# QUADRATIC CATEGORIES AND SQUARE RINGS

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# QUADRATIC CATEGORIES AND SQUARE RINGS

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We consider quadratic categories which generalize the classical additive categories. An additive category $\underline{\underline{A}}$ is a category for which morphism sets are abelian groups and the composition $fg$ is bilinear, and for which sums exist in $\underline{\underline{A}}$. A quadratic category $\underline{\underline{Q}}$ is slightly more general in the sense that morphism sets are groups and the composition $fg$ is linear in $g$ and quadratic in $f$. This implies that morphism sets are groups of nilpotency degree 2. We describe below many examples of quadratic categories in algebra and topology which motivate the systematic study of quadratic categories started here; it may be considered as an extension of the investigation of quadratic functors in [4].

The properties of a quadratic category and its subcategories lead to the new notion of a "square ring" which is exactly the quadratic analogue of the classical notion of a "ring". Indeed each object $X$ in an additive category $\underline{\underline{A}}$ yields an endomorphism ring given by all morphisms $X \to X$ in $A$; similarly each object in a quadratic category yields the endomorphism square ring $End(X)$ of $X$. The initial object in the category of rings is the ring $\mathbb{Z}$ of integers for which the category of modules is the category of abelian groups. We here determine the initial object $\mathbb{Z}_{nil}$ in the category of square rings for which the category of modules is the category of groups of nilpotency degree 2.

We compute various square rings explicitly, for example, the endomorphism square rings of the suspended projective planes $\Sigma\mathbb{R}P_2$ and $\Sigma\mathbb{C}P_2$. This yields as an application an algebraic description of the homotopy category of all Moore spaces $M(V,2)$ where $V$ is a $\mathbb{Z}/2$ -vector space; in fact this category is equivalent to the full category of free objects in the category of 2-restricted $nil(2)$ -groups.

There has been recently a lot of interest in operads [9]. In fact, operads $\mathcal{O} = \{\mathcal{O}_n\}$ with $\mathcal{O}_n = 0$ for $n \geq 3$ are the same as special square rings. Therefore the theory of square rings shows naturally how the theory of operads has to be modified in order to deal with nilpotent groups.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

## § 1 ADDITIVE CATEGORIES AND MODULES

We first recall some basic notation and facts concerning additive categories; compare [10]. We do this since we are going to introduce the analogous notation and facts for 'quadratic categories'; in fact, the theory of quadratic categories has to be a canonical extension of the theory of additive categories.

*(1.1) Definition.* A category $\underline{\underline{A}}$ is preadditive if the morphism sets $\underline{\underline{A}}(X,Y)$ are abelian groups and the composition law is bilinear. Moreover $\underline{\underline{A}}$ is an additive category if in addition for all objects $X, Y$ there is given a diagram

$$X \underset{r_1}{\overset{i_1}{\rightleftarrows}} X \vee Y \underset{r_2}{\overset{i_2}{\leftrightarrows}} Y$$

with $r_1 i_1 = 1_X$, $r_2 i_2 = 1_Y$ and $i_1 r_1 + i_2 r_2 = 1_{X \vee Y}$. Here $X \vee Y$ is called a biproduct; this is a sum and a product in $\underline{\underline{A}}$ [10]. Moreover $\underline{\underline{A}}$ has a zero object $*$. A zero morphism $0 \in \underline{\underline{A}}(X,Y)$ is given by $X \to * \to Y$; this is also the neutral element of the abelian group $\underline{\underline{A}}(X,Y)$. A preadditive category $\underline{\underline{R}}$ is the same as an $\underline{\underline{Ab}}$ -category (i.e. a category enriched over the monoidal category $(\underline{\underline{Ab}}, \otimes)$ of abelian groups). Such a category is also called a ringoid; in fact, if $\underline{\underline{R}}$ has only one object then $\underline{\underline{R}}$ is the same as a ring.

*(1.2) Definition.* Let $\underline{\underline{R}}$ be a ringoid. Then the biproduct completion of $\underline{\underline{R}}$,

$$i : \underline{\underline{R}} \subset \underline{\underline{Add}}(\underline{\underline{R}}),$$

is given as follows. The objects of $\underline{\underline{Add}}(\underline{\underline{R}})$ are the $n$-tuples $X = (X_1, \dots, X_n)$ of objects in $\underline{\underline{R}}$ with $0 \leq n < \infty$. The morphisms are the corresponding matrices of morphisms in $\underline{\underline{R}}$. The inclusion $i$ carries the object $X$ to the tuple of length 1 given by $X$. The category $\underline{\underline{Add}}(\underline{\underline{R}})$ is an additive category with distinguished biproducts given by

$$\begin{aligned} & X \amalg Y = (X_1, \dots, X_n, Y_1, \dots, Y_k) \\ & \text{for} \quad X = (X_1, \dots, X_n) \quad \text{and} \quad Y = (Y_1, \dots, Y_k). \end{aligned}$$

A functor $F : \underline{\underline{R}} \to \underline{\underline{B}}$ between ringoids is additive if $F(f+g) = F(f) + F(g)$ for $f, g \in \underline{\underline{R}}(X,Y)$. If $\underline{\underline{B}}$ is an additive category there is a unique additive functor

$$\bar{F} : \underline{\underline{Add}}(\underline{\underline{R}}) \to \underline{\underline{B}} \tag{1.3}$$

with $\bar{F} i = F$ and $\bar{F}(X \amalg Y) = \bar{F}(X) \vee \bar{F}(Y)$. This is the freeness property of $\underline{\underline{Add}}(\underline{\underline{R}})$. For example if $R$ is a ring then $\underline{\underline{Add}}(\underline{\underline{R}})$ in the category of finitely generated free $R$-modules.

Let $\underline{\underline{R}}$ be a full subcategory of an additive category $\underline{\underline{A}}$ and let $\hat{\underline{\underline{R}}}$ be the full subcategory of $\underline{\underline{A}}$ consisting of finite biproducts $(n \geq 0)$ $X_1 \vee \dots \vee X_n$ in $\underline{\underline{A}}$ with $X_i \in \underline{\underline{R}}$. Then one obtains an equivalence of additive categories

(*) $$\epsilon : \underline{\underline{Add}}(\underline{\underline{R}}) \xrightarrow{\sim} \hat{\underline{\underline{R}}}$$

which is the additive extension $\epsilon = \bar{j}$ of the inclusion $j : \underline{\underline{R}} \subset \hat{\underline{\underline{R}}}$. In particular for the additive category $\underline{\underline{A}}$ one has the equivalence of additive categories

(**) $$\epsilon : \underline{\underline{Add}}(\underline{\underline{A}}) \xrightarrow{\sim} \underline{\underline{A}}$$

which is the additive extension of the identity on $\underline{\underline{A}}$.

*(1.4)* *Remark.* Let $\underline{\underline{R}}$ be a ringoid. A <u>family</u> $X$ of objects in $\underline{\underline{R}}$ is a set I together with a function $X : I \to Ob(\underline{\underline{R}})$ where $Ob(\underline{\underline{R}})$ is the class of objects in $\underline{\underline{R}}$; we also write $X = \{X_i\}_{i \in I}$. Such families are the objects in the category $\underline{\underline{add}}(\underline{\underline{R}})$ with

$$\underline{\underline{R}} \subset \underline{\underline{Add}}(\underline{\underline{R}}) \subset \underline{\underline{add}}(\underline{\underline{R}})$$

Morphisms in $\underline{\underline{add}}(R)$ from $X = \{X_i\}_{i \in I}$ to $Y = \{Y_j\}_{j \in J}$ are the matrices $(\alpha^i_j)_{(i,j) \in I \times J}$ of elements $\alpha^i_j \in \underline{\underline{R}}(X_i, X_j)$ such that for $i \in I$ almost all elements in $(\alpha^i_j)_{j \in J}$ are zero morphisms. For example for a ring $R$ the category $\underline{\underline{add}}(\underline{\underline{R}}) = \underline{\underline{mod}}(\underline{\underline{R}})$ is the category of free $R$-modules. In particular $\underline{\underline{add}}(\mathbb{Z}) = \underline{\underline{ab}}$ is the category of free abelian groups. Clearly $\underline{\underline{add}}(\underline{\underline{R}})$ is again an additive category with the biproduct $X \amalg Y$ given by the family $\{X_i, Y_j\}_{i \in I, j \in J}$.

We now introduce some notations on theories. A <u>theory</u> $\underline{\underline{T}}$ is a category in which finite sums exist. A <u>model</u> of a theory $\underline{\underline{T}}$ is a functor

(1.5) $$F : \underline{\underline{T}}^{op} \to \underline{\underline{Set}}$$

from the opposite category $\underline{\underline{T}}^{op}$ of $\underline{\underline{T}}$ to the category of sets such that $F$ carries a sum in $\underline{\underline{T}}$ to a product in $\underline{\underline{Set}}$. Let $\underline{\underline{Model}}(\underline{\underline{T}})$ be the corresponding category of models; morphisms are natural transformations. For example for a ringoid $\underline{\underline{R}}$ the category $\underline{\underline{Add}}(\underline{\underline{R}})$ is a theory.

A <u>(right)</u> $\underline{\underline{R}}$ <u>-module</u> $M$ is an additive functor

(1.6) $$M : \underline{\underline{R}}^{op} \to \underline{\underline{Ab}}$$

Let $\underline{\underline{Mod}}(\underline{\underline{R}})$ be the category of such $\underline{\underline{R}}$ -modules. Module homomorphisms $M \to M'$ are the natural transformations. Then one has the canonical isomorphism of categories.

*(1.7)* *Lemma.* $\underline{\underline{Mod}}(\underline{\underline{R}}) = \underline{\underline{Model}}(\underline{\underline{Add}}(\underline{\underline{R}}))$

Here the isomorphism carries the module $M$ to the composition $\underline{\underline{Add}}(\underline{\underline{R}})^{op} \xrightarrow{\bar{M}}$

$\underline{\underline{Ab}} \xrightarrow{\phi} Set$ where $\bar{M}$ is the additive extension of $M$ in (1.3) and $\phi$ is the forgetful functor. Clearly if $R$ is a ring then $\underline{\underline{Mod}}(R)$ is the classical category of right $R$-modules.

We also shall use quadratic functors on additive categories. For this recall the following classical notation of Eilenberg-Mac Lane [7].

*(1.8) Definition.* Let $\underline{\underline{A}}$ be an additive category. A functor $T : \underline{\underline{A}} \to \underline{\underline{Ab}}$ is termed quadratic if $T(0) = 0$ and the cross-effect bifunctor

$$T(X \mid Y) = kernel\,(T(X \vee Y) \xrightarrow{r} T(X) \oplus T(Y)) \quad \text{with} \quad r = (r_{1*}, r_{2*}),$$

is biadditive. Equivalently the functor $T$ is quadratic iff the induced function

$$\underline{\underline{A}}(A, B) \xrightarrow{T} Hom(T(A), T(B))$$

is quadratic for all objects $A, B$ in $\underline{\underline{A}}$, see (2.1). The functor $T$ is additive iff $T(X \mid Y) = 0$ for all $X, Y$.

Examples of quadratic functors $\underline{\underline{Ab}} \to \underline{\underline{Ab}}$ are $\otimes^2$, $\Lambda^2$, $Sym^2$, $\Gamma$; compare [4] and [12, 13] where quadratic functors are studied. Examples of left additive and right quadratic bifunctors are the functors

$$Hom(-, T) : \underline{\underline{Ab}}^{op} \times \underline{\underline{Ab}} \to \underline{\underline{Ab}}$$

which carry $(A, B)$ to $Hom(A, T(B))$ where $T : \underline{\underline{Ab}} \to \underline{\underline{Ab}}$ is quadratic. In a similar way we define the bifunctor $Ext(-, T)$ on $\underline{\underline{Ab}}$ which is left additive and right quadratic.

## § 2 Quadratic categories

We introduce the notion of a quadratic category which is the "quadratic analogue" of an additive category. In a quadratic category the morphism sets are groups which need not to be abelian. Here we write the group structure (also of a non-abelian group) additively since we write the composition law multiplicatively.

We say that a function $\varphi : G \to G'$ between groups is linear if $(x, y \in G)$

$$\varphi(x + y) = \varphi(x) + \varphi(y).$$

Moreover $\varphi$ is quadratic if the function $(\mid)_\varphi : G \times G \to G'$ given by the "cross effect"

$$(x \mid y)_\varphi = \varphi(x + y) - \varphi(y) - \varphi(x) \tag{2.1}$$

is linear in $x$ and $y$ and $(x \mid y)_\varphi$ is central in $G'$. Clearly $\varphi$ is linear if and only if $(\mid)_\varphi = 0$.

*(2.2) Definition.* A category $\underline{\underline{Q}}$ is prequadratic if the morphism sets $\underline{\underline{Q}}(X, Y)$ are groups and if the composition law $f \circ g$ is linear in $g$ and quadratic in $f$. This

means more precisely the following. Consider for $X \xleftarrow{f} Y \xleftarrow{g} Z$ in $\underline{\underline{Q}}$ the induced functions

$$f_* : \underline{\underline{Q}}(Z,Y) \to \underline{\underline{Q}}(Z,X)$$
$$g^* : \underline{\underline{Q}}(Y,X) \to \underline{\underline{Q}}(Z,X)$$

with $f_*(g) = g^*(f) = f \circ g$. Then $f_*$ is linear for all $Z \in \underline{\underline{Q}}$ and $g^*$ is quadratic for all $X \in \underline{\underline{Q}}$. Hence for $f, f' \in \underline{\underline{Q}}(Y,X)$ the cross effect

$$(f \mid f')_g = (f + f') \circ g - f' \circ g - f \circ g \in \underline{\underline{Q}}(Z,X)$$

is linear in $f$ and $f'$ and central in $\underline{\underline{Q}}(Z,X)$. We say that a morphism $g$ in $\underline{\underline{Q}}$ is <u>linear</u> if $g^*$ is linear for all $X$.

The prequadratic category $\underline{\underline{Q}}$ is termed a <u>quadratic category</u> if $\underline{\underline{Q}}$ has a zero object $*$ and if for all objects $X, Y$ there is given a diagram of linear morphisms

$$X \underset{r_1}{\overset{i_1}{\rightleftarrows}} X \vee Y \underset{r_2}{\overset{i_2}{\leftrightarrows}} Y$$

with $r_1 i_1 = 1_X$, $r_2 i_2 = 1_Y$ and $i_1 r_1 + i_2 r_2 = 1_{X \vee Y}$. We call $X \vee Y$ a <u>quadratic biproduct</u> in $\underline{\underline{Q}}$.

We shall see that an additive category is the same as a quadratic category for which all morphisms are linear. Clearly a full subcategory $\underline{\underline{R}}$ of a quadratic category $\underline{\underline{Q}}$ is prequadratic. Let $\hat{\underline{\underline{R}}}$ be the <u>biproduct completion</u> of $\underline{\underline{R}}$ in $\underline{\underline{Q}}$, i.e. the full subcategory of $\underline{\underline{Q}}$ consisting of finite quadratic biproducts $X_1 \vee \ldots \vee X_n$ in $\underline{\underline{Q}}$ with $X_i \in \underline{\underline{R}}$. Then the structure of $\underline{\underline{R}}$ as a prequadratic category does not determine $\hat{\underline{\underline{R}}}$ so that the direct analogue of (1.3) (*) is not true. Therefore there arises the problem of adding "structure" to $\underline{\underline{R}}$ in such a way that $\underline{\underline{R}}$ together with the structure determines $\hat{\underline{\underline{R}}}$. We specify this additional structure of $\underline{\underline{R}}$ via the notion of "square ringoid" in § 3.

