DERIVED CATEGORIES AND KUMMER VARIETIES

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Abstract. We prove that if two abelian varieties have equivalent derived categories then the derived categories of the smooth stacks associated to the corresponding Kummer varieties are equivalent as well. The second main result establishes necessary and sufficient conditions for the existence of equivalences between the twisted derived categories of two Kummer surfaces in terms of Hodge isometries between the generalized transcendental lattices of the corresponding abelian surfaces.

1. Introduction

The *Kummer variety* of an abelian variety A is the quotient $\mathrm{K}(A) := A/\langle\iota\rangle$, where $\iota(a) = -a$ for any $a \in A$. The singular variety $\mathrm{K}(A)$ has an orbifold structure and it admits a minimal crepant resolution $\mathrm{Km}(A)$ if and only if the dimension of A is 2. In this case $\mathrm{Km}(A)$ is a K3 surface (i.e. it is simply connected and its dualizing sheaf is trivial) and it is called the *Kummer surface of A*. More generally, we can associate to the global quotient $\mathrm{K}(A)$ the smooth quotient stack $[A/\langle\iota\rangle]$.

In [9] Hosono, Lian, Oguiso and Yau proved that,

(A) *given two abelian surfaces A and B,* $\mathrm{D}^{\mathrm{b}}(A) \cong \mathrm{D}^{\mathrm{b}}(B)$ *if and only if*

$$\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A)) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{Km}(B)).$$

Their argument is quite easy: They notice that, due to the geometric construction of the Kummer surfaces $\mathrm{Km}(A)$ and $\mathrm{Km}(B)$, the transcendental lattices of A and B are Hodge-isometric if and only if the transcendental lattices of $\mathrm{Km}(A)$ and $\mathrm{Km}(B)$ are Hodge-isometric. Then, they apply a deep result of Orlov which says that two abelian or K3 surfaces have equivalent derived categories if and only if their transcendental lattices are Hodge-isometric (see Theorem 2.2). From this it is evident that (A) can be reformulated in the following way:

(B) *given two abelian surfaces A and B,* $\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A)) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{Km}(B))$ *if and only if there exists a Hodge isometry between the transcendental lattices of A and B.*

Since Mukai proved in [18] that two K3 surfaces with Picard number greater than 11 and with Hodge-isometric transcendental lattices are isomorphic, (A) and (B) are equivalent to the following statement:

(C) *given two abelian surfaces A and B,* $\mathrm{D}^{\mathrm{b}}(A) \cong \mathrm{D}^{\mathrm{b}}(B)$ *if and only if* $\mathrm{Km}(A) \cong \mathrm{Km}(B)$.

The aim of this paper is to address (A), (B) and (C) in two more general contexts. Indeed, Theorem 3.1 shows that if A_1 and A_2 are abelian varieties with equivalent derived categories, then the derived categories of the stacks $[A_1/\langle\iota\rangle]$ and $[A_2/\langle\iota\rangle]$ are equivalent as well. As we will show in Section 3.3, when we deal with abelian surfaces this result leads to a direct proof of one implication in (A).

According to (B), Theorem 4.3 proves that the twisted derived categories of two Kummer surfaces are equivalent if and only if the generalized transcendental lattices of the corresponding abelian surfaces are Hodge isometric. We will observe that the analogues of (A) and (C) in the twisted setting are no longer true (see Remark 4.5). Nevertheless we completely generalize the results in [9]

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about the number of Kummer structures on K3 surfaces in the twisted context (Proposition 4.6). A geometric example involving abelian surfaces with Picard number two is discussed.

2. Derived categories of abelian varieties and K3 surfaces

In this section we recall some facts and definitions concerning the derived categories of coherent sheaves on abelian varieties and K3 surfaces. In the following pages $\mathrm{D}^{\mathrm{b}}(X)$ will always mean the bounded derived category of coherent sheaves on the smooth projective variety X (we will also use the same notation for the bounded derived category of coherent sheaves on a smooth quotient stack according to [14]).

Suppose that X_1 and X_2 are smooth projective varieties. Let $\mathrm{D}^{\mathrm{b}}(X_1)$ and $\mathrm{D}^{\mathrm{b}}(X_2)$ be the bounded derived categories of coherent sheaves on X_1 and X_2. Orlov proved in [23] that any equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X_1) \to \mathrm{D}^{\mathrm{b}}(X_2)$ is a *Fourier-Mukai equivalence*, i.e. there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X_1 \times X_2)$ such that, for any $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X_1)$,

$$\Phi(\mathcal{F}) = \mathbf{R}p_{2*}(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_1^*\mathcal{F}), \tag{2.1}$$

where $p_i : X_1 \times X_2 \to X_i$ is the projection and $i \in \{1, 2\}$. The complex $\mathcal{E}$ is the *kernel* of Φ and it is uniquely (up to isomorphism) determined. We write $\Phi_{\mathcal{E}}$ for a Fourier-Mukai equivalence whose kernel is $\mathcal{E}$ and we say that two smooth projective varieties X and Y are *Fourier-Mukai partners* if $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$. In general, given $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X_1 \times X_2)$, we write $\Phi_{\mathcal{E}}$ for a functor defined as in (2.1) (notice that $\Phi_{\mathcal{E}}$ is not necessarily an equivalence).

2.1. **Derived categories of abelian varieties.** Assume that A_1 and A_2 are abelian varieties of dimension d. For $i \in \{1, 2\}$, let $\mathcal{P}_i$ be the Poicaré line bundle on $A_i \times \widehat{A_i}$, let $\mu_i : A_i \times A_i \to A_i \times A_i$ be the isomorphism such that $(a, b) \mapsto (a+b, b)$ and let $\Phi_i := \mu_{i*} \circ (\mathrm{id} \times \Phi_{\mathcal{P}_i})$. If $\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(A_1) \to \mathrm{D}^{\mathrm{b}}(A_2)$ is a Fourier-Mukai equivalence with kernel $\mathcal{E}$, we get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(A_1 \times \widehat{A_1}) \arrow[rrr, "F_{\mathcal{E}}"] \arrow[dr, "\mathrm{id}\times\Phi_{\mathcal{P}_1}"] \arrow[dd, "\Phi_1"'] & & & \mathrm{D}^{\mathrm{b}}(A_2 \times \widehat{A_2}) \arrow[dl, "\mathrm{id}\times\Phi_{\mathcal{P}_2}"'] \arrow[dd, "\Phi_2"] \\
& \mathrm{D}^{\mathrm{b}}(A_1 \times A_1) \arrow[dl, "\mu_{1*}"'] & \mathrm{D}^{\mathrm{b}}(A_2 \times A_2) \arrow[dr, "\mu_{2*}"] & \\
\mathrm{D}^{\mathrm{b}}(A_1 \times A_1) \arrow[rrr, "\Phi_{\mathcal{E}}\times\Phi_{\mathcal{E}_R}"] & & & \mathrm{D}^{\mathrm{b}}(A_2 \times A_2),
\end{tikzcd} \tag{2.2}$$

where $F_{\mathcal{E}}$ is the functor completing the diagram, $\mathcal{E}_R = \mathcal{E}^\vee[d]$ and $\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R}$ is the Fourier-Mukai equivalence whose kernel is $\mathcal{E} \boxtimes \mathcal{E}_R$. Observe that since $\Phi_{\mathcal{E}}$, Φ_1 and Φ_2 are equivalences, $\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R}$ and $F_{\mathcal{E}}$ are equivalences as well.

