Max-Planck-Institut für Mathematik
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by

A. Fialowski
L. Magnin
A. Mandal

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A. Fialowski
L. Magnin
A. Mandal

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Institute of Mathematics
Eötvös Loránd University
Pázmány Péter Sétány 1/C
1117 Budapest
Hungary

Institut de Mathématiques de Bourgogne
UMR CNRS 5584
Université de Bourgogne
BP 47870
21078 Dijon Cedex
France

University of Luxembourg
Campus Kirchberg
Math. Research Unit
6, rue Richard Coudenhove-Kelergi
1359 Luxembourg City
Luxembourg

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ABOUT LEIBNIZ COHOMOLOGY AND DEFORMATIONS OF LIE ALGEBRAS

A. FIALOWSKI, L. MAGNIN, AND A. MANDAL

Abstract. We compare the second adjoint and trivial Leibniz cohomology spaces of a Lie algebra to the usual ones by a very elementary approach. The comparison gives some conditions, which are easy to verify for a given Lie algebra, for deciding whether it has more Leibniz deformations than just the Lie ones. We also give the complete description of a Leibniz (and Lie) versal deformation of the 4-dimensional diamond Lie algebra, used in a WZW model, and study the case of its 5-dimensional analogue by computing Massey products.

1. Introduction

Leibniz algebras, along with their Leibniz cohomologies, were introduced in [8] as a non antisymmetric version of Lie algebras. Lie algebras are special Leibniz algebras, and Pirashvili introduced [17] a spectral sequence, that, when applied to Lie algebras, measures the difference between the Lie algebra cohomology and the Leibniz cohomology. Now, Lie algebras have deformations as Leibniz algebras and those are piloted by the adjoint Leibniz 2-cocycles. On Lie algebra cohomology we refer to [7, 6]. In the present paper, we focus on the second Leibniz cohomology groups $HL^2(\mathfrak{g},\mathfrak{g})$, $HL^2(\mathfrak{g},\mathbb{C})$ for adjoint and trivial representations of a complex Lie algebra $\mathfrak{g}$. We adopt a very elementary approach, not resorting to the Pirashvili sequence, to compare $HL^2(\mathfrak{g},\mathfrak{g})$ and $HL^2(\mathfrak{g},\mathbb{C})$ to $H^2(\mathfrak{g},\mathfrak{g})$ and $H^2(\mathfrak{g},\mathbb{C})$ respectively. In both cases, HL^2 is the direct sum of 3 spaces: $H^2 \oplus ZL_0^2 \oplus \mathcal{C}$ where H^2 is the Lie algebra cohomology group, ZL_0^2 is the space of symmetric Leibniz 2-cocycles and $\mathcal{C}$ is a space of coupled Leibniz 2-cocycles, the nonzero elements of which have the property that their symmetric and antisymmetric parts are not Leibniz cocycles. Our comparison gives some useful practical information about the structure of Lie and Leibniz cocycles. We analyse the case of Heisenberg algebras,

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the 4-dimensional diamond algebra which is used to construct a Wess-Zumino-Witten model, and its 5-dimensional analogue. We completely describe a versal Leibniz and Lie deformation of the diamond algebra by computing Massey products.

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2. Leibniz cohomology and deformations

Recall that a (right) Leibniz algebra is an algebra $\mathfrak{g}$ with a (non necessarily antisymmetric) bracket, such that the right adjoint operations $[-, Z]$ are required to be derivations for any $Z \in \mathfrak{g}$. In the presence of antisymmetry, that is equivalent to the Jacobi identity, hence any Lie algebra is a Leibniz algebra.

The Leibniz cohomology $HL^\bullet(\mathfrak{g}, \mathfrak{g})$ of a finite dimensional Leibniz algebra is defined from the complex $CL^\bullet(\mathfrak{g}, \mathfrak{g}) = \text{Hom }(\mathfrak{g}^{\otimes\bullet}, \mathfrak{g}) = \mathfrak{g} \otimes (\mathfrak{g}^*)^{\otimes\bullet}$ with the Leibniz coboundary δ defined for $\psi \in CL^n(\mathfrak{g}, \mathfrak{g})$ by

$$(\delta\psi)(X_1, X_2, \cdots, X_{n+1}) =$$
$$[X_1, \psi(X_2, \cdots, X_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [\psi(X_1, \cdots, \hat{X}_i, \cdots, X_{n+1}), X_i]$$
$$+ \sum_{1 \leqslant i < j \leqslant n+1} (-1)^{j+1} \psi(X_1, \cdots, X_{i-1}, [X_i, X_j], X_{i+1}, \cdots, \hat{X}_j, \cdots, X_{n+1}).$$

(If $\mathfrak{g}$ is a Lie algebra, δ coincides with the usual coboundary d on $C^\bullet(\mathfrak{g}, \mathfrak{g}) = \mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^*$.)

For $\psi \in CL^1(\mathfrak{g}, \mathfrak{g}) = C^1(\mathfrak{g}, \mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g}^*$

$$(\delta\psi)(X, Y) = [X, \psi(Y)] + [\psi(X), Y] - \psi([X, Y]).$$

For $\psi \in CL^2(\mathfrak{g}, \mathfrak{g}) = \mathfrak{g} \otimes (\mathfrak{g}^*)^{\otimes 2}$,

$$\begin{aligned}(\delta\psi)(X, Y, Z) = [X, \psi(Y, Z)] + [\psi(X, Z), Y] - [\psi(X, Y), Z] \\ - \psi([X, Y], Z) + \psi(X, [Y, Z]) + \psi([X, Z], Y).\end{aligned}$$

In the same way, the Leibniz cohomology $HL^\bullet(\mathfrak{g}, \mathbb{C})$ with trivial coefficients is defined from the complex $CL^\bullet(\mathfrak{g}, \mathbb{C}) = (\mathfrak{g}^*)^{\otimes\bullet}$ with the Leibniz coboundary

$\delta_\mathbb{C}$ defined for $\psi \in CL^n(\mathfrak{g}, \mathbb{C})$ by

$$(\delta_\mathbb{C}\psi)(X_1, X_2, \cdots, X_{n+1}) =$$
$$\sum_{1 \leqslant i < j \leqslant n+1} (-1)^{j+1} \psi(X_1, \cdots, X_{i-1}, [X_i, X_j], X_{i+1}, \cdots, \hat{X}_j, \cdots, X_{n+1}).$$

If $\mathfrak{g}$ is a Lie algebra, $\delta_\mathbb{C}$ is the usual coboundary $d_\mathbb{C}$ on $C^\bullet(\mathfrak{g}, \mathbb{C}) = \bigwedge^\bullet \mathfrak{g}^*$.

For $\psi \in CL^1(\mathfrak{g}, \mathbb{C}) = \mathfrak{g}^*$,

$$(\delta_{\mathbb{C}}\psi)(X,Y) = -\psi([X,Y]).$$

For $\psi \in CL^2(\mathfrak{g}, \mathbb{C}) = (\mathfrak{g}^*)^{\otimes 2}$,

$$(\delta_{\mathbb{C}}\psi)(X,Y,Z) = -\psi([X,Y],Z) + \psi(X,[Y,Z]) + \psi([X,Z],Y).$$

For computing Leibniz deformations, we need to consider the 2- and 3-dimensional cohomology cocycles.

Let $\mathbb{K}$ be a field of zero characteristic. A deformation of a Lie algebra with (nontrivial) base was introduced in [1]. We recall the notion of deformation of a Lie (Leibniz) algebra $\mathfrak{g}$ (L) over a commutative algebra base A with identity, with a fixed augmentation $\varepsilon : A \to \mathbb{K}$ and maximal ideal $\mathfrak{M}$. Assume $\dim(\mathfrak{M}^k/\mathfrak{M}^{k+1}) < \infty$ for every k (see [2, 4]).

Definition 1. *A deformation λ of a Lie algebra $\mathfrak{g}$ (or a Leibniz algebra L) with base $(A, \mathfrak{M})$, or simply with base A is an A-Lie algebra (or an A-Leibniz algebra) structure on the tensor product $A \otimes \mathfrak{g}$ (or $A \otimes L$) with the bracket $[,]_\lambda$ such that*

$$\varepsilon \otimes id : A \otimes \mathfrak{g} \to \mathbb{K} \otimes \mathfrak{g} \ (or\ \varepsilon \otimes id : A \otimes L \to \mathbb{K} \otimes L)$$

is an A-Lie algebra (A-Leibniz algebra) homomorphism.

