Local Class Field Theory for Metabelian Extensions

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1 Introduction

1.1

Let K be a local field with finite residue class field $\mathfrak{k}$ of cardinality q. We fix a prime element π of K. All algebraic extensions of K which we consider in the following are subfields of a fixed separable closure $\bar{K}$ of K. In particular K_f denotes the unramified extension of K of degree f and K_π the union of all finite abelian extensions F/K such that $\pi \in N_{F/K}(F^\times)$. A profinite group G is called metabelian if the second commutator group of G is trivial. A finite normal extension L of K will be called metabelian with respect to f and π if $\mathrm{Gal}(LK_fK_\pi/K_fK_\pi)$ is abelian. Since by class field theory every finite abelian extension of K is contained in K_fK_π for suitable f it is clear that every finite metabelian extension of K is metabelian with respect to a certain f.

The purpose of this paper is the development of a class field theory for metabelian extensions by means of two-step towers of Lubin-Tate extensions: For natural numbers f we construct a locally compact group $\mathcal{G}_f$ consisting of pairs $(a, \xi(X))$, where $a \in K^\times$ and $\xi(X)$ is a power series with coefficients in the algebraic closure $\bar{\mathfrak{k}}$ of $\mathfrak{k}$, such that there is a map Ψ_f from $\mathcal{G}_f$ onto $\mathrm{Gal}(L/K)$ which defines a one-to-one correspondence between all closed normal subgroups of finite index in $\mathcal{G}_f$ and all finite normal extensions L of K which are metabelian with respect to f and π.

There is a simple description of the ramification groups of $\mathrm{Gal}(L/K)$ in terms of the corresponding subgroups in $\mathcal{G}_f$.

I would like to thank S. Vostokov who read a preliminary version of the manuscript and made valuable remarks.

1.2

In this section we recall basic facts of Lubin-Tate theory ([LT], [S], [H]) in a form which is convenient for our purposes.

O denotes the ring of integers and U the group of units of K. Let $f(X) := X^q + \pi X$ and let $a \in O$. Then there is a unique power series $[a](X)$ in $O[[X]]$ such that $[a](X) = aX + a_2X^2 + \cdots$ and

$$f([a](X)) = [a](f(X)).$$

In particular $[\pi](X) = f(X)$.

Let $F(X,Y)$ be the formal group law corresponding to f. Then

$$\begin{aligned}[ab](X) &= [a]([b](X)),\\ [a+b](X) &= F([a](X),[b](X))\end{aligned}$$

for $a, b \in O$.

For any integer $n \geq 1$ the quotient

$$[\pi^n](X)/[\pi^{n-1}](X)$$

is a separable Eisenstein polynomial. Let $T = T_n$ be a root of this polynomial in $\bar{K}$. Then $K(T)/K$ is an abelian extension with

$$\left(\frac{u}{K(T)/K}\right) T = [u^{-1}](T) \quad \text{for} \quad u \in U$$

and

$$\left(\frac{\pi}{K(T)/K}\right) T = T.$$

Furthermore $[u](T) = T$ if and only if $u \in U^n$ where U^n denotes the n-th group of principal units of K.

If $u = 1 + v\pi^i$ with $v \in O$, then

$$[u](X) \equiv X + vX^{q^i} \pmod{X^{q^i+1}, \pi} \tag{1}$$

which shows that U^i is mapped by $\left(\frac{}{K(T)/K}\right)$ onto the i-th ramification group of $\mathrm{Gal}(K(T)/K)$ in the upper numeration. (1) can be proved by means of the formal group law $F(X,Y)$ corresponding to $f(X)$:

$$[1+v\pi^i](X) = F(X,[v\pi^i](X)) \equiv X + [v\pi^i](X) \pmod{X \cdot [v\pi^i](X)}.$$

But

$$[v\pi^i](X) = [v]([\pi^i](X)) \equiv vX^{q^i} \pmod{\pi, X^{q^i+1}}.$$

For the following it is useful to prove (1) without using $F(X,Y)$ by means of a refinement, which is easily proved by induction (see also Lemma 13):

Lemma 1 *Let* $u = 1 + v\pi^i$ *with* $v \in O$ *and*

$$[u](X) = uX + u_2X^2 + \cdots.$$

Then for any integer j *with* $1 \leq j \leq i$

$$u_\kappa \equiv 0 \pmod{\pi^{i-j+1}} \quad \text{if} \quad 2 \leq \kappa \leq q^j - 1$$

and

$$u_{q^j} \equiv v\pi^{i-j} \pmod{\pi^{i-j+1}}.$$

Now let K_{nr} be the maximal unramified extension of K in $\bar{K}$, let $\tilde{K}$ be the completion of K_{nr}, $\tilde{O}$ the ring of integers of $\tilde{K}$, and φ the Frobenius automorphism of $\tilde{K}/K$. Then φ induces an automorphism of $\tilde{K}[[X]]$ such that $\varphi(X) = X$. This extension of φ will be denoted by φ, too.

