ELLIPTIC GENERA, MODULAR FORMS OVER KO_* AND THE BROWN–KERVAIRE INVARIANT

and

A VANISHING THEOREM FOR THE ELLIPTIC GENUS

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by

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Let Ω_*^{SO} be the oriented cobordism ring and Λ any commutative $\mathbb{Q}$–algebra . An <u>elliptic genus</u> over Λ , as originally defined in [14], is a ring homomorphism

$$\varphi : \Omega_*^{SO} \longrightarrow \Lambda$$

satisfying

$$\sum_{i\geq 0} \varphi\,[\mathbb{C}P_{2i}]\; u^{2i} = (1 - 2\delta u^2 + \varepsilon u^4)^{-1/2} \,.$$

Here

$$\delta = \varphi[\mathbb{C}P_2] \quad \text{and} \quad \varepsilon = \varphi\,[\mathbb{H}P_2]$$

are two parameters in Λ which determine φ completely.

In the most interesting <u>universal</u> examples, Λ is the ring $\mathbb{Q}[[q]]$ of formal power series over $\mathbb{Q}$, and for any oriented manifold V , $\varphi[V]$ is the q–expansion of a level 2 modular form whose values at the two cusps are, up to an inessential factor, the

$\hat{A}$–genus $\hat{A}[V]$ and the signature $\sigma(V)$ (cf. [9], [5], [10], [23], [8]).

Though defined for oriented manifolds, the elliptic genera reveal their most striking properties, such as rigidity (constancy) under compact Lie group actions ([3], [15]) or integrality ([6]), on spin manifolds. Both rigidity and integrality rely on the fact noticed by E. Witten ([22]) that in the universal examples, the coefficients of $\varphi[V]$ are indices of twisted Dirac operators, therefore KO–characteristic numbers.

In this paper we consider a refined elliptic genus

$$\beta_q : \Omega_*^{spin} \longrightarrow KO_*[[q]]$$

whose values are q–expansions of level 2 modular forms over the coefficient ring KO_* of the real K–theory. In dimensions divisible by 4, $\beta_q[V]$ is essentially the above universal genus $\varphi[V]$. On the other hand, in dimensions $8m+1$, $8m+2$, $\beta_q[V]$ is a modular form over $\mathbb{F}_2$ (in the sense of J.–P. Serre [18]), and can be expressed as a polynomial in the basic form $\bar{\varepsilon} = \sum_{n\geq 1} q^{(2n-1)^2}$:

$$\beta_q[V] = a_0 + a_1\bar{\varepsilon} + \dots + a_m\bar{\varepsilon}^m .$$

It turns out that a_0 is the Atiyah invariant while a_m is the KO–part of the Brown–Kervaire invariant of V.

Being a refinement of an elliptic genus, β_q retains at least a few of the properties of the latter. For example, M. Bendersky ([2]) recently proved that $\beta_q[V] = 0$ for a spin manifold V admitting an odd type semi–free circle action, which implies the vanishing of both the Atiyah invariant and the KO–part of the Brown–Kervaire invariant*). It seems very likely that Bendersky's theorem can be reversed: we conjecture that $\beta_q[V] = 0$ if and only if V is spin cobordant to (or at least has the same KO–characteristic numbers as) a spin manifold admitting an odd type semi–free circle action.

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1. Definition of β_q. Let E be a real vector bundle over X. Writing $\Lambda^i(E)$ and $S^i(E)$ respectively for the exterior and the symmetric powers of E, and

$$\Lambda_t(E) = \sum_{i \geq 0} \Lambda^i(E)t^i ,$$

$$S_t(E) = \sum_{i \geq 0} S^i(E)t^i ,$$

one defines the Witten characteristic class Θ_q ([22], cf. [10]) by

$$\Theta_q(E) = \bigotimes_{n \geq 1} (\Lambda_{-q^{2n-1}}(E) \otimes S_{q^{2n}}(E)) .$$

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*) For a proof valid for all odd type actions see [16].

For any E, $\Theta_q(E)$ is a formal power series in q whose coefficients are virtual vector bundles over X. Moreover, one has

$$\Theta_q(E) = 1 - E \cdot q + \ldots$$

and

$$\Theta_q(E \oplus F) = \Theta_q(E) \cdot \Theta_q(F) .$$

Therefore Θ_q canonically extends to $KO(X)$:

$$\Theta_q : KO(X) \longrightarrow KO(X)[[q]] .$$

Let $\beta_q(E)$ be defined by

$$\beta_q(E) = \Theta_q(E - \dim E) .$$

Then

$$\beta_q(E) = b_0(E) + b_1(E)q + \ldots$$

where

$$b_0(E) = 1$$

$$b_i(E) \in \widetilde{KO}(X) \quad (i > 0)$$

and

$$\beta_q(E \oplus F) = \beta_q(E) \cdot \beta_q(F) .$$

It is easy to see that $b_i (i \geq 0)$ are stable KO–characteristic classes and can be expressed as polynomials in the Pontrjagin classes π_i defined by (cf. [21]):

$$\Sigma\ \pi_i(E) u^i = \Lambda_t(E - \dim E) ,$$

where

$$u = \frac{t}{(1+t)^2} .$$

For example

$$b_1 = -\pi_1$$

$$b_2 = \pi_2 - \pi_1$$

$$b_3 = -\pi_3 + 4\pi_2 - \pi_1^2 - 4\pi_1$$

and, more generally,

$$b_i = (-1)^i \pi_i + \text{lower terms} .$$

Let now V^n be a closed spin manifold, and $[V^n] \in KO_n(V^n)$ be the fundamental class of V^n in real K–theory.

Definition:

$$\beta_q[V^n] = \beta_q(TV)[V^n] = \sum_{i\geq 0} b_i[V^n]q^i ,$$

where TV is the tangent bundle of V^n and

$$b_i[V^n] = b_i(TV)[V^n] \in KO_n = KO_n\,(\text{point})$$

is the KO–characteristic number corresponding to b_i.