A deformation of the Lie (Leibniz) algebra $\mathfrak{g}$ (L) with base A is called *infinitesimal*, or *first order*, if in addition to this $\mathfrak{M}^2 = 0$. We call a deformation of *order k*, if $\mathfrak{M}^{k+1} = 0$. A deformation with base is called local if A is a local algebra over $\mathbb{K}$, which means A has a unique maximal ideal.

Suppose A is a complete local algebra ($A = \varprojlim_{n\to\infty} (A/\mathfrak{M}^n)$), where $\mathfrak{M}$ is the maximal ideal in A. Then a deformation of $\mathfrak{g}$ (L) with base A which is obtained as the projective limit of deformations of $\mathfrak{g}$ (L) with base $A/\mathfrak{M}^n$ is called a *formal deformation* of $\mathfrak{g}$ (L).

Definition 2. *(see [2]) Let C be a complete local algebra. A formal deformation η of a Lie algebra $\mathfrak{g}$ (Leibniz algebra L) with base C is called miniversal if*
(i) for any formal deformation λ of $\mathfrak{g}$ (L) with base A there exists a homomorphism $f : C \to A$ such that the deformation λ is equivalent to $f_*\eta$;
(ii) if A satisfies the condition $\mathfrak{M}^2 = 0$, then f is unique.

If only (i) is satisfied, we call the deformation versal.

Theorem 1. *([2, 4]) If $H^2(\mathfrak{g}; \mathfrak{g})$ is finite dimensional, then there exists a (mini)versal deformation of $\mathfrak{g}$ (similarly for L).*

In [1] a construction for a miniversal deformation of a Lie algebra was given and it was generalized to Leibniz algebras in [4]. The computation for a specific Leibniz algebra example was given in [3].

3. Comparison of the cohomology spaces HL^2 and H^2 for a Lie algebra

In [17] the relation between Chevalley-Eilenberg and Leibniz homology with coefficients in a right module is considered via a spectral sequence. The statements are valid in the cohomological version as well. As a corollary, one deduces

Proposition 1. [17] *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ and M be a right $\mathfrak{g}$-module. If*

$$H_*(\mathfrak{g}, M) = 0, \quad \textit{then} \quad HL_*(\mathfrak{g}, M) = 0.$$

As the similar statement is true for cohomologies, it implies that rigid Lie algebras are Leibniz rigid as well.

Now we describe the Leibniz 2-cohomology spaces with the help of Lie 2-cohomology space of a Lie algebra $\mathfrak{g}$.

Recall that a symmetric bilinear form $B \in S^2\mathfrak{g}^*$ is invariant, i.e. $B \in (S^2\mathfrak{g}^*)^{\mathfrak{g}}$ if and only if $B([Z,X],Y) = -B(X,[Z,Y])\ \forall X,Y,Z \in \mathfrak{g}$. The Koszul map [7] $\mathcal{I} : (S^2\mathfrak{g}^*)^{\mathfrak{g}} \to \left(\bigwedge^3 \mathfrak{g}^*\right)^{\mathfrak{g}} \subset Z^3(\mathfrak{g}, \mathbb{C})$ is defined by $\mathcal{I}(B) = I_B$, with $I_B(X,Y,Z) = B([X,Y],Z)\ \forall X,Y,Z \in \mathfrak{g}$. Since the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathcal{C}^2\mathfrak{g}$ induces an isomorphism

$$\varpi : \ker \mathcal{I} \to S^2\left(\mathfrak{g}/\mathcal{C}^2\mathfrak{g}\right)^*,$$

(where $\mathcal{C}^2\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$), $\dim (S^2\mathfrak{g}^*)^{\mathfrak{g}} = \frac{p(p+1)}{2} + \dim \operatorname{Im} \mathcal{I}$, with $p = \dim H^1(\mathfrak{g}, \mathbb{C})$. For reductive $\mathfrak{g}$, $\dim (S^2\mathfrak{g}^*)^{\mathfrak{g}} = \dim H^3(\mathfrak{g}, \mathbb{C})$. Note also that the restriction of $\delta_{\mathbb{C}}$ to $(S^2\mathfrak{g}^*)^{\mathfrak{g}}$ is $-\mathcal{I}$.

Definition 3. $\mathfrak{g}$ *is said to be $\mathcal{I}$-null (resp. $\mathcal{I}$-exact) if $\mathcal{I} = 0$ (resp. $Im\mathcal{I} \subset B^3(\mathfrak{g}, \mathbb{C})$).*

For more details on $\mathcal{I}$-null Lie algebras, see [13].

Example 1. The $(2N+1)$-dimensional complex Heisenberg Lie algebra $\mathcal{H}_N$ $(N \geqslant 1)$ with basis $(x_i)_{1 \leqslant i \leqslant 2N+1}$ and nonzero commutation relations (with anticommutativity) $[x_i, x_{N+i}] = x_{2N+1}$ $(1 \leqslant i \leqslant N)$ is $\mathcal{I}$-null since, for any $B \in ({S^2\mathcal{H}_N}^*)^{\mathcal{H}_N}$, $B(x_i, x_{2N+1}) = B(x_i, [x_i, x_{N+i}]) = -B([x_i, x_i], x_{N+i}) = 0$ (similarly with x_{N+i} instead of x_i) $(1 \leqslant i \leqslant N)$, and $B(x_{2N+1}, x_{2N+1}) = B(x_{2N+1}, [x_1, x_{N+1}]) = -B([x_1, x_{2N+1}], x_{N+1}) = 0$.

If $\mathfrak{c}$ denotes the center of $\mathfrak{g}$, then $\mathfrak{c} \otimes (S^2\mathfrak{g}^*)^{\mathfrak{g}}$ is the space of invariant $\mathfrak{c}$-valued symmetric bilinear mapS and we denote $F = Id \otimes \mathcal{I} : \mathfrak{c} \otimes (S^2\mathfrak{g}^*)^{\mathfrak{g}} \to C^3(\mathfrak{g}, \mathfrak{g}) = \mathfrak{g} \otimes \bigwedge^3 \mathfrak{g}^*$. Then $\operatorname{Im} F = \mathfrak{c} \otimes \operatorname{Im} \mathcal{I}$.

Theorem 2. *Let $\mathfrak{g}$ be any finite dimensional complex Lie algebra and $ZL_0^2(\mathfrak{g}, \mathfrak{g})$ (resp. $ZL_0^2(\mathfrak{g}, \mathbb{C})$) the space of symmetric adjoint (resp. trivial) Leibniz 2-cocycles.*

(i) $ZL_0^2(\mathfrak{g},\mathfrak{g}) = \mathfrak{c}\otimes\ker\mathcal{I}$. *In particular,* $\dim ZL_0^2(\mathfrak{g},\mathfrak{g}) = c\,\frac{p(p+1)}{2}$ *where* $c=\dim\mathfrak{c}$ *and* $p=\dim\mathfrak{g}/\mathcal{C}^2\mathfrak{g} = \dim H^1(\mathfrak{g},\mathbb{C})$.
(ii) $ZL^2(\mathfrak{g},\mathfrak{g})\,/(Z^2(\mathfrak{g},\mathfrak{g})\oplus ZL_0^2(\mathfrak{g},\mathfrak{g})) \cong (\mathfrak{c}\otimes Im\mathcal{I})\cap B^3(\mathfrak{g},\mathfrak{g})$.
(iii) $HL^2(\mathfrak{g},\mathfrak{g}) \cong H^2(\mathfrak{g},\mathfrak{g})\oplus(\mathfrak{c}\otimes\ker\mathcal{I})\oplus((\mathfrak{c}\otimes Im\mathcal{I})\cap B^3(\mathfrak{g},\mathfrak{g}))$.
(iv) $ZL_0^2(\mathfrak{g},\mathbb{C}) = \ker\mathcal{I}$.
(v) $ZL^2(\mathfrak{g},\mathbb{C})\,/(Z^2(\mathfrak{g},\mathbb{C})\oplus ZL_0^2(\mathfrak{g},\mathbb{C})) \cong Im\mathcal{I}\cap B^3(\mathfrak{g},\mathbb{C})$.
(vi) $HL^2(\mathfrak{g},\mathbb{C}) \cong H^2(\mathfrak{g},\mathbb{C})\oplus\ker\mathcal{I}\oplus(Im\mathcal{I}\cap B^3(\mathfrak{g},\mathbb{C}))$.