Furthermore let π' be a prime element of K and $f'(X) = X^q + \pi' X$. Then there exists a power series $\omega(X) \in \tilde{O}[[X]]$ such that $\omega(X) = \omega_1 X + \omega_2 X^2 + \cdots$, $\omega_1 \in \tilde{O}^\times$ and

$$f'(\omega(X)) = \varphi\omega(f(X)). \tag{2}$$

If ω_1 is a further solution of (2), then

$$\omega_1(X) = \omega([u](X)) \quad \text{with} \quad u \in U.$$

1.3

In this section we define the group $\mathcal{G}_f$.

Let $a \in K^\times$ and $u = \pi^{-\nu(a)}a$, where ν denotes the exponential valuation of K with $\nu(\pi) = 1$. Let

$$[u](X) = uX + u_2X^2 + \cdots$$

and let $\bar{b}$ be the residue class of $b \in O$ in $\mathfrak{k}$. We denote by $\{u\}(X)$ the power series

$$\bar{u}X + \bar{u}_2X^2 + \cdots$$

in $\mathfrak{k}[[X]]$ corresponding to $[u](X)$.

Furthermore let $\bar{\mathfrak{k}}$ be the algebraic closure of $\mathfrak{k}$ and let φ be the Frobenius automorphism of $\bar{\mathfrak{k}}/k$.

Then $\mathcal{G}_f$ is defined as the set of pairs $(a, \xi(X))$ such that $a \in K^\times$ and $\xi(X) \in \bar{\mathfrak{k}}[[X]]^\times$ satifies the equation

$$\varphi^f \xi(X) = \xi(X)\{u\}(X)X^{-1}. \tag{3}$$

It is well known (see e. g. [H, p. 48]) that (3) has always a solution and if $\xi_u(X)$ is a fixed solution, then the set of all solutions of (3) is

$$\{\xi_u(X)\eta(X) | \eta(X) \in \mathfrak{k}_f[[X]]^\times\},$$

where $\mathfrak{k}_f$ denotes the extension of $\mathfrak{k}$ of degree f.

The multiplication in $\mathcal{G}_f$ is defined by

$$(a_1, \xi_1(X))(a_2, \xi_2(X)) = \left(a_1a_2, \xi_1(X)\varphi^{-\nu(a_1)}\xi_2(\{u\}(X))\right)$$

with $a_1 = \pi^{\nu(a_i)}u_i$, $i = 1, 2$.

The pair $(a_1a_2, \xi_1(X)\varphi^{-\nu(a_1)}\xi_2(\{u\}(X)))$ belongs to $\mathcal{G}_f$. In fact

$$\varphi^f\xi_1(X)\varphi^{f-\nu(a_1)}\xi_2(\{u\}(X)) =$$
$$\xi_1(X)\{u_1\}(X)X^{-1} \cdot \varphi^{-\nu(a_1)}\xi_2(\{u_1\}(X))\{u_2\}(\{u_1\}(X))\{u_1\}(X)^{-1} =$$
$$\xi_1(X)\varphi^{-\nu(a_1)}\xi_2(\{u_1\}(X))\{u_1u_2\}(X)X^{-1}.$$

It is easy to see that the multiplication is associative. The unit element is (1,1) and the inverse of $(a,\xi(X))$ is $\left(a^{-1},\varphi^{\nu(a)}\xi\left(\{u^{-1}\}(X)\right)^{-1}\right)$. Hence $\mathcal{G}_f$ is a group.

We define a topology in $\mathcal{G}_f$ by means of a system of subgroups $\mathcal{V}_{n,h}$, $n,h\in\mathbb{N}$, in $\mathcal{G}_f$:

$$\mathcal{V}_{n,h} = \{(a,\xi(X)) \mid \nu(a)\equiv 0 \pmod{hl},\ u\equiv 1 \pmod{\pi^n},\ \xi(X)\equiv 1 \pmod{X^m}\},$$

where $m:=q^n-q^{n-1}$, $l:=mqf(q^f-1)$ and $u:=a\pi^{-\nu(a)}$.