For $i \in \{1, 2\}$, the Künneth formula yields a decomposition

$$H_1(A_i \times \widehat{A_i}, \mathbb{Z}) \cong H_1(A_i, \mathbb{Z}) \oplus H_1(\widehat{A_i}, \mathbb{Z}).$$

Since $H_1(\widehat{A_i}, \mathbb{Z}) \cong H_1(A_i, \mathbb{Z})^\vee$, the group $H_1(A_i \times \widehat{A_i}, \mathbb{Z})$ is endowed with a natural quadratic form. Indeed, if $(a_1, \alpha_1), (a_2, \alpha_2) \in H_1(A_i \times \widehat{A_i}, \mathbb{Z})$, we define

$$\langle (a_1, \alpha_1), (a_2, \alpha_2) \rangle_i := \alpha_1(a_2) + \alpha_2(a_1),$$

where $i \in \{1, 2\}$. Consider the set of isomorphisms

$$\mathrm{U}(A_1, A_2) := \{ f \in \mathrm{Isom}(A_1 \times \widehat{A_1}, A_2 \times \widehat{A_2}) : \langle f_*(a_1, \alpha_1), f_*(a_2, \alpha_2) \rangle_2 = \langle (a_1, \alpha_1), (a_2, \alpha_2) \rangle_1 \}.$$

Theorem 2.1. **([24], Theorem 2.19 and Proposition 4.12.)** *Let A_1 and A_2 be abelian varieties. If $\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(A_1) \to \mathrm{D}^{\mathrm{b}}(A_2)$ is an equivalence, then, for any $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(A_1)$,*

$$F_{\mathcal{E}}(\mathcal{F}) = f_{\mathcal{E}*}(\mathcal{F}) \otimes N_{\mathcal{E}},$$

where $F_{\mathcal{E}}$ *is the equivalence in (2.2),* $f_{\mathcal{E}} \in \mathrm{U}(A_1, A_2)$ *and* $N_{\mathcal{E}} \in \mathrm{Pic}(A_2 \times \widehat{A_2})$*. Moreover, there exists a surjective map*

$$\gamma : \mathrm{Eq}(\mathrm{D^b}(A_1), \mathrm{D^b}(A_2)) \longrightarrow \mathrm{U}(A_1, A_2) \tag{2.3}$$

such that $\gamma(\Phi_{\mathcal{E}}) = f_{\mathcal{E}}$*, where* $\mathrm{Eq}(\mathrm{D^b}(A_1), \mathrm{D^b}(A_2))$ *is the set of the equivalences between* $\mathrm{D^b}(A_1)$ *and* $\mathrm{D^b}(A_2)$*.*

2.2. **Hodge structures and derived categories.** If X is a smooth projective variety of dimension d, we denote by $\widetilde{H}(X, \mathbb{Q})$ the total cohomology group $H^*(X, \mathbb{Q})$ with the weight-$2d$ Hodge structure defined as follows:

$$\widetilde{H}^{p,q}(X) = \bigoplus_{\frac{p-q}{2}=r-s} H^{r,s}(X), \tag{2.4}$$

where $H^{r,s}(X)$ is the (r, s)-part of the usual Hodge decomposition of $H^{r+s}(X, \mathbb{C})$. An equivalent way to put on $H^*(X, \mathbb{C})$ such a Hodge structure could be obtained considering the natural grading on the Hochschild homology of X (see, for example, [5]).

Suppose now that X is either an abelian or a K3 surface, $H^{2,0}(X) = \langle \sigma_X \rangle$ and B is any class in $H^2(X, \mathbb{Q})$. Then

$$\varphi := \exp(B)(\sigma_X) = \sigma_X + B \wedge \sigma_X \in H^2(X, \mathbb{C}) \oplus H^4(X, \mathbb{C})$$

is a *generalized Calabi-Yau structure on* X (for a complete picture see [10]). Let $T(X, B)$ be the minimal primitive sublattice of $H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ such that $\varphi \in T(X, B) \otimes \mathbb{C}$. The lattice $T(X, B)$ is the *generalized transcendental lattice of* φ (see [10] and [12]). Let $\widetilde{H}(X, \mathbb{Z})$ be the $\mathbb{Z}$-module $H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ endowed with the *Mukai pairing*

$$\langle (a_0, a_2, a_4), (b_0, b_2, b_4) \rangle = a_2 \cdot b_2 - a_0 \cdot b_4 - a_4 \cdot b_0,$$

where $(a_0, a_2, a_4), (b_0, b_2, b_4) \in H^{2*}(X, \mathbb{Z})$ and "$\cdot$" is the cup-product. We write $\widetilde{H}(X, B, \mathbb{Z})$ for the lattice $\widetilde{H}(X, \mathbb{Z})$ with the weight-two Hodge structure such that

$$\widetilde{H}^{2,0}(X, B) := \exp(B)(\widetilde{H}^{4,0}(X))$$

and $\widetilde{H}^{1,1}(X, B)$ is its orthogonal complement in $H^2(X, \mathbb{C})$. It is clear that $T(X, B)$ inherits from $\widetilde{H}(X, B, \mathbb{Z})$ a weight-two Hodge structure. By definition, $T(X) = T(X, 0)$ is the *transcendental lattice of* X and

$$\mathrm{NS}(X) := T(X)^{\perp} \subset H^2(X, \mathbb{Z})$$

is the *Néron-Severi group of* X. The number $\mathrm{rkNS}(X)$ is the *Picard number* of X. If L_1 and L_2 are lattices endowed with a weight-k Hodge structure, then an isometry $f : L_1 \to L_2$ is a *Hodge isometry* if it preserves the Hodge structures.

For abelian and K3 surfaces, Orlov proved in [23] (using results of Mukai) the following theorem:

Theorem 2.2. ([23], Theorem 3.3.) *Let* X_1 *and* X_2 *be either abelian or K3 surfaces. Then the following two conditions are equivalent:*

(i) X_1 *and* X_2 *are Fourier-Mukai partners;*
(ii) *there exists a Hodge isometry* $T(X_1) \cong T(X_2)$*.*

3. Derived categories of the smooth stacks

In this section we prove our first main result:

Theorem 3.1. *Let* A_1 *and* A_2 *be abelian varieties. If* $\mathrm{D^b}(A_1) \cong \mathrm{D^b}(A_2)$*, then there exists a Fourier-Mukai equivalence* $\mathrm{D^b}([A_1/\langle\iota\rangle]) \cong \mathrm{D^b}([A_2/\langle\iota\rangle])$*.*

Conversely, if $\mathrm{D^b}([A_1/\langle\iota\rangle])$ *and* $\mathrm{D^b}([A_2/\langle\iota\rangle])$ *are equivalent, then there is an isomorphism of Hodge structures* $\widetilde{H}(A_1, \mathbb{Q}) \cong \widetilde{H}(A_2, \mathbb{Q})$*.*

As it will turn out, the proof of this theorem, which will be given in Section 3.2, relies on some results about equivariant derived categories of abelian varieties explained in Section 3.1. In Section 3.3 some geometric applications are discussed.

3.1. **Equivariant derived categories and abelian varieties.** Consider the simple case of an abelian variety A with the action of $G := \mathbb{Z}/2\mathbb{Z}$ induced by the automorphism $\iota : A \to A$ such that $\iota(a) = -a$, for any $a \in A$. A G-linearization for a coherent sheaf $\mathcal{E} \in \mathbf{Coh}(A)$ is an isomorphism $\lambda : \mathcal{E} \to \iota^*\mathcal{E}$ such that $\iota^*(\lambda) = \lambda$ and $\iota^*(\lambda) \circ \lambda = \lambda \circ \lambda = \mathrm{id}$.

$\mathbf{Coh}^G(A)$ is the abelian category whose objects are the pairs $(\mathcal{E}, \lambda)$, where $\mathcal{E} \in \mathbf{Coh}(A)$ admits a G-linearization and λ is a G-linearization for $\mathcal{E}$. The morphisms in $\mathbf{Coh}^G(A)$ are the morphisms in $\mathbf{Coh}(A)$ compatible with the G-linearizations. We define $\mathrm{D}^{\mathrm{b}}_G(A) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}^G(A))$ to be the bounded derived category of $\mathbf{Coh}^G(A)$. A complete discussion about the general case when G is any finite group acting on a smooth projective variety can be found in [2].

If A_1 and A_2 are abelian varieties and $G_\Delta \cong \mathbb{Z}/2\mathbb{Z}$ acts on $A_1 \times A_2$ via the automorphism $\iota \times \iota$, the set of G_Δ-invariant equivalences has the following description:

$$\mathrm{Eq}(\mathrm{D}^{\mathrm{b}}(A_1), \mathrm{D}^{\mathrm{b}}(A_2))^{G_\Delta} = \{\Phi_{\mathcal{G}} \in \mathrm{Eq}(\mathrm{D}^{\mathrm{b}}(A_1), \mathrm{D}^{\mathrm{b}}(A_2)) : \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(A_1 \times A_2) \text{ is } G_\Delta\text{-invariant}\}.$$

An equivalence $\Phi : \mathrm{D}^{\mathrm{b}}_G(A_1) \cong \mathrm{D}^{\mathrm{b}}_G(A_2)$ is a Fourier-Mukai equivalence if it satisfies an equation of type (2.1), where $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_G(A_1)$ and the kernel $\mathcal{E}$ is in $\mathrm{D}^{\mathrm{b}}_{G\times G}(A_1 \times A_2)$. $\mathrm{Eq}(\mathrm{D}^{\mathrm{b}}_G(A_1), \mathrm{D}^{\mathrm{b}}_G(A_2))$ is the set whose elements are the equivalences of this type.