Proof. (i) The Leibniz 2-cochain space $CL^2(\mathfrak{g},\mathfrak{g}) = \mathfrak{g}\otimes(\mathfrak{g}^*)^{\otimes 2}$ decomposes as $\left(\mathfrak{g}\otimes\bigwedge^2\mathfrak{g}^*\right)\oplus\left(\mathfrak{g}\otimes S^2\,\mathfrak{g}^*\right)$ with $\mathfrak{g}\otimes S^2\,\mathfrak{g}^*$ the space of symmetric elements in $CL^2(\mathfrak{g},\mathfrak{g})$. By definition of the Leibniz coboundary δ, one has for $\psi\in CL^2(\mathfrak{g},\mathfrak{g})$ and $X,Y,Z\in\mathfrak{g}$

$$(\delta\psi)(X,Y,Z) = u+v+w+r+s+t \tag{1}$$

with $u=[X,\psi(Y,Z)]$, $v=[\psi(X,Z),Y]$, $w=-[\psi(X,Y),Z]$, $r=-\psi([X,Y],Z)$, $s=\psi(X,[Y,Z])$, $t=\psi([X,Z],Y)$. δ coincides with the usual coboundary operator on $\mathfrak{g}\otimes\bigwedge^2\mathfrak{g}^*$. Now, let $\psi=\psi_1+\psi_0\in CL^2(\mathfrak{g},\mathfrak{g})$, $\psi_1\in\mathfrak{g}\otimes\bigwedge^2\mathfrak{g}^*$, $\psi_0\in\mathfrak{g}\otimes S^2\,\mathfrak{g}^*$.

Suppose $\psi\in ZL^2(\mathfrak{g},\mathfrak{g}) : \delta\psi = 0 = \delta\psi_1+\delta\psi_0 = d\psi_1+\delta\psi_0$. Then $\delta\psi_0 = -d\psi_1\in\mathfrak{g}\otimes\bigwedge^3\mathfrak{g}^*$ is antisymmetric. Then permuting X and Y in formula (1) for ψ_0 yields $(\delta\psi_0)(Y,X,Z) = -v-u+w-r+t+s$. As $\delta\psi_0$ is antisymmetric, we get

$$w+s+t=0. \tag{2}$$

Now, the circular permutation (X,Y,Z) in (1) for ψ_0 yields $(\delta\psi_0)(Y,Z,X) = -v-w+u-s-t+r$. Again, by antisymmetry,

$$v+w+s+t=0, \tag{3}$$

i.e. $(\delta\psi_0)(X,Y,Z)=u+r$.

From (2) and (3), $v=0$. Applying twice the circular permutation (X,Y,Z) to v, we get first $w=0$ and then $u=0$. Hence $(\delta\psi_0)(X,Y,Z) = r = -\psi_0([X,Y],Z)$. Note first that $u=0$ reads $[X,\psi_0(Y,Z)]=0$. As X,Y,Z are arbitrary, ψ_0 is $\mathfrak{c}$-valued. Now the permutation of Y and Z changes r to $-t=s$ (from (3)). Again, by antisymmetry of $\delta\psi_0$, $r=t=-s$. As X,Y,Z are arbitrary, one gets $\psi_0\in\mathfrak{c}\otimes(S^2\mathfrak{g}^*)^{\mathfrak{g}}$. Now $F(\psi_0) = -r = -\delta\psi_0 = d\psi_1\in B^3(\mathfrak{g},\mathfrak{g})$. Hence

$$\psi_0\in ZL_0^2(\mathfrak{g},\mathfrak{g}) \Leftrightarrow F(\psi_0)=0 \Leftrightarrow \psi_1\in Z^2(\mathfrak{g},\mathfrak{g}) \Leftrightarrow \psi_0\in\mathfrak{c}\otimes\ker\mathcal{I}.$$

Consider now the linear map $\Phi\ :\ ZL^2(\mathfrak{g},\mathfrak{g})\to F^{-1}(B^3(\mathfrak{g},\mathfrak{g}))\,/\ker F$ defined by $\psi\mapsto[\psi_0]\,(\mathrm{mod}\ker F)$. Φ is onto: for any $[\varphi_0]\in F^{-1}(B^3(\mathfrak{g},\mathfrak{g}))\,/\ker F$, $\varphi_0\in\mathfrak{c}\otimes(S^2\mathfrak{g}^*)^{\mathfrak{g}}$, one has $F(\varphi_0)\in B^3(\mathfrak{g},\mathfrak{g})$, hence $F(\varphi_0)=d\varphi_1$, $\varphi_1\in C^2(\mathfrak{g},\mathfrak{g})$, and then $\varphi=\varphi_0+\varphi_1$ is a Leibniz cocycle such that $\Phi(\varphi)=[\varphi_0]$. Now $\ker\Phi = Z^2(\mathfrak{g},\mathfrak{g})\oplus ZL_0^2(\mathfrak{g},\mathfrak{g})$, since condition $[\psi_0]=[0]$ reads $\psi_0\in\ker F$ which is equivalent to $\psi\in Z^2(\mathfrak{g},\mathfrak{g})\oplus ZL_0^2(\mathfrak{g},\mathfrak{g})$.

Hence Φ yields an isomorphism $ZL^2(\mathfrak{g},\mathfrak{g})\,/(Z^2(\mathfrak{g},\mathfrak{g})\oplus ZL_0^2(\mathfrak{g},\mathfrak{g}))\cong F^{-1}(B^3(\mathfrak{g},\mathfrak{g}))\,/\,\ker F$. The latter is isomorphic to $\operatorname{Im} F\cap B^3(\mathfrak{g},\mathfrak{g})\cong(\mathfrak{c}\otimes\operatorname{Im}\mathcal{I})\cap B^3(\mathfrak{g},\mathfrak{g})$.
(ii) results from the invariance of $\psi_0\in ZL_0^2(\mathfrak{g},\mathfrak{g})$.
(iii) results immediately from (i) and (ii) since $BL^2(\mathfrak{g},\mathfrak{g})=B^2(\mathfrak{g},\mathfrak{g})$ as the Leibniz differential on $CL^1(\mathfrak{g},\mathfrak{g})=\mathfrak{g}^*\otimes\mathfrak{g}=C^1(\mathfrak{g},\mathfrak{g})$ coincides with the usual one.
(iv)-(vi) are similar. □

Remark 1. Since $\ker\mathcal{I}\oplus(\operatorname{Im}\mathcal{I}\cap B^3(\mathfrak{g},\mathbb{C}))\cong\ker h$ where h denotes $\mathcal{I}$ composed with the projection of $Z^3(\mathfrak{g},\mathbb{C})$ onto $H^3(\mathfrak{g},\mathbb{C})$, the result (vi) is the same as in [14].

Remark 2. Any supplementary subspace to $Z^2(\mathfrak{g},\mathbb{C})\oplus ZL_0^2(\mathfrak{g},\mathbb{C})$ in $ZL^2(\mathfrak{g},\mathbb{C})$ consists of coupled Leibniz 2-cocycles, i.e. the nonzero elements have the property that their symmetric and antisymmetric parts are not cocycles. To get such a supplementary subspace, pick any supplementary subspace W to $\ker\mathcal{I}$ in $(S^2\mathfrak{g}^*)^{\mathfrak{g}}$ and take $\mathcal{C}=\{B+\omega\,;B\in W\cap\mathcal{I}^{-1}(B^3(\mathfrak{g},\mathbb{C})),I_B=d\omega\}$.