*(2.3)* *<u>Remark</u>*. We call $\underline{\underline{Q}}$ in (2.2) also a <u>left</u> quadratic category since $\underline{\underline{Q}}$ has a left quadratic composition law. Using duality we can define a <u>right</u> quadratic category $\underline{\underline{P}}$ by the condition that the opposite category $\underline{\underline{P}}^{op}$ is a left quadratic category. Then the composition $f \circ g$ in $\underline{\underline{P}}$ is linear in $f$ and quadratic in $g$ and for biproducts in $\underline{\underline{P}}$ the maps $i_1, i_2, r_1, r_2$ are linear. All results below refer to (left) quadratic categories; there are obvious dual results for right quadratic categories.

We now describe various examples of quadratic categories. Let $\underline{\underline{Top}}^*/\simeq$ be the homotopy category of pointed topological spaces. Suspensions and loop spaces give rise to the following quadratic categories of the "metastable range" of homotopy theory.

*(2.4)* *<u>Example</u>*. Let $n \geq 2$ and let

$$\underline{\underline{\Sigma}}(n, 3n-3) \subset \underline{\underline{Top}}^* / \simeq$$

be the full subcategory consisting of suspensions $\Sigma X$ which are $(n-1)$ -connected $(3n-3)$ -dimensional CW-spaces. Then $\underline{\underline{\Sigma}}(n, 3n-3)$ is a (left) quadratic category. The group structure for the set $[\Sigma X, \Sigma Y]$ of morphisms is given by the suspension $\Sigma X$. The left distributivity law of homotopy theory shows that the composition in $\underline{\underline{\Sigma}}(n, 3n-3)$ is left quadratic, see Appendix [5]. Quadratic biproducts are one point unions $(\Sigma X) \vee (\Sigma Y) = \Sigma(X \vee Y)$ of suspensions.

*(2.5)* *Example*. Let $n \geq 2$ and let

$$\underline{\underline{\Omega}}(n, 3n-1) \subset \underline{\underline{Top}}^* / \simeq$$

be the full subcategory consisting of loop spaces $\Omega X$ which are $(n-1)$ -connected CW-spaces with $\pi_i \Omega X = 0$ for $i > 3n-1$. Then $\underline{\underline{\Omega}}(n, 3n-1)$ is a right quadratic category. The group structure for the set $[\Omega X, \Omega Y]$ of morphisms is given by the loop space $\Omega Y$. Quadratic biproducts are products $(\Omega X) \times (\Omega Y) = \Omega(X \times Y)$ of loop spaces.

*(2.6)* *Example*. Let $\underline{\underline{Gr}}$ be the category of groups. A group $G$ has nilpotency degree 2 if all triple commutators in $G$ vanish. Then $G$ is also termed a nil-group. Let $\underline{\underline{Nil}} \subset \underline{\underline{Gr}}$ be the full subcategory of nil-groups. The free nil group $\langle M \rangle_{nil}$ generated by a set $M$ is given by the quotient $\langle M \rangle_{nil} = \langle M \rangle / \Gamma_3 \langle M \rangle$ where $\langle M \rangle$ is the free group generated by $M$ and where $\Gamma_3 \langle M \rangle$ is its subgroup of triple commutators. Let $\underline{\underline{nil}} \subset \underline{\underline{Nil}}$ be the subcategory of free nil groups. Then $\underline{\underline{nil}}$ is a quadratic category. The group structure of $\underline{\underline{Gr}}(\langle M \rangle_{nil}, \langle N \rangle_{nil})$ is given by $(f+g)(m) = f(m) + g(m)$ for $m \in M$. One readily checks that the disjoint union $M \dot{\cup} N$ yields the quadratic biproduct $\langle M \rangle_{nil} \vee \langle N \rangle_{nil} = \langle M \dot{\cup} N \rangle_{nil}$.

We now describe some basic properties of prequadratic, resp. quadratic, categories $\underline{\underline{Q}}$. The zero morphism $0 \in \underline{\underline{Q}}(Y, X)$ is given by the neutral element in the group $\underline{\underline{Q}}(Y, X)$. For $g \in \underline{\underline{Q}}(Z, Y)$ let $-g$ be the inverse of $g$. Moreover let $2 = 2_X \in \underline{\underline{Q}}(X, X)$ be the double of the identity; i.e. $2_X = 1_X + 1_X$ where $1_X$ is the identity of $X$.

**(2.7) Lemma.** *In a prequadratic category $\underline{\underline{Q}}$ we have the formulas*

$$\begin{aligned}
&f \circ 0 = 0 \quad \textit{and} \quad 0 \circ g = 0, \\
&r_1 i_2 = 0 \quad \textit{and} \quad r_2 i_1 = 0 \quad \textit{for a quadratic biproduct}, \\
&(-f)g = -(fg) + (f \mid f)_g, \\
&(f \mid f')_{2_Y} = f' + f - f' - f = -f' - f + f' + f
\end{aligned}$$

*where $f, f' \in \underline{\underline{Q}}(Y, X)$ and $g \in \underline{\underline{Q}}(Z, Y)$.*

If $\underline{\underline{Q}}$ has a zero obejct $*$ then the first formula implies that $0 \in \underline{\underline{Q}}(Y, X)$ coincides with $Y \to * \to X$. Moreover the last formula shows that commutators in $\underline{\underline{Q}}(Y, X)$ are central. Therefore one gets

**(2.7) Addendum.** *All morphism groups $\underline{\underline{Q}}(Y,X)$ in a prequadratic category $\underline{\underline{Q}}$ are groups of nilpotency degree 2.*

*Proof of (2.7).* We have $f0 = f(0+0) = f0 + f0$ so that $f0 = 0$. Moreover since $(|)_g$ is bilinear we get

$$(0 \mid 0)_g = 0 = (0+0)g - 0g - 0g = 0g.$$

For a quadratic biproduct we have

$$\begin{aligned} r_2(i_1 r_1 + i_2 r_2) &= r_2 \, 1_{X \vee Y} = r_2 \\ r_2 i_1 r_1 + r_2 i_2 r_2 &= r_2 i_1 r_1 + r_2 \end{aligned}$$

so that $r_2 i_1 r_1 = 0$. Therefore $r_2 i_1 = r_2 i_1 r_1 i_1 = 0\, i_1$. Next we have $0 = f + (-f)$ and therefore

$$\begin{aligned} -(f \mid f)_g = (f \mid -f)_g &= (f + (-f))g - (-f)g - fg \\ &= -(-f)g - fg \end{aligned}$$

Finally we get

$$\begin{aligned} (f \mid f')_{2_Y} &= (f + f')(1_Y + 1_Y) - f'(1_Y + 1_Y) - f(1_Y + 1_Y) \\ &= f + f' + f + f' - f' - f' - f - f \\ &= f + (f' + f - f' - f) - f \end{aligned}$$

This yields the commutator formula since $(f \mid f')_{2_Y} = (-f \mid -f')_{2_Y}$ is central.

*q.e.d.*

**(2.8) Lemma.** *Linear morphisms in a prequadratic category $\underline{\underline{Q}}$ form a subcategory which we denote by Linear $(\underline{\underline{Q}})$.*

*Proof.* Let $g, g'$ be linear. Then $gg'$ is linear since

$$\begin{aligned} & (f_1 + f_2)gg' - f_2 gg' - f_1 gg' \\ (*) \quad & = ((f_1 + f_2)g - f_2 g - f_1 g)g' \\ (**) \quad & = 0\, g' = 0 \end{aligned}$$

Here (*) holds since $g'$ is linear and (**) is true since $g$ is linear.

q.e.d.

*Remark.* For example in (2.4) the linear maps are the co-$H$ -maps and in (2.5) the linear maps are the $H$-maps. The linear maps in $\underline{\underline{nil}}$ are obtained by all homomorphisms $E_M \to E_N$ given by functions $M \to N \cup \{0\}$ so that Linear $(\underline{\underline{nil}}) = \underline{\underline{Set}}^*$ is the category of pointed sets.

**(2.9) Lemma.** *A quadratic biproduct $X \vee Y$ is a sum in $\underline{\underline{Q}}$, that is*

$$(i_1^*, i_2^*) : \underline{\underline{Q}}(X \vee Y, Z) = \underline{\underline{Q}}(X, Z) \times \underline{\underline{Q}}(Y, Z)$$

*is a bijection. Moreover $(i_1^*, i_2^*)$ is an isomorphism of groups.*

*Proof.* $i_1^*, i_2^*$ are homomorphisms since $i_1$ and $i_2$ are linear. The inverse $j$ of $(i_1^*, i_2^*)$ carries $(a, b)$ to $ar_1 + br_2$. In fact

$$\begin{aligned} j(i_1^*, i_2^*)(u) &= j(ui_1, ui_2) \\ &= ui_1r_1 + ui_2r_2 \\ &= u(i_1r_1 + i_2r_2) \\ &= u\, 1_{X\vee Y} = u \end{aligned}$$

$$\begin{aligned} (i_1^*, i_2^*)j(a, b) &= (i_1^*, i_2^*)(ar_1 + br_2) \\ &= (ar_1i_1 + br_2i_1, ar_1i_2 + br_2i_2) \\ &= (a, b) \end{aligned}$$

q.e.d.

A quadratic biproduct in $\underline{\underline{Q}}$ in general is not a product but we have the following property of the morphism set $\underline{\underline{Q}}(Z, X \vee Y)$. For objects $X, Y, Z$ let $\underline{\underline{Q}}(Z, X \mid Y)$ be the kernel of

$$r = (r_{1*}, r_{2*}) : \underline{\underline{Q}}(Z, X \vee Y) \to \underline{\underline{Q}}(Z, X) \times \underline{\underline{Q}}(Z, Y)$$

**(2.10) Lemma.** *This kernel defines a functor*

$$\underline{\underline{Q}}(\quad, \mid) : \underline{\underline{Q}}^{op} \times \underline{\underline{Q}} \times \underline{\underline{Q}} \to \underline{\underline{Ab}}$$

*which we call the cross effect functor on the quadratic category $\underline{\underline{Q}}$ and*

$$\underline{\underline{Q}}(Z, X \mid Y) \overset{i_{12}}{\rightarrowtail} \underline{\underline{Q}}(Z, X \vee Y) \overset{r}{\twoheadrightarrow} \underline{\underline{Q}}(Z, X) \times \underline{\underline{Q}}(Z, Y)$$

*is a central extension of groups which is natural in $Z, X$ and $Y$. Here $i_{12}$ is the inclusion. Moreover the functor $\underline{\underline{Q}}(\quad, \mid)$ is additive in each variable $Z, X, Y$.*

*Proof.* $r$ is surjective since $r(i_1a + i_2b) = (r_1(i_1a + i_2b),\, r_2(i_1a + i_2b)) = (a, b)$. We define

$$r_{12} : \underline{\underline{Q}}(Z, X \vee Y) \to \underline{\underline{Q}}(A, X \mid Y)$$

by $r_{12}(u) = i_{12}^{-1}(u - i_2r_2u - i_1r_1u)$. In fact $r_{12}(u) \in M(Z, X \mid Y)$ since

$$\begin{aligned} r\, r_{12}(u) &= (r_1(u - i_2r_2u - i_1r_1u),\, r_2(u - i_2r_2u - i_1r_1u)) \\ &= (r_1u - r_1u,\, r_2u - r_2u) = (0, 0) \end{aligned}$$

Moreover $r_{12}$ is surjective since for $v \in \underline{\underline{Q}}(Z, X \mid Y)$ we have $r_1 v = 0$ and $r_2 v = 0$ and hence $r_{12}(v) = v - i_2 r_2 v - i_1 r_1 v = v$. Now we can write

$$r_{12}(u) = (i_1 r_1 + i_2 r_2)u - i_1 r_1 u - i_2 r_2 u = (i_1 r_1 \mid i_2 r_2)_u$$

and hence $r_{12}(u)$ is central in the group $\underline{\underline{Q}}(Z, X \vee Y)$ since cross effects are central. Next we see that $\underline{\underline{Q}}(Z, X \mid Y)$ is linear in $Z$. In fact, for $f + f' : Z \to Z'$ we have $(w = i_{12} v)$

$$\begin{aligned} i_{12}(f + f')^* v &= (f + f')^* i_{12} v = w_*(f + f') \\ &= w_* f + w_* f' = i_{12}(f^* v + f'^* v) \end{aligned}$$

since $w_*$ is linear. Moreover we show that $\underline{\underline{Q}}(Z, X \mid Y)$ is linear in $X$ and $Y$. For this we observe that $f \vee g : X \vee Y \to X' \vee Y'$ satisfies the formula

$$f \vee g = i_1 f r_1 + i_2 g r_2$$

so that for $v \in \underline{\underline{Q}}(Z, X \mid Y)$ with $w = i_{12} v$

$$\begin{aligned} i_{12}(f, g)_* v &= (f \vee g)_* i_{12} v \\ &= (i_1 f r_1 + i_2 g r_2) w \\ &= i_1 f r_1 w + i_2 g r_2 w + (i_1 f r_1 \mid i_2 g r_2)_w \\ &= (i_1 f r_1 \mid i_2 g r_2)_w \end{aligned}$$

since $r_1 w = 0$ and $r_2 w = 0$. Here the cross effect is linear in $f$ and $g$ since $r_1$ and $r_2$ are linear and since the cross effect is bilinear.

q.e.d.

**(2.11) Corollary.** *One has a bijection of sets*

$$\underline{\underline{Q}}(Z, X \vee Y) = \underline{\underline{Q}}(Z, X) \times \underline{\underline{Q}}(Z, Y) \times \underline{\underline{Q}}(Z, X \mid Y)$$

*which carries $u$ to $(r_1 u, r_2 u, r_{12}(u))$ and the inverse carries $(a, b, v)$ to $i_{12} v + i_1 a + i_2 b$. The bijectioon is natural in $X$ and $Y$.*

In an additive category a biproduct is a sum and a product. In a quadratic category a quadratic biproduct $X \vee Y$ is a sum and satisfies property (2.11) so that $X \vee Y$ is a product iff for all $Z$ the group $\underline{\underline{Q}}(Z, X \mid Y)$ is trivial.

*(2.12) Definition.* The cross effect functor $\underline{\underline{Q}}(\ \ , |)$ of a quadratic category $\underline{\underline{Q}}$ is endowed with the following structure maps $H, P, T$. For $X \vee X$ we have the morphisms

$$\begin{cases} \mu : X \to X \vee X, \ \mu = i_1 + i_2 \\ \nabla : X \vee X \to X, \ \nabla = (1_X, 1_X) \end{cases}$$

We define functions $H$ and $P$,

$$\underline{\underline{Q}}(Z,X) \xrightarrow{H} \underline{\underline{Q}}(Z,X \mid X) \xrightarrow{P} \underline{\underline{Q}}(Z,X),$$

by $H(w) = r_{12}(\mu_* w)$ and $P(v) = \nabla_*(i_{12}v)$. Moreover we define the <u>interchange map</u>

$$T : \underline{\underline{Q}}(Z,X \mid Y) \approx \underline{\underline{Q}}(Z,Y \mid X)$$

by the commutative diagram

$$\begin{tikzcd}
\underline{\underline{Q}}(Z,X \mid Y) \arrow[r, "i_{12}"] \arrow[d, "T"] & \underline{\underline{Q}}(Z,X \vee Y) \arrow[d, "t_*"] \\
\underline{\underline{Q}}(Z,Y \mid X) \arrow[r, "i_{12}"] & \underline{\underline{Q}}(Z,Y \vee X)
\end{tikzcd}$$

where $t : X \vee Y \to Y \vee X$ is defined by $ti_1 = i_2$, $ti_2 = i_1$. Since $t_*$ is a homomorphism we see that $T$ is an isomorphism of abelian groups and clearly $TT = 1$ since $tt = 1$.

