Anabelian Geometry

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The aim of the talk was to give a sketch of the proof of a result as follows. First let k be the algebraic closure of some finite field or of the rationals $\mathbb{Q}$. For any function field $K|k$ and a prime $\ell \neq \mathrm{char} k$ we denote by K^ℓ a maximal pro-ℓ Galois extension of K, and set $G^\ell_K = \mathrm{Gal}(K^\ell|K)$. Then we have the following:

Main Result. *There exists a group theoretic recipe by which we can recover the field* $(K, +, .)$ *from* G^ℓ_K *in a functorial way, provided* $\mathrm{td}(K|k) > 1$. *This recipe is invariant under isomorphisms of profinite groups. Thus given a further function field* $K'|k'$ *as above, we have:*

$$G^\ell_K \cong G^\ell_{K'} \;\Leftrightarrow\; K^{\mathrm{i}} \cong K'^{\mathrm{i}},$$

where $(\ )^{\mathrm{i}}$ *means pure inseparable closure. More precisely, one has a canonical bijection:*

$$\mathrm{Isom}^i(K'^{\mathrm{i}}, K^{\mathrm{i}}) \;\cong\; \mathrm{Out}(G^\ell_K, G^\ell_{K'})$$

where Isom^i *means up to Frobenius twists, and* $\mathrm{Out}(G^\ell_K, G^\ell_{K'})$ *means outer isomorphisms of profinite groups.*

Comments/Remarks.

1) First we remark that the above result completes the Program proposed by Bogomolov [Bo] over k as above. See also Bogomolov–Tschinkel [B–T].

2) "By taking limits" the above result implies a corresponding assertion for the full Galois groups.

Furthermore, the above result implies *Grothendieck's birational anabelian Conjecture* for finitely generated infinite fields of transcendence degree > 1. This follows from the fact that for such fields $K_0|k_0$, where k_0 is the constant field, the canonical projection $G_{K_0} \to G_{k_0}$ is encoded in G_{K_0}.

Finally, if $\mathrm{td}(K|k) = 1$, then by a result of Harbater and Pop, it follows that G^ℓ_K is the pro-ℓ free group on countably many generators. Therefore it carries no information about the isomorphy type of K.

Because of the same reason one cannot expect a *Hom-form* of the Main Result above, as proved by Mochizuki in the case of function fields $K_1|k_1$ over a sub-p-adic base field k_1. Indeed for any two function fields $K|k$ and $K'|k'$ as above, one can find "many" surjective homomorphisms as follows: $G^\ell_K \to G^\ell_{k(t)} \to G^\ell_{K'}$, and naturally, these cannot be of geometric nature.

Motivation/Consequence.

The above result is related in an intimate way to the so-called Ihara/Oda–Matsumoto Conjecture, which itself is related to the following idea of Grothendieck about looking for a (non-tautological) description of the absolute Galois group of the rationals $G_{\mathbb{Q}}$, as follows:

Let $\mathcal{V}$ be a category of (geometrically connected) $\mathbb{Q}$-varieties. For every $X \in \mathcal{V}$ one considers $X(\mathbb{C})$ the complex points of X, and $\pi_1^{\mathrm{top}}\big(X(\mathbb{C})\big)$ its topological fundamental group (up to inner conjugation). We finally denote by $\pi = \widehat{\pi^{\rm top}}$ and $\pi^\ell = \widehat{\pi^{\rm top}}$ the profinite and the pro-ℓ completion of the topological fundamental group, respectively. These are exactly the geometric, respectively the pro-ℓ quotient of the geometric fundamental group functor of $\mathcal{V}$. Thus we get two functors

$$\pi_{\mathcal{V}} : \mathcal{V} \to (\text{proGroups}), \quad \pi^\ell_{\mathcal{V}} : \mathcal{V} \to (\text{pro-}\ell\text{-Groups}).$$

Clearly, by the functoriality of the fundamental group, since all varieties from $\mathcal{V}$ are defined over $\mathbb{Q}$, the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ acts on the homotopy category of $\mathcal{V}$, i.e., there exist group homomorphisms

$$\imath_{\mathcal{V}} : G_{\mathbb{Q}} \to \mathrm{Aut}(\pi_{\mathcal{V}}), \quad \imath^\ell_{\mathcal{V}} : G_{\mathbb{Q}} \to \mathrm{Aut}(\pi^\ell_{\mathcal{V}}).$$

Now the problem / question is the following:

1) Give "interesting" categories $\mathcal{V}$ such that $\mathrm{Aut}(\pi_{\mathcal{V}})$ and/or $\mathrm{Aut}(\pi^\ell_{\mathcal{V}})$ can be computed.

2) Find $\mathcal{V}$ as above such that the resulting $\imath_{\mathcal{V}}$ and/or $\imath^\ell_{\mathcal{V}}$ is an isomorphism.

Specifically, Grothendieck proposed to work with categories $\mathcal{V}$ consisting of moduli spaces $\mathcal{M}_{g,n}$, e.g. $\mathcal{V}_0 = \{\mathcal{M}_{0,4}\, , \mathcal{M}_{0,5}\}$. In this last case, $\mathrm{Aut}(\pi_{\mathcal{V}_0}) =: \widehat{\mathrm{GT}}$ is the celebrated Grothendieck–Teichmüller group... There is a lot of research in this area of mathematics by several people (Drinfeld, Ihara, Deligne, Belyi, Nakamura, Schneps, Lochak, Harbater, Matsumoto, etc.). In particular, if the category $\mathcal{V}$ contains $\mathcal{M}_{0,4} = \mathbb{P}^1\backslash\{0,1,\infty\}$, then both $\imath_{\mathcal{V}}$ and $\imath^\ell_{\mathcal{V}}$ are injective.

It is a question by Ihara/Oda–Matsumoto, whether there do exist categories $\mathcal{V}$ for which $\imath_{\mathcal{V}}$ or $\imath^\ell_{\mathcal{V}}$ are isomorphisms. This question was answered in a positive sense by the present author in 1998 (unpublished).

The remark we want to make here is that actually the Main Result above implies a very strong form of the "positive answer" to the above question. Indeed, let $k = \overline{\mathbb{Q}}$, and X some $\mathbb{Q}$-variety as above. Set $K = k(X)$. Then denoting by $\mathcal{V}_X$ a basis of Zariski open neighbourhoods of the generic point of X and inclusions, we have:

- $\mathrm{Aut}(\pi^\ell_{\mathcal{V}_X}) = \mathrm{Out}(G^\ell_K)$ and $\mathrm{Aut}(\pi_{\mathcal{V}_X}) = \mathrm{Out}(G_K)$.

On the other hand, $\mathrm{Out}(G^\ell_K) \cong \mathrm{Aut}(K)$ by the Main Result above. Therefore, since X is defined over $\mathbb{Q}$, we have an exact sequence of the form

$$1 \to \mathrm{Aut}_k(K) \to \mathrm{Aut}(K) \to \mathrm{Aut}(\overline{\mathbb{Q}}) = G_{\mathbb{Q}} \to 1\,.$$

In particular, if K is rigid, i.e., $\mathrm{Aut}_k(K) = \{1\}$, then $\mathrm{Aut}(\pi^\ell_{\mathcal{V}_X}) = G_{\mathbb{Q}}$.

References:

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