Proposition 3.2. *Let A_1 and A_2 be abelian varieties and let $G = \mathbb{Z}/2\mathbb{Z}$ act on A_1 and A_2 as above. Then the restriction*

$$\gamma : \mathrm{Eq}(\mathrm{D}^{\mathrm{b}}(A_1), \mathrm{D}^{\mathrm{b}}(A_2))^{G_\Delta} \longrightarrow \mathrm{U}(A_1, A_2)$$

of the map in (2.3) is surjective and $\mathrm{Eq}(\mathrm{D}^{\mathrm{b}}_G(A_1), \mathrm{D}^{\mathrm{b}}_G(A_2))$ *is non-empty if* $\mathrm{U}(A_1, A_2)$ *is non-empty.*

Proof. By definition, we can think of any $f \in \mathrm{U}(A_1, A_2)$ as represented by a matrix

$$\begin{pmatrix} x_f & y_f \\ z_f & w_f \end{pmatrix}.$$

Define $S(A_1, A_2) := \{f \in \mathrm{U}(A_1, A_2) : y_f \text{ is an isogeny}\}$ and let $f \in S(A_1, A_2)$. Using results from [17], Orlov proved in [24] (see, in particular, Proposition 4.12) that there exists a vector bundle $\mathcal{E}$ on $A_1 \times A_2$ with the following properties:

(a) $\mathcal{E}$ is simple and $\Phi_{\mathcal{E}}$ is an equivalence;
(b) for any $(a, b) \in A_1 \times A_2$, if $T_{(a,b)}$ is the translation with respect to the point (a, b), then $T_{(a,b)*}\mathcal{E} \cong \mathcal{E} \otimes P$ for some $P \in \mathrm{Pic}^0(A_1 \times A_2)$;
(c) $\gamma(\Phi_{\mathcal{E}}) = f$.

Consider now the sheaf $\mathcal{F} := (\iota \times \iota)^*\mathcal{E}$. It is clear that $\gamma(\Phi_{\mathcal{F}}) = \gamma(\Phi_{\mathcal{E}}) = f$.

For a brief proof of this fact, consider the maps $\Phi_{\mathcal{P}_i}$ and μ_i in (2.2). A straightforward calculation shows that $(\iota \times \iota)^*\mathcal{P}_i \cong \mathcal{P}_i$. Moreover, μ_i is a morphism of abelian varieties. Hence $(\iota \times \iota)^* \circ (\mathrm{id} \times \Phi_{\mathcal{P}_i}) \circ (\iota \times \iota)^* = \mathrm{id} \times \Phi_{\mathcal{P}_i}$ and $(\iota \times \iota)^* \circ \mu_{i*} \circ (\iota \times \iota)^* = \mu_{i*}$. This implies that $(\iota \times \iota)^* \circ \Phi_i \circ (\iota \times \iota)^* = \Phi_i$, for $i \in \{1, 2\}$. Since $\Phi_{\mathcal{F}} = \iota^* \circ \Phi_{\mathcal{E}} \circ \iota^*$ and $\Phi_{\mathcal{F}} \times \Phi_{\mathcal{F}_R} = (\iota \times \iota)^* \circ (\Phi_{\mathcal{E}} \times \Phi_{\mathcal{E}_R}) \circ (\iota \times \iota)^*$, we rewrite the commutative diagram (2.2) in the following way:

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(A_1 \times \widehat{A_1}) \arrow[r, "(\iota\times\iota)^*"'] \arrow[d, "\Phi_1"'] \arrow[rrr, bend left, "F_{\mathcal{F}}"] & \mathrm{D}^{\mathrm{b}}(A_1 \times \widehat{A_1}) \arrow[r, "F_{\mathcal{E}}"] \arrow[d, "\Phi_1"'] & \mathrm{D}^{\mathrm{b}}(A_2 \times \widehat{A_2}) \arrow[r, "(\iota\times\iota)^*"'] \arrow[d, "\Phi_2"'] & \mathrm{D}^{\mathrm{b}}(A_2 \times \widehat{A_2}) \arrow[d, "\Phi_2"'] \\
\mathrm{D}^{\mathrm{b}}(A_1 \times A_1) \arrow[r, "(\iota\times\iota)^*"] \arrow[rrr, bend right, "\Phi_{\mathcal{F}}\times\Phi_{\mathcal{F}_R}"'] & \mathrm{D}^{\mathrm{b}}(A_1 \times A_1) \arrow[r, "\Phi_{\mathcal{E}}\times\Phi_{\mathcal{E}_R}"] & \mathrm{D}^{\mathrm{b}}(A_2 \times A_2) \arrow[r, "(\iota\times\iota)^*"] & \mathrm{D}^{\mathrm{b}}(A_2 \times A_2).
\end{tikzcd} \tag{3.1}$$

By Theorem 2.1, for any $\mathcal{G} \in \mathrm{D^b}(A_1 \times \widehat{A_1})$, $F_{\mathcal{F}}(\mathcal{G}) = f_{\mathcal{F}*}(\mathcal{G}) \otimes N_{\mathcal{F}}$, for some $f_{\mathcal{F}} \in \mathrm{U}(A_1, A_2)$ and $N_{\mathcal{F}} \in \mathrm{Pic}(A_2 \times \widehat{A_2})$. Hence, from (3.1) we deduce that

$$\begin{aligned} F_{\mathcal{F}}(\mathcal{G}) &= ((\iota \times \iota)^* \circ F_{\mathcal{E}} \circ (\iota \times \iota)^*)(\mathcal{G}) \\ &= ((\iota \times \iota)^* \circ f_{\mathcal{E}*} \circ (\iota \times \iota)^*)(\mathcal{G}) \otimes M \\ &= f_{\mathcal{E}*}(\mathcal{G}) \otimes M, \end{aligned}$$

for some $M \in \mathrm{Pic}(A_2 \times \widehat{A_2})$. Observe that the last equality holds true because $f_{\mathcal{E}}$ is a morphism of abelian varieties. This proves that $\gamma(\Phi_{\mathcal{F}}) = \gamma(\iota^* \circ \Phi_{\mathcal{E}} \circ \iota^*) = \gamma(\Phi_{\mathcal{E}})$ which is what we claimed.

Due to this last remark and to Corollary 3.4 in [24], there exist $a \in A_1$ and $\alpha \in \widehat{A_1}$ such that

$$\mathcal{F} = T_{(a,0)*}\mathcal{E} \otimes p^* P_\alpha[i], \tag{3.2}$$

where $p : A_1 \times A_2 \to A_1$ is the projection, i is an integer and P_α is the degree zero line bundle on A_1 corresponding to α. In the following arguments, without loss of generality, we will forget about the shift $[i]$ in (3.2).

Since $\mathcal{E}$ satisfies (b), from (3.2) we get $\mathcal{F} = \mathcal{E} \otimes Q$, where Q is a degree zero line bundle on $A_1 \times A_2$. Let $N \in \mathrm{Pic}^0(A_1 \times A_2)$ be such that $N^2 = Q$ and consider the sheaf $\mathcal{E}_f := \mathcal{E} \otimes N$. It is easy to see that

$$\begin{aligned} (\iota \times \iota)^*(\mathcal{E}_f) &= (\iota \times \iota)^*(\mathcal{E} \otimes N) \\ &\cong \mathcal{E} \otimes Q \otimes N^\vee \\ &\cong \mathcal{E} \otimes N \\ &= \mathcal{E}_f. \end{aligned}$$

Due to Proposition 3.3 in [24] and to (c), $\gamma(\Phi_{\mathcal{E}_f}) = \gamma(\Phi_{\mathcal{E}}) = f$.

Let $f \in \mathrm{U}(A_1, A_2)$. Orlov observed in Section 4 of [24] that there exist $g_1 \in S(A_1, A_2)$ and $g_2 \in S(A_2, A_2)$ such that $f = g_2 \circ g_1$. From its very definition, the map γ in Theorem 2.1 preserves the compositions. Hence γ restricts to a surjective map $\gamma : \mathrm{Eq}(\mathrm{D^b}(A_1), \mathrm{D^b}(A_2))^{G_\Delta} \to \mathrm{U}(A_1, A_2)$.