One can easily see that β_q defines a ring homomorphism (genus)

$$\beta_q : \Omega_*^{spin} \longrightarrow KO_*[[q]] .$$

Considered as $\mathbb{Z}/8$–graded, the ring KO_* is generated by two elements η and ω of degree 1 and 4 respectively subject to the relations

$$2\eta = \eta^3 = \eta\omega = 0 , \ \ \omega^2 = 4 .$$

Clearly, β_q preserves the degree mod 8.

Let

$$\mathrm{ph} : KO^*(X) \longrightarrow H^{**}(X;\mathbb{Q})$$

be the Pontrjagin character defined as the composition

$$KO^*(X) \xrightarrow{\otimes \mathbb{C}} K^*(X) \xrightarrow{\text{Chern char.}} H^{**}(X;\mathbb{Q}) .$$

For $X = \text{point}$ one has $KO^*(X) \cong KO_*$ and $H^{**}(X;\mathbb{Q}) \cong \mathbb{Q}$, and ph is entirely determined by

$$\mathrm{ph}(\eta) = 0 , \ \mathrm{ph}(\omega) = 2 .$$

In particular, ph is integral:

$$\mathrm{ph} : KO_* \longrightarrow \mathbb{Z} .$$

Composing β_q with ph leads to a genus

$$\varphi_q = \mathrm{ph} \circ \beta_q : \Omega_*^{spin} \longrightarrow \mathbb{Z}[[q]]$$

such that

$$\varphi_q[V^n] = \sum_{i\geq 0} \mathrm{ph}(b_i[V^n])q^i = \sum_{i\geq 0} \mathrm{ph}(b_i(TV))\hat{\mathfrak{A}}(TV)[V^n]q^i ,$$

where $\hat{\mathfrak{A}}(TV)$ is the total $\hat{\mathfrak{A}}$–class of V^n . In particular, the constant term of $\varphi_q[V^n]$ is the $\hat{A}$–genus $\hat{A}[V^n]$.

Theorem 1 ([10], [23]). φ_q is the restriction to Ω_*^{spin} of an elliptic genus

$$\varphi_q : \Omega_*^{SO} \longrightarrow \mathbb{Q}[[q]]$$

with parameters

$$\delta = -\frac{1}{8} - 3 \sum_{n\geq 1} \Big(\sum_{\substack{d|n \\ d \text{ odd}}} d \Big) q^n$$

$$\varepsilon = \sum_{n\geq 1} \Big(\sum_{\substack{d|n \\ n/d \text{ od}}} d^3 \Big) q^n \qquad \square$$

2. Modular forms over graded rings. It turns out that $\beta_q[V^n]$ can be interpreted as a modular form of degree n over the graded ring KO_*.

If Γ is a subgroup of $SL_2(\mathbb{Z})$ of finite index, let $M_*^\Gamma(\mathbb{C})$ be the graded ring of modular forms over $\mathbb{C}$ for Γ. Thus $M_w^\Gamma(\mathbb{C})$ is the group of forms of weight w. We will always identify a modular form from $M_*^\Gamma(\mathbb{C})$ with its q–expansion. This way $M_*^\Gamma(\mathbb{C})$ becomes a subring in $\mathbb{C}[[q^{1/h}]]$, where h is the smallest positive integer such that $\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ belongs to Γ.

Let now $M_*^\Gamma(\mathbb{Z})$ be the subring of $M_*^\Gamma(\mathbb{C})$ of forms having integral q–expansions

$$M_*^\Gamma(\mathbb{Z}) = M_*^\Gamma(\mathbb{C}) \cap \mathbb{Z}[[q^{1/h}]] .$$

For any graded commutative ring with unit

$$R_* = \bigoplus_n R_n ,$$

the canonical injection

$$M_*^\Gamma(\mathbb{Z}) \longrightarrow \mathbb{Z}[[q^{1/h}]]$$

extends to a ring homomorphism

$$R_* \otimes_{\mathbb{Z}} M_*^\Gamma(\mathbb{Z}) \longrightarrow R_*[[q^{1/h}]] .$$

We define $M^\Gamma(R_*)$ to be the image of this homomorphism, and will call its elements <u>modular forms over R_* for Γ</u>.

Notice that $M^\Gamma(R_*)$ is canonically a graded R_*–algebra :

$$M^\Gamma(R_*) = \bigoplus_n M^\Gamma(R_n) ,$$

where $M^\Gamma(R_n)$ is the set of forms from $M^\Gamma(R_*)$ whose coefficients are in R_n. We refer to the elements of $M^\Gamma(R_n)$ as forms of <u>degree</u> n .

If for a certain n , R_n has no torsion, then

$$R_n \otimes M_*^\Gamma(\mathbb{Z}) \longrightarrow M^\Gamma(R_n)$$

is an isomorphism. In this case,

$$M^{\Gamma}(R_n) = \bigoplus_{w} M_w^{\Gamma}(R_n) ,$$

where

$$M_w^{\Gamma}(R_n) \cong R_n \otimes M_w^{\Gamma}(\mathbb{Z}) .$$

We will say that forms from $M_w^{\Gamma}(R_n)$ have <u>weight</u> w.

In the general situation, a form $F \in M^{\Gamma}(R_n)$ may come from integral forms of different weights, and the weight of F cannot be defined correctly. Instead, one defines an increasing <u>filtration</u> of $M^{\Gamma}(R_n)$ as follows: a form $F \in M^{\Gamma}(R_n)$ has filtration $\leq f$ if F is the image of an element of

$$R_n \otimes \left[\bigoplus_{w \leq f} M_w^{\Gamma}(\mathbb{Z}) \right] ,$$

i.e. if

$$F = \Sigma\, r_j F_j ,$$

where $F_j \in M_*^{\Gamma}(\mathbb{Z})$ are forms of weight $\leq f$.