It is clear that the intersection of the groups $\mathcal{V}_{n,h}$ is $\{1\}$. Hence we can take them as a fundamental system of neighbourhoods of 1 for the topology of $\mathcal{G}_f$. Later on (2.2) we will see that the groups $\mathcal{V}_{n,h}$ are normal subgroups of $\mathcal{G}_f$.

Remark. Let $(a,\xi(X))$ be a pair in $\mathcal{G}_f$ and $u=a\pi^{-\nu(a)}$. Then u is uniquely determined by $\xi(X)$.

In fact,

$$\{u\}(X)+X\varphi^f\xi(X)\cdot\xi(X)^{-1}$$

by (3). Furthermore let R be the system of representatives of $\mathfrak{k}$ in O consisting of 0 and the $q-1$-th roots of unity and let

$$u=\alpha_0+\alpha_1\pi+\cdots,\quad \alpha_i\in R.$$

We write u in the form $u=b_n+c_n+d_n$ with

$$\begin{aligned} b_n &= \alpha_0+\alpha_1\pi+\cdots+\alpha_{n-1}\pi^{n-1},\\ c_n &= \alpha_n\pi^n,\\ d_n &= \alpha_{n+1}\pi^{n+1}+\alpha_{n+2}\pi^{n+2}+\cdots \quad . \end{aligned}$$

Then

$$[u](X)=[a_n](X)+[b_n](X)+[c_n](X)+H_n,$$

where H_n is a power series in $[a_n](X),[b_n](X),[c_n](X)$ without linear terms. Since

$$[b_n](X)\equiv\alpha_nX^{q^n} \pmod{\pi},$$

we have

$$u_{q^n}\equiv\alpha_n+g_n(\alpha_0,\ldots,\alpha_{n-1}) \pmod{\pi},\quad n=0,1,\ldots,$$

where $g_n(\alpha_0,\ldots,\alpha_{n-1})$ is a polynomial in $\alpha_0,\ldots,\alpha_{n-1}$. Now the assertion follows by induction over n.

1.4

We define standard extensions $K^{(n)}/K$ such that every finite extension of K which is metabelian with respect to f and π is contained in $K^{(n)}K_{nr}$ for a certain n (Theorem 3).

Let $f(X) = X^q + \pi X$ as in 1.2 and let $\{T_n \mid n = 1, 2, \dots\}$ be a sequence of elements of $\bar{K}$ such that

$$T_n \neq \{0\}, \quad f^{(n)}(T_n) = [\pi^n](T_n) = 0, \quad T_n = f(T_{n+1}), \quad n = 1, 2, \dots$$

where $f^{(n)}(X)$ denotes the n-th iteration of the power series $f(X)$. Furthermore let

$$g_n(Y) := Y^{q^f} + T_n Y$$

and let V_n be an element of $\bar{K}$ such that

$$g_n^{(m)}(V_n) = 0, \quad g_n^{(m-1)}(V_n) \neq 0$$

with $m = q^n - q^{n-1}$. We put

$$K^{(n)} := K_l(T_n, V_n), \quad l := mqf(q^f - 1).$$

By Lubin-Tate theory V_n is a prime element of the fully ramified extension $K^{(n)}/K_l$. Hence $K_l(T_n, V_n) = K_l(V_n)$.

Proposition 2 *$K^{(n)}/K$ is a normal extension.*

Proof. By Lubin-Tate theory over $K_f(T_n)$, the extension $K^{(n)}/K_f(T_n)$ belongs to the subgroup $H := (T_n)^{l/f}(1 + \mathfrak{P}^m)$ of $K_f(T_n)^\times$, where $\mathfrak{P}$ denotes the prime ideal of $K_f(T_n)$. Let η be an automorphism of $K_f(T_n)/K$. For the proof of Proposition 2 it is sufficient to show that $(\eta T_n)^{l/f}$ belongs to H. More generally for any prime element π_n of $K_f(T_n)$ one has

$$(\pi_n)^{l/f} \in H. \tag{4}$$

In fact

$$\pi_n^{q^f - 1} = T_n^{q^f - 1} W, \quad W \in 1 + \mathfrak{P},$$

and

$$W^{mq} = W^{q^n(q-1)} \in 1 + \mathfrak{P}^m. \quad \square$$

We call $K^{(n)}$ a standard extension of K with respect to f.

1.5

Now we formulate our main results. Proofs will be given in the following chapters.