Let $\underline{\underline{C}}$ be a category with a zero object and finite sums. We recall that a <u>cogroup</u> in $\underline{\underline{C}}$ is a tuple $(X, \mu, \nu)$ where $X$ is an object in $\underline{\underline{C}}$ and where $\mu : X \to X \vee X$ $\nu : X \to X$ are morphisms with the following properties.

$$(2.13) \qquad \begin{cases} (1,0)\mu = 1,\ (0,1)\mu = 1 & \text{(counit property)} \\ (1 \vee \mu)\mu = (\mu \vee 1)\mu & \text{(coassociativity)} \\ (1,\nu)\mu = 0 & \text{(coinverse)} \end{cases}$$

A cogroup $X$ induces the structure of a group on the morphism set $\underline{\underline{C}}(X,Z)$ for all $Z$. The group structure is obtained by $a + b = (a,b)\mu$ with inverse $-a = a\nu$. A map $f : Y \to X$ between cogroups is a <u>co-H-map</u> if $\mu f = (f \vee f)\mu$. Such a map induces a homomorphism between groups $f^* : \underline{\underline{C}}(X,Z) \to \underline{\underline{C}}(Y,Z)$.

**(2.14) <u>Lemma</u>.** *Each object $X$ in a quadratic category $\underline{\underline{Q}}$ is canonically a cogroup such that the group structure of $\underline{\underline{Q}}(X,Z)$ coincides with the induced group structure. A map $f : X \to Y$ in $\underline{\underline{Q}}$ is linear iff $f$ is a co-H-map, this is the case, if and only if $H(f) = 0$.*

*<u>Proof</u>.* We obtain the cogroup structure of $X$ by $\mu = i_1 + i_2 : X \to X \vee X$ and $\nu = -1_X : X \to X$. Now $H(f) = 0$ iff $i_{12}H(f) = 0$ where

$$\begin{aligned} i_{12}H(f) &= i_{12}r_{12}\mu_*(f) \\ &= \mu f - i_2 r_2 \mu f - i_1 r_1 \mu f \\ &= (i_1 + i_2)f - i_2 f - i_1 f \\ &= (i_1 + i_2)f - (f \vee f)(i_1 + i_2) \\ &= \mu f - (f \vee f)\mu \end{aligned}$$

This completes the proof of (2.14).

q.e.d.

## § 3 Square ringoids

Quadratic categories $\underline{\underline{Q}}$ with cross effect $M = \underline{\underline{Q}}(\ \ ,|)$ and structure maps $T, H, P$ in (2.12) satisfy properties which are condensed in the following notion of a 'square ringoid'.

*(3.1) Definition.* A square ringoid

$$(\underline{\underline{R}}, M, T, H, P)$$

is given by a category $\underline{\underline{R}}$ together with the following data. All morphism sets $\underline{\underline{R}}(X,Y)$ are groups (written additively) and

$$\text{(i)} \qquad M : \underline{\underline{R}}^{op} \times \underline{\underline{R}} \times \underline{\underline{R}} \to \underline{\underline{Ab}}$$

is a functor which is linear in each variable. That is, for morphisms $f, g, h$ in $\underline{\underline{R}}$ the function $M$ which carries $(f,g,h)$ to $M(f,g,h) = f^*(g,h)_*$ is linear in each variable $f, g$ and $h$ respectively. Next

$$\text{(ii)} \qquad T : M(X,Y,Z) \cong M(X,Z,Y)$$

is a natural isomorphism with $TT = 1$. Moreover $H$ and $P$ denote functions

$$\text{(iii)} \qquad \underline{\underline{R}}(X,Y) \xrightarrow{H} M(X,Y,Y) \xrightarrow{P} \underline{\underline{R}}(X,Y)$$

for all objects $X, Y$ in $\underline{\underline{R}}$. These data satisfy the following properties (1)...(7).

(1) $P$ is a homomorphism which maps to the center of the group $\underline{\underline{R}}(X,Y)$ and $P$ is natural in $X$ and $Y$, that is for $x : X \to X'$ and $y : Y \to Y'$ in $\underline{\underline{Q}}$ we have

$$x^*P = Px^* \qquad \text{and} \qquad P(y,y)_* = y_*P$$

Moreover for $\alpha \in M(X,X',X')$ and $\beta \in M(Y,Y',Y')$ the induced maps

$$(x, P\beta)_*,\ (P\alpha, y)_* : M(Z,X,Y) \to M(Z,X',Y')$$

are trivial, that is

$$(x, P\beta)_* = (P\alpha, y)_* = 0.$$

(2) For $a, b \in \underline{\underline{R}}(X,Y)$ we have $a + b \in \underline{\underline{R}}(X,Y)$ by the group structure of $\underline{\underline{R}}(X,Y)$ and $H$ satisfies

$$H(a+b) = H(a) + H(b) + (a,b)_*H(2_X).$$

Moreover $H$ is a derivation, that is, for $X \xleftarrow{f} Y \xleftarrow{g} Z$ in $\underline{\underline{R}}$ one has the formula

$$H(fg) = (f,f)_* H(g) + g^* H(f).$$

(3) $T = HP - 1$ on $M(X,Y,Y)$
(4) $PT = P$ on $M(X,Y,Y)$
(5) $TH = H + \nabla_H$ where for $a \in \underline{\underline{R}}(X,Y)$

$$\nabla_H(a) = a^* H(2_Y) - (a,a)_* H(2_X)$$

(6) For $X \overset{f,f'}{\longleftarrow} Y \overset{g}{\longleftarrow} Z$ in $\underline{\underline{R}}$ we have the 'quadratic left distributivity law'

$$(f+f')\circ g = f\circ g + f'\circ g + P(f,f')_* H(g)$$

(7) For $X \overset{f}{\longleftarrow} Y \overset{g,g'}{\longleftarrow} Z$ in $\underline{\underline{R}}$ we have the 'linear right distributivity law'

$$f\circ (g+g') = f\circ g + f\circ g'.$$

By (7) and (6) we see that $\underline{\underline{R}}$ is a prequadratic category.

*(3.2)* *Remark*. Let $\underline{\underline{R}}$ be a square ringoid. Then beside (1)...(7) above the following equations hold. By (6) one has for $X \overset{f,f'}{\longleftarrow} Y \overset{g}{\longleftarrow} Z$ the cross effect formula

(a) $$(f \mid f')_g = P(f,f')_* H(g)$$

This implies by (2.7) the formula

(b) $$(-f)g = -(fg) + P(f,f)_* H(g)$$

and for $a, b \in \underline{\underline{R}}(X,Y)$ we get

(c) $$b + a - b - a = -b - a + b + a = P(a,b)_* H(2_X).$$

Moreover 'double cross effects' vanish in $\underline{\underline{R}}$, that is, for $W \overset{u,v}{\longleftarrow} X \overset{f}{\longleftarrow} Y \overset{g}{\longleftarrow} Z$ and $W \overset{y}{\longleftarrow} Y$ in $\underline{\underline{R}}$ we have

(d) $$((u \mid v)_f \mid y)_g = 0 = (y \mid (u \mid v)_f)_g$$

This follows from (a) since we have $(P\alpha, y)_* = 0 = (x, P\beta)_*$ by (1) above.

**(3.3) Theorem.** *Each quadratic category $\underline{\underline{Q}}$ with cross effect $M = \underline{\underline{Q}}(\ \ ,|)$ and structure maps $T, H, P$ as defined in § 2 is a square ringoid.*

The proposition implies that each full subcategory $\underline{\underline{R}}$ of a quadratic category $\underline{\underline{Q}}$ has the structure of a square ringoid.

*Proof of (3.3).* (1) We obtain $P$ by the composition

$$P : \underline{\underline{Q}}(Z, X \mid X) \overset{i_{12}}{\subset} \underline{\underline{Q}}(Z, X \vee X) \xrightarrow{\nabla_*} \underline{\underline{Q}}(Z, X)$$

where $i_{12}$ is central and $\nabla_*$ is surjective since $\nabla i_1 = 1_X$. Hence $P$ is central. Moreover we get the naturality of $P$ since $i_{12}$ is natural in $Z$ (by the definition of $\underline{\underline{Q}}(\ \ ,|)$ in (2.10)) and since $\nabla(f \vee f) = f\nabla$, $r_\tau(f \vee f) = fr_\tau$ for $\tau = 1,2$. For the proof of $(P\alpha, y)_* = 0$ we first observe that for $\xi \in \underline{\underline{Q}}(X, X' \mid X'')$ with $i_{12}\xi \in \underline{\underline{Q}}(X, X' \vee X'')$ the induced map

$$0 = (i_{12}\xi, 1)_* : \underline{\underline{Q}}(Z, X \mid Y) \to \underline{\underline{Q}}(Z, X' \vee X'' \mid Y)$$

is trivial. This follows since by (2.10) the map

$$\underline{\underline{Q}}(Z, X' \vee X'' \mid Y) \xrightarrow{\cong} \underline{\underline{Q}}(Z, X' \mid Y) \oplus \underline{\underline{Q}}(Z, X'' \mid Y)$$

given by $(r_1, 1)_*$ and $(r_2, 1)_*$ is an isomorphism. Hence we get $(i_{12}\xi, 1)_* = 0$ since $r_1 i_{12} = 0$ and $r_2 i_{12} = 0$. Since $P\alpha = \nabla_* i_{12}\alpha$ we obtain $(P\alpha, y)_* = (\nabla, y)_*(i_{12}\alpha, 1)_* = 0$. Similarly one gets $(X, P\beta)_* = 0$.

(4) $PT = P$ is a consequence of $\nabla t = \nabla$.

(7) is part of the definition of a prequadratic category.

(6) This formula is obtained by

$$\begin{aligned} P(f,f')_* H(g) &= P(f,f')_* r_{12}(\mu g) \\ &= \nabla_*(f \vee f')_*(i_1 r_1 \mid i_2 r_2)_{\mu g}, \quad \text{see proof (2.10)}, \\ &= \nabla(f \vee f')[(i_1 r_1 + i_2 r_2)\mu g - i_2 r_2 \mu g - i_1 r_1 \mu g] \\ &= (f + f')g - f'g - fg. \end{aligned}$$

(3) We have the commutative diagram in $\underline{\underline{Q}}$

$$\begin{array}{ccccc} Y \vee Y & \xrightarrow{\nabla} & Y & \xrightarrow{\mu} & Y \vee Y \\ {\scriptstyle \mu\vee\mu}\downarrow & & & & \uparrow{\scriptstyle \nabla\vee\nabla} \\ Y \vee Y \vee Y \vee Y & & \xrightarrow{1\vee t\vee 1} & & Y \vee Y \vee Y \vee Y \end{array}$$

which we use in the following equations with $v \in \underline{\underline{Q}}(X, Y \mid Y)$.

$$\begin{aligned} HPv &= r_{12}\mu_* \nabla_* i_{12}(v) \\ &= r_{12}(\nabla \vee \nabla)_*(1 \vee t \vee 1)_*(\mu \vee \mu)_* i_{12} v \\ &= r_{12}(\nabla \vee \nabla)_*(1 \vee t \vee 1)_* i_{12}(\mu, \mu)_* v \end{aligned}$$

Since $\underline{\underline{Q}}(X, Y \mid Z)$ is linear in $Y$ and $Z$ by (2.10) we get

$$\begin{aligned} (\mu, \mu)_* v &= (i_1 + i_2, i_1 + i_2)_* v \\ &= (i_1, i_1)_* v + (i_1, i_2)_* v + (i_2, i_1)_* v + (i_2, i_2)_* v \end{aligned}$$

Observe that

$$\begin{aligned} r_{12}(\nabla \vee \nabla)_*(1 \vee t \vee 1)_* i_{12}(i_1, i_1)_* &= r_{12}(\nabla \vee \nabla)_*(1 \vee t \vee 1)_*(i_1 \vee i_1)_* i_{12} \\ &= r_{12} i_1 \nabla_* i_{12} \\ &= 0 \end{aligned}$$

since $r_{12} i_{1*} = 0$. Similarly

$$r_{12}(\nabla \vee \nabla)_*(1 \vee t \vee 1)_* i_{12}(i_2, i_2)_* = 0$$

On the other hand we get

$$\begin{aligned} r_{12}(\nabla \vee \nabla)_*(1 \vee t \vee 1)_* i_{12}(i_1, i_2)_* &= \text{identity} \\ r_{12}(\nabla \vee \nabla)_*(1 \vee t \vee 1)_* i_{12}(i_2, i_1)_* &= T \end{aligned}$$

This completes the proof of (3).

(5) For $a \in \underline{\underline{Q}}(X, Y)$ we have

$$i_{12} TH(a) = i_{12} T r_{12} \mu_* a = t_* i_{12} r_{12} \mu_* a$$

Here we can use for $v = \mu_* a$ the formula

$$i_{12} r_{12} v + (i_1 r_1)_* v + (i_2 r_2)_* v = v$$

which follows from the definition of $r_{12}$ in (2.10). Hence we obtain

$$\begin{aligned} i_{12} TH(a) &= t_*(id - (i_2 r_2)_* - (i_1 r_1)_*)\mu_* a \\ &= t_*(\mu_* a - (i_2 r_2 \mu)_* a - (i_1 r_1 \mu)_* a) \\ &= t_*(\mu_* a - i_2 a - i_1 a) \\ &= (i_2 + i_1)_* a - i_1 a - i_2 a \end{aligned}$$

On the other hand we have

$$i_{12}(H + \nabla_H)(a) = i_{12}H(a) + i_{12} \nabla_H (a)$$

where

$$i_{12}H(a) = i_{12}r_{12}\mu_* a = (i_1 + i_2)_* a - i_{2*}a - i_{1*}a$$

Hence we have to show

$$(i_2 + i_1)_* a - i_1 a - i_2 a + i_1 a + i_2 a - (i_1 + i_2)_* a = i_{12} \nabla_H (a)$$

Here the commutator rule (8) shows

$$\begin{aligned} -i_1 a - i_2 a + i_1 a + i_2 a &= -(-i_2 a - i_1 a + i_2 a + i_1 a) \\ &= -P(i_1 a, i_2 a)_* H(2_X) \\ &= -i_{12}(a, a)_* H(2_X) \end{aligned}$$

since $P(i_1, i_2)_* = i_{12}$. Moreover

$$\begin{aligned} (i_2 + i_1)_* a - (i_1 + i_2)_* a &= a^*(i_2 + i_1 - i_1 - i_2) \\ &= a^* P(i_1, i_2)_* H(2_Y) \\ &= a^* i_{12} H(2_X) = i_{12} a^* H(2_Y). \end{aligned}$$

This completes the proof of (5).