Definition 4. $\mathfrak{g}$ *is said to be an* adjoint (resp. trivial) ZL^2-uncoupling *if*

$$(\mathfrak{c}\otimes Im\mathcal{I})\cap B^3(\mathfrak{g},\mathfrak{g})=\{0\}\ \big(resp.\ \ Im\mathcal{I}\cap B^3(\mathfrak{g},\mathbb{C})=\{0\}\big)\,.$$

The class of adjoint ZL^2-uncoupling Lie algebras is rather extensive since it contains all zero-center Lie algebras and all $\mathcal{I}$-null Lie algebras. For non zero-center, adjoint ZL^2-uncoupling implies trivial ZL^2-uncoupling, since $\mathfrak{c}\otimes(\operatorname{Im}\mathcal{I}\cap B^3(\mathfrak{g},\mathbb{C}))\subset(\mathfrak{c}\otimes\operatorname{Im}\mathcal{I})\cap B^3(\mathfrak{g},\mathfrak{g})$. The reciprocal holds obviously true for $\mathcal{I}$-exact Lie algebras. However we do not know if it holds true in general (e.g. we do not know of a nilpotent Lie algebra which is not $\mathcal{I}$-exact).

Corollary 1. *(i)* $HL^2(\mathfrak{g},\mathfrak{g})\cong H^2(\mathfrak{g},\mathfrak{g})\oplus(\mathfrak{c}\otimes\ker\mathcal{I})$ *if and only if* $\mathfrak{g}$ *is adjoint* ZL^2*-uncoupling.*
(ii) $HL^2(\mathfrak{g},\mathbb{C})\cong H^2(\mathfrak{g},\mathbb{C})\oplus\ker\mathcal{I}$ *if and only if* $\mathfrak{g}$ *is trivial* ZL^2*-uncoupling.*

Corollary 2. *For any Lie algebra* $\mathfrak{g}$ *with trivial center* $\mathfrak{c}=\{0\}$, $HL^2(\mathfrak{g},\mathfrak{g})=H^2(\mathfrak{g},\mathfrak{g})$.

Remark 3. *This fact also follows from the cohomological version of Theorem A in* [17].

Proof. Let $\mathfrak{g}$ be a Lie algebra and M be a right $\mathfrak{g}$-module. Consider the product map $m:\mathfrak{g}\otimes\Lambda^n\mathfrak{g}\longrightarrow\Lambda^{n+1}$ in the exterior algebra. This map yields an epimorphism of chain complexes

$$C_*(\mathfrak{g},\mathfrak{g})\longrightarrow C_*(\mathfrak{g},\mathbb{K})[-1],$$

where $C_*(\mathfrak{g},\mathbb{K})$ is the reduced chain complex:

$$C_0(\mathfrak{g},\mathbb{K})=0,\quad C_i(\mathfrak{g},\mathbb{K})=C_i(\mathfrak{g},\mathbb{K})\ \text{ for }\ i>0.$$

Define the reduced chain complex $CR_*(\mathfrak{g})$ such that $CR_*(\mathfrak{g}[1])$ is the kernel of the epimorphism $C_*(\mathfrak{g},\mathfrak{g}) \longrightarrow C_*(\mathfrak{g},\mathbb{K})[-1]$. Denote the cohomology of $CR_*(\mathfrak{g})$ by $HR_*(\mathfrak{g})$.

Let us recall Theorem A in [17]. It states that there exists a spectral sequence

$$E^2_{pq} = HR_p(\mathfrak{g} \otimes HL_q(\mathfrak{g}, M)) \Longrightarrow H^{rel}_{p+q}(\mathfrak{g}, M).$$

As the center of our Lie algebra is 0, it follows that $E^2_{00} = 0$, and so we get $H^{rel}_0(\mathfrak{g},\mathfrak{g}) = 0$.

But then from the exact sequence in [17]

$$0 \leftarrow H_2(\mathfrak{g}, M) \leftarrow HL_2(\mathfrak{g}, M) \leftarrow H^{rel}_0(\mathfrak{g}, M) \leftarrow H_3(\mathfrak{g}, M) \leftarrow ...$$

we get

$$HL_2(\mathfrak{g}, M) = H_2(\mathfrak{g}, M).$$

□

Corollary 3. *For any reductive Lie algebra* $\mathfrak{g}$ *with center* $\mathfrak{c}$, $HL^2(\mathfrak{g},\mathfrak{g}) = H^2(\mathfrak{g},\mathfrak{g}) \oplus (\mathfrak{c} \otimes S^2\mathfrak{c}^*)$, *and* $\dim H^2(\mathfrak{g},\mathfrak{g}) = \frac{c^2(c-1)}{2}$ *with* $c = \dim \mathfrak{c}$.

Proof. $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{c}$ with $\mathfrak{s} = \mathcal{C}^2\mathfrak{g}$ semisimple. We first prove that $\mathfrak{g}$ is adjoint ZL^2-uncoupling. $\mathfrak{c} \otimes (S^2\mathfrak{g}^*)^{\mathfrak{g}} = \left(\mathfrak{c} \otimes (S^2\mathfrak{s}^*)^{\mathfrak{s}}\right) \oplus (\mathfrak{c} \otimes S^2\mathfrak{c}^*) = c\ (S^2\mathfrak{s}^*)^{\mathfrak{s}} \oplus c\ (S^2\mathfrak{c}^*)$. Suppose first $\mathfrak{s}$ simple. Then any bilinear symmetric invariant form on $\mathfrak{s}$ is some multiple of the Killing form K. Hence $\mathfrak{c} \otimes (S^2\mathfrak{g}^*)^{\mathfrak{g}} = c\,(\mathbb{C}K) \oplus c\ (S^2\mathfrak{c}^*)$. For any $\psi_0 \in \mathfrak{c} \otimes (S^2\mathfrak{g}^*)^{\mathfrak{g}}$, $F(\psi_0)$ is then some linear combination of copies of I_K. As is well-known, I_K is no coboundary. Hence if we suppose that $F(\psi_0)$ is a coboundary, necessarily $F(\psi_0) = 0$. $\mathfrak{g}$ is adjoint ZL^2-uncoupling when $\mathfrak{s}$ is simple. Now, if $\mathfrak{s}$ is not simple, $\mathfrak{s}$ can be decomposed as a direct sum $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m$ of simple ideals of $\mathfrak{s}$. Then $(S^2\mathfrak{s}^*)^{\mathfrak{s}} = \bigoplus_{i=1}^m (S^2\mathfrak{s}_{\mathfrak{i}}{}^*)^{\mathfrak{s}_{\mathfrak{i}}} = \bigoplus_{i=1}^m \mathbb{C}\,K_i$ (K_i Killing form of $\mathfrak{s}_i$.) The same reasoning then applies and shows that $\mathfrak{g}$ is adjoint ZL^2-uncoupling. ¿From (ii) in theorem 2, $ZL^2_0(\mathfrak{g},\mathfrak{g}) = \mathfrak{c} \otimes S^2\mathfrak{c}^*$. Now, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{c}$ with $\mathfrak{s} = \mathcal{C}^2\mathfrak{g}$ semisimple. $\mathfrak{s}$ can be decomposed as a direct sum $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m$ of ideals of $\mathfrak{s}$ hence of $\mathfrak{g}$. Then $H^2(\mathfrak{g},\mathfrak{g}) = \bigoplus_{i=1}^m H^2(\mathfrak{g},\mathfrak{s}_i) \oplus H^2(\mathfrak{g},\mathfrak{c})$. As $\mathfrak{s}_i$ is a nontrivial $\mathfrak{g}$-module, $H^2(\mathfrak{g},\mathfrak{s}_i) = \{0\}$ ([6], Prop. 11.4, page 154). Hence $H^2(\mathfrak{g},\mathfrak{g}) = H^2(\mathfrak{g},\mathfrak{c}) = c\,H^2(\mathfrak{g},\mathbb{C})$. By the Künneth formula and Whitehead's lemmas,

$$\begin{aligned} H^2(\mathfrak{g},\mathbb{C}) = &\left(H^2(\mathfrak{s},\mathbb{C}) \otimes H^0(\mathfrak{c},\mathbb{C})\right) \oplus \left(H^1(\mathfrak{s},\mathbb{C})\right. \\ &\left.\otimes H^1(\mathfrak{c},\mathbb{C})\right) \oplus \left(H^0(\mathfrak{s},\mathbb{C}) \otimes H^2(\mathfrak{c},\mathbb{C})\right) \\ =&H^0(\mathfrak{s},\mathbb{C}) \otimes H^2(\mathfrak{c},\mathbb{C}) \\ =&\mathbb{C} \otimes H^2(\mathfrak{c},\mathbb{C}). \end{aligned}$$

.
Hence

$$\dim H^2(\mathfrak{g},\mathfrak{g}) = \frac{c^2(c-1)}{2}.$$

□

4. Examples

For $\omega, \pi \in \mathfrak{g}^*$, $\odot$ stands for the symmetric product $\omega \odot \pi = \omega \otimes \pi + \pi \otimes \omega$.