3. <u>Modular forms over KO_*</u> . From now on Γ will designate the group $\Gamma_0(2)$ of matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$$

such that $c \equiv 0 (\mathrm{mod}\ 2)$. The series δ and ε of theorem 1 are the basic examples of modular forms for $\Gamma_0(2)$. More precisely, let

$$\delta_0 = -8\delta = 1 + 24q + 24q^2 + 96q^3 + \dots$$

<u>Proposition 1</u> (cf. [8], Anhang I).

(i) $\delta_0 \in M_2^\Gamma(\mathbb{Z})$, $\varepsilon \in M_4^\Gamma(\mathbb{Z})$;

(ii) $M_*^\Gamma(\mathbb{Z}) = \mathbb{Z}[\delta_0, \varepsilon]$. □

Consider now $M^\Gamma(KO_*)$. For $n \equiv 0 (\mathrm{mod}\ 4)$, one has $KO_n \cong \mathbb{Z}$. Thus

$$M^\Gamma(KO_n) \cong KO_n \otimes M_*^\Gamma(\mathbb{Z}) .$$

It follows that:

(a) a modular form of degree $n = 8m$ and weight w over KO_* can be written in a unique way as a polynomial $P(\delta_0, \varepsilon)$ of weight w with integer coefficients;

(b) a modular form of degree $n = 8m + 4$ and weight w over KO_* can be written in a unique way as $\omega P(\delta_0,\varepsilon)$, where $P(\delta_0,\varepsilon)$ is a polynomial of weight w with integer coefficients.

Notice now that one has $\delta_0 \equiv 1 (\mathrm{mod}\ 2)$. Let $\bar{\varepsilon}$ be the reduction mod 2 of $\varepsilon \in \mathbb{Z}[[q]]$. It is easy to see that

$$\bar{\varepsilon} = \sum_{n\geq 1} q^{(2n-1)^2} = q + q^9 + q^{25} + \dots$$

For $n = 8m + r (r = 1,2)$, one has $KO_n = \mathbb{F}_2\eta^r$ and the map

$$KO_n \otimes M_*^\Gamma(\mathbb{Z}) \longrightarrow KO_n[[q]]$$

is essentially the reduction mod 2 :

$$\eta^r \otimes P(\delta_0,\varepsilon) \longmapsto \eta^r \bar{P}(1,\bar{\varepsilon}),$$

where $P(\delta_0,\varepsilon)$ is a polynomial with integer coefficients and $\bar{P}$ is its reduction mod 2. As $\bar{\varepsilon} = q + \dots$, the powers of $\bar{\varepsilon}$ are linearly independent over $\mathbb{F}_2$. Therefore:

(c) a modular form F of degree $n = 8m + r (r = 1,2)$ and filtration $\leq f$ over KO_* can be written in a unique way as $\eta^r Q(\bar{\varepsilon})$, where

$$Q(\bar{\varepsilon}) = a_0 + a_1\bar{\varepsilon} + \dots + a_s\bar{\varepsilon}^s \quad (a_i \in \mathbb{F}_2)$$

and $4s \leq f$. The filtration of F is <u>exactly</u> $4s$ if and only if $a_s \neq 0$.

The additive structure of $M^\Gamma(KO_*)$ is completely described by (a), (b), and (c). The ring structure is given by the following theorem.

<u>Theorem 2</u>.

(i) The kernel of

$$KO_* \otimes M_*^\Gamma(\mathbb{Z}) \longrightarrow M^\Gamma(KO_*)$$

is the principal ideal generated by $\eta \otimes (\delta_0 - 1)$.

(ii) The commutative KO_*–algebra $M^\Gamma(KO_*)$ is generated by δ_0 and ε subject to the single relation $\eta\,\delta_0 = \eta$.

The proof is immediate from the above description of

$$KO_* \otimes M_*^\Gamma(\mathbb{Z}) \longrightarrow KO_*[[q]] .$$

4. $\beta_q[V^n]$ <u>as a modular form</u>. We will now see that $\beta_q[V^n]$ is a modular form of degree n over KO_*.

Theorem 3.

(i) If $n = 4s$, then $\beta_q(\Omega_n^{spin})$ is the set of all modular forms of degree n and weight $2s$ over KO_*.

(ii) If $n = 8m + r$ $(r = 1,2)$, then $\beta_q(\Omega_n^{spin})$ is the set of all modular forms of degree n and filtration $\le 4m$ over KO_*.

(iii) $\beta_q(\Omega_*^{spin})$ is the subring of $M^\Gamma(KO_*)$ generated by η, $\omega\delta_0$, δ_0^2 and ε.

Proof. Part (iii) clearly follows from (i), (ii) and the above description of $M^\Gamma(KO_*)$.

Part (i) is a simple consequence of the definition of φ_q, the description of ph and the following theorem:

Theorem 4 ([6], cf. [10]). For any spin manifold V^{4s}, $\varphi_q[V^{4s}]$ is a modular form from $M_{2s}^\Gamma(\mathbb{Z})$. More precisely,

$$\varphi_q(\Omega_{8m}^{spin}) = M_{4m}^\Gamma(\mathbb{Z})$$

$$\varphi_q(\Omega_{8m+4}^{spin}) = 2M_{4m+2}^\Gamma(\mathbb{Z}) \qquad \square$$

The proof of the remaining part (ii) relies on the following construction due to R.E. Stong (cf. [21], p. 341, for the details):

Let $\bar{S}^1$ be the circle equipped with its non–trivial spin structure. $\bar{S}^1$ represents the non–zero element of $\Omega_1^{spin} \cong \mathbb{F}_2$. If V is an (8m + 2)–dimensional spin manifold, then $\bar{S}^1 \times V$ is the boundary of a compact spin manifold U. On the other hand, $2\bar{S}^1$ is the boundary of a compact spin manifold M^2. Therefore one can form a closed (8m + 4)–dimensional spin manifold T(V) by glueing together two copies of U and $-M^2 \times V$ along

$$\partial(2U) = 2\bar{S}^1 \times V = \partial(M^2 \times V) .$$

Though involving arbitrary choices of M^2 and U, this construction induces a well–defined homomorphism

$$T : \Omega_{8m+2}^{spin} \longrightarrow \Omega_{8m+4}^{spin} \otimes \mathbb{F}_2 .$$

Let

$$t : KO_2 \longrightarrow KO_4 \otimes \mathbb{F}_2$$

be the isomorphism which sends η^2 to $\omega \otimes 1$.