(2) We use the formula (see (2.14))

$$i_{12}H(a) = (i_1 + i_2)a - i_2 a - i_1 a$$

Thus we get

$$\begin{aligned} i_{12}H(a + b) &= (i_1 + i_2)(a + b) - i_2(a + b) - i_1(a + b) \\ &= (i_1 + i_2)a + (i_1 + i_2)b - i_2 b - i_2 a - i_1 b - i_1 a \\ &= i_{12}H(a) + i_1 a + i_2 b + i_{12}H(b) + i_1 b + i_2 b - i_2 b - i_2 a - i_1 b - i_1 a \\ &= i_{12}(H(a) + H(b)) + i_1 a + (i_2 a + i_1 b - i_2 a - i_1 b) - i_1 a \\ &= i_{12}(H(a) + H(b)) + i_1 a + P(i_1 a, i_2 b)_* H(2_X) - i_1 a \\ &= i_{12}(H(a) + H(b) + (a, b)_* H(2_X)) \end{aligned}$$

In the last equation we use $P(i_1, i_2)_* = i_{12}$. This completes the proof of (2).

For the proof of the derivation property of $H$ we first obtain the following formulas.

$$
\begin{aligned}
i_{12}H(fg) &= (i_1+i_2)fg - i_2fg - i_1fg\\
i_{12}(f,f)_*H(g) &= (f\vee f)_* i_{12}H(g) = (f\vee f)_*((i_1+i_2)g - i_2g - i_1g)\\
&= (i_1fr_1 + i_2fr_2)(i_1+i_2)g - (i_1fr_1+i_2fr_2)i_2g - (i_1fr_1+i_2fr_2)i_1g\\
&= (i_1f+i_2f)g - i_2fg - i_1fg\\
i_{12}g^*H(f) &= g^*i_{12}H(f) = ((i_1+i_2)f - i_2f - i_1f)g
\end{aligned}
$$

These formulas imply

$$
\begin{aligned}
&i_{12}(H(fg) - (f,f)_*H(g) - g^*H(f)) =\\
&\quad = (i_1+i_2)fg - i_2fg - i_1fg + i_1fg + i_2fg - (i_1f+i_2f)g - ((i_1+i_2)f - (i_1f+i_2f))g\\
&\quad = (i_1+i_2)fg - \bar{f}g - [(i_1+i_2)fg + (-\bar{f})g + P((i_1+i_2)fg - \bar{f})Hg] \quad (\text{with} \quad \bar{f} = i_1f + i_2f)\\
&\quad = (i_1+i_2)fg - \bar{f}g - [-\bar{f}g + P(\bar{f},\bar{f})_*H(g)] - (i_1+i_2)fg - P((i_1+i_2)f_1 - \bar{f})_*H(g)\\
&\quad = -P(\bar{f},\bar{f})_*H(g) + P((i_1+i_2)f,\bar{f})_*H(g)\\
&\quad = P((i_1+i_2)f - \bar{f},\bar{f})_*H(g)\\
&\quad = P(P(i_1,i_2)_*H(f),\bar{f})_*H(g) = 0 \quad \text{by (1).}
\end{aligned}
$$

Hence $H$ is a derivation since $i_{12}$ is injective. This completes the proof of (3.3).

q.e.d.

## § 4 <u>Biproduct completion of square ringoids</u>

In this section we describe the quadratic analogue of the biproduct completion of a ringoid in (1.2).

*(4.1)* <u>*Definition*</u>. A functor $F : \underline{\underline{Q}} \to \underline{\underline{Q}}'$ between prequadratic categories is <u>linear</u> if $F$ induces a homomorphism between groups

$$F : \underline{\underline{Q}}(X,Y) \to \underline{\underline{Q}}'(FX,FY)$$

for $X, Y \in \underline{\underline{Q}}$ and if $F$ carries linear maps to linear maps.

Hence a linear functor carries a quadratic biproduct to a quadratic biproduct. This implies that a linear functor $F$ between quadratic categories induces a natural transformation

$$F_\sharp : \underline{\underline{Q}}(X,Y \mid Z) \to \underline{\underline{Q}}'(FX,FY \mid FZ)$$

compatible with $T, H, P$ in (3.2). Hence $(F, F_\sharp)$ is a morphism of square ringoids defined as follows.

*(4.2)* <u>*Definition*</u>. A <u>morphism</u> $F : \underline{\underline{R}} \to \underline{\underline{R}}'$ <u>between square ringoids</u> is a linear functor $F : \underline{\underline{R}} \to \underline{\underline{R}}'$ of the underlying prequadratic categories together with a natural transformation in $\underline{\underline{Ab}}$

$$F_\sharp : M(X,Y,Z) \to M'(FX,FY,FZ)$$

such that $F_\sharp$ is compatible with $T, H$ and $P$ respectively, that is:

$$\begin{aligned} F_\sharp T &= T' F_\sharp \quad \text{on} \quad M(X,Y,Z) \\ F_\sharp H &= H' F \quad \text{on} \quad \underline{\underline{R}}(X,Y) \\ FP &= P' F_\sharp \quad \text{on} \quad M(X,Y,Y) \end{aligned}$$

for all $X, Y, Z \in \underline{\underline{R}}$.

We now are able to describe the universal property of the biproduct completion $\underline{\underline{Add}}(\underline{\underline{R}})$ of a square ringoid $\underline{\underline{R}}$. First $\underline{\underline{Add}}(\underline{\underline{R}})$ is a quadratic category and $i : \underline{\underline{R}} \to \underline{\underline{Add}}(\underline{\underline{R}})$ is a morphism of square ringoids such that for any quadratic category $\underline{\underline{Q}}$ and any morphism $F : \underline{\underline{R}} \to \underline{\underline{Q}}$ between square ringoids there is a unique linear functor

(4.3) $$\bar{F} : \underline{\underline{Add}}(\underline{\underline{R}}) \to \underline{\underline{Q}} \quad \text{with} \quad \bar{F} i = F.$$

Here $\bar{F}$ is the quadratic analogue of (1.3). The following results justifies the selection of properties used in the definition of a square ringoid.

**(4.4) Theorem.** *For a square ringoid there exists the biproduct completion $i : \underline{\underline{R}} \to \underline{\underline{Add}}(\underline{\underline{R}})$.*

If $\underline{\underline{Q}}$ is a quadratic category then any full subcategory $j : \underline{\underline{R}} \subset \underline{\underline{Q}}$ has the structure of a square ringoid. Let $\hat{\underline{\underline{R}}}$ be the full subcategory of $\underline{\underline{Q}}$ consisting of finite quadratic biproducts $X_1 \vee \ldots \vee X_r$ with $X_i \in \underline{\underline{R}}$. Then

(4.5) $$\epsilon : \underline{\underline{Add}}(\underline{\underline{R}}) \to \hat{\underline{\underline{R}}}$$

with $\epsilon = \bar{j}$ is a linear equivalence between quadratic categories. Compare (1.3) (*). As in (1.4) one can extend the definition of $\underline{\underline{Add}}(\underline{\underline{R}})$ in (4.7) below for 'families of objects in $\underline{\underline{R}}$' and one obtains this way

(4.6) $$\underline{\underline{R}} \subset \underline{\underline{Add}}(\underline{\underline{R}}) \subset \underline{\underline{add}}(\underline{\underline{R}}).$$

We leave this to the reader. The proof of (4.4) relies on the following construction of $\underline{\underline{Add}}(\underline{\underline{R}})$.

*(4.7) Definition.* Given a square ringoid $\underline{\underline{R}}$ we define the quadratic category $\underline{\underline{Q}} = \underline{\underline{Add}}(\underline{\underline{R}})$ as follows. The objects of $\underline{\underline{Q}}$ are the finite tuple of objects in $\underline{\underline{R}}$ which we denote by

$$X_1 \amalg X_2 \amalg \ldots \amalg X_x = (X_1, \ldots, X_x),\ x \geq 1.$$

We define for $Y \in \underline{\underline{R}}$ the group

$$\underline{\underline{Q}}(Y, X_1 \amalg \ldots \amalg X_x) = \left(\mathop{\times}_{i=1}^{x} \underline{\underline{R}}(Y, X_i)\right) \times \left(\mathop{\times}_{1 \leq i < j \leq x} M(Y, X_i, X_j)\right)$$

where $\times$ denotes the product of sets. The group structure on this set is given by the formula

$$\begin{cases} (f_i, f_{ij}) + (f'_i, f'_{ij}) & = (f_i + f'_i, f_{ij} + f'_{ij} + \delta_{ij}) \\ \qquad\qquad \delta_{ij} & = (f_i, f'_j)_* H(2_Y) \end{cases}$$

Moreover we define the group

$$\underline{\underline{Q}}(Y_1 \amalg \ldots \amalg Y_y,\ X_1 \amalg \ldots \amalg X_x) = \mathop{\times}_{k=1}^{y} \underline{\underline{Q}}(Y_k,\ X_1 \amalg \ldots \amalg X_x)$$

as a product of groups. An element in this group is denoted by $f = (f_i^k, f_{ij}^k)$ with $1 \leq k \leq y$ and $1 \leq i < j \leq x$. Now let $g = (g_k^s, g_{kl}^s)$ be an element in $\underline{\underline{Q}}(Z_1 \amalg \ldots \amalg Z_z,\ Y_1 \amalg \ldots \amalg Y_y)$. Then the composition is defined by

$$fg = ((fg)_i^s, (fg)_{ij}^s)$$

where the coordinates are given as follows.

$$\begin{aligned} (fg)_i^s &= f_i^1 g_1^s + f_i^2 g_2^s + \ldots + f_i^y g_y^s + \sum_{k<l} P(f_i^k, f_i^l)_* g_{kl}^s \\ (fg)_{ij}^s &= \sum_k (g_k^s)^* f_{ij}^k \\ &\quad + \sum_{k<l} \left((f_i^k, f_j^l)_* g_{kl}^s + (f_i^l, f_j^k)_* T\, g_{kl}^s + (f_i^l g_l^s,\ f_j^k g_k^s)_* H(2_{Z_s})\right) \end{aligned}$$

Using the properties of a square ringoid one now can check that the composition is associative and that $\underline{Add}(\underline{\underline{R}})$ is a well defined quadratic category with the universal property of the biproduct completion of $\underline{\underline{R}}$ in (4.4).

## § 5 Quadratic categories as linear extensions of additive categories

We show that all quadratic categories can be obtained by certain linear extensions of additive categories. This gives rise to many examples of quadratic categories and it also yields a kind of classification of quadratic categories.

*(5.1)* *Definition*. Let $\underline{\underline{C}}$ be a category and let $D : \underline{\underline{C}}^{op} \times \underline{\underline{C}} \to \underline{\underline{Ab}}$ be a bifunctor (also termed $\underline{\underline{C}}$ -bimodule). We say that

$$D \overset{+}{\rightarrowtail} \underline{\underline{E}} \overset{p}{\twoheadrightarrow} \underline{\underline{C}}$$

is a linear extension of the category $\underline{\underline{C}}$ by $D$ if (a), (b) and (c) hold; compare [6].

(a) $\underline{\underline{E}}$ and $\underline{\underline{C}}$ have the same objects and $p$ is a full functor which is the identity on objects.
(b) For each $f : A \to B$ in $\underline{\underline{C}}$ the abelian group $D(A,B)$ acts transitively and effectively on the subset $p^{-1}(f)$ of morphism in $\underline{\underline{E}}$. We write $f_0 + \alpha$ for the action of $\alpha \in D(A,B)$ on $f_0 \in p^{-1}(f)$. Any $f_0 \in p^{-1}(f)$ is called a lift of $f$.
(c) The action satisfies the linear distributivity law:

$$(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_* \beta + g^* \alpha$$

A map between linear extensions is a diagram

$$\begin{tikzcd} D \arrow[r, "+"] \arrow[d, "d"] & \underline{\underline{E}} \arrow[r, "p"] \arrow[d, "\epsilon"] & \underline{\underline{C}} \arrow[d, "\varphi"] \\ D' \arrow[r] & \underline{\underline{E}}' \arrow[r, "p"] & \underline{\underline{C}}' \end{tikzcd}$$

where $\epsilon, \varphi$ are functors with $p'\epsilon = \varphi p$ and $d : D(AB) \to D'(\varphi A, \varphi B)$ is a natural transformation satisfying $\epsilon(f_0 + \alpha) = \epsilon(f_0) + d(\alpha)$. If $\varphi$ and $d$ are the identity then $\epsilon$ is called an equivalence of linear extensions.

We call $D \rightarrowtail \underline{\underline{E}} \overset{q}{\twoheadrightarrow} \underline{\underline{K}}$ a weak linear extension if there is a linear extension $D \rightarrowtail \underline{\underline{E}} \to \underline{\underline{C}}$ as above together with an equivalence of categories $\underline{\underline{C}} \overset{\sim}{\to} \underline{\underline{K}}$ such that $E \to \underline{\underline{C}} \overset{\sim}{\to} \underline{\underline{K}}$ coincides with $q$.

There is a canonical bijection

$$\pi : M(\underline{\underline{C}}, D) \cong H^2(\underline{\underline{C}}, D) \tag{5.2}$$

Here $M(\underline{\underline{C}}, D)$ is the set of equivalence classes of linear extensions and $H^2(\underline{\underline{C}}, D)$ is the cohomology of $\underline{\underline{C}}$ with coefficients in $D$; [6]. We now describe examples of linear extensions of categories

*(5.3) Example.* Recall that $\underline{\underline{ab}}$ and $\underline{\underline{nil}}$ denote the categories of free abeliean groups and free nil-groups respectively; see (2.6). Then there is a linear extension

$$\mathrm{Hom}(-, \Lambda^2) \overset{+}{\rightarrowtail} \underline{\underline{nil}} \overset{p}{\twoheadrightarrow} \underline{\underline{ab}}$$

obtained as follows. The functor $p$ carries $\langle M \rangle_{nil}$ to the abelianisation $\mathbb{Z}[M]$ which is the free abelian group generated by $M$. One has the classical central extension

$$\Lambda^2(\mathbb{Z}[M]) \overset{w}{\rightarrowtail} \langle M \rangle_{nil} \overset{q}{\longrightarrow} \mathbb{Z}[M]$$

where $q$ is the abelianization and where $w$ is the commutator map. Now the action of $\alpha \in \mathrm{Hom}(\mathbb{Z}[N], \Lambda^2\mathbb{Z}[M])$ on $f_0 : \langle N \rangle_{nil} \to \langle M \rangle_{nil} \in \underline{\underline{nil}}$ is given by $(f_0 + \alpha)(x) = f_0(x) + w\alpha q(x)$. In this example $\underline{\underline{ab}}$ is an additive category and $\underline{\underline{nil}}$ is a quadratic category; see (2.6).