Example 2. For $\mathfrak{g} = \mathfrak{gl}(n)$,

$$HL^2(\mathfrak{g},\mathfrak{g}) = ZL_0^2(\mathfrak{g},\mathfrak{g}) = \mathbb{C}x_{n^2} \oplus \mathbb{C}(\omega^{n^2} \odot \omega^{n^2}),$$

where $(x_i)_{1\leqslant i\leqslant n^2}$ is a basis of $\mathfrak{g}$ such that $(x_i)_{1\leqslant i\leqslant n^2-1}$ is a basis of $\mathfrak{sl}(n)$ and x_{n^2} is the identity matrix, and $(\omega^i)_{1\leqslant i\leqslant n^2}$ the dual basis to $(x_i)_{1\leqslant i\leqslant n^2}$. Hence there is a unique Leibniz deformation of $\mathfrak{gl}(n)$.

Corollary 4. *Let $\mathfrak{g} = \mathcal{H}_N$ be the $(2N+1)$-dimensional complex Heisenberg Lie algebra ($N \geqslant 1$) as in example 1.*
(i) *$ZL_0^2(\mathcal{H}_N, \mathcal{H}_N)$ has basis $(x_{2N+1}\otimes(\omega^i\odot\omega^j))_{1\leqslant i\leqslant j\leqslant 2N}$ with $(\omega^i)_{1\leqslant i\leqslant 2N+1}$ the dual basis to $(x_i)_{1\leqslant i\leqslant 2N+1}$*
(ii)

$$\dim ZL_0^2(\mathcal{H}_N, \mathcal{H}_N) = \dim B^2(\mathcal{H}_N, \mathcal{H}_N) = N(2N+1);$$

$$\dim HL^2(\mathcal{H}_N, \mathcal{H}_N) = \dim Z^2(\mathcal{H}_N, \mathcal{H}_N) = \begin{cases} \frac{N}{3}(8N^2+6N+1) & \text{if } N \geqslant 2 \\ 8 & \text{if } N = 1\,. \end{cases}$$

Proof. (i) This follows from $\ker \mathcal{I} = S^2\left(\mathfrak{g}/\mathcal{C}^2\mathfrak{g}\right)^*$.

(ii) $\mathcal{H}_N$ consists of adjoint ZL^2-uncouplings since it is $\mathcal{I}$-null. The result then follows from the fact that ([9]) $\dim B^2(\mathcal{H}_N, \mathcal{H}_N) = N(2N+1)$ and for $N \geqslant 2$, $\dim H^2(\mathcal{H}_N, \mathcal{H}_N) = \frac{2N}{3}(4N^2-1)$. □

Example 3. The case $N = 1$ has been studied in [3]. In that case, $\dim ZL_0^2(\mathcal{H}_1, \mathcal{H}_1) = 3$ and the 3 Leibniz deformations are nilpotent, in contradistinction with the 5 Lie deformations. The authors completely describe a Leibniz versal deformation of the 3-dimensional Heisenberg algebra.

Example 4. The 4-dimensional complex solvable "diamond" Lie algebra $\mathfrak{d}$ has basis (x_1, x_2, x_3, x_4) and nonzero commutation relations (with anticommutativity)

$$[x_1, x_2] = x_3, [x_1, x_3] = -x_2, [x_2, x_3] = x_4. \quad (4)$$

The relations show that $\mathfrak{d}$ is an extension of the one-dimensional abelian Lie algebra $\mathbb{C}x_1$ by the Heisenberg algebra $\mathfrak{n}_3$ with basis x_2, x_3, x_4. It is also known as the Nappi-Witten Lie algebra [15] or the central extension of the Poincaré Lie algebra in two dimensions. It is a solvable quadratic Lie algebra, as it admits a nondegenerate bilinear symmetric invariant form. Because of these properties, it plays an important role in conformal field theory. We can use $\mathfrak{d}$ to construct

a Wess-Zumino-Witten model, which describes a homogeneous four-dimensional Lorentz-signature space time [15]. It is easy to check that $\mathfrak{d}$ is $\mathcal{I}$-exact. In fact, one verifies that all other solvable 4-dimensional Lie algebras are $\mathcal{I}$-null (for a list, see e.g. [16]).

Consider $\mathfrak{d}$ as Leibniz algebra with basis $\{e_1,\ e_2,\ e_3,\ e_4\}$ over $\mathbb{C}$. Define a bilinear map $[\ ,\ ] : \mathfrak{d} \times \mathfrak{d} \longrightarrow \mathfrak{d}$ by $[e_2, e_3] = e_1$, $[e_3, e_2] = -e_1$, $[e_2, e_4] = e_2$, $[e_4, e_2] = -e_2$, $[e_3, e_4] = e_2 - e_3$ and $[e_4, e_3] = e_3 - e_2$, all other products of basis elements being 0.

We get a basis satisfying the usual commutation relations (4) by letting

$$x_1 = ie_4, \quad x_2 = e_3, \quad x_3 = i(-e_2 + e_3), \quad x_4 = ie_1. \tag{5}$$

One should mention that even though these two forms are equivalent over $\mathbb{C}$, they represent the two nonisomorphic real forms of the complex diamond algebra.

We found that by considering Leibniz algebra deformations of $\mathfrak{d}$ one gets more structures. Indeed it gives not only extra stuctures but also keeps the Lie structures obtained by considering Lie algebra deformations. To get the precise deformations we need to consider the cohomology groups.

We compute cohomologies necessary for our purpose. First consider the Leibniz cohomology space $HL^2(L; L)$. Our computation consists of the following steps:
(i) determine a basis of the space of cocycles $ZL^2(L; L)$,
(ii) determine a basis of the coboundary space $BL^2(L; L)$,
(iii) determine a basis of the quotient space $HL^2(L; L)$.

(i) Let $\psi \in ZL^2(L; L)$. Then $\psi : L \otimes L \longrightarrow L$ is a linear map and $\delta\psi = 0$, where

$$\begin{aligned}\delta\psi(e_i, e_j, e_k) = {}& [e_i, \psi(e_j, e_k)] + [\psi(e_i, e_k), e_j] - [\psi(e_i, e_j), e_k] - \psi([e_i, e_j], e_k)\\ & + \psi(e_i, [e_j, e_k]) + \psi([e_i, e_k], e_j) \text{ for } 0 \le i, j, k \le 4.\end{aligned}$$

Suppose $\psi(e_i, e_j) = \sum_{k=1}^{4} a_{i,j}^k e_k$ where $a_{i,j}^k \in \mathbb{C}$; for $1 \le i, j, k \le 4$. Since $\delta\psi = 0$, equating the coefficients of e_1, e_2, e_3 and e_4 in $\delta\psi(e_i, e_j, e_k)$

we get the following relations:

(*i*) $a^1_{1,1} = a^2_{1,1} = a^3_{1,1} = a^4_{1,1} = a^1_{1,2} = a^3_{1,2} = a^4_{1,2} = 0$;

(*ii*) $a^4_{1,3} = a^3_{1,4} = a^4_{1,4} = a^1_{2,1} = a^3_{2,1} = a^4_{2,1} = a^1_{2,2} = a^2_{2,2} = a^3_{2,2} = a^4_{2,2} = 0$;

(*iii*) $a^4_{3,1} = a^2_{3,3} = a^3_{3,3} = a^4_{3,3} = a^3_{4,1} = a^4_{4,1} = a^2_{4,4} = a^3_{4,4} = a^4_{4,4} = 0$;

(*iv*) $a^2_{1,2} = -a^2_{2,1} = a^2_{1,3} = -a^3_{1,3} = -a^2_{3,1} = a^3_{3,1}$;

(*v*) $a^1_{1,3} = -a^1_{3,1} = a^2_{1,4} = -a^2_{4,1}$;

(*vi*) $a^3_{2,3} = -a^3_{3,2} = -a^4_{2,4} = a^4_{4,2};\ a^4_{2,3} = -a^4_{3,2};\ a^2_{2,3} = -a^2_{3,2}$;

(*vi*) $a^1_{2,4} = -a^1_{4,2};\ a^2_{2,4} = -a^2_{4,2};\ a^3_{2,4} = -a^3_{4,2}$;

(*vii*) $a^1_{3,4} = -a^1_{4,3};\ a^2_{3,4} = -a^2_{4,3};\ a^3_{3,4} = -a^3_{4,3};\ a^4_{3,4} = -a^4_{4,3}$

(*ix*) $a^3_{3,4} = (a^1_{14} - a^2_{24});\ a^4_{3,4} = (a^2_{14} + a^2_{23})$

$(x) a^1_{33} = \frac{1}{2}(a^1_{23} + a^1_{32});\ a^1_{41} = -(a^1_{14} + a^1_{23} + a^1_{32})$.