To prove the second claim in Proposition 3.2, consider the set

$$\mathrm{Ker}(A_1, A_2, G_\Delta) := \{(\mathcal{G}, \lambda) \in \mathrm{D}^{\mathrm{b}}_{G_\Delta}(A_1 \times A_2) : \Phi_{\mathcal{G}} \in \mathrm{Eq}(\mathrm{D^b}(A_1), \mathrm{D^b}(A_2))\}.$$

Since the group cohomology $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*)$ is trivial, Theorem 6 in [25] shows the existence of two maps

$$\begin{aligned} \psi_1 &: \mathrm{Ker}(A_1, A_2, G_\Delta) \longrightarrow \mathrm{Eq}(\mathrm{D^b}(A_1), \mathrm{D^b}(A_2))^{G_\Delta} \\ \psi_2 &: \mathrm{Ker}(A_1, A_2, G_\Delta) \longrightarrow \mathrm{Eq}(\mathrm{D}^{\mathrm{b}}_G(A_1), \mathrm{D}^{\mathrm{b}}_G(A_2)) \end{aligned}$$

such that, for any $(\mathcal{G}, \lambda) \in \mathrm{Ker}(A_1, A_2, G_\Delta)$, $\psi_1((\mathcal{G}, \lambda)) = \Phi_{\mathcal{G}}$ and $\psi_2((\mathcal{G}, \lambda)) = \Phi_{\mathcal{H}}$, where $\mathcal{H} := (\mathcal{G} \oplus (\iota, \mathrm{id})^*\mathcal{G}, \lambda')$ and λ' is the natural $(G \times G)$-linearization induced by λ.

We previously proved that for any $f \in S(A_1, A_2)$, there exists $\Phi_{\mathcal{E}_f} \in \mathrm{Eq}(\mathrm{D^b}(A_1), \mathrm{D^b}(A_2))^{G_\Delta}$ such that $\gamma(\Phi_{\mathcal{E}_f}) = f$. Form [25] it follows that ψ_1 is surjective and that the set $\mathrm{Ker}(A_1, A_2, G_\Delta)$ is non-empty if $\mathrm{Eq}(\mathrm{D^b}(A_1), \mathrm{D^b}(A_2))^{G_\Delta}$ is non-empty. Hence, there exists $\Psi_f \in \mathrm{Ker}(A_1, A_2, G_\Delta)$ such that $\psi_1(\Psi_f) = \Phi_{\mathcal{E}_f}$. The functor $\psi_2(\Psi_f)$ is in $\mathrm{Eq}(\mathrm{D}^{\mathrm{b}}_G(A_1), \mathrm{D}^{\mathrm{b}}_G(A_2))$. □

The special case $A_1 = A_2$ is also treated in [25].

3.2. **Proof of Theorem 3.1.** Let A_1 and A_2 be abelian varieties and suppose that $\mathrm{D^b}(A_1) \cong \mathrm{D^b}(A_2)$. Due to Theorem 2.1, the set $\mathrm{U}(A_1, A_2)$ is non-empty. Therefore, if $G = \mathbb{Z}/2\mathbb{Z}$ acts on A_1 and A_2 as prescribed at the beginning of Section 3.1, then Proposition 3.2 yields an equivalence $\Psi : \mathrm{D}^{\mathrm{b}}_G(A_1) \cong \mathrm{D}^{\mathrm{b}}_G(A_2)$.

Consider the stacks $[A_1/G]$ and $[A_2/G]$ (see [7] and [14]). For any $i \in \{1, 2\}$, let $\mathrm{D^b}([A_i/G])$ be the bounded derived category of the abelian category $\mathbf{Coh}([A_i/G])$ of coherent sheaves on $[A_i/G]$ (see [14]). Obviously $\mathrm{D^b}([A_i/G]) \cong \mathrm{D}^{\mathrm{b}}_G(A_i)$, because $\mathbf{Coh}([A_i/G]) \cong \mathbf{Coh}^G(A_i)$, for any $i \in \{1, 2\}$. This implies that Ψ can be rewritten as $\Phi : \mathrm{D^b}([A_1/G]) \cong \mathrm{D^b}([A_2/G])$. Due to [14], Φ is of Fourier-Mukai type (i.e. it is as in (2.1)). Hence, the first part of Theorem 3.1 is proved.

Assume that an equivalence $\Phi : \mathrm{D}^{\mathrm{b}}([A_1/G]) \cong \mathrm{D}^{\mathrm{b}}([A_2/G])$ is given. As before, the results in [14] imply that we can think of Φ as a Fourier-Mukai equivalence whose kernel is a $(G \times G)$-linearized complex $(\mathcal{E}, \lambda)$. Obviously, the inverse Φ^{-1} is a Fourier-Mukai equivalence as well. Suppose that its kernel is $(\mathcal{F}, \lambda')$. It is an easy exercise to show that the kernel of the identity functor $\mathrm{id} = \Phi \circ \Phi^{-1} : \mathrm{D}^{\mathrm{b}}_G(A_i) \to \mathrm{D}^{\mathrm{b}}_G(A_i)$ is the $(G \times G)$-linearized sheaf $(\mathcal{O}_\Delta \oplus (\iota, \mathrm{id})^* \mathcal{O}_\Delta, \mu)$, where μ is the natural linearization and $\Delta \hookrightarrow A_i \times A_i$ is the diagonal embedding.

Consider the functors $\Phi_{\mathcal{E}}$, $\Phi_{\mathcal{F}}$ and $\Phi_{\mathcal{O}_\Delta \oplus (\iota, \mathrm{id})^* \mathcal{O}_\Delta}$. Although they are no longer equivalences, they induce the commutative diagram

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}(A_1) \arrow[r, "\Phi_{\mathcal{E}}"] \arrow[d, "\mathrm{ch}(\ )"'] \arrow[rr, bend left, "\Phi_{\mathcal{O}_\Delta \oplus (\iota, \mathrm{id})^* \mathcal{O}_\Delta}"] & \mathrm{D}^{\mathrm{b}}(A_2) \arrow[r, "\Phi_{\mathcal{F}}"] \arrow[d, "\mathrm{ch}(\ )"'] & \mathrm{D}^{\mathrm{b}}(A_1) \arrow[d, "\mathrm{ch}(\ )"'] \\ H^*(A_1, \mathbb{Q}) \arrow[r, "\Phi^H_{\mathcal{E}}"] \arrow[rr, bend right, "\Phi^H_{\mathcal{O}_\Delta \oplus (\iota, \mathrm{id})^* \mathcal{O}_\Delta}"'] & H^*(A_2, \mathbb{Q}) \arrow[r, "\Phi^H_{\mathcal{F}}"] & H^*(A_1, \mathbb{Q}), \end{tikzcd} \tag{3.3}$$

where $\Phi^H_{\mathcal{E}} : H^*(A_1, \mathbb{Q}) \to H^*(A_2, \mathbb{Q})$ is such that $\Phi^H_{\mathcal{E}}(a) = p_{2*}(\mathrm{ch}(\mathcal{E}) \cdot p_1^*(a))$ and $p_i : A_1 \times A_2 \to A_i$ is the projection. Take analogous definitions for $\Phi^H_{\mathcal{F}}$ and $\Phi^H_{\mathcal{O}_\Delta \oplus (\iota, \mathrm{id})^* \mathcal{O}_\Delta}$.

Observe that

$$\mathrm{ch}(\mathcal{O}_\Delta \oplus (\iota, \mathrm{id})^* \mathcal{O}_\Delta) = 2\mathrm{ch}(\mathcal{O}_\Delta).$$

Since $\Phi^H_{\mathcal{O}_\Delta} = \mathrm{id}$, from (3.3) we deduce that $\Phi^H_{\mathcal{F}} \circ \Phi^H_{\mathcal{E}} = 2\mathrm{id}$. Hence $\Phi^H_{\mathcal{E}}$ is injective. Exchanging the rôles of $\Phi^{\mathcal{E}}$ and $\Phi_{\mathcal{F}}$ in (3.3), we see that $\Phi^H_{\mathcal{E}}$ is an isomorphism of $\mathbb{Q}$-vector spaces. In particular, $\dim(A_1) = \dim(A_2) = n$.

The fact that the Hodge structures defined in (2.4) are preserved follows from the standard argument for Fourier-Mukai equivalences (see [11], Proposition 5.38). Indeed, one just needs to observe that $\mathrm{ch}(\mathcal{E}) \in \widetilde{H}^{2n,2n}(A_1 \times A_2)$. This concludes the proof of Theorem 3.1.

Remark 3.3. Of course, in general, $\Phi^H_{\mathcal{E}}$ does not preserve the Mukai pairing naturally defined on $H^*(A_i, \mathbb{Q})$ by means of the cup product ([11], Section 5). Indeed, it is easy to see that the Mukai pairing is preserved up to a factor 2.