*(5.4) Example.* Let $A$ be an abelian group and let $\mathbb{Z}[N] \overset{d}{\rightarrowtail} \mathbb{Z}[M] \twoheadrightarrow A$ be a free resolution of $A$. We choose a map

$$\partial : \bigvee_N S^1 \to \bigvee_M S^1$$

between one point unions of 1- spheres which induces $d$ in homology, $H_1(\partial) = d$. Let $M_A$ be the mapping cone of $\partial$. Then the suspension $M(A,n) = \sum^{n-1} M_A, n \geq 2$, is a <u>Moore space of</u> $A$ in degree $n$. Let $\underline{\underline{M}}^n$ be the full homotopy category of such Moore spaces $M(A,n), A \in \underline{\underline{Ab}}$, and let $p : \underline{\underline{M}}^n \to \underline{\underline{Ab}}$ be the homology functor which carries $M(A,n)$ to $A$. The suspension functor $\sum : \underline{\underline{M}}^n \to \underline{\underline{M}}^{n+1}$ is full for $n = 2$ and is an isomorphism of categories for $n = 3$. The category $\underline{\underline{M}}^2$ is quadratic and the category $\underline{\underline{M}}^n, n \geq 3$, is additive. Moreover one has the following diagram in which the rows and the column are weak linear extensions; compare V.3a in [2].

$$\begin{tikzcd}
 & & \mathrm{Ext}(-, \otimes \mathbb{Z}/2) \arrow[d, "+"] \\
P_* \mathrm{Ext}(-, \otimes^2) \arrow[r, "+"] \arrow[d, "d"] & \underline{\underline{M}}^2 \arrow[r, "\Sigma"] \arrow[d, equal, "\bar{\epsilon}"] & \underline{\underline{M}}^3 \arrow[d, "q"] \\
\mathrm{Ext}(-, \Gamma) \arrow[r, "+"] & \underline{\underline{M}}^2 \arrow[r, "p"] & \underline{\underline{Ab}}
\end{tikzcd}$$

Here we use for $B \in \underline{\underline{Ab}}$ the natural exact sequence

$$\otimes^2 B \xrightarrow{P} \Gamma B \xrightarrow{\sigma} B \otimes \mathbb{Z}/2 \to 0$$

which induces for $A \in \underline{\underline{Ab}}$ the binatural exact sequence

$$\mathrm{Ext}(A, \otimes^2 B) \xrightarrow{P_*} \mathrm{Ext}(A, \Gamma B) \xrightarrow{\sigma_*} \mathrm{Ext}(A, B \otimes \mathbb{Z}/2) \to 0$$

Hence the image $P_* Ext(A, \otimes^2 B)$ is a $\underline{\underline{Ab}}$-bimodule which via $q$ is also an $\underline{\underline{M}}^3$-bimodule. The map $d$ in the diagram is the inclusion such that $(d, \bar{\epsilon}, q)$ is a map between linear extensions.

Motivated by such examples of quadratic categories we prove the following classification of quadratic categories in terms of linear extensions.

**(5.6) <u>Theorem</u>.** *Each quadratic category $\underline{\underline{Q}}$ is canonically part of a linear extension of categories*

$$D_\Delta \overset{+}{\rightarrowtail} \underline{\underline{Q}} \twoheadrightarrow \underline{\underline{Q}}^{add}$$

*Here $\underline{\underline{Q}}^{add}$ is an additive category and $D_\Delta$ is an $\underline{\underline{Q}}^{add}$-bimodule which is left additive and right quadratic. We call $\underline{\underline{Q}}^{add}$ the <u>additive quotient</u> of $\underline{\underline{Q}}$.*

*<u>Proof</u>.* We define $\underline{\underline{Q}}^{add}$ and $D_\Delta$ as follows. The objects in $\underline{\underline{Q}}^{add}$ are the same as in $\underline{\underline{Q}}$. Morphism sets in $\underline{\underline{Q}}^{add}$ are given by the cokernel

$$\underline{\underline{Q}}^{add}(X,Y) = \text{cokernel}\,(P : \underline{\underline{Q}}(X,Y \mid Y) \to \underline{\underline{Q}}(X,Y))$$

This cokernel also defines the projection $\underline{\underline{Q}} \to \underline{\underline{Q}}^{add}$. The composition law in $\underline{\underline{Q}}^{add}$ is induced by the composition law in $\underline{\underline{Q}}$. Using the properties in (3.1) and (3.2) one readily checks that $\underline{\underline{Q}}^{add}$ is an additive category. We define the $\underline{\underline{Q}}^{add}$-bimodule $D_\Delta$ by

$$D_\Delta(X,Y) = \text{image}\,(P : \underline{\underline{Q}}(X,Y|Y) \to \underline{\underline{Q}}(X,Y))$$

Then the additive $\underline{\underline{Q}}^{add}$- trifunctor $\underline{\underline{Q}}(\ \ ,|)$ shows that $D_\Delta$ is left additive and right quadratic since $\overline{\overline{P}}$ is a natural homomorphism. Moreover using the short exact sequence of groups

$$0 \to D_\Delta(X,Y) \to \underline{\underline{Q}}(X,Y) \to \underline{\underline{Q}}^{add}(X,Y) \to 0$$

obtained by the definitions above we obtain the action of $P_\Delta(X,Y)$ on $\underline{\underline{Q}}(X,Y)$ such that the linear extension of categories in (5.6) is well defined. The linear distributivity law follows from property $(x, P\beta)_* = (P\alpha, y)_* = 0$ in (3.2)(1) by use of (3.2)(6). q.e.d.

**(5.7)<u>Theorem</u>.** *Suppose that a linear extension of categories*

$$(*) \qquad D \overset{+}{\rightarrowtail} \underline{\underline{E}} \twoheadrightarrow \underline{\underline{A}}$$

*is given where $\underline{\underline{A}}$ is an additive category and where $D$ is an $\underline{\underline{A}}$-bimodule which is left additive and right quadratic. Let $\underline{\underline{R}}$ be a full subcategory of $\underline{\underline{A}}$ for which the additive functor $\varepsilon : \underline{\underline{Add}}(\underline{\underline{R}}) \to \underline{\underline{A}}$ is given by (1.3)(*). Then there is a quadratic category $\underline{\underline{Q}}$ together with a map between linear extensions*

$$(**) \qquad \begin{array}{ccccc} \varepsilon^* D & \longrightarrow & \underline{\underline{Q}} & \longrightarrow & \underline{\underline{Add}}(\underline{\underline{R}}) \\ \| & & \downarrow{\scriptstyle \bar\varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\ D & \longrightarrow & \underline{\underline{E}} & \longrightarrow & \underline{\underline{A}} \end{array}$$

*If $\varepsilon$ is an equivalence, for example if $\underline{\underline{R}} = \underline{\underline{A}}$, then also $\bar\varepsilon$ is an equivalence. Quadratic biproducts in $\underline{\underline{Q}}$ are lifts of biproducts in $\underline{\underline{Add}}(\underline{\underline{R}})$.*

We prove this result in (6.11) below.

Since the eqivalence $\varepsilon : \underline{\underline{Add}}(\underline{\underline{A}}) \to \underline{\underline{A}}$ induces an isomorphism $\varepsilon^* : H^2(\underline{\underline{A}}, D) \cong H^2(\underline{\underline{Add}}(\underline{\underline{A}}), \varepsilon^* D)$ we see by (5.2) that the equivalence class of the extension $\underline{\underline{E}}$ in (5.7)(*) can be identified with the equivalence class of the extension $\underline{\underline{Q}}$ in (5.7)(**) with $\underline{\underline{R}} = \underline{\underline{A}}$.

*(5.8)Addendum.* For the extension $\underline{\underline{Q}}$ in (5.7)(**) one has the following diagram in which the rows and the column are linear extensions of categories.

$$\begin{array}{ccccc} & & & & \varepsilon^*(D/D') \\ & & & & \downarrow + \\ D_\Delta & \xrightarrow{+} & \underline{\underline{Q}} & \longrightarrow & \underline{\underline{Q}}^{add} \\ \downarrow d & & \| & & \downarrow q \\ \varepsilon^* D & \xrightarrow{+} & \underline{\underline{Q}} & \longrightarrow & \underline{\underline{Add}}(\underline{\underline{R}}) \end{array}$$

Here $D_\Delta = \varepsilon^* D'$ is given by

$$D'(A,B) = \mathrm{image}(D(A,B|B) \subset D(A,B \vee B) \stackrel{(1,1)_*}{\rightarrow} D(A,B))$$

for $A, B \in Ob(\underline{\underline{A}})$ so that $D' \to D \to D/D'$ is a short exact sequence of $\underline{\underline{A}}$-bimodules. The functor $q$ is an additive functor and the quotient $D/D'$is biadditive.

*(5.9)Example.* For the quadratic category $\underline{\underline{Q}} = \underline{\underline{nil}}$ in (5.4) we have

$$\begin{array}{ccccc} D_\Delta & \longrightarrow & \underline{\underline{Q}} & \longrightarrow & \underline{\underline{Q}}^{add} \\ \| & & \| & & \| \\ \mathrm{Hom}(-,\Lambda^2) & \longrightarrow & \underline{\underline{nil}} & \longrightarrow & \underline{\underline{ab}} \end{array}$$

For the quadratic category $\underline{\underline{Q}} = \underline{\underline{M}}^2$ in (4.4) we have

$$\begin{array}{ccccc} D_\Delta & \longrightarrow & \underline{\underline{Q}} & \longrightarrow & \underline{\underline{Q}}^{add} \\ \| & & \| & & \| \\ P_* \mathrm{Ext}(-,\oplus^2) & \longrightarrow & \underline{\underline{M}}^2 & \xrightarrow[\Sigma]{} & \underline{\underline{M}}^3 \end{array}$$

Moreover the diagram in (5.4) is (up to equivalences of categories) an example of the diagram in (5.8).

## § 6 LIFTING SUMS IN LINEAR EXTENSIONS

A sum of objects $X_1, X_1$ in a category $\underline{\underline{C}}$ is an object $X_1 \vee X_2$ together with morphisms $i_k : X_k \to X_1 \vee X_2 (k = 1,2)$ such that

$$(i_1^*, i_2^*) : \underline{\underline{C}}(X_1 \vee X_2, Z) = \underline{\underline{C}}(X_1, Z) \times \underline{\underline{C}}(X_2, Z)$$

is a bijection for all $Z$. Linear extensions behave very well with respect to sums:

**(6.1) Lemma.** *Let $D \rightarrowtail \underline{\underline{E}} \twoheadrightarrow \underline{\underline{C}}$ be a linear extension and let $(X_1 \vee X_2, i_1, i_2)$ be a sum in $\underline{\underline{C}}$ such that*

$$(i_1^*, i_2^*) : D(X_1 \vee X_2, Z) \cong D(X_1, Z) \oplus D(X_2, Z)$$

*is an isomorphism. Then also*

$$(X_1 \vee X_2, \tilde{i}_1, \tilde{i}_2)$$

*is a sum in* $\underline{\underline{E}}$ *for any lift* $\tilde{i}_k$ *of* $i_k (k = 1, 2)$.

The proof is an easy exercise, compare 3.4 [8]. Now let $\underline{\underline{A}}$ be an additive category and consider a linear extension

$$D \rightarrowtail \underline{\underline{E}} \xrightarrow{p} \underline{\underline{A}} \tag{6.2}$$

Clearly 0 is a zero object in $\underline{\underline{E}}$ if and only if $D(0, A) = D(A, 0) = 0$ for all objects $A \in \underline{\underline{A}}$. We derive from (6.1) and the dual of (6.1).

**(6.3) <u>Proposition</u>.** *If $D$ is left additive then sums exist in $\underline{\underline{E}}$ and if $D$ is right additive then products exist in $\underline{\underline{E}}$. Moreover if $D$ is biadditive then $\underline{\underline{E}}$ has in a cannonical way the structure of an additive category such for all objects $X, Y \in \underline{\underline{A}}$ the sequence*

$$D(X, Y) \overset{i}{\rightarrowtail} \underline{\underline{E}}(X, Y) \overset{p}{\twoheadrightarrow} \underline{\underline{A}}(X, Y)$$

*is a short exact sequence of abelian groups. Here $i$ carries $\alpha$ to $0 + \alpha$. In addition the functor $p$ respects sums, products and biproducts respectively.*

*(6.4)<u>Addendum</u>.* Let $D$ be left additive and $D(A, 0) = 0$ for all $A \in \underline{\underline{A}}$. Then $\underline{\underline{E}}$ has sums and a zero object. Hence for $X, Y \in \underline{\underline{E}}$ one has inclusions and retractions

$$X \xrightarrow{i_X} X \vee Y \xrightarrow{r_X} X \qquad \text{and} \qquad Y \xrightarrow{i_Y} X \vee Y \xrightarrow{r_Y} Y$$

with $r_X i_X = 1, r_Y i_Y = 1, r_X i_Y = 0, r_Y i_X = 0$. Moreover the following formulas are satisfied for $f : X \to Z, g : Y \to Z, h : Y \to W \in \underline{\underline{E}}$

$$(f + \alpha, g + \beta) = (f, g) + r_X^* \alpha + r_Y^* \beta : X \vee Y \to Z$$
$$(f + \alpha) \vee (h + \beta) = f \vee h + r_X^* i_{Z*} \alpha + r_Y^* i_{W*} \beta : X \vee Y \to Z \vee W$$

We now consider the case when D in (6.3) is left additive and right quadratic. Then 0 is a zero object in $\underline{\underline{E}}$. Moreover for a sum $Y \vee Z$ in $\underline{\underline{E}}$ the sequence

$$D(X, Y|Z) \xrightarrow{+} \underline{\underline{E}}(X, Y \vee Z) \overset{r}{\twoheadrightarrow} \underline{\underline{E}}(X, Y) \times \underline{\underline{E}}(X, Z) \tag{6.5}$$

is exact, that is, the group $D(X, Y|Z)$ acts effectively on the set $\underline{\underline{E}}(X, Y \vee Z)$ and the set of orbits is $\underline{\underline{E}}(X, Y) \times \underline{\underline{E}}(X, Z)$ via $r = (t_{Y*}, r_{Z*})$. This is an immediate consequence of the definition of the cross effect $D(X, Y|Z)$, see (1.8). Since this cross effect is additive in $Y$ and $Z$ we derive from (6.5) that the map $(r_{12*}, r_{13*}, r_{23*})$:

$$\underline{\underline{E}}(X, X_1 \vee X_2 \vee X_3) \rightarrowtail \underline{\underline{E}}(X, X_1 \vee X_2) \times \underline{\underline{E}}(X, X_1 \vee X_3) \times \underline{\underline{E}}(X, X_2 \vee X_3) \tag{6.6}$$

is injective. Here $r_{ij}$ is the canonical retraction $X_1 \vee X_2 \vee X_3 \to X_i \vee X_j$ for $i < j$. We now consider cogroups in the category $\underline{\underline{E}}$, see (2.13).

**(6.7)** **Lemma.** *If $\mu : X \to X \vee X$ in $\underline{\underline{E}}$ satisfies the counit property then there is a unique $\nu$ such that $(X, \mu, \nu)$ is a cogroup in $\underline{\underline{E}}$.*

Hence we may call a morphism $\mu \to X \vee X$ a cogroup structure of $X$ if $\mu$ satisfies the counit property.

*Proof of (6.7).* The coassociativity follows from (6.6) since $r_{ij}(\mu \vee 1)\mu = 1_{X \vee X} = r_{ij}(1 \vee \mu)\mu$. In order to find $\nu$ we take $\nu' : X \to X$ in $\underline{\underline{E}}$ which is a lift of $-1 : X \to X$ in $\underline{\underline{A}}$. Then there exist $\alpha \in D(X,X)$ such that $(1,\nu')\mu = 0_{X,X} + \alpha$ where $0_{X,X}$ is the zero morphism $X \to X$ in $\underline{\underline{E}}$. Using (6.4) we have $(1, \nu' - \alpha)\mu = ((1,\nu') - r_1^*\alpha)\mu = (1,\nu')\mu - \alpha = 0$. Hence $\nu = \nu' - \alpha$ is a coinverse. q.e.d.