Theorem 3 *For every finite metabelian extension L of K with respect to f there exists a standard extension $K^{(n)}$ with respect to f such that L is contained in $K^{(n)}K_{nr}$.*

Theorem 4 *Let $K^{(n)}$ be a standard extension of K with respect to f. Then there is a surjective homomorphism Φ_f of $\mathcal{G}_f$ onto $\mathrm{Gal}(K^{(n)}/K)$ with kernel $V_{n,1}$. The restriction of $\Phi_f(a,\xi(X))$ to $K_f(T_n)$ is the automorphism corresponding to a^{-1} by class field theory.*

Theorem 5 *Let $\mathfrak{V}_i$ be the i-th ramification group of $K^{(n)}/K$ in the lower numeration. Then the jumps λ of the series*

$$\mathfrak{V}_0 \supseteq \mathfrak{V}_1 \supseteq \mathfrak{V}_2 \supseteq \dots$$

appear for $\lambda = q^{fj} - 1$, $j = 0, 1, \dots, m-1$, $m = q^n - q^{n-1}$. Furthermore,

$$\Phi_f^{-1}(\mathfrak{V}_{q^{fj}-1}) =$$
$$\left\{(u,\xi(X)) \middle| \quad u \equiv 1 \pmod{\pi^i} \quad \text{with} \quad j \le q^i - 1,\ \xi(X) \equiv 1 \pmod{X^j}\right\}.$$

2 The Homomorphism Φ_f

In this chapter we prove Theorem 4.

2.1

First we prove two propositions, which will be used in the following.

Proposition 6 *Let $\varphi - 1$ be the homomorphism of $\tilde{O}[[X]]^\times$ into itself which sends $\alpha(X)$ to $\varphi\alpha(X)/\alpha(X)$. Then $\varphi - 1$ is surjective and its kernel is $O[[X]]^\times$.*

Proof. Let $u_0 + u_1 X + \cdots$ be in the kernel of $\varphi - 1$. Then $u_i = \varphi u_i$ for $i = 0, 1, \dots$, and it follows (see e. g. [SCL, p. 208]) that $u_i \in O$. On the other hand it is clear that every element of $O[[X]]^\times$ is in the kernel of $\varphi - 1$.

Now we prove that $\varphi - 1$ is surjective. Let $\alpha_0 + \alpha_1 X + \cdots$ be an arbitrary element of $\tilde{O}[[X]]^\times$. The elements of $\tilde{O}[[X]]^\times$ can be written uniquely in the form

$$\xi_0(1 + \xi_1 X)(1 + \xi_2 X^2)\cdots, \quad \xi_0 \in \tilde{O}^\times, \quad \xi_i \in \tilde{O}.$$

We show by induction on i that there are $\xi_0, \xi_1, \dots, \xi_i$ such that

$$\sum_{\kappa=0}^{i} \alpha_\kappa X^\kappa \equiv \varphi\xi_0 \cdot \xi_0^{-1} \prod_{\kappa=1}^{i} (1 + \varphi\xi_\kappa X^\kappa)(1 + \xi_\kappa X^\kappa)^{-1} \pmod{X^{i+1}}.$$

For $i = 0$ we have to solve the equation

$$\alpha_0 = \varphi\xi_0 \cdot \xi_0^{-1},$$

which has a solution (see e. g. [SCL, p. 209]). Now assume that $\xi_0, \dots, \xi_i$ exists and we are looking for ξ_{i+1} with

$$\sum_{\kappa=0}^{i+1} \alpha_\kappa X^\kappa \equiv \varphi\xi_0 \cdot \xi_0^{-1} \prod_{\kappa=1}^{i+1} (1 + \varphi\xi_\kappa X^\kappa)(1 + \xi_\kappa X^\kappa)^{-1} \pmod{X^{i+2}}.$$

Since

$$(1 + \varphi\xi_{i+1} X^{i+1})(1 + \xi_{i+1} X^{i+1})^{-1} \equiv 1 + (\varphi\xi_{i+1} - \xi_{i+1}) X^{i+1} \pmod{X^{i+2}},$$

ξ_{i+1} has to satisfy an equation of the form

$$\varphi\xi_{i+1} - \xi_{i+1} = \beta_{i+1} \tag{5}$$

with

$$\beta_{i+1} X^{i+1} \equiv \sum_{\kappa=0}^{i+1} \alpha_\kappa X^\kappa - \varphi\xi_0 \cdot \xi_0^{-1} \prod_{\kappa=1}^{i} (1 + \varphi\xi_\kappa X^\kappa)(1 + \xi_\kappa X^\kappa)^{-1} \pmod{X^{i+2}}.$$