3.3. **Geometric applications.** Assume that A_1 and A_2 are abelian surfaces. The main result in [2] produces an equivalence $\Psi_i : \mathrm{D}^{\mathrm{b}}([A_i/\langle\iota\rangle]) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A_i))$, for any $i \in \{1, 2\}$. Thus, if $\Upsilon : \mathrm{D}^{\mathrm{b}}(A_1) \cong \mathrm{D}^{\mathrm{b}}(A_2)$ is a Fourier-Mukai equivalence, we immediately get a second Fourier-Mukai equivalence

$$\Phi : \mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A_1)) \xrightarrow{\Psi_1^{-1}} \mathrm{D}^{\mathrm{b}}([A_1/\langle\iota\rangle]) \cong \mathrm{D}^{\mathrm{b}}([A_2/\langle\iota\rangle]) \xrightarrow{\Psi_2} \mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A_2)), \tag{3.4}$$

where the middle equivalence is produced by Theorem 3.1 and the kernel of Φ can be easily computed using [2]. This leads to a different and explicit proof of the "only if" implication in (A) without using the lattice theoretical description of the transcendental lattices of an abelian surface and of the associated Kummer surface.

Let us discuss a second geometric application. Assume that A is an abelian surface. We indicate with $\mathrm{K}^n(A)$ the *n-th generalized Kummer variety of A*. Recalling the construction in [1], we see that $\mathrm{K}^n(A)$ is the fiber over 0 with respect to the map Ψ which is the composition of the morphisms in the following diagram:

$$\Psi : \mathrm{Hilb}^{n+1}(A) \xrightarrow{\rho} \mathrm{Sym}^{n+1}(A) \xrightarrow{\sigma} A,$$

where ρ is the Hilbert-Chow morphism and $\sigma(a_1,\ldots,a_{n+1}) = a_1+\ldots+a_{n+1}$. It is easy to see that $\mathrm{K}^n(A)$ is smooth and that $\mathrm{K}^1(A) = \mathrm{Km}(A)$. Furthermore, in [1] Beauville proved that these varieties are examples of irreducible symplectic manifolds.

Proposition 3.4. *Let A be an abelian surface and let $n \geq 2$ be an integer. The number of generalized Kummer varieties $\mathrm{K}^n(B)$ birational to $\mathrm{K}^n(A)$ is finite up to isomorphisms. Moreover if $\mathrm{K}^n(A)$ and $\mathrm{K}^n(B)$ are birational, then $\mathrm{Km}(A) \cong \mathrm{Km}(B)$ and A and B are isogenous.*

Proof. Let A_1 and A_2 be abelian surfaces and let φ be a birational morphism between $\mathrm{K}^n(A_1)$ and $\mathrm{K}^n(A_2)$. Obviously, φ induces an isomorphism $g : H^2(\mathrm{K}^n(A_1),\mathbb{Z}) \cong H^2(\mathrm{K}^n(A_2),\mathbb{Z})$. Furthermore, there exists an isometry of lattices $H^2(\mathrm{K}^n(A_i),\mathbb{Z}) \cong H^2(A_i,\mathbb{Z}) \oplus \mathbb{Z}[E_i]$, where E_i is the restriction to $\mathrm{K}^n(A)$ of the exceptional locus of $\mathrm{Hilb}^{n+1}(A_i)$. The left hand side of the isomorphism is endowed with the Beauville-Bogomolov form while the quadratic form on $H^2(A_i,\mathbb{Z})$ is the cup-product (see [1] and Lemma 4.10 and Proposition 4.11 in [28]).

Since E_1 and E_2 are algebraic, g yields an isomorphism $T(A_1) \cong T(A_2)$. Using Theorem 2.2, we get an equivalence $\mathrm{D^b}(A_1) \cong \mathrm{D^b}(A_2)$. To prove that A_1 is isogenous to A_2 observe that if $\mathrm{D^b}(A_1) \cong \mathrm{D^b}(A_2)$, then $A_1 \times \widehat{A_1} \cong A_2 \times \widehat{A_2}$ (Theorem 2.1). Hence $A_1 \times A_1$ and $A_2 \times A_2$ are isogenous and A_1 and A_2 are isogenous as well. On the other hand, as there are only finitely many isomorphism classes of abelian surfaces A such that $\mathrm{D^b}(A) \cong \mathrm{D^b}(A_1)$ (Proposition 5.3 in [3]), the number of generalized Kummer varieties $\mathrm{K}^n(A_2)$ birational to $\mathrm{K}^n(A_1)$ is finite up to isomorphism. Moreover, Theorem 3.1 yields an equivalence $\mathrm{D^b}([A_1/\langle\iota\rangle]) \cong \mathrm{D^b}([A_2/\langle\iota\rangle])$. Due to (3.4) and Theorem 3.1, $\mathrm{D^b}(\mathrm{Km}(A)) \cong \mathrm{D^b}(\mathrm{Km}(B))$ and then $\mathrm{Km}(A) \cong \mathrm{Km}(B)$ (see [18]). □

An analogous result for Hilbert schemes of points on K3 surfaces was proved in [25].

Remark 3.5. Observe that, in general, if A and B are abelian surfaces such that $\mathrm{Km}(A) \cong \mathrm{Km}(B)$, then $\mathrm{K}^n(A)$ and $\mathrm{K}^n(B)$ are not necessarily birational. Indeed, consider an abelian surface A such that $A \ncong \widehat{A}$ and $\mathrm{NS}(A) = \langle H\rangle$ with $H^2 = 6$. Obviously $\mathrm{D^b}(A) \cong \mathrm{D^b}(\widehat{A})$. Due to Theorem 3.1, $\mathrm{D^b}(\mathrm{Km}(A)) \cong \mathrm{D^b}(\mathrm{Km}(\widehat{A}))$ and $\mathrm{Km}(A) \cong \mathrm{Km}(\widehat{A})$ ([18]). On the other hand, Namikawa ([19], Section 5) proved that $\mathrm{K}^2(A)$ and $\mathrm{K}^2(\widehat{A})$ are not birational.

Furthermore, Example 4.3 yields very explicit examples of isogenous abelian surfaces A and B which are not Fourier-Mukai partners. In particular $\mathrm{Km}(A) \ncong \mathrm{Km}(B)$ and $\mathrm{K}^n(A)$ is not birational to $\mathrm{K}^n(B)$ for any positive integer n.

4. Derived categories of twisted Kummer surfaces

In this section we prove the second main result of this paper which relates the existence of equivalences between the twisted derived categories of two Kummer surfaces and the existence of Hodge isometries between the generalized transcendental lattices of the corresponding abelian surfaces. We also discuss a geometric example and an application to the problem of determining the number of possible twisted Kummer structures on a twisted K3 surface.

4.1. **Brauer groups and twisted sheaves.** Recall that the *Brauer group* $\mathrm{Br}(X)$ of a smooth projective variety X is the torsion part of $H^2(X,\mathcal{O}_X^*)$.

Assume that X is either a K3 or an abelian surface. It is known that any $\alpha \in \mathrm{Br}(X)$ is determined (not uniquely) by some $B \in H^2(X,\mathbb{Q})$ (see Chapter 1 of [4] for the case of K3 surfaces and use a similar argument to deal with abelian surfaces). This follows from the fact that $H^2(X,\mathbb{Z})$ is unimodular and $H_1(X,\mathbb{Z})$ is torsion free. More precisely, we deduce the existence of natural isomorphisms $\mathrm{Br}(X) \cong T(X)^\vee \otimes \mathbb{Q}/\mathbb{Z} \cong \mathrm{Hom}(T(X),\mathbb{Q}/\mathbb{Z})$ and for any $t \in T(X)$, $\alpha : t \longmapsto t\cdot B$ (mod $\mathbb{Z}$), where "$\cdot$" is the cup-product. From this we get a surjective map

$$\kappa_X : H^2(X,\mathbb{Q}) \longrightarrow \mathrm{Br}(X).$$

Lemma 4.1. *If A is an abelian surface, there exists an isomorphism $\Theta_A : \mathrm{Br}(A) \to \mathrm{Br}(\mathrm{Km}(A))$.*

Proof. The K3 surface $\mathrm{Km}(A)$ is the crepant resolution of $\mathrm{K}(A) = A/\langle\iota\rangle$. Hence there exists a rational map $\pi : A \dashrightarrow \mathrm{Km}(A)$. Furthermore, as it was observed in Remark 2 of [20] (see also Section 4 in [15]), the homomorphism π_* induces a Hodge isometry

$$\pi_* : T(A)(2) \longrightarrow T(\mathrm{Km}(A)). \tag{4.1}$$

(Recall that, given a lattice L with quadratic form b_L, the lattice $L(m)$, with $m \in \mathbb{Z}$, coincides with L as a group but its quadratic form $b_{L(m)}$ is such that $b_{L(m)}(l_1, l_2) = mb_L(l_1, l_2)$, for any $l_1, l_2 \in L$.)