Proposition 2 (cf. [21], p. 343). If ξ is a polynomial in the Pontrjagin classes π_i, then one has in $KO_4 \otimes \mathbb{F}_2$:

$$\xi\,[T(V)] \otimes 1 = t(\xi\,[V]) . \qquad \square$$

Roughly speaking, $\xi\,[V]$ is the reduction mod 2 of $\xi\,[T(V)]$.

Let $I_* \subset \Omega_*^{spin}$ be the ideal of classes with vanishing Pontrjagin KO–characteristic numbers. Proposition 2 implies that T induces a homomorphism

$$\tilde{T} : \Omega^{spin}_{8m+2} / I_{8m+2} \longrightarrow (\Omega^{spin}_{8m+4} / I_{8m+4}) \otimes \mathbb{F}_2 .$$

Proposition 3 (cf. [21], p. 344). $\tilde{T}$ is an isomorphism. □

The coefficients of $\beta_q[V]$ are Pontrjagin KO–characteristic numbers. Therefore one has:

$$\beta_q[T(V)] \otimes 1 = t(\beta_q[V])$$

in $(KO_4 \otimes \mathbb{F}_2)[[q]]$. By theorem 3 (i),

$$\beta_q[T(V)] = \omega P(\delta_0,\varepsilon) ,$$

where $P(\delta_0,\varepsilon)$ is a polynomial of weight $4m + 2$ in δ_0,ε with integer coefficients. Therefore

$$\beta_q[V] = \eta^2 \overline{P}(1,\overline{\varepsilon})$$

is a modular form of degree $8m + 2$ and filtration $\leq 4m$ over KO_* . Proposition 3 implies that all such forms can be obtained from spin manifolds V , and this settles the case of manifolds of dimension $8m + 2$.

The proof in the case of $(8m + 1)$–dimensional manifolds is similar. Instead of T one considers the multiplication by $\overline{S}^1$ homomorphism

$$S : \Omega_{8m}^{spin} \longrightarrow \Omega_{8m+1}^{spin} .$$

If ξ is a polynomial in the classes π_i , then

$$\xi[\overline{S}^1 \times M] = \eta \cdot \xi[M]$$

for any spin manifold M . Thus S induces a homomorphism

$$\tilde{S} : \Omega_{8m}^{spin} / I_{8m} \longrightarrow \Omega_{8m+1}^{spin} / I_{8m+1} .$$

Proposition 4 (cf. [21], p. 344). $\tilde{S}$ is onto. □

It follows that

$$\beta_q(\Omega_{8m+1}^{spin}) = \eta \cdot \beta_q(\Omega_{8m}^{spin})$$

and the result follows from (i) and the description of $M^\Gamma(KO_*)$.

5. Characteristic classes a_i . Let $h(q) = q + \ldots$ be any series from $\mathbb{Z}[[q]]$ whose reduction mod 2 is

$$\sum_{n\geq 1} q^{(2n-1)^2} = q + q^9 + q^{25} + \ldots$$

For example, one can take $h(q) = \varepsilon(q)$. Another possible choice for $h(q)$ is the Ramanujan series

$$\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - \dots \quad ^{*)}$$

For any real vector bundle E over X define

$$\alpha_t(E) \in KO(X)[[t]]$$

by

$$\alpha_t(E) = \beta_q(E) ,$$

where

$$t = h(q) .$$

Since the leading term of $h(q)$ is q, this series is invertible in $\mathbb{Z}[[q]]$, therefore $\alpha_t(E)$ is well–defined. Clearly, one has

$$\alpha_t(E \oplus F) = \alpha_t(E)\alpha_t(F) .$$

If

*) It is an amusing exercise to show that $\Delta \equiv \varepsilon (\mathrm{mod}\ 2)$ and even, as noticed by P. Landweber, $\Delta \equiv \varepsilon (\mathrm{mod}\ 16)$.

$$\alpha_t(E) = a_0(E) + a_i(E)t + a_2(E)t^2 + \dots ,$$

then $a_i(E)$ is a polynomial in the Pontrjagin classes $\pi_i(E)$ such that

$$a_0(E) = 1$$

$$a_i(E) \in \widetilde{KO}(X) \quad (i > 0)$$

and

$$a_i(E) = (-1)^i \pi_i(E) + \text{lower terms} .$$

Notice that while $a_i(E)$ depends on the choice of $h(q)$, its reduction mod 2, that is its image in $KO(X) \otimes \mathbb{F}_2$ is independent of any choice.

By definition of a_i, for any spin manifold V^n one has:

$$\beta_q[V^n] = a_0[V^n] + a_1[V^n]t + a_2[V^n]t^2 + \dots ,$$

where

$$a_i[V^n] = a_i(TV)[V^n] .$$

On the other hand, the reduction mod 2 of $\beta_q[V^n]$ is of the form (cf. Section 3):

$$a_0 + a_1\bar{\varepsilon} + \dots + a_m\bar{\varepsilon}^m ,$$

where $a_i \in KO_n \otimes \mathbb{F}_2$ and $m = [n/8]$. Comparing these two expressions leads to the following:

Theorem 5.