**(6.8)** **Proposition.** *Consider the linear extension $\underline{\underline{E}}$ as in (6.2) where $\underline{\underline{A}}$ is an additive category and where $D$ is left additive and right quadratic. Then each object $X$ in $\underline{\underline{E}}$ has a cogroup structure and the group $D(X, X|X)$ acts on the set of cogroup structures of $X$ transitively and effectively.*

*(6.9)Addendum.* With the assumption on $\underline{\underline{E}}$ in (6.8) let $\mu_X : X \to X \vee X$ be a cogroup structure for $X \in \mathrm{Ob}\,\underline{\underline{E}}$. Then $\mu_X$ yields a group structure $+$ on the set $\underline{\underline{E}}(X,Y)$ by $x + y = (x,y)\mu_X$. This structure is compatible with the ation of $D$ on $\underline{\underline{E}}$ since we show

$$(*) \qquad (x+\alpha) + (y+\beta) = (x+y) + (\alpha+\beta)$$

for $x, y \in \underline{\underline{E}}(X,Y), \alpha, \beta \in D(X,Y)$. Indeed by (5.4) we get

$$\begin{aligned}(x+\alpha) + (y+\beta) &= (x+\alpha, y+\beta)\mu_X \\ &= ((x,y) + r_1^*\alpha + r_2^*\beta))\mu_X \\ &= (x,y)\mu_x + \alpha + \beta = (x+y) + (\alpha+\beta).\end{aligned}$$

Now (*) implies that

$$(**) \qquad 0 \to D(X,Y) \xrightarrow{0^+} \underline{\underline{E}}(X,Y) \to \underline{\underline{A}}(X,Y) \to 0$$

is a central extension of groups and $\underline{\underline{E}}(X,Y)$ is a nil-group. Here $0^+$ carries $\alpha$ to $0_{X,Y} + \alpha$.

With the assumptions on $D, \underline{\underline{E}}, \underline{\underline{A}}$ in (6.8) we consider the following diagram in $\underline{\underline{E}}$

$$\begin{array}{ccc} X & \xrightarrow{\mu_X} & X \vee X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \vee f} \\ Y & \xrightarrow{\mu_Y} & Y \vee Y \end{array}$$

where $\mu_X$ and $\mu_Y$ are cogroup structures. Then the induced diagram in $\underline{\underline{A}}$ commutes so that there is a unique element

$$(6.10) \qquad \alpha = \mathcal{O}_{\mu_X,\mu_Y}(f) \in D(X, Y \mid Y)$$

with $(f \vee f)\mu_X = (\mu_Y f) + \alpha$. This is the <u>obstruction</u> for $f$ of being a cogroup morphism since $\alpha = 0$ if and only if the diagram commutes.

*<u>Remark</u>.* Let $D'$ be the bifunctor on $\underline{\underline{A}}$ given by $D'(X,Y) = D(X,Y \mid Y)$ and let $\mathrm{Cogr}(\underline{\underline{E}})$ be the category of cogroups in $\underline{\underline{E}}$ and cogroup morphisms. Then

$$\mathrm{Cogr}(\underline{\underline{E}}) \xrightarrow{\mathcal{O}} D'$$

is a linear covering of $\underline{\underline{E}}$ by $D'$ in the sense of IV.§4 [2]. Here $\mathcal{O}$ is the obstruction operator given by (6.10) and $\phi$ is the faithful forgetful functor.

*(6.11) <u>Proof of</u> (5.7).* The linear extension $\underline{\underline{Q}}$ in (5.7) (**) is the pull back of $\underline{\underline{E}}$ via the functor $\varepsilon : \underline{\underline{Add}}(\underline{\underline{R}}) \to \underline{\underline{A}}$. Hence for $X, Y \in \underline{\underline{Add}}(\underline{\underline{R}})$ we have

$$\underline{\underline{Q}}(X,Y) = \underline{\underline{E}}(\varepsilon X, \varepsilon Y)$$

and composition in $\underline{\underline{Q}}$ is given by the composition in $\underline{\underline{E}}$. We now choose by (6.8) for each object $A$ in $\mathrm{Ob}(\underline{\underline{R}}) \subset \mathrm{Ob}(\underline{\underline{E}})$ a cogroup structure $\mu_A$ in $\underline{\underline{E}}$. Hence we obtain for each object $X$ in $\underline{\underline{Q}}$ a cogroup structure by setting (see (1.2))

$$(1) \qquad \mu_{X \amalg Y} = t_{23}(\mu_X \amalg \mu_Y)$$

Here $t_{23} : X \amalg X \amalg Y \amalg Y \to X \amalg Y \amalg X \amalg Y$ is the interchange for the second and third factor. The cogroup structure $\mu_X$ yields the group structure for the set $\underline{\underline{Q}}(X,Y)$ by setting as in (6.9)

$$(2) \qquad x + y = (x,y)\mu_X \quad \text{for} \quad x, y \in \underline{\underline{Q}}(X,Y).$$

Then clearly $g_* : \underline{\underline{Q}}(X,Y) \to \underline{\underline{Q}}(X,Z)$ is linear for $g : Y \to Z$. On the other hand we have

$$(3) \qquad g^*(x+y) - g^*(y) - g^*(x) = (x \mid y)_* \mathcal{O}_{\mu_X,\mu_Y}(g)$$

where $(x|y)_* : D(X,Y \mid Y) \to D(X,Z \mid Z)$ is given by the right quadratic functor $D$ on $\underline{\underline{A}}$ so that $(x \mid y)_*$ is linear in $x$ and $y$ and hence also (3) is linear in (x) and (y). By (6.9) (**) also (3) is central in $\underline{\underline{Q}}(X,Z)$. According to (1) the natural map $i_1 : X \to X \amalg Y$ and $i_2 : Y \to X \amalg Y$ are morphisms of cogroups and this implies the equality $i_1 r_1 + i_2 r_2 = 1_{X \amalg Y}$.

q.e.d.

## § 7 Square rings

Ringoids with only one object are the same as rings. Therefore square ringoids with only one object are termed square rings. Each object $X$ in a quadratic category $\underline{\underline{Q}}$ determines a square ring $End(X)$ which is the endomorphism square ring of $X$. The examples of quadratic categories in § 2 yield therefore many examples of square rings. In particular we get the square ring

$$\mathbb{Z}_{nil} = End(\mathbb{Z}) \tag{7.1}$$

which is the endomorphism square ring of the object $\mathbb{Z}$ in the quadratic category $\underline{\underline{nil}}$. We shall see that $\mathbb{Z}_{nil}$ is completely described by

$$\mathbb{Z}_{nil} = (\mathbb{Z} \xrightarrow{H} \mathbb{Z} \xrightarrow{P} \mathbb{Z})$$

with $P = 0$ and $H(x) = x(x-1)/2$. In fact $\mathbb{Z}_{nil}$ is the initial object in the category of square rings.

Each square ring $Q$ yields a theory $\underline{\underline{Add}}(Q)$ and hence a category of models $\underline{\underline{Mod}}(Q)$ which is the category of (right) $Q$-modules if $Q$ is a ring. For the initial object $\mathbb{Z}_{nil}$ of the category of square rings the category $\underline{\underline{Mod}}(\mathbb{Z}_{nil})$ coincides with the category $\underline{\underline{Nil}}$ of groups of nilpotency degree 2; compare (7.11) below.

We now describe in more detail the algebraic notion of a square ring; this is the specialization of the axioms of a square ringoid for the case of a single object. We introduce a square ring in three steps. First we define a square group which describes the basic linear structure of square ring. A 'square ring' will be a 'square group over a ring $R$' with additional multiplicative structure.

*(7.2) Definition.* A square group

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

is given by a group $M_e$ and an abelian group $M_{ee}$. Both groups are written additively. Moreover $P$ is a homomorphism and $H$ is a quadratic function, that is the cross effect

$$(a \mid b)_H = H(a+b) - H(b) - H(a)$$

is linear in $a, b \in Q_e$. In addition the following properties are satisfied $(x, y \in M_{ee})$

(1) $\quad (Px \mid b)_H = 0 \quad \text{and} \quad (a \mid Py)_H = 0$

(2) $\quad P(a \mid b)_H = a + b - a - b$

(3) $\quad PHP(x) = P(x) + P(x)$

(4) $\quad \Delta(a) = HPH(a) + H(a+a) - 4H(a) \quad$ is linear in $a$

By (1) and (2) $P$ maps to the center of $M_e$ and by (2) cokernel of $P$ is abelian. Hence $M_e$ is a group of nilpotency degree 2. Let *Square* be the category of square groups.

*(7.3) Definition.* A square group over a ring

$$Q = (1 \in Q_e \xrightarrow{H} Q_{ee} \xrightarrow{P} Q_e \xrightarrow{\epsilon} R)$$

is given by a ring $R$, a square group $(H, P)$ as in (7.2), a homomorphism $\epsilon$ (denoted by $\epsilon a = \bar{a}$ for $a \in Q_e$) from the group $Q_e$ to the underlying abelien group of $R$ and an element $1 \in Q_e$ with $\epsilon(1) = 1$. Moreover the abelian group $Q_{ee}$ is an $R \otimes R \otimes R^{op}$ -module with action denoted by $(t \otimes s) \cdot x \cdot r \in Q_{ee}$ for $t, s, r \in R$, $x \in Q_{ee}$. The following additional properties hold where $H(2) = H(1+1)$.

(1) $\quad (a \mid b)_H = (\bar{b} \otimes \bar{a}) \cdot H(2),$

(2) $\quad \Delta(a) = HPH(a) + H(a+a) - 4H(a) = H(2) \cdot \bar{a}$

(3) $\quad T = HP - 1$ is an isomorphism of abelian groups satisfying

$$T((t \otimes s) \cdot x \cdot r) = (s \otimes t) \cdot T(x) \cdot r.$$

*(7.4) Definition.* A square ring

$$Q = (Q_e \xrightarrow{H} Q_{ee} \xrightarrow{P} Q_e)$$

is given by a square group $(H, P)$ for which $Q_e$ has the additional structure of a monoid with unit $1 \in Q_e$ and multiplication $a \cdot b \in Q_e$. This monoid structure induces on $R = \text{cokernel}\,(P)$ a ring structure such that

$$(1 \in Q_e \xrightarrow{H} Q_{ee} \xrightarrow{P} Q_e \xrightarrow{\epsilon} R)$$

is a square group over the ring $R$. Here $\epsilon$ is the quotient map for the cokernel of $P$ with $\epsilon a = \bar{a}$. Moreover the multiplication $a \cdot b$ in $Q_e$ satisfies the following equations

(1) $\quad (Py) \cdot a = P(y \cdot \bar{a})$

(2) $\quad a \cdot (Py) = P((\bar{a} \otimes \bar{a}) \cdot y)$

(3) $\quad H(a \cdot b) = (\bar{a} \otimes \bar{a}) \cdot H(b) + H(a) \cdot \bar{b}$

(4) $\quad (a + b) \cdot c = a \cdot c + b \cdot c + P((\bar{a} \otimes \bar{b}) \cdot H(c))$

(5) $\quad a \cdot (b + c) = a \cdot b + a \cdot c$

We also call $Q$ a square ring extension of the ring $R$ and $R$ is the ring associated to $Q$.

**(7.5) Lemma.** *A square ring as defined in (7.4) is the same as a square ringoid in (3.1) with only one object.*

Morphisms $Q \to Q'$ between square rings are given by homomorphisms $Q_e \to Q'_e$, $Q_{ee} \to Q''_{ee}$ which respect all the structure described above. We point out that a square ring $Q$ with $Q_{ee} = 0$ is the same as a ring so that the category of rings is a full subcategory in the category of square rings.

We now consider the square ring $\mathbb{Z}_{nil}$ in (7.1). In fact $\mathbb{Z}_{nil}$ is the initial object in the category of square rings since there is a unique morphism $\mathbb{Z}_{nil} \to Q$ which carries $1 \in \mathbb{Z} = (\mathbb{Z}_{nil})_e$ to $1 \in Q_e$ and $1 \in \mathbb{Z} = (\mathbb{Z}_{nil})_{ee}$ to $H(2) \in Q_{ee}$. By (7.4) (4) we have in any square ring

$$PH(2) = 0 \tag{7.6}$$

so that $\mathbb{Z}_{nil} \to Q$ is well defined. For a square ring $Q$ we obtain the quadratic categories $\underline{\underline{Add}}(Q)$ and $\underline{\underline{add}}(Q)$ in the same way as in (4.6). If $Q$ is the endomorphism square ring of an object $X$ in a quadratic category $\underline{\underline{Q}}$ then $\underline{\underline{Add}}(Q)$ coincides with the full subcategory of $\underline{\underline{Q}}$ consisting of finite sums $X \vee \ldots \vee X$ with all summands given by $X$. This implies the next proposition on the category $\underline{\underline{nil}}$ of free nil-groups in (2.6). Let $fg - \underline{\underline{nil}}$ be the full subcategory of finitely generated free nil-groups.

**(7.7) Proposition.** *One has equivalences of categories*

$$\begin{aligned} fg - \underline{\underline{nil}} &= \underline{\underline{Add}}(\mathbb{Z}_{nil}) \\ \underline{\underline{nil}} &= \underline{\underline{add}}(\mathbb{Z}_{nil}) \end{aligned}$$

Next we introduce for a square ring $Q$ the notion of $Q$-module which generalizes the classical notion of a (right) $R$-module for a ring $R$.

*(7.8)* *Definition.* Given a square ring $Q$ we obtain the category $\underline{\underline{Add}}(Q)$ in (4.7) which is a theory in the sense of (1.5). A $Q$-module $M$ is a model of this theory, that is

$$M : \underline{\underline{Add}}(Q)^{op} \to \underline{\underline{Set}}$$

is a functor which carries a sum in $\underline{\underline{Add}}(Q)$ to a product of sets. Let

$$\underline{\underline{Mod}}(Q) = \underline{\underline{Model}}(\underline{\underline{Add}}(Q))$$

be the category of $Q$-modules; compare (1.7). We now describe a $Q$-module more explicitely in terms of operators on a group.

*(7.9)* *Definition.* A $Q$-module $M$ as defined in (7.8) is given by a group $M$ (which we write additively) and by $Q$-operations which are functions

$$\begin{cases} M \times Q_e \longrightarrow M, & (m,a) \longmapsto m \cdot a \\ M \times M \times Q_{ee} \longrightarrow M, & (m,n,x) \longmapsto [m,n] \cdot x \end{cases}$$

For $a, b \in Q_e$, $x, y \in Q_{ee}$, $m, n \in M$ the following relations hold where $[M] = \{[m,n] \cdot x;\ m, n \in M,\ x \in Q_{ee}\} \subset M$.

$$\begin{aligned}
&m \cdot 1 = m,\ (m \cdot a) \cdot b = m \cdot (a \cdot b),\ m \cdot (a + b) = m \cdot a + m \cdot b \\
&(m + n) \cdot a = m \cdot a + n \cdot a + [m,n] \cdot H(a) \\
&m \cdot Px = [m,m] \cdot x \\
&[m,n] \cdot Tx = [n,m] \cdot x \\
&[m \cdot a, n \cdot b] \cdot x = [m,n] \cdot (a \otimes b) \cdot x \quad \text{and} \quad ([m,n] \cdot x) \cdot a = [m,n] \cdot (x \cdot a) \\
&[m,n] \cdot x \quad \text{is linear in} \quad m, n \quad \text{and} \quad x \\
&[m,n] \cdot x = 0 \quad \text{for} \quad m \in [M]
\end{aligned}$$

These equations imply that the commutator in $M$ satisfies

$$n + m - n - m = -n - m - n + m = [m,n] \cdot H(2)$$

Hence $M$ is a group of nilpotency degree 2 and $[M]$ is central in $M$. Morphisms in the category $\underline{\underline{Mod}}(Q)$ of $Q$-modules are homomorphisms $M \to M'$ which are compatible with the $Q$-operations.