(5) has a solution by Hensel's Lemma. □

Proposition 7 *Let $u(X) \in O[[X]]$ be of the form*

$$u(X) = u_1 X + u_2 X^2 + \cdots \quad \textit{with} \quad u_1 \in O^\times.$$

Then there is a series

$$\omega(X, Y) = \omega_1(X) Y + \omega_2(X) Y^2 + \cdots$$

with $\omega_1(X) \in \tilde{O}[[X]]^\times$ and $\omega_i(X) \in \tilde{O}((X))$ for $i = 1, 2, \dots$ (i. e. $\omega_i(X)$ has the form

$$\omega_i(X) = \sum_{j=j_0}^{\infty} \alpha_{ij} X^j, \quad \alpha_{ij} \in \tilde{O},$$

with $j_0 \in \mathbb{Z}$), such that

$$\omega(X, Y)^q + u(X)\omega(X, Y) = \varphi\omega(X, Y^q + XY). \tag{6}$$

$\omega_1(X)$ is uniquely determined by (6) up to an arbitrary factor in $O[[X]]^\times$. If $\omega_1(X)$ is fixed, then $\omega_i(X)$ is uniquely determined for $i = 2, 3, \dots$.

Proof. We determine $\omega_i(X)$ by induction over i. For $i = 1$ the assertion is Proposition 6. Assume that $\omega_1(X), \dots, \omega_i(X)$ are already determined such that for

$$\omega^{(i)}(X, Y) := \omega_1(X) Y + \cdots + \omega_i(X) Y^i$$

the congruence

$$\omega^{(i)}(X,Y)^q + u(X)\omega^i(X,Y) \equiv \varphi\omega^{(i)}(X,Y^q+XY) \pmod{Y^{i+1}}$$

is satisfied. Then $\omega_{i+1}(X)$ has to satisfy the equation

$$\varphi\omega_{i+1}(X)X^{i+1} - \omega_{i+1}(X)u(X) = \Phi(X), \tag{7}$$

where $\Phi(X)$ denotes the coefficient of Y^{i+1} in the series

$$\omega^{(i)}(X,Y)^q + u(X)\omega^{(i)}(X,Y) - \varphi\omega^{(i)}(X,Y^q+XY).$$

One can rewrite (7) as follows:

$$(1-\delta)\omega_{i+1}(X) = -u^{-1}(X)\Phi(X),$$

where the operator δ is given by

$$\delta = X^{i+1}u^{-1}(X)\varphi.$$

Then

$$\omega_{i+1}(X) = -(1+\delta+\delta^2+\cdots)u^{-1}(X)\Phi(X),$$

where the series on the right-hand is well defined since the series $X^{i+1}u(X)^{-1}$ is of order $i \geq 1$. □

2.2

Now we consider the extension $K^{(n)}/K$ defined in 1.4. We fix n and write

$$T := T_n, \quad V := V_n.$$

It follows from Lubin-Tate theory that $K(T)/K$ and $K_f(V)/K_f$ are fully ramified abelian extensions of degree $q^n - q^{n-1}$ and $q^{fm} - q^{f(m-1)}$, $m = q^n - q^{n-1}$.

Let $u \in U$ and $\omega(X,Y)$ a power series of the form

$$\omega(X,Y) = \omega_1(X)Y + \omega_2(X)Y^2 + \cdots$$

with $\omega_1(X) \in \tilde{O}[[X]]^\times$, $\omega_i(X) \in \tilde{O}((X))$, $i = 2,3,\ldots$, such that

$$\omega(X,Y)^{q^f} + [u](X)\omega(X,Y) = \varphi^f\omega(X,Y^{q^f}+XY). \tag{8}$$

By Proposition 7, $\omega(X,Y)$ exists and is uniquely determined up to a factor in $O_f[[X]]^\times$ of $\omega_1(X)$. Furthermore the proof of Proposition 7 shows that $\omega_i(T)$ lies in the ring of integers of the field $\tilde{K}(T)$, $i = 1,2,\ldots$.

Let η_u be the automorphism of $K_l(T)/K$ with

$$\eta_u(T) = [u](T)$$

and

$$\eta_u|_{K_l} = 1.$$

We put

$$g_T(Y) = Y^{q^f} + TY.$$

Then (8) implies

$$g_{[u](T)}(\omega(T,Y)) = \varphi^f \omega(T, g_T(Y)),$$

hence

$$g^{(m)}_{[u](T)}(\omega(T,Y)) = \varphi^{fm}\omega(T, g^{(m)}_T(Y))$$

and

$$g^{(m)}_{[u](T)}(\omega(T,V)) = 0$$

$\omega(T,V)$ lies in $\tilde{K}(V)$ and in the normal closure of $K(V)$. Therefore $\omega(T,V)$ is contained in $K_l(V) = K^{(n)}$ and

$$\begin{aligned} \tilde{\eta}_u(V) &= \omega(T,V), \\ \tilde{\eta}_u|_{K_l} &= 1 \end{aligned}$$

defines an automorphism of $K^{(n)}/K$ whose restriction to $K_l(T)$ is η_u.