(i) For $i > [n/8]$, one has $a_i[V^n] \otimes 1 = 0$ in $KO_n \otimes \mathbb{F}_2$.

(ii) One has in $(KO_n \otimes \mathbb{F}_2)[[q]]$:

$$\beta_q[V^n] \equiv a_0[V^n] + a_1[V^n]\bar{\varepsilon} + \dots + a_m[V^n]\bar{\varepsilon}^m ,$$

where $m = [n/8]$.

6. The Brown–Kervaire invariant. Notice that for $n = 8m + 2$, the constant term $a_0[V^n] = 1[V^n]$ is the so-called Atiyah invariant ([1]). We will see now that $a_m[V^n]$ has an interpretation in terms of the Brown–Kervaire invariant of V^n .

Let V^n , $n = 8m + 2$, be a spin manifold. As mentioned earlier, $\overline{S^1} \times V = \partial U$, where U is a compact spin manifold. It is shown in [13] that the signature $\sigma(U)$ is divisible by 8, and that

$$k(V) = \sigma(U)/8 \in \mathbb{F}_2$$

is a spin cobordism invariant satisfying

$$k(\bar{S}^1 \times \bar{S}^1 \times M) = \sigma(M) \bmod 2$$

for all 8m–dimensional spin manifolds M. For a large class of manifolds, including all complex–spin manifolds ([20]), $k(V)$ agrees with the Brown–Kervaire invariant ([4]). For a general spin manifold V, $k(V)$ can be thought of as the KO–part of the Brown–Kervaire invariant (cf. [13] for the details).

More generally, one defines an invariant $\kappa(V^n) \in KO_n \otimes \mathbb{F}_2$ by

$$\kappa(V^n) = \begin{cases} \sigma(V) & , n \equiv 0 \pmod 8 \\ k(\bar{S}^1 \times V)\eta & , n \equiv 1 \pmod 8 \\ k(V)\eta^2 & , n \equiv 2 \pmod 8 \\ (\sigma(V)/16)\omega & , n \equiv 4 \pmod 8 \end{cases}$$

The multiplicative properties of k are summarized by saying that κ defines a ring homomorphism

$$\kappa : \Omega_*^{spin} \longrightarrow KO_* \otimes \mathbb{F}_2 .$$

A new proof of this will be given later.

<u>Theorem 6</u>. Let V^n be a spin manifold. Then

$$a_m[V^n] = \kappa(V^n)$$

in $KO_n \otimes \mathbb{F}_2$, where $m = [n/8]$.

<u>Proof</u>. Consider first the case when $n = 8m + 4$. According to theorem 3,

$$\beta_q[V^n] = \omega(a_0\delta_0^{2m+1} + a_1\delta_0^{2m-1}\varepsilon + \dots + a_m\delta_0\varepsilon^m),$$

where $a_i \in \mathbb{Z}$. Then

$$\varphi_q[V^n] = 2(a_0\delta_0^{2m+1} + a_1\delta_0^{2m-1}\varepsilon + \dots + a_m\delta_0\varepsilon^m).$$

If we consider φ_q as an elliptic genus over $\mathbb{Z}[\delta,\varepsilon]$, the signature $\sigma(V^n)$ is obtained by specializing $\delta = 1$, $\varepsilon = 1$, or $\delta_0 = -8$, $\varepsilon = 1$. Thus,

$$\sigma(V^n) = 2(a_0(-8)^{2m+1} + a_1(-8)^{2m-1} + \dots + a_m(-8))$$

$$\equiv 16\, a_m (\text{mod } 32),$$

and

$$\kappa(V^n) = a_m\omega \bmod 2 .$$

On the other hand, by theorem 5,

$$a_m\omega = a_m[V^n] \bmod 2 ,$$

therefore

$$\kappa(V^n) = a_m[V^n] \bmod 2 .$$

If $n = 8m + 2$, proposition 2 gives

$$t(a_m[V^n]) = a_m[T(V)] \bmod 2$$

$$= (\sigma(T(V))/16)\omega \bmod 2$$

by the previous case.

By definition,

$$T(V) = (2U) \cup (-M^2 \times V) ,$$

where $\partial U = \bar{S}^1 \times V$. Thus

$$\sigma(T(V)) = 2\sigma(U) .$$

On the other hand,

$$k(V) = \frac{\sigma(U)}{8} = \frac{\sigma(T(V))}{16} \bmod 2 .$$

Comparing with the above expression for $t(a_m[V^n])$, we obtain:

$$a_m[V^n] = k(V^n)\eta^2 = \kappa(V^n) .$$

If $n = 8m + 1$,

$$a_m[V^n]\eta = a_m[\bar{S}^1 \times V^n] = k(\bar{S}^1 \times V^n)\eta^2 ,$$

therefore

$$a_m[V^n] = k(\bar{S}^1 \times V^n)\eta = \kappa(V^n)$$

since the multiplication by η is an isomorphism $KO_1 \xrightarrow{\cong} KO_2$.

Finally, if $n = 8m$, then

$$a_m[V^n]\eta^2 = a_m[\bar{S}^1 \times \bar{S}^1 \times V^n] = k(\bar{S}^1 \times \bar{S}^1 \times V^n)\eta^2 = \sigma(V^n)\eta^2 \text{ ,and}$$

$$a_m[V^n] \equiv \sigma(V^n) \pmod 2 \qquad \square$$

<u>Corollary 1</u>. $\kappa : \Omega_*^{spin} \longrightarrow KO_* \otimes \mathbb{F}_2$ is a ring homomorphism.