In particular, we get a natural morphism $\Xi : H^2(A, \mathbb{Q}) \to T(\mathrm{Km}(A)) \otimes \mathbb{Q}$ defined by

$$\Xi : B \longmapsto \frac{\pi_*(p(B))}{2}, \tag{4.2}$$

where $p : H^2(A, \mathbb{Q}) \to T(A) \otimes \mathbb{Q}$ is the orthogonal projection. This yields a morphism $\Theta_A : \mathrm{Br}(A) \longrightarrow \mathrm{Br}(\mathrm{Km}(A))$ of Brauer groups defined by the commutative diagram

$$\begin{tikzcd} H^2(A,\mathbb{Q}) \arrow[r, "\Xi"] \arrow[d, "\kappa_A"'] & T(\mathrm{Km}(A)) \otimes \mathbb{Q} \arrow[d, "\kappa_{\mathrm{Km}(A)}"] \\ \mathrm{Br}(A) \arrow[r, "\Theta_A"] & \mathrm{Br}(\mathrm{Km}(A)). \end{tikzcd} \tag{4.3}$$

Observe that Θ_A is well-defined because, obviously, the restriction $\kappa_{\mathrm{Km}(A)}|_{T(\mathrm{Km}(A))\otimes\mathbb{Q}}$ is still surjective. An easy check then shows that Θ_A is an isomorphism. $\square$

Any $\alpha \in \mathrm{Br}(X)$ can be represented by a Čech 2-cocycle $\{\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$ on an analytic open cover $X = \bigcup_{i\in I} U_i$. An *α-twisted coherent sheaf* E is a collection of pairs $(\{E_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$ where E_i is a coherent sheaf on the open subset U_i and $\varphi_{ij} : E_j|_{U_i\cap U_j} \to E_i|_{U_i\cap U_j}$ is an isomorphism such that $\varphi_{ii} = \mathrm{id}$, $\varphi_{ji} = \varphi_{ij}^{-1}$ and $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$. Given $\alpha \in \mathrm{Br}(X)$, we indicate with $\mathbf{Coh}(X, \alpha)$ the abelian category of α-twisted coherent sheaves on X while $\mathrm{D^b}(X, \alpha) := \mathrm{D^b}(\mathbf{Coh}(X, \alpha))$ is the bounded derived category of $\mathbf{Coh}(X, \alpha)$ (see [4] and [13] for details).

If X and Y are smooth projective varieties and $\alpha \in \mathrm{Br}(X)$ while $\beta \in \mathrm{Br}(Y)$, an equivalence $\Phi : \mathrm{D^b}(X, \alpha) \to \mathrm{D^b}(Y, \beta)$ is a *twisted Fourier-Mukai equivalence* if and only if it satisfies an equation of type (2.1) whose kernel $\mathcal{E}$ is in $\mathrm{D^b}(X \times Y, \alpha^{-1} \boxtimes \beta)$.

4.2. **The second main result.** As in [12], a *twisted variety* is a pair (X, α), where X is a smooth projective variety and $\alpha \in \mathrm{Br}(X)$. An isomorphism $f : (X, \alpha) \to (Y, \beta)$ of the twisted varieties (X, α) and (Y, β) is an isomorphism $f : X \to Y$ such that $f^*\beta = \alpha$. In [12] two equivalence relations were introduced:

Definition 4.2. Let (X_1, α_1) and (X_2, α_2) be twisted K3 or abelian surfaces.

(i) They are *D-equivalent* if there exists a twisted Fourier-Mukai equivalence
$$\Phi : \mathrm{D^b}(X_1, \alpha_1) \to \mathrm{D^b}(X_2, \alpha_2).$$

(ii) They are *T-equivalent* if there exist $B_i \in H^2(X_i, \mathbb{Q})$ such that $\alpha_i = \kappa_{A_i}(B_i)$ and a Hodge isometry
$$\varphi : T(X_1, B_1) \to T(X_2, B_2).$$

We use Lemma 4.1 to prove the main result of this section:

Theorem 4.3. *Let A_1 and A_2 be abelian surfaces. Then the following two conditions are equivalent:*

(i) *there exist $\alpha_1 \in \mathrm{Br}(\mathrm{Km}(A_1))$ and $\alpha_2 \in \mathrm{Br}(\mathrm{Km}(A_2))$ such that $(\mathrm{Km}(A_1), \alpha_1)$ and $(\mathrm{Km}(A_2), \alpha_2)$ are D-equivalent;*

(ii) *there exist $\beta_1 \in \mathrm{Br}(A_1)$ and $\beta_2 \in \mathrm{Br}(A_2)$ such that (A_1, β_1) and (A_2, β_2) are T-equivalent.*

Furthermore, if one of these two equivalent conditions holds true, then A_1 and A_2 are isogenous.

Proof. First of all, observe that, if X is either a K3 or an abelian surface and $\alpha \in \mathrm{Br}(X)$, the lattice $T(X,\alpha) := \ker(\alpha)$ inherits from $T(X)$ a weight-two Hodge structure. Secondly, if $\Theta_{A_i} : \mathrm{Br}(A_i) \to \mathrm{Br}(\mathrm{Km}(A_i))$ is the isomorphism in Lemma 4.1(ii), the isometry $\pi_{i*} : T(A_i)(2) \to T(\mathrm{Km}(A_i))$ defined in (4.1) yields a Hodge isometry $f_i : T(A_i,\alpha)(2) \longrightarrow T(\mathrm{Km}(A_i), \Theta_{A_i}(\alpha))$, for any $\alpha \in \mathrm{Br}(A_i)$ and $i \in \{1,2\}$.

Proposition 4.7 in [10], originally proved for K3 surfaces, works perfectly in the case of abelian surfaces as well. Therefore if X is either a K3 or an abelian surface, $\alpha \in \mathrm{Br}(X)$ and $B \in H^2(X,\mathbb{Q})$ is such that $\alpha = \kappa_X(B)$, then there exists a Hodge isometry

$$\begin{array}{rcl} \exp(B) : T(X,\alpha)(k) & \longrightarrow & T(X,B)(k) \\ \gamma & \longmapsto & (\gamma, B \wedge \gamma), \end{array} \tag{4.4}$$

for any $k \in \{1,2\}$. Given $B_i \in H^2(A_i,\mathbb{Q})$, let $\widetilde{B}_i \in H^2(\mathrm{Km}(A_i),\mathbb{Q})$ be such that

$$\Theta_{A_i}(\kappa_{A_i}(B_i)) = \kappa_{\mathrm{Km}(A_i)}(\widetilde{B}_i).$$

Define $\widetilde{\alpha}_i := \Theta_{A_i}(\kappa_{A_i}(B_i))$. If $g : T(A_1,B_1) \to T(A_2,B_2)$ is a Hodge isometry, the diagram

$$\begin{tikzcd}
T(A_1,B_1)(2) \arrow[rrr, "g"] \arrow[dddd] \arrow[dr, "\exp(-B_1)"'] & & & T(A_2,B_2)(2) \arrow[dddd] \arrow[dl, "\exp(-B_2)"] \\
& T(A_1,\kappa_{A_1}(B_1))(2) \arrow[r] \arrow[d, "f_1"'] & T(A_2,\kappa_{A_2}(B_2))(2) \arrow[d, "f_2"] & \\
& T(\mathrm{Km}(A_1),\widetilde{\alpha}_1) \arrow[r] & T(\mathrm{Km}(A_2),\widetilde{\alpha}_2) & \\
& & & \\
T(\mathrm{Km}(A_1),\widetilde{B}_1) \arrow[rrr, "f"] \arrow[uur, "\exp(\widetilde{B}_1)"] & & & T(\mathrm{Km}(A_2),\widetilde{B}_2) \arrow[uul, "\exp(\widetilde{B}_2)"']
\end{tikzcd}$$

commutes and yields a Hodge isometry $f : T(\mathrm{Km}(A_1),\widetilde{B}_1) \to T(\mathrm{Km}(A_2),\widetilde{B}_2)$. Conversely, since Θ_i is an isomorphism (Lemma 4.1), the same diagram and remarks show that any Hodge isometry between the generalized transcendental lattices of $\mathrm{Km}(A_1)$ and $\mathrm{Km}(A_2)$ determined by some $\widetilde{B}_i \in H^2(\mathrm{Km}(A_i),\mathbb{Q})$ induces a Hodge isometry of the generalized transcendental lattices of A_1 and A_2 corresponding to $B_i \in H^2(A_i,\mathbb{Q})$ such that $\kappa_{A_i}(B_i) = \Theta_{A_i}^{-1}(\kappa_{\mathrm{Km}(A_i)}(\widetilde{B}_i)) \in \mathrm{Br}(A_i)$.