By Proposition 7 $\omega(X,Y)$ is uniquely determined by u and $\omega_1(X)$. Let $\mathcal{H}_f$ be the set of all pairs

$$(a, \omega_1(X))$$

such that $a \in K^\times$, $\omega_1(X) \in \tilde{O}[[X]]^\times$ and

$$\varphi^f \omega_1(X)X = \omega_1(X)[u](X), \quad u = a\pi^{-\nu(a)}.$$

We define a group structure in $\mathcal{H}_f$ by means of

$$(a,\omega_1(X))\,(a',\omega_1'(X)) = \left(aa', \omega_1(X)\varphi^{-\nu(a)}\omega_1'([u](X))\right)$$

(compare 1.3).

Futhermore we associate to $(a,\omega_1(X))$ the automorphism $\tau = \Psi_f(a,\omega_1(X))$ with

$$\begin{aligned} \tau V &= \omega(T,V), \\ \tau|_{K_l} &= \varphi^{-\nu(a)}. \end{aligned}$$

Theorem 8 *Ψ_f is a surjective homomorphism from $\mathcal{H}_f$ onto* $\mathrm{Gal}(K^{(n)}/K)$.

Proof. Let $(a,\omega_1(X)),(a',\omega_1'(X))$ be two elements of $\mathcal{H}_f$ and let τ,τ' be the corresponding automorphisms of $K^{(n)}/K$ by Ψ_f. Then

$$\tau\tau' V = \varphi^{-\nu(a)}\omega'(\tau T, \tau V) = \varphi^{-\nu(a)}\omega'([u](T), \omega(T,V)),$$

where $\omega(X,Y)$, $\omega'(X,Y)$ are the power series determined by $(a,\omega_1(X))$, $(a',\omega'_1(X))$. The power series

$$\varphi^{-\nu(a)}\omega'([u](X),\omega(X,Y))$$

has initial coefficient

$$\omega'_1(X)\varphi^{-\nu(a)}\omega'_1([u](X))$$

and satisfies (8).

This shows that Ψ_f is a homomorphism. Let η'_u be an automorphism of $K^{(n)}/K$ whose restriction to $K_l(T)$ is η_u. Then $\eta'_u\tilde{\eta}_u^{-1}$ is an automorphism of $K^{(n)}/K_l(T)$ and contained in $\mathrm{Im}\Psi_f$ by Lubin-Tate theory over $K_f(T)$. □

Theorem 9 *The kernel of Ψ_f consists of the elements $(a,\omega_1(X))$ with*

$$\nu(a)\equiv 0 \pmod{l},\quad u\equiv 1 \pmod{\pi^n} \tag{9}$$

and

$$\omega_1(X)\equiv 1 \pmod{X^m,\pi}. \tag{10}$$

Proof. (9) means that $\Psi_f(a,\omega_1(X))$ fixes $K_f(T)$. Then, by class field theory over $K_f(T)$, $\omega_a(T)^{-1}$ corresponds to $\Psi_f(a,\omega_1(X))$. Therefore $\Psi_f(a,\omega_1(X))$ is the identity if and only if

$$\omega_1(T)\equiv 1 \pmod{T^m}.$$

This is equivalent to (10) since $K(T)/K$ is fully ramified of degree $m=q^n-q^{n-1}$. □

Theorem 9 shows that Ψ_f factors through the natural projection of $\mathcal{H}_f$ onto $\mathcal{G}_f$. We define Φ_f as in the corresponding homomorphism of $\mathcal{G}_f$ onto $\mathrm{Gal}(K^{(n)}/K)$. Then according to Theorem 9 the kernel of Φ_f is $\mathcal{V}_{n,1}$. This shows Theorem 4.

2.3

Let $K^{(n,h)} := K^{(n)}K_{lh}$. Then repeating 2.2 for lh instead of l one proves the following extension of Theorem 4.

Theorem 10 *Let $\Psi_f^{(h)}$ be the mapping from $\mathcal{G}_f$ into $\mathrm{Gal}(K^{(n,h)}/K)$ with*

$$\Psi_f^{(h)}t|_{K^{(n)}}=\Psi_f(t),\quad \Psi_f^{(h)}t|_{K_{lh}}=\varphi^{-\nu(a)},$$

where $t:=(a,\xi(X))$. Then $\Psi_f^{(h)}$ is a surjective homomorphism with kernel $\mathcal{V}_{n,h}$. □

3 The Ramification Groups

In this chapter we determine the ramification groups of the extensions $K^{(n)}/K$. But first we consider more general extension of K.