*(7.10) Example.* Given an object $X$ in a quadratic category $\underline{\underline{Q}}$ we obtain the endomorphism square ring $Q = End(X)$. Any object $Y$ in $\underline{\underline{Q}}$ therefore yields the representable functor

$$M_Y : \underline{\underline{Add}}(Q) \subset \underline{\underline{Q}} \to \underline{\underline{Set}}$$

which carries the object $X \vee \ldots \vee X$ to the set $\underline{\underline{Q}}(X \vee \ldots \vee X, Y)$ of morphisms in $\underline{\underline{Q}}$. The functor $M_Y$ is obviously a model of the theory $\underline{\underline{Add}}(Q)$ and hence a $Q$-module. We can define $M_Y$ as well by the $Q$-operations

$$\begin{aligned}
&M_Y = \underline{\underline{Q}}(X,Y) \\
&m \cdot a = m \circ a \quad (\text{composition in} \quad \underline{\underline{Q}}) \\
&[m,n] \cdot x = P(m,n)_* x
\end{aligned}$$

given by the square ringoid structure of $\underline{\underline{Q}}$. This shows that the equations in (7.9) are given by the corresponding equations in a square ringoid.

*(7.11) Example.* Recall that

$$\mathbb{Z}_{nil} = (\mathbb{Z} \xrightarrow{H} \mathbb{Z} \xrightarrow{P} \mathbb{Z})$$

is the endomorphism square ring of $\mathbb{Z}$ in $\underline{\underline{nil}}$ with $P = 0$ and $H(a) = a(a-1)/2$ for $a \in \mathbb{Z}$. We now show that a $\mathbb{Z}_{nil}$ -module can be identified with a group of nilpotency degree 2 so that we have an isomorphism of categories

$$\underline{\underline{Mod}}(\mathbb{Z}_{nil}) = \underline{\underline{Nil}}$$

In fact, any object $M$ in $\underline{\underline{Nil}}$ has canonically the structure of a $\mathbb{Z}_{nil}$ -module by the $\mathbb{Z}_{nil}$ -operations

$$\begin{aligned} &m \cdot a = m + \ldots + m \quad (a - \text{fold sum in } M) \\ &[m,n] \cdot x = (n + m - n - m) \cdot x \quad (x - \text{fold sum in } M) \end{aligned}$$

for $m, n \in M$ and $a \in (\mathbb{Z}_{nil})_e = \mathbb{Z}$, $x \in (\mathbb{Z}_{nil})_{ee} = \mathbb{Z}$. One readily checks that the equations (7.9) for the $\mathbb{Z}_{nil}$ -operations are satisfied.

*(7.12)* *Remark*. For each square ringoid $\underline{\underline{Q}}$ with finitely many objects $X_1, \ldots, X_r$ we obtain the square ring of the object $X_1 \amalg \ldots \amalg X_r$ in $\underline{\underline{Add}}(\underline{\underline{Q}})$. One can check that $\underline{\underline{Q}}$ -modules and $Q$-modules can be identified so that one has a canonical isomorphism of categories

$$\underline{\underline{Mod}}(\underline{\underline{Q}}) = \underline{\underline{Mod}}(Q)$$

This shows that for many purposes square ringoids can be replaced by square rings.

## § 8 Examples of square rings

We here describe some examples of square rings which arise naturally in algebra and topology.

*(8.1)* *Factor square rings of* $\mathbb{Z}_{nil}$. Let $r, s \geq 1$ be integers with $r \mid s$ if $s$ is odd and $2r \mid s$ if $s$ is even. Then

$$\mathbb{Z}_{nil}^{r,s} = (\mathbb{Z}/r \xrightarrow{H} \mathbb{Z}/s \xrightarrow{P} \mathbb{Z}/r)$$

is the square ring with $H(a) = a(a-1)/2$ and $P = 0$. These are all square rings $Q$ for which there exists a surjection $\mathbb{Z}_{nil} \twoheadrightarrow Q$. Let $\underline{\underline{Nil}}^{r,s}$ be the category of $nil^{r,s}$ -groups which are the groups of nilpotency degree 2 satisfying the relations $(m, n \in M)$

$$\begin{aligned} &0 = m \cdot r = m + \ldots + m \quad (r - \text{fold sum of } m) \\ &0 = (-m - n + m + n) \cdot s \end{aligned}$$

This is a free $nil^{r,s}$ -group if $M$ is obtained by dividing out these relations in a free $nil$ -group; see (2.6). Let

$$fg - \underline{\underline{nil}}^{r,s} \subset \underline{\underline{nil}}^{r,s} \subset \underline{\underline{Nil}}^{r,s}$$

be the full subcategory of free $nil^{r,s}$ -groups and finitely generated free $nil^{r,s}$ -groups respectively. Then we obtain as in (7.7) and (7.11) equivalences of categories

$$\begin{aligned} fg-\underline{\underline{nil}}^{r,s} &= \underline{\underline{Add}}(\mathbb{Z}^{r,s}_{nil}) \\ \underline{\underline{nil}}^{r,s} &= \underline{\underline{add}}(\mathbb{Z}^{r,s}_{nil}) \\ \underline{\underline{Nil}}^{r,s} &= \underline{\underline{Mod}}(\mathbb{Z}^{r,s}_{nil}) \end{aligned}$$

The $\mathbb{Z}^{r,s}_{nil}$ -operations on a group $M \in \underline{\underline{Nil}}^{r,s}$ are defined by the same formulas as the $\mathbb{Z}_{nil}$ -operations in (7.11). As an example we obtain the $nil^{4,2}$ -groups which are exactly the groups $M$ for which the lower 2-central series $\Gamma_r M$ satisfies $\Gamma_3 M = 0$; they play a role for the unstable Adams spectral sequence [11]. Moreover we obtain the following result which is an application of the theory of this paper.

**(8.2) Theorem.** *Let $\underline{\underline{M}}^2(\mathbb{Z}/2)$ be the homotopy category of Moore spaces $M(V,2)$ in degree 2 of $\mathbb{Z}/2$ -vector spaces $V$. Then there is an equivalence of categories*

$$\underline{\underline{M}}^2(\mathbb{Z}/2) = \underline{\underline{nil}}^{4,2}$$

*Proof.* Let $\Sigma P_2$ be the suspension of the real projective plane; then $\Sigma P_2 = M(\mathbb{Z}/2, 2)$ is the Moore space of $\mathbb{Z}/2$ in degree 2. Moreover for a $\mathbb{Z}/2$ -vector space $V$ with basis B the one point union

$$\bigvee_B \Sigma P_2 = M(V,2)$$

is a Moore space of $V$. This shows that

(8.3) $$\underline{\underline{M}}^2(\mathbb{Z}/2) = \underline{\underline{add}}(End(\Sigma P_2))$$

by (2.4). Here the endomorphism square ring of $\Sigma P_2$ satisfies by a result of Barratt [1]

(8.2) $$End(\Sigma P_2) = \mathbb{Z}^{4,2}_{nil}$$

Hence the result in (8.2) follows from (8.1).

q.e.d.

*(8.4) Endomorphism square rings of suspended pseudo projective planes* $\Sigma P_n$. Here a pseudo projective plane

(1) $$P_n = S^1 \cup_n e^2$$

is obtained by attaching a 2-cell to a 1-sphere by a map of degree $n$. For $n = 2$ this is the real projective plane. Clearly $\Sigma P_n = M(\mathbb{Z}/n, 2)$ is a Moore space of the cyclic group $\mathbb{Z}/n$. Using results in [3] we obtain the endomorphism square ring

(2) $$End(\Sigma P_n) = (\mathbb{Z}/n \times \mathbb{Z}/n \xrightarrow{H} \mathbb{Z}/n \xrightarrow{P} \mathbb{Z}/n \times \mathbb{Z}/n)$$

where $End(\Sigma P_n)_e = \mathbb{Z}/n \times \mathbb{Z}/n$ as a set with the monoid structure

$$(a, \alpha) \cdot (b, \beta) = (ab,\ a^2 \cdot \beta + b \cdot \alpha)$$

and the (abelian) group structure

$$(a, \alpha) + (b, \beta) = (a + b, \alpha + \beta + abn(n-1)/2).$$

Moreover $End(\Sigma P_n)_{ee} = \mathbb{Z}/n$ as an abelian group and $H(a, \alpha) = \alpha$ and $P(x) = (0, 2x)$. The cokernel of $P$ is the ring $R = \mathbb{Z}/n$ which acts on $End(\Sigma P_n)_{ee} = \mathbb{Z}/n$ in the canonical way. One now can show that for $n = 2$ this square ring coincides with (8.2) and as in (8.3) we obtain the equivalences of categories

(3) $$\underline{\underline{M}}^2(\mathbb{Z}/n) = \underline{\underline{add}}(End(\Sigma P_n))$$

Here $\underline{\underline{M}}^2(\mathbb{Z}/n)$ is the full homotopy category of Moore spaces $M(V, 2)$ for which $V$ is a free $\mathbb{Z}/n$ -module. By (2) we see that the right hand side of (3) is a purely algebraic category.

*(8.5) <u>The R-localization of nil-groups</u>.* A ring $R$ is termed 2-binomial if for all $r \in R$ the element $r(r-1) \in R$ is uniquely 2-divisible so that $r(r-1)/2 \in R$. Clearly if 2 is invertible then $R$ is 2-binomial. Also any subring $R \subset \mathbb{Q}$ of the rationals is 2-binomial. Given a 2-binomial ring $R$ we obtain the square ring

(1) $$R_{nil} = (R \xrightarrow{H} R \xrightarrow{P} R)$$

with $H(r) = r(r-1)/2$ and $P = 0$. This generalizes the square ring $\mathbb{Z}_{nil}$. Therefore we may consider $R_{nil}$ -modules as generalizations of nilpotent groups of order 2. The morphism $\mathbb{Z}_{nil} \to R_{nil}$ induces $\underline{\underline{Add}}(\mathbb{Z}_{nil}) \to \underline{\underline{Add}}(R_{nil})$ by the universal property of $\underline{\underline{Add}}$ in (4.3). Hence we obtain the induced functor

$$\underline{\underline{Mod}}(R_{nil}) \to \underline{\underline{Mod}}(\mathbb{Z}_{nil}) = \underline{\underline{Nil}}$$

which has a left adjoint

$$\underline{\underline{Nil}} \to \underline{\underline{Mod}}(R_{nil})$$

which carries $G \in \underline{\underline{Nil}}$ to $G_R \in \underline{\underline{Mod}}(R_{nil})$. Here $G_R$ is the $R$-<u>localization</u> of $G$ which for $R \subset \mathbb{Q}$ is the classical localization of $G$; see for example [14], [16].

*(8.6) <u>Square rings with P = 0</u>.* Let $R$ be a ring and $M$ be an $R \otimes R \otimes R^{op}$ -module satisfying

$$(s \otimes t) \cdot x \cdot r = (t \otimes s) \cdot x \cdot r$$

for $s, t, r \in R$ and $x \in M$. Moreover let $H : R \to M$ be a function for which

$$H(s+t) = Hs + Ht + (t \otimes s) \cdot H(2)$$
$$H(s \cdot t) = (s \otimes s) \cdot H(t) + H(s) \cdot t$$

holds. Then

$$(1) \qquad R^H_{nil} = (R \xrightarrow{H} M \xrightarrow{P=0} R)$$

is a square ring with $P = 0$ and conversely each square ring with $P = 0$ is obtained this way. This generalizes the square ring $R_{nil}$ of a 2-binomial ring $R$.

As an example of a square ring with $P = 0$ we describe the automorphim square ring $End(\Sigma\mathbb{C}P_2)$ where $\mathbb{C}P_2$ is the complex projective plane. Let $\mathbb{Z} \times_2 \mathbb{Z}$ be the subring of $\mathbb{Z} \times \mathbb{Z}$ consisting of all pairs $a = (a_0, a_1)$ with $a_0 - a_1 \equiv 0 \, mod \, 2$. Then we have

$$(2) \qquad End(\Sigma\mathbb{C}P_2) = (\mathbb{Z} \times_2 \mathbb{Z} \xrightarrow{H} \mathbb{Z} \xrightarrow{P=0} \mathbb{Z} \times_2 \mathbb{Z})$$

where $H$ is defined by $H(1,1) = 0$, $H(0,2) = 1$ and

$$H(a+b) - H(a) - H(b) = a_0 \cdot b_0$$

The $R \otimes R \otimes R^{op}$ -modules $\mathbb{Z}$ with $R = \mathbb{Z} \times_2 \mathbb{Z}$ is given by

$$(a \otimes b) \cdot k \cdot c = a_0 \cdot b_0 \cdot k \cdot c_1$$

where $a, b, c \in R$ and $k \in \mathbb{Z}$. The isomorphism

$$[\Sigma\mathbb{C}P_2, \, \Sigma\mathbb{C}P_2] = \mathbb{Z} \times_2 \mathbb{Z}$$

carries a map $F$ to the degree $(a_0, a_1)$ in homology where $a_0 =$ degree $(H_3 F)$ and $a_1 =$ degree $(H_5 F)$. Clearly the algebraic description of $End(\Sigma\mathbb{C}P_2)$ above yields an algebraic characterization of the subcategory

$$\underline{\underline{Add}}(End \, \Sigma\mathbb{C}P_2) \subset \underline{\underline{Top}}^* / \simeq$$

which is the full homotopy category consisting of finite one point unions $\Sigma\mathbb{C}P_2 \vee \ldots \vee \Sigma\mathbb{C}P_2$. This category was computed in different terms by Unsöld [17] who showed that for $\underline{\underline{Q}} = \underline{\underline{Add}}(\Sigma\mathbb{C}P_2)$ the associated linear extension $\underline{\underline{Q}} \to \underline{\underline{Q}}^{add} = \underline{\underline{Add}}(R)$ is non-split.

*(8.7) Square rings arising from operads.* Let $K$ be a commutative ring and let $P$ be an operad in the monoidal catgeory of $K$-modules with the monoidal structure given by the tensor product. Recall that $P$ consists of $K$-modules $P(n)$, $n \geq 0$, with an action of the symmetric group $\Sigma_n$ and of composition laws $\mu(i_1, \ldots, i_k; k)$:

$$P(i_1) \otimes \ldots \otimes P(i_k) \otimes P(k) \to P(i_1 + \ldots + i_k)$$

for $k, i_1, \ldots, i_k \geq 0$ where $P(0) = K$. Moreover certain associativity and symmetry properties hold [9]. It is well known that an operad $P$ with $P(n) = 0$ for $n \geq 2$ is the same as a $K$-algebra. An operad with $P(n) = 0$ for $n \geq 3$ actually yields canonically a square ring

$$Q(P) = (P(2)_{\Sigma_2} \oplus P(1) \xrightarrow{H} P(2) \xrightarrow{P} P(2)_{\Sigma_2} \oplus P(1))$$

where $P(2)_{\Sigma_2} = P(2)/(x - x^t \sim 0)$ is the module of coinvariants of the $\Sigma_2$ -action with $t$ a generator of $\Sigma_2$. The function $H$ is given by $H(\bar{x}, y) = x + x^t$ where $\bar{x} \in P(2)_{\Sigma_2}$ is the class of $x \in P_2$, $y \in P(1)$. Moreover $P$ is defined by $P(x) = (\bar{x}, 0)$. Hence the cokernel of $P$ is the $K$-module $P(1)$ which is a ring $R$ via the multiplication $\mu(1;1)$. Moreover $P(2)$ is an $R \otimes R \otimes R^{op}$ -module by $\mu(1,1;2)$ and $\mu(2;1)$. The structure of $P(2)_{\Sigma_2} \oplus P(1)$ as a monoid is defined by

$$(\bar{x}_1, y_1) \cdot (\bar{x}_2, y_2) = (\bar{x}_1 \cdot y_2 + (y_1 \otimes y_1) \cdot \bar{x}_2,\ y_1 \cdot y_2).$$

One can check that the axioms of an operad show that $Q(P)$ is in this way a well defined square ring. Let $\underline{\underline{niloperad}}(K)$ be the category of operads $P$ with $P(n) = 0$ for $n \geq 3$ and let $\underline{\underline{squarering}}$ be the category of square rings. Then the construction of $Q(P)$ above yields for $K \subset \mathbb{Q}$ a full embedding

$$\underline{\underline{niloperad}}(K) \subset \underline{\underline{squarering}}.$$

This shows that a square ring is in a canonical way a non-abelian version of a nil-operad. Therefore there exists a more general theory of "non-abelian operads" generalizing both the concept of square ring and the concept of operad.