Since the Picard number of $\mathrm{Km}(A_i)$ is greater than 11, the equivalence between item (i) and item (ii) of Theorem 4.3 follows from Theorem 0.4 in [12]. Indeed such a result proves that, for any $B_i \in H^2(\mathrm{Km}(A_i),\mathbb{Q})$, there exists a twisted Fourier-Mukai equivalence

$$\mathrm{D^b}(\mathrm{Km}(A_1),\kappa_{\mathrm{Km}(A_1)}(B_1)) \cong \mathrm{D^b}(\mathrm{Km}(A_2),\kappa_{\mathrm{Km}(A_2)}(B_2))$$

if and only if there exists a Hodge isometry $T(\mathrm{Km}(A_1),B_1) \cong T(\mathrm{Km}(A_2),B_2)$.

Due to what we have just proved, any twisted Fourier-Mukai equivalence $\mathrm{D^b}(\mathrm{Km}(A_1),\alpha_1) \cong \mathrm{D^b}(\mathrm{Km}(A_2),\alpha_2)$ induces a Hodge isometry $T(\mathrm{Km}(A_1)) \otimes \mathbb{Q} \cong T(\mathrm{Km}(A_2)) \otimes \mathbb{Q}$ which extends to a Hodge isometry $H^2(A_1,\mathbb{Q}) \cong H^2(A_2,\mathbb{Q})$. Consider the Kuga-Satake varieties $\mathrm{KS}(A_1)$ and $\mathrm{KS}(A_2)$ of A_1 and A_2 (see Section 4 in [16] for the definition). Theorem 4.3 in [16] shows that, for any $i \in \{1,2\}$,

$$\mathrm{KS}(A_i) \sim (A_i \times \widehat{A_i})^4,$$

where "$\sim$" denotes an isogeny of abelian varieties. Since there is a Hodge isometry $H^2(A_1,\mathbb{Q}) \cong H^2(A_2,\mathbb{Q})$, by construction, $\mathrm{KS}(A_1) \sim \mathrm{KS}(A_2)$ and then

$$(A_1 \times \widehat{A_1})^4 \sim (A_2 \times \widehat{A_2})^4.$$

In particular, A_1 and A_2 are isogenous. $\square$

Corollary 4.4. (i) $(\mathrm{Km}(A_1),1)$ *is D-equivalent to* $(\mathrm{Km}(A_2),1)$ *if and only if* $(A_1,1)$ *and* $(A_2,1)$ *are T-equivalent.*

(ii) *If* (A_1,α_1) *and* (A_2,α_2) *are D-equivalent twisted abelian surfaces, then* $(\mathrm{Km}(A_1),\Theta_{A_1}(\alpha_1))$ *and* $(\mathrm{Km}(A_2),\Theta_{A_2}(\alpha_2))$ *are D-equivalent.*

Proof. Due to the isomorphism in Lemma 4.1, (i) follows trivially from Theorem 4.3. The machinery in [12] applied to the case of abelian surfaces shows that if (A_1,α_1) and (A_2,α_2) are D-equivalent, then they are T-equivalent as well. Then use Theorem 4.3. □

Notice that part (i) of Corollary 4.4 is exactly (B) in the introduction.

Remark 4.5. (i) Due to Proposition 8.1 in [12], if $\alpha_j\in\mathrm{Br}(\mathrm{Km}(A_j))$ is non-trivial for any $j\in\{1,2\}$, then the existence of an equivalence $\mathrm{D^b}(\mathrm{Km}(A_1),\alpha_1)\cong\mathrm{D^b}(\mathrm{Km}(A_2),\alpha_2)$ does not imply that $\mathrm{Km}(A_1)\cong\mathrm{Km}(A_2)$ (see also Example 4.3). This is one of the main differences with the untwisted case treated by Hosono, Lian, Oguiso and Yau in [9] (see (A) and (C) in the introduction).

(ii) As suggested by Corollary 4.4, we would expect (ii) in Theorem 4.3 to be equivalent to the existence of a twisted Fourier-Mukai equivalence $\mathrm{D^b}(A_1,\beta_1)\cong\mathrm{D^b}(A_2,\beta_2)$, where $\beta_i\in\mathrm{Br}(A_i)$. This would lead to a twisted version of (A). Actually this is not the case. Indeed, since the period map is surjective for abelian surfaces ([26]), one can produce a counterexample to this expectation by adapting Example 4.11 in [12].

(iii) Let A_1 and A_2 be two abelian surfaces with $\mathrm{NS}(A_1)=\langle H_1\rangle$ and $\mathrm{NS}(A_2)=\langle H_2\rangle$. If there exist $\alpha_1\in\mathrm{Br}(\mathrm{Km}(A_1))$ and $\alpha_2\in\mathrm{Br}(\mathrm{Km}(A_2))$ such that $\mathrm{D^b}(\mathrm{Km}(A_1),\alpha_1)\cong\mathrm{D^b}(\mathrm{Km}(A_2),\alpha_2)$ then H_1^2/H_2^2 is a square in $\mathbb{Q}$. Indeed, by Theorem 4.3 (and by Section 7 in [12]), if $\mathrm{D^b}(\mathrm{Km}(A_1),\alpha_1)\cong\mathrm{D^b}(\mathrm{Km}(A_2),\alpha_2)$ then there exists an isogeny $\varphi:A_1\to A_2$ inducing a Hodge isometry $\varphi^*:H^2(A_2,\mathbb{Q})\to H^2(A_1,\mathbb{Q})$ such that $\varphi^*(H_2)=qH_1$, for some $q\in\mathbb{Q}$. In particular $H_2^2=q^2H_1^2$.

4.3. **An explicit example.** In this example, we use Theorem 4.3 to establish a connection between the twisted derived categories of some nice Kummer surfaces with Picard number 2. Recall that the lattices U and $U(n)$ are the free abelian group $\mathbb{Z}\oplus\mathbb{Z}$ endowed respectively with the quadratic forms represented by the matrices

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0&n\\n&0\end{pmatrix}.$$

Let A be an abelian surface such that $\mathrm{NS}(A)\cong U(n)$, for some positive integer n. We first show that there exist two elliptic curves E and F and a subgroup $C_n\cong\mathbb{Z}/n\mathbb{Z}$ of $E\times F$ such that either $A\cong(E\times F)/C_n$ or $\widehat{A}\cong(E\times F)/C_n$.

To see this, let us first observe that, since $\mathrm{NS}(A)\cong U(n)$, the transcendental lattice $T(A)$ is isometric to $U(n)\oplus U$. Indeed for any abelian surface A, $H^2(A,\mathbb{Z})$, endowed with the cup-product, is isometric to the lattice $U\oplus U\oplus U$ (see [15] for more details).

We choose a basis $\langle e_1,e_2,f_1,f_2\rangle=U\oplus U(n)\hookrightarrow U^3$, an isometry $\varphi:H^2(A,\mathbb{Z})\to U^3$ and $c\in\mathbb{C}$ such that

$$\varphi(c\sigma_A)=e_1-n\omega_1\omega_2e_2+\omega_1f_1+\omega_2f_2, \tag{4.5}$$

where $H^{2,0}(A)=\langle\sigma_A\rangle$. We define in $\mathbb{C}$ the lattices $\Gamma_1=\mathbb{Z}+\omega_1\mathbb{Z}$ and $\Gamma_2=\mathbb{Z}+\omega_2\mathbb{Z}$ and the elliptic curves $E:=\mathbb{C}/\Gamma_1$ and $F:=\mathbb{C}/\Gamma_2$. Notice that, since $T(A)\cong U(n)\oplus U$ and $\sigma_A^2=0$, 1, ω_1, ω_2 and $\omega_1\omega_2$ are linearly independent over $\mathbb{Q}$. So, in particular, E and F are not isogenous. If $H_1(E\times F,\mathbb{Z})=\langle\gamma_1,\gamma_2,\delta_1,\delta_2\rangle$, then we consider the subgroup C_n of $E\times F$ such that

$$H_1((E\times F)/C_n,\mathbb{Z})=\left\langle\frac{\gamma_1+\delta_1}{n},\gamma_2,\delta_1,\delta_2\right\rangle.$$