3.1

Let n, m and f be arbitrary natural numbers, let $L_n = K_f(T_n)$ with T_n as in 1.4, and let $L_n^{(m)}/L_n$ be the extension corresponding to the subgroup $(T_n^h)(1+\mathfrak{P}^m)$ of $L_n^\times$ with $h := (q^f - 1)q^m$. In particular if $m = q^n - q^{n-1}$, then $L_n^{(m)} \subseteq K^{(n)}K_{fh(q-1)}$.

One shows as in 1.4 that the extension $L_n^{(m)}/K$ is normal. We want to determine the orders of the ramification groups of $L_n^{(m)}/K$.

Theorem 11 *The jumps $r > 0$ of the ramification filtration $\{\mathfrak{V}^r|\quad r \in \mathbb{R}^+\}$ of* $\mathrm{Gal}(L_n^{(m)}/K)$ *in the upper numeration are*

$$r = i - (q^i - 1 - \mu)/(q^i - q^{i-1})$$

with $\mu = 1, \ldots, m-1$, where i is determined by the condition

$$q^{i-1} - 1 < \mu \le q^i - 1,$$

and $r \in \mathbb{Z}$ with $r < n$, $q^r - 1 \ge m$.

The order of $\mathfrak{V}^r$ for such r is given as follows:

$$|\mathfrak{V}^r| = \begin{cases} q^{n-i}q^{f(m-\psi(r))} & \text{if} \quad i \le n, \quad \psi(r) \le m, \\ q^{f(m-\psi(r))} & \text{if} \quad i > n, \quad \psi(r) \le m, \\ q^{n-r} & \text{if} \quad r \in \mathbb{Z}, \quad r \le n, \quad \psi(r) > m, \end{cases}$$

where $\psi(r)$ denotes the inverse of the Herbrand function of L_n/K.

Proof. Let $r \in \mathbb{R}^+$. Any $\tau \in \mathfrak{V}^r$ can be written uniquely in the form

$$\tau = \tilde{\eta}\sigma,$$

where $\tilde{\eta}$ is a fixed extension of some $\eta \in \mathfrak{V}^r(L_n/K)$ and $\sigma \in \mathfrak{V}^{\psi(r)}(L_n^{(m)}/L_n)$. The function $\psi(r)$ is given by

$$\psi(r) = q^r - 1 \quad \text{for} \quad r \in \mathbb{N} \cup \{0\}$$

and linear interpolation for $r \notin \mathbb{N} \cup \{0\}$. Now the Theorem follows from

$$\begin{aligned} |\mathfrak{V}^r(L_n/K)| &= q^{n\{r\}}, \\ |\mathfrak{V}^{\psi(r)}(L_n^{(m)}/L_n)| &= q^{f(m-\{\psi(r)\})}, \end{aligned}$$

where $\{s\}$ denotes the smallest integer i with $s \le i$. □

3.2

In this section we prove Theorem 3. We have the following characterization of the fields $L_n^{(m)}$:

Proposition 12 *$L_n^{(m)}$ is maximal in the set of all normal extensions M/K with the properties:*

1. *M contains L_n.*
2. *M/L_n is abelian and its intertia degree divides h with $h := (q^f - 1)q^m$.*
3. *The ramification group $\mathfrak{V}^{\varphi(m)}(M/K)$ is trivial, where φ denotes the Herbrand function of L_n/K.*

Proof. The subgroup of $L_n^\times$ corresponding to M by class field theory over L_n contains $1 + \mathfrak{P}^m$ and S^h for some prime element S of L_n. In particular, $L_n^{(m)}/L_n$ belongs to $(T_n^h)(1 + \mathfrak{P}^m)$ and contains therefore S^h.

Since $L_n^{(m)}/L_n$ is normal and has property 3, we see that $L_n^{(m)}$ is maximal in the set of extensions M/K with the properties 1.–3. □

Now we come to the proof of Theorem 3.

Every extension M of K which is metabelian with respect to f and π is contained in $K_{nr}L_n^{(m)}$ for some n and m. Furthermore the ramification group $\mathfrak{V}^r(L_n^{(m)}(T_{n'})/K)$ is trivial for $r > \max\{\varphi(m), n' - 1\}$. Hence $M \subseteq K^{(n')}K_{nr}$ for $n' \geq \varphi(m) + 1$ by Proposition 12.