*(8.8)* <u>*Square rings arising from nilpotent algebras*</u>. Let $R$ be a commutative ring. Then one has the following square rings where $R$ and $R \oplus R$ are groups given by the additive structure of $R$ and where $R \oplus R$ is a monoid by

$$(x, y) \cdot (u, v) = (xu, x^2 y + yv)$$

We now define:

$$\begin{cases} \Lambda_R = (R \xrightarrow{0} R \xrightarrow{0} R) \\ H = P = 0 \quad \text{is trivial.} \end{cases}$$

$$\begin{cases} \otimes_R = (R \oplus R \xrightarrow{H} R \oplus R \xrightarrow{P} R \oplus R) \\ H(x, y) = (y, y) \quad \text{and} \quad P(x, y) = (0, x + y) \end{cases}$$

$$\begin{cases} S_R = (R \oplus R \xrightarrow{H} R \xrightarrow{P} R \oplus R) \\ H(x, y) = 2y \quad \text{and} \quad P(x) = (0, x) \end{cases}$$

$$\begin{cases} \Gamma_R = (R \oplus R \xrightarrow{H} R \xrightarrow{P} R \oplus R) \\ H(x, y) = y \quad \text{and} \quad P(x) = (0, 2x) \end{cases}$$

The corresponding modules are $R$-algebras of nilpotency degree 2 as in the following table:

| $Q$ | $Q$-modules |
|---|---|
| $\Lambda_R$ | Lie algebras |
| $\otimes_R$ | associative algebras, Leibniz-algebras |
| $S_R$ | commutative algebras |
| $\Gamma_R$ | divided power algebras |

*(8.9)* <u>*Restriction of square rings*</u>. Let $Q$ be a square ring with associated ring $R$ and let $R'$ be a subring of $R$. Then we obtain a square ring $Q \mid R'$ which we call the <u>restriction of</u> $Q$ <u>to</u> $R'$. Let $p : Q_e \twoheadrightarrow R$ be the projection and let $Q_e \mid R' = p^{-1}(R')$ be the inverse image of $R' \subset R$. Then

$$Q \mid R = (Q_e \mid R' \xrightarrow{H} Q_{ee} \xrightarrow{P} Q_e \mid R')$$

is given by the structure maps $H$ and $P$ in $Q$. This is a subobject of the square ring $Q$.

*(8.10)* <u>*Monoid square rings*</u>. The free abelian group $\mathbb{Z}[M]$ generated by a monoid $M$ has the structure of a ring with multiplication induced by the multiplication of $M$. This is the classical <u>monoid ring</u> of $M$ which is the <u>group ring</u> if $M$ is a group. This construction has the following analogue for square rings. Let $\langle M\rangle_{nil}$ be the free nil-group generated by the set $M$, that is $\langle M\rangle_{nil} = \langle M\rangle/\Gamma_3\langle M\rangle$. We now consider the $M$-objects in the category $\underline{\underline{Nil}}$ which form the category $M - \underline{\underline{Nil}}$ with the subcategory $M - \underline{\underline{nil}}$ of free objects. In fact $\langle M\rangle_{nil}$ is the free object in $M - \underline{\underline{Nil}}$ with one generator. Again $M - \underline{\underline{nil}}$ is a quadratic category so that the endomorphism square ring

$$\mathbb{Z}_{nil}[M] = End(\langle M\rangle_{nil})$$

is defined. This is the <u>monoid square ring</u> given by the monoid $M$. More explicitly

$$\mathbb{Z}_{nil}[M] = (\langle M\rangle_{nil} \xrightarrow{H} \mathbb{Z}[M] \otimes \mathbb{Z}[M] \xrightarrow{P} \langle M\rangle_{nil})$$

is the unique square ring for which the following holds.

$$\begin{aligned}
&H(m) = 0 \\
&(a,b)_H = \{a\} \otimes \{b\} \\
&P(\{a\} \otimes \{b\}) = a + b - a - b
\end{aligned}$$

Here $\{a\} \in \mathbb{Z}[M]$ is the abelianization of $a \in \langle M\rangle_{nil}$. The underlying group of $\mathbb{Z}_{nil}[M]_e$ is the group $\langle M\rangle_{nil}$, the underlying monoid structure of $\mathbb{Z}_{nil}[M]_e$ is uniquely gievn by

$$m \cdot n = mn \in M \quad \text{for} \quad m, n \in M$$

and (4), (5) in (7.4). The $\mathbb{Z}[M] \otimes \mathbb{Z}[M] \otimes \mathbb{Z}[M]^{op}$ -module structure of $\mathbb{Z}_{nil}[M]_{ee} = \mathbb{Z}[M] \otimes \mathbb{Z}[M]$ is given by

$$(a \otimes b) \cdot (u \otimes v) \cdot m = (aum) \otimes (bvm)$$

for $a, b, n, v \in \mathbb{Z}[M]$. One readily checks that one has equivalences of categories

$$\underline{\underline{Mod}}(\mathbb{Z}_{nil}[M]) = M - \underline{\underline{Nil}}$$
$$\underline{\underline{add}}(\mathbb{Z}_{nil}[M]) = M - \underline{\underline{nil}}$$

which coincide with the corresponding equivalences in (7.11) if $M$ is a point.

*(8.11)* *Square rings arising from restricted Lie algebras.* Let $K$ be a commutative $\mathbb{Z}/2$ -algebra and let

$$\Lambda_K^{restr} = \left( R \xrightarrow{H} R \xrightarrow{P=0} R \right)$$

be the following square ring with $P = 0$ as in (8.6). Here $R$ is the abelian group given by the free $K$-module

$$R = \bigoplus_{i \geq 0} K\, t^i$$

generated by the monomials $1, t, t^2, \ldots$. This is a ring by the multiplication rules

$$tk = k^2 t, \; t^n t^m = t^{n+m}$$

for $k \in K$, $n, m \geq 0$, with $t^0 = 1 \in K$. Moreover $R$ as an $R \otimes R \otimes R^{op}$ -module is obtained by the action $(a, b, c, x \in R)$

$$(a \otimes b) \cdot x \cdot c = a_0 \cdot b_0 \cdot x \cdot c$$

where $a_0$ is the constant term of the polynomial $a$. Now $H$ is the unique function with properties as in (8.6) satisfying $H(t) = 1$. One readily verifies that the category of $\Lambda_K^{restr}$ -modules coincides with the category of 2-restricted Lie $K$-algebras satisfying $[[x, y], z] = 0$. Here the action of $t$ corresponds to the operation $x \longmapsto x^{[2]}$ of a restricted Lie algebra. The modules over the factor square ring

$$\Lambda_K^{restr}/(t^2, t) = (R/(t^2) \to R/(t) \to R/(t^2))$$

are the 2-restricted Lie $K$-algebras satsifying the relations $[[x, y], z] = 0$, $(x^{[2]})^{[2]} = 0$ and $[x, y]^{[2]} = 0$.

## § 9 <u>Equivalences of square rings</u>

It is clear that two square rings $Q$ and $Q'$ are <u>isomorphic</u>, $Q \cong Q'$, if and only if there is an isomorphism

$$(9.1) \qquad \psi : \underline{\underline{Add}}(Q) \cong \underline{\underline{Add}}(Q')$$

of quadratic categories which is the identity on objects. Here the isomorphism $\psi$ is an isomorphism of categories which is linear in the sense of (4.1). We say that $Q$ and $Q'$ are <u>equivalent</u> if there is an isomorphism $\psi$ as in (9.1) of categories which not necessarily needs to be linear. Such an equivalence induces an isomorphism of module categories

$$(9.2) \qquad \underline{\underline{Mod}}(Q) \cong \underline{\underline{Mod}}(Q')$$

since an equivalence $\psi$ is an isomorphism of theories; compare (1.5) and (7.8). We now study explicit conditions which show that square rings $Q$ and $Q'$ are equivalent. For this we need the following construction.

*(9.3) <u>Definition</u>.* Given a square ring

$$Q = (Q_e \xrightarrow{H} Q_{ee} \xrightarrow{P} Q_e)$$

and an element $\xi \in Q_{ee}$ we define a new square ring

$$Q^\xi = (Q_e^\xi \xrightarrow{H^\xi} Q_{ee} \xrightarrow{P} Q_e^\xi)$$

as follows. Here $Q_e^\xi$ as a monoid in the same as $Q_e$. Yet the group structure of $Q_e^\xi$, denoted by $a \oplus b$, is defined by

$$(1) \qquad a \oplus b = a + b + P((\bar{a} \otimes \bar{b}) \cdot \xi)$$

for $a, b \in Q_e$. Moreover $H^\xi$ is given by the formula

$$(2) \qquad H^\xi(a) = H(a) + \xi \cdot \bar{a} - (\bar{a} \otimes \bar{a}) \cdot \xi$$

The function $P$ for $Q^\xi$ coincides with $P$ in $Q$. This shows that the associated ring $R$ of $Q^\xi$ coincides with the associated ring of $Q$. Moreover $M_{ee}$ in $Q^\xi$ is the same $R \otimes R \otimes R^{op}$ -module as in $Q$. We point out that the element $2 = 1 + 1$ in $Q_e$ does not coincide with the element $2^\xi = 1 \oplus 1$ in $Q_e^\xi$, in fact, $2^\xi = 2 + P\xi$. A straightforward but somewhat tedious proof shows:

**(9.4) <u>Lemma</u>.** *$Q^\xi$ is a well defined square ring for any $\xi \in Q_{ee}$.*

We point out that for $Q = \mathbb{Z}_{nil}^{4,2}$ and $\xi = 1 \in Q_{ee}$ we have $Q^\xi = Q$. Using $Q^\xi$ above we can characterize equivalence of square rings as follows.

**(9.5) Proposition.** *Two square rings $Q$ and $Q'$ are equivalent if and only if there is $\xi \in Q_{ee}$ such that $Q^\xi$ is isomorphic to $Q'$.*

This in particular implies by (9.2) that one has an isomorphism of categories

$$\underline{\underline{Mod}}(Q) \cong \underline{\underline{Mod}}(Q^\xi) \tag{9.6}$$

There is a nice classical example of this isomorphism obtained by the Malcev correspondence between rational nilpotent Lie algebras and uniquely divisible nilpotent groups. For nilpotency degree 2 this correspondence in the sense of Lazard gives us an isomorphism

$$\underline{\underline{Mod}}(R_{nil}) \cong \underline{\underline{Mod}}(\Lambda_R) \tag{9.7}$$

for $1/2 \in R \subset \mathbb{Q}$. Here by (8.5) the left hand side is the category of $R$-local groups $G$ in $\underline{\underline{Nil}}$ and the right hand side is by (8.8) the category of $R$-Lie algebras $L$ of nilpotency degree 2. The Malcev correspondence (9.7) carries $L$ to the group $G$ given by the set $L$ with the group law

$$x \cdot y = x + y + (1/2)[x, y]$$

This is the nil-case of the classical Baker-Campbell-Hausdorff formula, see [15]. We now obtain a new interpretation of this correspondence by use of the notion of equivalence of square rings, namely:

*(9.8) Lemma.* For $\xi = -1/2 \in R$ there is a canonical isomorphism $(\Lambda_R)^\xi = R_{nil}$.

For this compare the definitions of $\Lambda_R$ and $R_{nil}$ above. Now one can check that the isomorphism $(\Lambda_R)^\xi = R_{nil}$ yields via (9.6) exactly the Malcev correspondence (9.7). In this sense we can consider the isomorphism of categories in (9.6) as a generalization of the Malcev correspondence.

*(9.9) Proof of (9.5).* The objects of $\underline{\underline{Add}}(Q)$ and $\underline{\underline{Add}}(Q')$ are given by numbers $0, 1, 2, \dots$ where $n \in \mathbb{N}$ corresponds to the $n$-fold sum $1 \amalg 1 \amalg \dots \amalg 1$. Let

$$\psi : \underline{\underline{Add}}(Q') \cong \underline{\underline{Add}}(Q) = \underline{\underline{Q}}$$

be an isomorphism of categories which is the identity on objects. The cogroup structure $\mu' : 1 \to 1 \amalg 1$ in $\underline{\underline{Add}}(Q')$ is carried via $\psi$ to a cogroup structure $\psi(\mu') : 1 \to 1 \amalg 1$ in $\underline{\underline{Add}}(Q)$ where $\psi(\mu')$ needs not to coincide with $\mu = i_1 + i_1$. Hence there is $\xi \in Q_{ee} = \underline{\underline{Q}}(1, 1 \mid 1)$ with

$$i_{12}(\xi) = -\mu + \psi(\mu')$$

We claim that there is now an isomorphism $Q^\xi \cong Q'$ of square rings.

q.e.d.

*(9.10) Definition.* We say that a square ring $Q$ is abelian if each $Q$-module $M \in \underline{\underline{Mod}}(Q)$ is an abelian group or equivalently $Add(\underline{\underline{Q}})$ has abelian $Hom$-sets. This

is the case if and only if $H(2) = 0$. We say that $Q$ is of abelian type if there is an equivalence $Q \sim Q'$ where $Q'$ is abelian. This is the case if and only if there is $\xi \in H(2)$ such that the equation

$$H(2) = 2\xi - HP(\xi) = \xi - T(\xi)$$

holds. Hence if $Q_{ee}$ is 2-divisible and $P = 0$ then $Q$ is of abelian type. For example for $1/2 \in R \subset \mathbb{Q}$ the square ring $\Lambda_R$ is of abelian type. One can check that $Q = End(\Sigma P_n)$ in (8.4) for $2 \mid n$ is not of abelian type though $Q_e$ is an abelian group in this case. Moreover $End(\Sigma P_n)$ is abelian if $n$ is odd. We point out that for $n$ even and $\alpha = [i_n, i_n] \in \pi_{2n-1} S^n$ the square ring $End(\Sigma C_\alpha)$ is not abelian but of abelian type since $\Sigma\alpha = 0$.

*(9.11) Example.* Let $K = \mathbb{Z}[1/2] \subset \mathbb{Q}$ and let

$$\underline{\underline{niloperad}}(K) \subset \underline{\underline{squarering}}$$

be the inclusion in (8.7) which carries the nil-operad $P$ to $Q(P)$. Given any square ring $Q$ such that the associated ring $R$ contains $1/2$ there is a niloperad $P$ with $Q(P)$ equivalent to $Q$. Compare the Malcev correspondence in (9.7).

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