Therefore, in terms of the ordered basis $\{e_i \otimes e_j\}_{1\leq i,j\leq 4}$ of $L \otimes L$ and $\{e_i\}_{1\leq i\leq 4}$ of L, the transpose of the matrix corresponding to ψ is of the form

$$M^t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ x_2 & x_1 & -x_1 & 0 \\ x_3 & x_2 & 0 & 0 \\ 0 & -x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_4 & x_5 & x_6 & x_7 \\ x_8 & x_9 & x_{10} & -x_6 \\ -x_2 & -x_1 & x_1 & 0 \\ x_{11} & -x_5 & -x_6 & -x_7 \\ \frac{1}{2}(x_4 + x_{11}) & 0 & 0 & 0 \\ x_{12} & x_{13} & (x_3 - x_9) & (x_2 + x_5) \\ -(x_4 + x_3 + x_{11}) & -x_2 & 0 & 0 \\ -x_8 & -x_9 & -x_{10} & x_6 \\ -x_{12} & -x_{13} & -(x_3 - x_9) & -(x_2 + x_5) \\ x_{14} & 0 & 0 & 0 \end{pmatrix}.$$

where $x_1 = a^2_{1,2};\ x_2 = a^1_{1,3};\ x_3 = a^1_{1,4};\ x_4 = a^1_{2,3};\ x_5 = a^2_{2,3};\ x_6 = a^3_{2,3};$
$x_7 = a^4_{2,3};\ x_8 = a^1_{2,4};\ x_9 = a^2_{2,4}; x_{10} = a^3_{2,4};\ x_{11} = a^1_{3,2};\ x_{12} = a^1_{3,4};$
$x_{13} = a^2_{3,4}$ and $x_{14} = a^1_{4,4}$

are in $\mathbb{C}$. Let $\phi_i \in ZL^2(L; L)$ for $1 \leq i \leq 14$, be the cocyle with $x_i = 1$ and $x_j = 0$ for $i \neq j$ in the above matrix of ψ. It is easy to check that $\{\phi_1, \cdots, \phi_{14}\}$ forms a basis of $ZL^2(L; L)$.

(ii) Let $\psi_0 \in BL^2(L;L)$. We have $\psi_0 = \delta g$ for some 1-cochain $g \in CL^1(L;L) = \mathrm{Hom}\,(L;L)$. Suppose the matrix associated to ψ_0 is same as the above matrix M.

Let $g(e_i) = a_i^1 e_1 + a_i^2 e_2 + a_i^3 e_3 + a_i^4 e_4$ for $i = 1,2,3,4$. The matrix associated to g is given by

$$(a_i^j)_{i,j=1,\ldots,4}$$

From the definition of coboundary we get

$$\delta g(e_i, e_j) = [e_i, g(e_j)] + [g(e_i), e_j] - \psi([e_i, e_j])$$

for $0 \leq i, j \leq 4$. The transpose matrix of δg can be written as

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
-a_1^3 & -a_1^4 & 0 & 0 \\
a_1^2 & -a_1^4 & a_1^4 & 0 \\
0 & (a_1^2 + a_1^3) & -a_1^3 & 0 \\
a_1^3 & a_1^4 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-(a_1^1 - a_2^2 - a_3^3) & -(a_1^2 + a_2^4 - a_3^4) & -(a_1^3 - a_2^4 & -a_1^4 \\
-(a_2^1 - a_4^3) & (a_2^3 + a_4^4) & -2a_2^3 & -a_2^4 \\
-a_1^2 & a_1^4 & -a_1^4 & 0 \\
(a_1^1 - a_2^2 - a_3^3) & (a_1^2 + a_2^4 - a_3^4) & (a_1^3 - a_2^4) & a_1^4 \\
0 & 0 & 0 & 0 \\
-(a_2^1 - a_3^1 + a_4^2) & -(a_2^2 - 2a_3^2 - a_3^3 - a_4^4) & -(a_2^3 + a_4^4) & -(a_2^4 - a_3^4) \\
0 & -(a_1^2 + a_1^3) & a_1^3 & 0 \\
(a_2^1 - a_4^3) & -(a_2^3 + a_4^4) & 2a_2^3 & a_2^4 \\
(a_2^1 - a_3^1 + a_4^2) & (a_2^2 - 2a_3^2 - a_3^3 - a_4^4) & (a_2^3 + a_4^4) & (a_2^4 - a_3^4) \\
0 & 0 & 0 & 0
\end{pmatrix}.$$

Since $\psi_0 = \delta g$ is also a cocycle in $CL^2(L;L)$, comparing matrices δg and M we conclude that the transpose matrix of ψ_0 is of the form

$$M^t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ x_2 & x_1 & -x_1 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & -x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_4 & x_5 & x_6 & x_1 \\ x_8 & x_9 & x_{10} & -x_6 \\ -x_2 & -x_1 & x_1 & 0 \\ -x_4 & -x_5 & -x_6 & -x_1 \\ 0 & 0 & 0 & 0 \\ x_{12} & x_{13} & -x_9 & (x_2+x_5) \\ 0 & -x_2 & 0 & 0 \\ -x_8 & -x_9 & -x_{10} & x_6 \\ -x_{12} & -x_{13} & x_9 & -(x_2+x_5) \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let ${\phi_i}' \in BL^2(L;L)$ for $i = 1,2,4,5,6,8,9,10,12,13$ be the coboundary with $x_i = 1$ and $x_j = 0$ for $i \neq j$ in the above matrix of ψ_0. It follows that $\{\phi_1', \phi_2', \phi_4', \phi_5', \phi_6', \phi_8', \phi_9', \phi_{10}', \phi_{12}', \phi_{13}'\}$ forms a basis of the coboundary space $BL^2(L;L)$.

(iii) It is straightforward to check that

$$\{[\phi_3], [\phi_7], [\phi_{11}], [\phi_{14}]\}$$

span $HL^2(L;L)$ where $[\phi_i]$ denotes the cohomology class represented by the cocycle ϕ_i.

Thus $\dim(HL^2(L;L)) = 4$.

The representative cocycles of the cohomology classes forming a basis of $HL^2(L;L)$ are given explicitely as the following.

(1) $\phi_3 : \phi_3(e_1,e_4) = e_1,\ \phi_3(e_4,e_1) = -e_1;\ \phi_3(e_3,e_4) = e_3;\ \phi_3(e_4,e_3) = -e_3;$

(2) $\phi_7 : \phi_7(e_2,e_3) = e_4,\ \phi_7(e_3,e_2) = -e_4;$

(3) $\phi_{11} : \phi_{11}(e_3,e_2) = e_1,\ \phi_{11}(e_3,e_3) = \frac{1}{2}e_1,\ \phi_{11}(e_4,e_1) = -e_1;$

(4) $\phi_{14} : \phi_{14}(e_4,e_4) = e_1.$

Here ϕ_3 and ϕ_7 are skew-symmetric, so $\phi_i \in Hom(\Lambda^2 L; L) \subset Hom(L^{\otimes 2}; L)$ for $i = 3$ and 7.

Consider $\mu_i = \mu_0 + t\phi_i$ for $i = 3,7,11,14$, where μ_0 denotes the original bracket in L.