3.3

In this section we consider the ramification groups of standard extensions. We fix n and we write $T := T_n, V := V_n$.

Let $u := 1 + v\pi^i$ with $v \in O$ and $k := q^i - 1$. Then

$$[u](T) \equiv T + vT^{k+1} \pmod{T^{k+2}} \tag{11}$$

by Lemma 1. Let $\xi \in \tilde{O}$ such that

$$\xi^{q^f} - \xi \equiv v \pmod{\pi}.$$

Then there exists $\omega_1 \in \tilde{O}[[T]]$ such that

$$\omega_1 \equiv 1 + \xi T^k \pmod{T^{k+1}} \tag{12}$$

and

$$\omega_1[u](T) = \varphi^f\omega_1 \cdot T. \tag{13}$$

We need the following lemma, which is a generalization of Lemma 1.

Lemma 13 *Let*

$$\omega(Y) = \omega_1 Y + \omega_2 Y^2 + \cdots$$

be a power series with coefficients in $\tilde{O}[[T]]$ *such that*

$$\omega(Y)^{q^f} + [u](T)\omega(Y) = \varphi^f \omega(Y^{q^f} + TY), \tag{14}$$

where ω_1 *satisfies (12).*

Then for all j *with* $1 \leq j \leq k$

$$\omega_\kappa \equiv 0 \pmod{T^{k-j+1}} \quad \text{if} \quad 2 \leq \kappa \leq q^{fj} - 1$$

and

$$\omega_{q^{fj}} \equiv \xi^{q^{fj}} T^{k-j} \pmod{T^{k-j+1}}.$$

Proof. $\omega(Y)$ with (14) exists by usual Lubin-Tate theory (1.2). First of all let $j = 1$. Then $\omega_\kappa = 0$ for $2 \leq \kappa \leq q^{fj} - 1$ and

$$\omega_{q^f} \equiv (\varphi^f \omega_1 - \omega_1^{q^f})T^{-1} \equiv \xi^{q^f} T^{k-1} \pmod{T^k}.$$

Now we assume that Lemma 13 is proved for all $\kappa' < \kappa$ and $q^{fj} < \kappa \leq q^{f(j+1)}$, $\kappa > q^f$. Then one has the following congruence for ω_κ:

$$\begin{aligned} \omega_{\kappa/q^f}^{q^f} + T\omega_\kappa &\equiv \varphi^f \omega_{\kappa/q^f} + \varphi^f \omega_\kappa T^\kappa \pmod{T^{k-j+1}} \quad \text{if} \quad q^f | \kappa, \\ T\omega_\kappa &\equiv \varphi^f \omega_\kappa T^\kappa \pmod{T^{k-j+1}} \quad \text{if} \quad q^f \nmid \kappa. \end{aligned}$$

If $\kappa < q^{j+1}$, then $\omega_{\kappa/q^f} \equiv 0 \pmod{T^{k-j+1}}$, hence

$$\omega_\kappa \equiv 0 \pmod{T^{k-j}}.$$

If $\kappa = q^{j+1}$, then

$$\omega_{\kappa/q^f} \equiv \xi^{q^{fj}} T^{k-j} \pmod{T^{k-j+1}},$$

hence

$$T\omega_\kappa \equiv \xi^{q^{f(j+1)}} T^{k-j} \pmod{T^{k-j+1}}. \quad \square$$

Now we come to the proof of Theorem 5.

Let $u = 1 + v\pi^i$, $v \in O$. Then there is $\xi(X) \in \bar{\mathfrak{k}}[[X]]$ with

$$\xi(X) \equiv 1 + \bar{\xi} X^{q^i - 1} \pmod{X^{q^i}},$$

where

$$\xi^{q^f} - \xi \equiv v \pmod{\pi},$$

such that $(u, \xi(X)) \in \mathcal{G}_f$ and

$$\Phi_f(u, \xi(X))(V) \equiv V + \xi^{q^{fk}} V^{q^{fk}} \pmod{V^{q^{fk}+1}}$$

with $k := q^i - 1$. Hence

$$\Phi_f(u, \xi(X)) \in \mathfrak{V}_{q^{fk}-1}$$

and if $v \in U$, then

$$\Phi_f(u, \xi(X)) \notin \mathfrak{V}_{q^{fk}}.$$

Since any $(u, \xi(X)) \in \mathcal{G}_f$ has the form $(u, \xi(X)\eta(X))$ with $\eta(X) \in \mathfrak{k}_f[[X]]$. Theorem 5 follows now from Lubin-Tate theory over $K_f(T)$. □

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