<u>Proof</u>. Let V_1 and V_2 be two spin manifolds of dimension n_1 and n_2 respectively, and let

$$m_1 = [n_1/8] \ , \ m_2 = [n_2/8] \ , \ m = [(n_1 + n_2)/8] \ .$$

By theorem 6,

$$\kappa(V_1 \times V_2) = a_m[V_1 \times V_2] = \sum_{i_1+i_2=m} a_{i_1}[V_1]\, a_{i_2}[V_2] .$$

Notice that $m \geq m_1 + m_2$. If $m = m_1 + m_2$, then theorem 5 (i) and theorem 6 imply:

$$\kappa(V_1 \times V_2) = a_{m_1}[V_1]\, a_{m_2}[V_2] = \kappa(V_1)\kappa(V_2) .$$

If $m > m_1 + m_2$, then theorem 5 (i) gives

$$\kappa(V_1 \times V_2) = 0$$

and one has to check that

$$\kappa(V_1)\kappa(V_2) = 0 .$$

But $m > m_1 + m_2$ is possible only in one of the following cases:

(1) $n_1 \equiv n_2 \equiv 4 \pmod 8$. In this case

$$\kappa(V_1)\kappa(V_2) = 0$$

since $\omega^2 \equiv 0 \pmod 2$.

(2) $n_1 \equiv 5,6,7 \pmod 8$ or $n_2 \equiv 5,6,7 \pmod 8$.

In this case $\kappa(V_1)$ or $\kappa(V_2)$ is zero. □

Corollary 2. Let V^n, $n = 8m + r$ $(r = 1,2)$ be a spin manifold. The filtration of $\beta_q[V^n]$ is exactly $4m$ if and only if $\kappa(V^n) \neq 0$.

This follows from theorem 6 and the description of $M^\Gamma(KO_{8m+r})$ in section 3. □

7. The SU–case. Theorem 3 describes the subring $M_* = \beta_q(\Omega_*^{spin}) \subset M^\Gamma(KO_*)$. Using the results of [6] one can easily determine the image of the special unitary cobordism ring Ω_*^{SU} under β_q. We will focus on the dimensions $8m + 1$, $8m + 2$ leaving the easier remaining cases to the reader.

Theorem 7.

(i) If $n = 8m + 1$, then $\beta_q(\Omega_n^{SU}) \subset \beta_q(\Omega_n^{spin})$ is the subgroup of forms of the form $\eta P(\varepsilon^2)$ where P is a polynomial of degree $\leq m/2$ over $\mathbb{F}_2$.

(ii) If $n = 8m + 2$, then $\beta_q(\Omega_n^{SU}) = \beta_q(\Omega_n^{spin})$.

Corollary. If M^n, $n = 8m + 1$, is an SU–manifold, then

$$a_i[M^n] = 0$$

for all odd i. For instance,

$$\pi_1[M^n] = 0,$$

$$(\pi_3 + \pi_1^2)[M^n] = 0.$$

Proof.

(i) According to [6], an element from $\varphi_q(\Omega^{SU}_{8m})$ can be written as

$$2P(\delta_0^2,\varepsilon) + Q(\delta_0^2,\varepsilon^2)\ ,$$

where P , Q are two polynomials with integer coefficients. On the other hand, one has

$$\Omega^{SU}_{8m+1} = [\bar{S}^1] \cdot \Omega^{SU}_{8m}$$

where $\bar{S}^1$ is the circle S^1 equipped with its non–trivial SU–structure (cf. [21], chap. X). Therefore,

$$\beta_q(\Omega^{SU}_{8m+1}) = \eta \cdot \beta_q(\Omega^{SU}_{8m})$$

and the result follows.

Part (ii) is an immediate consequence of the following proposition.

Proposition 5. The canonical map

$$\Omega^{SU}_{8m+2} \longrightarrow \Omega^{spin}_{8m+2} / I_{8m+2}$$

is onto. In other words, any spin manifold of dimension $8m+2$ has the same KO–characteristic numbers as an SU–manifold .

Proof. Notice first that the homomorphism T used in the proof of theorem 4 can be defined using SU–manifolds : there is a homomorphism

$$T^c : \Omega^{SU}_{8m+2} \longrightarrow \Omega^{SU}_{8m+4} \otimes \mathbb{F}_2$$

which preserves the mod 2 KO–characteristic numbers. Let $I^c_* \subset \Omega^{SU}_*$ be the ideal of classes with vanishing KO–characteristic numbers. Then T^c induces a homomorphism

$$\tilde{T}^c : \Omega^{SU}_{8m+2} / I^c_{8m+2} \longrightarrow (\Omega^{SU}_{8m+4} / I^c_{8m+4}) \otimes \mathbb{F}_2 ,$$

and there is a commutative diagram

$$\begin{array}{ccc} \Omega^{SU}_{8m+2} / I^c_{8m+2} & \xrightarrow{\tilde{T}^c} & (\Omega^{SU}_{8m+4} / I^c_{8m+4}) \otimes \mathbb{F}_2 \\ \lambda \downarrow & & \downarrow \mu \\ \Omega^{spin}_{8m+2} / I_{8m+2} & \xrightarrow{\tilde{T}} & (\Omega^{spin}_{8m+4} / I_{8m+4}) \otimes \mathbb{F}_2 \end{array}$$

in which λ and μ are induced by the forgetful homomorphism. One has to show that λ is onto. It is well known (cf. [19]) that

$$\Omega^{SU}_{8m+4} \longrightarrow \Omega^{spin}_{8m+4} / \mathrm{Tors}$$

is onto. As $I_{8m+4} = \mathrm{Tors}\, \Omega^{spin}_{8m+4}$, this implies that μ is onto. Thus to prove the proposition, it will suffice to show that $\tilde{T}^c$ is onto.

Let $B_* \subset \Omega_*^{SO}/\mathrm{Tors}$ be the subring of classes represented by U–manifolds with spherical determinant. According to Stong ([21], p. 282), B_* is a polynomial algebra and $\Omega_{8m+4}^{SU} / I_{8m+4}^c \subset B_{8m+4}$ is exactly the subgroup $2B_{8m+4}$.