This gives 4 non-equivalent infinitesimal deformations of the Leibniz bracket μ_0 with μ_3 and μ_7 giving the Lie algebra structure on the factor space $L[[t]]/ < t^2 >$.

Now we have to compute the nontrivial Massey brackets which give relations on the base of the miniversal deformation.

Let us start to compute the nonzero brackets $[\phi_i, \phi_i]$ which are the obstructions to extending infinitesimal deformations. We find

$$[\phi_3, \phi_3] = 0, \quad [\phi_7, \phi_7] = 0.$$

That means that these two infinitesimal Lie deformations can be extended. In fact, they can be extended to real Lie deformations as follows.

We give the new nonzero Lie brackets (and their anticommutative analogue).

The first deformation

$$\begin{aligned} [e_2, e_3]_t &= e_1 + te_4 \\ [e_2, e_4]_t &= e_2 \\ [e_3, e_4]_t &= e_2 - e_3 \end{aligned}$$

is isomorphic to $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$ for every nonzero value of t, see [5].

The second deformation represents a 2-parameter projective family $d(\lambda, \mu)$, for which each projective parameter (λ, μ) defines a nonisomorphic Lie algebra (in fact, the diamond algebra is a member of this family with $(\lambda, \mu) = (1, -1)$):

$$\begin{aligned} [e_2, e_3]_{\lambda,\mu} &= e_1 \\ [e_2, e_4]_{\lambda,\mu} &= \lambda e_2 \\ [e_3, e_4]_{\lambda,\mu} &= e_2 + \mu e_3 \\ [e_1, e_4]_{\lambda,\mu} &= (\lambda + \mu) e_1. \end{aligned}$$

Furthermore, we also have $[\phi_{14}, \phi_{14}] = 0$ which means that ϕ_{14} defines a real Leibniz deformation:

$$\begin{aligned} [e_2, e_3]_t &= e_1 \\ [e_2, e_4]_t &= e_2 \\ [e_3, e_4]_t &= e_2 - e_3 \\ [e_4, e_4]_t &= te_1. \end{aligned}$$

We note that this Leibniz algebra is not nilpotent.

For the bracket $[\phi_{11}, \phi_{11}]$ we get a nonzero 3-cocycle, so the infinitesimal Leibniz deformation with infinitesimal part being ϕ_{11} can not be extended even to the next order. That means it gives a relation on the base of the versal deformation.

The nontrivial mixed brackets $[\phi_i, \phi_j]$ also determine relations on the base of the versal deformation.

Among the six possible cases $[\phi_3, \phi_{11}]$, $[\phi_3, \phi_{14}]$ and $[\phi_{11}, \phi_{14}]$ are nontrivial 3-cocycles, the others are represented by 3-coboundaries.

Thus we need to check the Massey 3-brackets which are defined, namely

$< \phi_3, \phi_3, \phi_7 >$

$< \phi_3, \phi_7, \phi_7 >$
$< \phi_7, \phi_7, \phi_{11} >$
$< \phi_7, \phi_7, \phi_{14} >$
$< \phi_7, \phi_{14}, \phi_{14} >$

In these five possible Massey 3-brackets, only $< \phi_3, \phi_3, \phi_7 >$ is represented by nontrivial cocycle.

So we now proceed to compute the possible Massey 4-brackets. We get that four of them are nontrivial:

$< \phi_3, \phi_7, \phi_7, \phi_{11} >$
$< \phi_3, \phi_7, \phi_7, \phi_{14} >$
$< \phi_7, \phi_7, \phi_{14}, \phi_{11} >$
$< \phi_7, \phi_7, \phi_{14}, \phi_{14} >$.

At the next step, we get that all the Massey 5-brackets which are defined are trivial.

So we can write the versal Leibniz deformation of our Lie algebra:

$$\begin{aligned}
&[e_1, e_2]_v = [e_2, e_1]_v = [e_1, e_3]_v = [e_3, e_1]_v = 0,\\
&[e_1, e_4]_v = te_1, \quad [e_4, e_1]_v = -(t+u)e_1,\\
&[e_2, e_3]_v = e_1 + se_4, \quad [e_3, e_2]_v = (u-1)e_1 - se_4,\\
&[e_2, e_4]_v = e_2, \quad [e_4, e_2] = -e_2,\\
&[e_3, e_4]_v = e_2 + (t-1)e_3, \quad [e_4, e_3]_v = -e_2 + (1-t)e_3,\\
&[e_1, e_1]_v = [e_2, e_2]_v = 0, \quad [e_3, e_3]_v = 1/2ue_1,\\
&[e_4, e_4]_v = we_1.
\end{aligned}$$

With the nontrivial Massey brackets and the identification $t = \phi_3, s = \phi_7, u = \phi_{11}, w = \phi_{14}$, we get that the base of the versal deformation is

$$\mathbb{C}[[t, s, u, w]]/\{u^2, tu, tw, uw; t^2 s; ts^2 u, ts^2 w, s^2 uw, s^2 w^2\}.$$

Example 5. The quadratic 5-dimensional nilpotent Lie algebra $\mathfrak{g}_{5,4}$ [11] has commutation relations $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$, $[x_2, x_3] = x_5$.

This is an extension of the trivial Lie algebra $\mathbb{C}x_1$ by the 4-dimensional Lie algebra $\mathbb{C}x_4 \times \mathfrak{n}_3$ ($\mathfrak{n}_3$ the 3-dimensional Heisenberg Lie algebra $[x_2, x_3] = x_5$). As it is moreover the only 5-dimensional indecomposable nilpotent Lie algebra which is not $\mathcal{I}$-null, it can be considered as a 5-dimensional analogue of the diamond algebra $\mathfrak{d}$.

Let us first compute its trivial Leibniz cohomology. We here denote simply d for $d_{\mathbb{C}}$, and $\omega^{i,j}$ for $\omega^i \wedge \omega^j$ (see also [10],[12]).
$B^2(\mathfrak{g}, \mathbb{C}) = \langle d\omega^3 = -\omega^{1,2}, d\omega^4 = -\omega^{1,3}, d\omega^5 = -\omega^{2,3}\rangle$, $\dim Z^2(\mathfrak{g}, \mathbb{C}) = 6$, $\dim H^2(\mathfrak{g}, \mathbb{C}) = 3$, $Z^2(\mathfrak{g}, \mathbb{C}) = \langle \omega^{1,4}, \omega^{2,5}, \omega^{1,5} + \omega^{2,4}\rangle \oplus B^2(\mathfrak{g}, \mathbb{C})$, $\dim ZL_0^2(\mathfrak{g}, \mathbb{C}) = 3$, $ZL_0^2(\mathfrak{g}, \mathbb{C})(\cong \ker \mathcal{I}) = \langle \omega^1 \otimes \omega^1, \omega^1 \odot \omega^2, \omega^2 \otimes \omega^2\rangle$, $\dim ZL^2(\mathfrak{g}, \mathbb{C}) = 10$, $\dim HL^2(\mathfrak{g}, \mathbb{C}) = 7$, and

$$\begin{aligned}
ZL^2(\mathfrak{g}, \mathbb{C}) &= Z^2(\mathfrak{g}, \mathbb{C}) \oplus ZL_0^2(\mathfrak{g}, \mathbb{C}) \oplus \mathbb{C}g_1,\\
HL^2(\mathfrak{g}, \mathbb{C}) &= H^2(\mathfrak{g}, \mathbb{C}) \oplus ZL_0^2(\mathfrak{g}, \mathbb{C}) \oplus \mathbb{C}g_1
\end{aligned}$$

with $g_1 = B + \omega^{1,5}$ and $B = \omega^1 \odot \omega^5 - \omega^2 \odot \omega^4 + \omega^3 \otimes \omega^3$. (Here $\mathrm{Im}\,\mathcal{I} = \mathbb{C}I_B = \mathbb{C}d\omega^{1,5}$ and $\mathrm{Im}\,\mathcal{I} \cap B^3(\mathfrak{g},\mathbb{C}) = \mathrm{Im}\,\mathcal{I}$ is one-dimensional.) $\mathfrak{g}_{5,4}$ is not trivial ZL^2-uncoupling (hence not adjoint ZL^2-uncoupling either), and g_1 is a coupled Leibniz 2-cocycle.

