_4(u),\quad \psi_5=f_5(u)+\frac{\widehat{f}_5(u)}{(z_5-\alpha)},\quad \psi_6=e^{u^3z_6}f_6(u),$$
$$\psi_7=\frac{f_7(u)}{z_7}+\widehat{f}_7(u),\quad \psi_8=\frac{f_8(u)}{z_8}+\widehat{f}_8(u),\quad \psi_9=f_9(u)+\frac{\widehat{f}_9(u)}{z_9-\alpha}.$$

We have the gluing conditions (for brevity,we skip the u-variables):

$$\psi_1(0)=\psi_2(0),\quad \psi_2(a)=\psi_3(b_1),\quad \psi_2(-a)=\psi_4(b_2),\quad \psi_3(c_1)=\psi_5(d),$$
$$\psi_4(c_2)=\psi_5(-d),\quad \psi_5(0)=\psi_6(0),\quad \psi_6(a)=\psi_7(b_1),\quad \psi_6(-a)=\psi_8(b_2),$$
$$\psi_7(c_1)=\psi_9(d),\quad \psi_8(c_2)=\psi_9(-d),$$

and the normalization condition is taken as follows:

$$\psi_1(u,-1)=1,\ \psi_3(u,\infty)=0,\ \psi_4(u,\infty)=0,\ \psi_7(u,\infty)=0,\ \psi_8(u,\infty)=0.$$

Let a,b_1,b_2,c_1,c_2,d take the same values as in the case of polar coordinates and then the regular form Ω is also constructed as in the this case. By straightforward computations, we obtain

$$x^1=\psi_5(Q_1)=r\sin\varphi,$$

$$x^2 = \psi_9(Q_2) = r\cos\varphi\sin\theta, \quad x^3 = \psi_9(Q_3) = r\cos\varphi\cos\theta$$

where $r = e^{u^1}, \varphi = u^2$, and $\theta = u^3$.

The spherical coordinates in $\mathbb{R}^n$. Given the data $\Gamma^{(n-1)}$ and $\psi^{(n-1)}$ for the $(n-1)$-dimensional spherical coordinates, the spectral curve $\Gamma^{(n)}$ for the n-dimensional spherical coordinates is the union of $\Gamma^{(n-1)}$ and the curve exposed on Fig. 7. Therewith these curves intersect at the points $0 \in \Gamma_{4n-7} \subset \Gamma^{(n-1)}$ and $0 \in \Gamma_{4n-6}$ (we remark that the number of irreducible components of $\Gamma^{(k)}$ equals $4k-3$). Moreover we have

$$P_n = \infty \in \Gamma_{4n-6}, \quad Q_{n-1} = \infty, \quad Q_n = 0 \in \Gamma_{4n-3},$$

and on the additional components the Baker–Akhiezer function is defined as follows:

$$\psi_{4n-6} = e^{u^n z_{4n-6}} f_{4n-6}(u), \quad \psi_{4n-5} = \frac{f_{4n-5}(u)}{z_{4n-5}},$$

$$\psi_{4n-4} = \frac{f_{4n-4}(u)}{z_{4n-4}}, \quad \psi_{4n-3} = f_{4n-3}(u) + \frac{\widehat{f}_{4n-3}(u)}{z_{4n-3} - \alpha}.$$

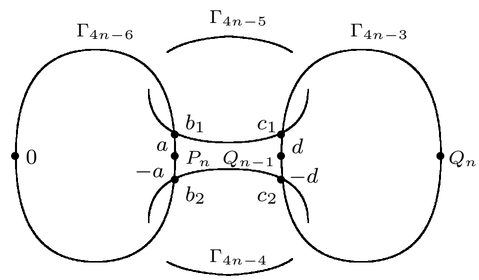

Fig. 7

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