Let M^{8m+4} be an SU–manifold, and let W^{8m+4} be a U–manifold with spherical determinant such that $[M] = 2[W]$ in B_{8m+4}. Dualizing the determinant of W gives an SU–manifold V^{8m+2} and we have

$$W = U \cup (-D^2 \times V)$$

where U is an SU–manifold with boundary $\bar{S}^1 \times V$, namely the complement of a tubular neighbourhood of V in W (cf. [13]).

By definition, $T^c([V])$ is represented by the manifold $Z = (2U) \cup (-M^2 \times V)$, where M^2 is an SU–manifold such that $\partial M^2 = 2\bar{S}^1$. It is easy to see that Z is cobordant to 2W as a U–manifold. Therefore Z and 2W have the same rational Pontrjagin numbers. Hence Z and M have the same KO–characteristic numbers, that is represent the same element in $\Omega_{8m+4}^{SU} / I_{8m+4}^c$. □

8. <u>Final remarks</u>. 1°. According to theorem 6, the reduction mod 2 of the class a_m measures the KO–part of the Brown–Kervaire invariant in dimension $8m + 2$. For instance,

$$k(V^{10}) = \pi_1 [V^{10}]$$

$$k(V^{18}) = (\pi_2 + \pi_1)[V^{18}]$$

$$k(V^{26}) = (\pi_3 + \pi_1^2)[V^{26}] .$$

Other sequences $a_0, a_1, \dots$ having the same property have been constructed in [13]. For example,

$$a_m = L_{2m}(\pi_1, \dots, \pi_{2m}) + (\pi_1^3 + \pi_1\pi_2 + \pi_3)L_{2m-2}(\pi_1, \dots, \pi_{2m-2}) ,$$

where L_{2m} is the reduced mod 2 Hirzebruch's polynomial, is such a sequence. A simple comparison of the first few terms shows that the new classes a_m have far fewer terms. Besides, they have better multiplicative properties. The classes a_m have been used in [17] to represent $k(V)$ as the index of a twisted Dirac operator on V.

Notice that the mod 2 reduction of $h(q)$ is of the form $q + o(q^8)$. Therefore one has

$$a_m \equiv b_m (\text{mod } 2)$$

for $m \le 8$. Thus in dimensions $n \le 71$, $\kappa(V)$ is measured by the Witten class $b_{[n/8]}$

2°. <u>The genus</u>

$$\varphi : \Omega_*^{SO} \longrightarrow M^{\Gamma}(\mathbb{Z}[1/2])$$

was used by Landweber, Ravenel and Stong ([12]) to construct an elliptic (co)homology theory $E\ell\ell_*$ ([10] , [11]). Namely they showed that

$$E\ell\ell_*(\ ) = \Omega_*^{SO}(\ ) \otimes_{\varphi} M^{\Gamma}(\mathbb{Z}[1/2])[\varepsilon^{-1}]$$

is a homology theory. Here $M^{\Gamma}(\mathbb{Z}[1/2])$ is considered as an Ω_*^{SO} – module via φ.

By analogy with the Conner–Floyd isomorphism ([7])

$$KO_*(\ ) \cong \Omega_*^{Sp}(\ ) \otimes KO_*$$

one can ask whether the functor

$$\Omega_*^{Sp}(\ ) \otimes_{\beta_q} M_*[\varepsilon^{-1}] ,$$

where $M_* \subset M^{\Gamma}(KO_*)$ is the image of β_q described in Theorem 3 (iii), is a homology theory. A positive answer to this question would provide a way of eliminating the undesirable 1/2 in the definition of $E\ell\ell_*$ ( ).

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A VANISHING THEOREM FOR THE ELLIPTIC GENUS

by

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Let

$$\varphi : \Omega_*^{SO} \longrightarrow \mathbb{Q}[\delta,\varepsilon]$$

be the universal rational elliptic genus defined by

$$\sum_{i \geq 0} \varphi[\mathbb{C}P_{2i}]\, u^{2i} = (1 - 2\delta u^2 + \varepsilon u^4)^{-1/2} .$$

It is a simple consequence of the rigidity theorem of Bott and Taubes [3] that $\varphi[V] = 0$ for any spin manifold V admitting an odd type circle action. Indeed, substituting for δ and ε two algebraically independent complex numbers gives an embedding $\mathbb{Q}[\delta,\varepsilon] \hookrightarrow \mathbb{C}$ hence a non degenerate elliptic genus over $\mathbb{C}$. The corresponding equivariant genus $\varphi_{S^1}[V]$ is an elliptic function $\varphi(u)$ for any oriented S^1–manifold V (cf. [5]). Moreover, if V is a spin manifold and the action is odd, then

$$\varphi(u + \omega) = -\varphi(u)$$

for a certain half period ω ([5], proposition 7 (ii)). On the other hand, according to [3], $\varphi(u)$ is <u>constant</u>. Therefore $\varphi[V] = \varphi_{S^1}[V] = 0$.

In the present note we extend the above vanishing theorem to the refined elliptic genus

$$\beta_q : \Omega_*^{spin} \longrightarrow KO_*[[q]]$$

introduced in [6]. The first results in this direction were obtained by M. Bendersky [1] who proved that $\beta_q[V] = 0$ for any spin manifold V admitting an odd type <u>semifree</u> circle action. Bendersky's proof follows from a detailed study of Borsari's exact sequence [2]. Our proof, valid for any odd type action, is based on a simple geometrical construction and on the strict multiplicativity of elliptic genera.