Now let us turn to the adjoint Leibniz cohomology, which represents nonequivalent infinitesimal Leibniz deformations.
$\dim Z^2(\mathfrak{g},\mathfrak{g}) = 24$; $ZL^2_0(\mathfrak{g},\mathfrak{g}) = \mathfrak{c} \otimes \ker \mathcal{I}$ has dimension 6, $\dim ZL^2(\mathfrak{g},\mathfrak{g}) = 32$,

$$\begin{aligned} ZL^2(\mathfrak{g},\mathfrak{g}) &= Z^2(\mathfrak{g},\mathfrak{g}) \oplus ZL^2_0(\mathfrak{g},\mathfrak{g}) \oplus \mathbb{C}G_1 \oplus \mathbb{C}G_2, \\ HL^2(\mathfrak{g},\mathfrak{g}) &= H^2(\mathfrak{g},\mathfrak{g}) \oplus ZL^2_0(\mathfrak{g},\mathfrak{g}) \oplus \mathbb{C}G_1 \oplus \mathbb{C}G_2, \end{aligned}$$

where G_1, G_2 are the following Leibniz 2-cocycles, each of which is coupled:

$$\begin{aligned} G_1 &= x_5 \otimes (B + \omega^{1,5}) \\ G_2 &= x_4 \otimes (B + \omega^{1,5}) \end{aligned}$$

Here $H^2(\mathfrak{g},\mathfrak{g})$ has dimension 9.
Of course these spaces are too huge to compute, but we would like to point out some structural similarity with the diamond algebra.

One may observe that the coupled cocycle ϕ_{11} of $\mathfrak{d}$ reads in the basis (5)

$$\phi_{11} = -ix_4 \otimes (C - \omega^{2,3} + \omega^{1,4})$$

with $C = \omega^1 \odot \omega^4 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$ the non degenerate invariant bilinear form, a similarity with G_1, G_2. The similarity extends to the fact that G_1, G_2 cannot be extended to the second level.

As for Lie deformations, $\mathfrak{g}_{5,4}$ has a number of deformations. Without identifying all of them, we list some:
1. A three-parameter solvable projective family $d(p : q : r)$ where $\mathfrak{g}_{5,4}$ belongs (it is its nilpotent element, with $p = q = r = 0$) with nonzero brackets

$$\begin{aligned} &[x_3, x_4]_{p,q,r} = x_2 \\ &[x_1, x_5]_{p,q,r} = rx_1 \\ &[x_2, x_5]_{p,q,r} = (p+q)x_2 \\ &[x_3, x_5]_{p,q,r} = px_3 + x_1 \\ &[x_4, x_5]_{p,q,r} = x_3 + qx_4. \end{aligned}$$

2. A solvable Lie algebra with nonzero brackets

$$[x_3, x_4] = 2x_4$$
$$[x_3, x_5] = -2x_5$$
$$[x_4, x_5] = x_3$$
$$[x_1, x_2] = x_1.$$

3. Another solvable Lie algebra with nonzero brackets

$$[x_3, x_4] = 2x_4$$
$$[x_3, x_5] = -2x_5$$
$$[x_4, x_5] = x_3$$
$$[x_1, x_3] = x_1$$
$$[x_2, x_5] = x_1$$
$$[x_2, x_3] = -x_2$$
$$[x_1, x_4] = x_2.$$

4. A 2-parameter solvable projective family with nonzero brackets

$$[x_2, x_5]_{p,q} = x_1 + px_2$$
$$[x_3, x_5]_{p,q} = x_2 + qx_3$$
$$[x_4, x_5]_{p,q} = x_3 + (p+q)x_4$$
$$[x_1, x_5]_{p,q} = (p+q)x_1$$
$$[x_2, x_3]_{p,q} = pqx_1$$
$$[x_2, x_4]_{p,q} = qx_1$$
$$[x_3, x_4]_{p,q} = x_1.$$

5. Another 2-parameter solvable projective family with nonzero brackets

$$[x_3, x_4]_{p,q} = x_2$$
$$[x_2, x_5]_{p,q} = (p+q)x_2$$
$$[x_3, x_5]_{p,q} = x_1 + px_3$$
$$[x_4, x_5]_{p,q} = x_3 + qx_4$$
$$[x_1, x_5]_{p,q} = (q+2p)x_1$$
$$[x_2, x_3]_{p,q} = (p-q)x_1$$
$$[x_2, x_4]_{p,q} = x_1.$$

References

[1] Fialowski, A., Deformations of Lie algebras, Mat.Sbornyik USSR, 127 (169), (1985), pp. 476–482; English translation: Math. USSR-Sb., 55, (1986), no. 2, 467–473

[2] Fialowski, A., An example of formal deformations of Lie algebras, NATO Conference on Deformation Theory of Algebras and Applications, Il Ciocco, Italy, 1986, Proceedings. Kluwer, Dordrecht, 1988, 375–401
[3] Fialowski, A., Mandal, A., Leibniz algebra deformations of a Lie algebra, *Journal of Math. Physics*, **49**, 2008, 093512, 10 pp.
[4] Fialowski, A., Mandal, A., Mukherjee, G., Versal Deformations of Leibniz Algebras, *Journal of K-Theory*, 2008, doi:10.1017/is008004027jkt049.
[5] Fialowski, A., Penkava, M., Versal deformations of four dimensional Lie algebras, *Commun. in Contemporary Math*, **9**, 2007, 41–79
[6] Guichardet, A., *Cohomologie des groupes topologiques et des algèbres de Lie*, Cedic/Fernand Nathan, Paris, 1980.
[7] Koszul, J.L. Homologie et cohomologie des algèbres de Lie, *Bull. Soc. Math. France*, **78**, 1950, 67-127.
[8] Loday, J.L., Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Ens. Math.*, **39**, 1993, 269-293.
[9] Magnin, L., Cohomologie adjointe de algèbres de Heisenberg, *Comm. Algebra*, **21**, 1993, 2101-2129.
[10] Magnin, L., Adjoint and trivial cohomologies of nilpotent complex Lie algebras of dimension $\leqslant 7$, *Int. J. Math. math. Sci.*, volume 2008, Article ID 805305, 12 pages.
[11] Magnin, L., Determination of 7-dimensional indecomposable nilpotent complex Lie algebras by adjoining a derivation to 6-dimensional Lie algebras, *Algebras and Representation Theory*, DOI: 10.1007/s10468-009-9172-3 (OnlineFirst), 2009.
[12] Magnin, L., *Adjoint and trivial cohomology tables for indecomposable nilpotent Lie algebras of dimension* ≤ 7 *over* $\mathbb{C}$, online book, 2d Corrected Edition 2007, (*Postcript, .ps file*) (810 pages + vi), accessible at `http://www.u-bourgogne.fr/monge/l.magnin`
or `http://math.u-bourgogne.fr/IMB/magnin/public_html/index.html`
[13] Magnin, L., On $\mathcal{I}$-null Lie algebras, arXiv, math.RA 1010.4660, 2010.
[14] Hu, N., Pei, Y., Liu, D., A cohomological characterization of Leibniz central extensions of Lie algebras, *Proc. Amer. Math. Soc.*, **136**, 2008, 437-477.
[15] Nappi, C.R., Witten E., Wess-Zumino-Witten model based on a nonsemisimple Lie group, *Phys. Rev. Lett.*, **71**, 1993, 3751.
[16] Ovando, G., Complex, symplectic and Kähler structures on 4 dimensional Lie groups, *Rev. Un. Mat. Argentina*, **45**, 2004, 55-67.
[17] Pirashvili, T., On Leibniz homology, *Ann. Instit. Fourier*, **44**, 1994, 401-411.

Institute of Mathematics, Eötvös Loránd University, Pázmány Péter sétány 1/C, H-1117 Budapest, Hungary
E-mail address: `fialowsk@cs.elte.hu`

Institut de Mathématiques de Bourgogne, UMR CNRS 5584, Université de Bourgogne, BP 47870,, 21078 Dijon Cedex, France
E-mail address: `magnin@u-bourgogne.fr`

University of Luxembourg, Campus Kirchberg, Math. Research Unit, 6, rue Richard Coudenhove-Kelergi, L-1359 Luxembourg City
E-mail address: `ashis.mandal@uni.lu`