We recall briefly the definition of β_q (cf. [6]). Let E be any real vector bundle over X. The Witten characteristic class $\Theta_q(E) \in KO(X)[[q]]$ is defined by

$$\Theta_q(E) = \bigotimes_{n\geq 1} (\Lambda_{-q^{2n-1}}(E) \otimes S_{q^{2n}}(E)),$$

where

$$\Lambda_t(E) = \sum_{i\geq 0} \Lambda^i(E)t^i$$

and

$$S_t(E) = \sum_{i\geq 0} S^i(E)t^i .$$

If V is a closed spin n–manifold , $\beta_q[V]$ is defined by

$$\beta_q[V] = \Theta_q(TV - n)[V] \in KO_n[[q]] ,$$

where $KO_n = KO_n(\text{point})$. One has

$$\beta_q[V] = b_0(TV)[V] + b_1(TV)[V]\, q + \dots ,$$

where $b_i \in KO(BSO)$ are certain stable KO–characteristic classes and $b_i(TV)[V]$ are the corresponding characteristic numbers. The map

$$V \longmapsto \beta_q[V]$$

defines a ring homomorphism (genus)

$$\beta_q : \Omega_*^{spin} \longrightarrow KO_*[[q]] ,$$

which is a refinement of a rational elliptic genus in the following sense. Let

$$ph : KO_* \longrightarrow \mathbb{Z}$$

be the Pontrjagin character, i.e. the composition of the complexification $KO_* \longrightarrow K_*$ and the Chern character. Then

$$\varphi_q = \mathrm{ph} \circ \beta_q : \Omega_*^{spin} \longrightarrow \mathbb{Z}[[q]]$$

is the restriction to Ω_*^{spin} of an elliptic genus over $\mathbb{Q}[[q]]$ with invariants

$$\delta = -\frac{1}{8} - 3 \sum_{n \geq 1} \Big(\sum_{\substack{d|n \\ d \text{ odd}}} d \Big) q^n$$

$$\varepsilon = \sum_{n \geq 1} \Big(\sum_{\substack{d|n \\ n/d \text{ odd}}} d^3 \Big) q^n .$$

Let now V be any connected closed spin n–manifold .

Theorem. If V admits an odd type circle action, then $\beta_q[V] = 0$.

Proof. The vanishing of the universal genus φ implies the vanishing of $\varphi_q[V]$. As $\varphi_q[V] = \mathrm{ph}(\beta_q[V])$, this in turn implies $\beta_q[V] = 0$ for $n \equiv 0 (\mathrm{mod}\ 4)$, for

$$\mathrm{ph} : KO_n[[q]] \longrightarrow \mathbb{Z}[[q]]$$

is then injective.

The case of dimensions $n \equiv 1(\mathrm{mod}\ 8)$ is easily reduced to that of dimensions $n \equiv 2(\mathrm{mod}\ 8)$ by multiplying V by the circle with its non–trivial spin structure and trivial S^1–action .

The proof in dimensions $n = 8m + 2$ is based on the following construction. Let M^{8m+4} be a closed oriented manifold and suppose we are given an embedding

$D^2 \times V \hookrightarrow M$ and a spin structure on

$$W = M - \mathrm{int}(D^2 \times V)$$

inducing the non–trivial spin structure on each circle

$$S^1 \times \{p\} \subset S^1 \times V = \partial W .$$

Then V has a canonical spin structure and we have:

<u>Proposition</u> (cf. [4], § 16). For any $\alpha \in KO(BSO)$ one has

$$\alpha[V] = (\mathrm{ph}(\alpha(TM))\hat{\mathfrak{A}}(TM)[M]) \cdot \eta^2$$

where $\hat{\mathfrak{A}}(TM)$ is the total $\hat{\mathfrak{A}}$–class of M and $\eta \in KO_1 = \mathbb{F}_2$ is the generator. □

In fact, M admits a spin^c–structure and the coefficient of η^2 is an integer.

Let now V^{8m+2} be a connected spin manifold with an odd type circle action

$$\mu : S^1 \times V \longrightarrow V .$$

Consider $M = S^3 \times_{S^1} V$, the total space of the fiber bundle associated with the Hopf bundle $S^3 \longrightarrow S^2$, and fiber V. M can be obtained by glueing together two copies of $D^2 \times V$, say $D^2_+ \times V$ and $D^2_- \times V$, using the map

$$f : S^1 \times V \longrightarrow S^1 \times V$$

given by

$$f(z,p) = (z,\mu(z,p)) .$$

The manifold

$$W = D^2_- \times V = M - \mathrm{int}(D^2_+ \times V)$$

has a unique spin structure compatible with the given spin structure on V . The map f restricted to the circle $S^1 = S^1 \times \{p\}$ is given by

$$z \longmapsto (z,\mu(z,p)) .$$

It can therefore be viewed as the inclusion of an orbit of the diagonal circle action on $S^1 \times V = \partial W$. This action is even type. Indeed, the standard circle action on S^1 equipped with the trivial spin structure is odd type, and so is the given action on V . It follows that the spin structure on W induces the non–trivial spin structure on each circle $S^1 \times \{p\} \subset \partial W$. On the other hand, it obviously induces the given spin structure on V . The proposition above gives

$$\alpha[V] = (\mathrm{ph}(\alpha(TM))\hat{\mathfrak{A}}(TM)[M]) \cdot \eta^2$$

for any $\alpha \in KO(BSO)$; in particular, one has:

$$\beta_q[V] = \varphi_q[M] \cdot \eta^2 .$$

The rigidity theorem of Bott and Taubes [3] implies the strict multiplicativity of elliptic genera over $\mathbb{Q}$–algebras (cf. [5]), therefore

$$\varphi_q[M] = \varphi_q[S^2] \cdot \varphi_q[V] = 0$$

and

$$\beta_q[V] = 0 . \qquad \square$$

Corollary. If a spin manifold V^{8m+2} admits an odd type circle action, then both the Atiyah invariant a(V) and the KO–part of the Brown–Kervaire invariant, k(V), vanish.

Indeed, a(V) and k(V) are two of the coefficients of $\beta_q[V]$ when expressed as a polynomial in the series

$$\bar{\varepsilon} = \sum_{n\geq 1} q^{(2n-1)^2}$$

(cf. [6]).

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