Let $S:=(E\times F)/C_n$. In terms of the dual bases of the bases of $H_1(E\times F,\mathbb{Z})$ and $H_1(S,\mathbb{Z})$ just described, we write $H^1(S,\mathbb{Z})=\langle\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}w_1,\mathrm{d}w_2\rangle$ and $H^1(E\times F,\mathbb{Z})=\langle\mathrm{d}x_1,\mathrm{d}x_2,\mathrm{d}y_1,\mathrm{d}y_2\rangle$. If

$\pi : E \times F \to S$ is the natural surjection, the map $\theta := \pi^* : H^1(S,\mathbb{Z}) \to H^1(E \times F,\mathbb{Z})$ is such that:

$$\begin{array}{ll} \theta(\mathrm{d}z_1) = n\mathrm{d}x_1, & \theta(\mathrm{d}w_1) = -\mathrm{d}x_1 + \mathrm{d}y_1, \\ \theta(\mathrm{d}z_2) = \mathrm{d}x_2, & \theta(\mathrm{d}w_2) = \mathrm{d}y_2. \end{array} \tag{4.6}$$

Observe that $\mathrm{NS}(E \times F) = \langle \mathrm{d}x_1 \wedge \mathrm{d}x_2, \mathrm{d}y_1 \wedge \mathrm{d}y_2 \rangle$. Furthermore, due to the properties in (4.6) which characterize the morphism $\overset{2}{\wedge}\,\theta : H^2(S,\mathbb{Z}) \to H^2(E \times F,\mathbb{Z})$ and due to the fact that $\overset{2}{\wedge}\,\theta$ preserves the Hodge structures on $H^2(S,\mathbb{Z})$ and $H^2(E \times F,\mathbb{Z})$,

$$\begin{array}{rcl} \mathrm{NS}(S) & = & \langle \mathrm{d}z_1 \wedge \mathrm{d}z_2, n\mathrm{d}w_1 \wedge \mathrm{d}w_2 + \mathrm{d}z_1 \wedge \mathrm{d}w_2 \rangle, \\ T(S) & = & \langle \mathrm{d}z_1 \wedge \mathrm{d}w_1, \mathrm{d}z_2 \wedge \mathrm{d}w_2, \mathrm{d}z_1 \wedge \mathrm{d}w_2, -n\mathrm{d}w_1 \wedge \mathrm{d}z_2 + \mathrm{d}z_1 \wedge \mathrm{d}w_2 \rangle. \end{array}$$

In particular, $\mathrm{NS}(S) \cong U(n)$ and $T(S) \cong U \oplus U(n)$.

Consider the two cohomology classes

$$\begin{array}{rcl} \sigma_{E\times F} & = & \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \omega_2 \mathrm{d}x_1 \wedge \mathrm{d}y_2 + \omega_1 \mathrm{d}x_2 \wedge \mathrm{d}y_1 + \omega_1\omega_2 \mathrm{d}x_2 \wedge \mathrm{d}y_2; \\ \sigma_S & = & \mathrm{d}z_1 \wedge \mathrm{d}w_1 + \omega_2 \mathrm{d}z_1 \wedge \mathrm{d}w_2 + \omega_1(n\mathrm{d}w_1 \wedge \mathrm{d}z_2 - \mathrm{d}z_1 \wedge \mathrm{d}w_2) + n\omega_1\omega_2 \mathrm{d}z_2 \wedge \mathrm{d}w_2. \end{array}$$

Obviously, $\sigma_{E\times F} \in T(E \times F) \otimes \mathbb{C}$ and $\sigma_S \in T(S) \otimes \mathbb{C}$. Since $\langle \sigma_{E\times F} \rangle = H^{2,0}(E \times F)$ and since an easy calculation shows that $\overset{2}{\wedge}\,\theta(\sigma_S) = n\sigma_{E\times F}$, $\langle \sigma_S \rangle = H^{2,0}(S)$. This implies that, due to (4.5), there exists an isometry $\eta : H^2(S,\mathbb{Z}) \to U^3$ such that $\eta^{-1} \circ \varphi : H^2(A,\mathbb{Z}) \to H^2(S,\mathbb{Z})$ is a Hodge isometry (see [26]). The Torelli Theorem for abelian surfaces shows that either $A \cong (E \times F)/C_n$ or $\widehat{A} \cong (E \times F)/C_n$.

Observe that, since $\mathrm{NS}(A) \cong U(n)$, the abelian surface A is principally polarized if and only if $n = 1$. This means that, if $n \neq 1$, $(E \times F)/C_n$ and its dual are not isomorphic. Furthermore, A and $E \times F$ are isogenous but $T(A) \not\cong T(E \times F)$. Therefore, due to Theorem 2.2, $\mathrm{D}^{\mathrm{b}}(A) \not\cong \mathrm{D}^{\mathrm{b}}(E \times F)$. This proves that there are isogenous abelian surfaces which are not (untwisted) Fourier-Mukai partners (see Remark 3.5).

Choose the standard basis $\{g_1, g_2, k_1, k_2\}$ for $U \oplus U$. Due to the explicit description of $T(A)$ that we have previously given, it is straightforward to see that there exists an inclusion $i_1 : T(A) \to U \oplus U$ where $i_1(e_j) = g_j$ ($j \in \{1,2\}$), $i_1(f_1) = nk_1$ and $i_1(f_2) = nk_2$. Let $\sigma := i(\sigma_A) \in U^2 \otimes \mathbb{C}$. Due to (4.5), we can write $\sigma = g_1 - n\omega_1\omega_2 g_2 + n\omega_1 h_1 + \omega_2 h_2$.

Consider in $\mathbb{C}$ the lattice $\Gamma_3 = \mathbb{Z} + n\omega_1\mathbb{Z}$ and the elliptic curve $E_1 := \mathbb{C}/\Gamma_3$. Of course, E and E_1 are isogenous. Reasoning as before and using the surjectivity of the period map and the Torelli Theorem for abelian surfaces ([26]), we get an isometry $\varphi_1 : T(E_1 \times F) \to U^2$ fitting in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[dr] & & T(E_1 \times F) \arrow[dd, "\varphi_1"] \arrow[dr, "\alpha"] & & 0 \\
& T(A) \arrow[ur, "i"] \arrow[dr, "i_1"'] & & \mathbb{Z}/n\mathbb{Z} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & U \oplus U \arrow[ur, "\alpha_1"'] & & 0.
\end{tikzcd}$$

Of course, i preserves the Hodge structures and $\alpha \in \mathrm{Br}(E_1 \times F)$. Proposition 4.7 in [10] yields $B \in H^2(E \times F, \mathbb{Q})$ such that $(A, 0)$ and $(E \times F, B)$ are T-equivalent. By Theorem 4.3, there exist $\beta \in \mathrm{Br}(\mathrm{Km}(E_1 \times F))$ of order n and a twisted Fourier-Mukai equivalence $\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A)) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{Km}(E_1 \times F), \beta)$.

4.4. **The number of twisted Kummer structures.** As an easy corollary of Lemma 4.1, we get a surjective map

$$\Psi : \{\text{Twisted abelian surfaces}\}/\text{isom} \longrightarrow \{\text{Twisted Kummer surfaces}\}/\text{isom}$$

which sends the isomorphism class $[(A,\alpha)]$ to the isomorphism class $[(\mathrm{Km}(A),\Theta_A(\alpha))]$. The main result in [9] proves that the preimage of $[(\mathrm{Km}(A),1)]$ is finite, for any abelian surface A and $1\in\mathrm{Br}(A)$ the trivial class (see Theorem 0.1 in [9]). On the other hand [9] shows that the cardinality of the preimages of Ψ can be arbitrarily large. This answers an old question by Shioda. Namely, there can be many non-isomorphic (untwisted) abelian surfaces giving rise to isomorphic (untwisted) Kummer surfaces (a partial result is also contained in [8]). This is usually rephrased saying that on a K3 surface one can put many non-isomorphic (untwisted) Kummer structures.

The picture in [9] can be completely generalized to the twisted case.

Proposition 4.6. (i) *For any twisted Kummer surface* $(\mathrm{Km}(A),\alpha)$*, the preimage* $\Psi^{-1}([(\mathrm{Km}(A),\alpha)])$ *is finite.*

(ii) *For positive integers N and n, there exists a twisted Kummer surface* $(\mathrm{Km}(A),\alpha)$ *with α of order n in* $\mathrm{Br}(\mathrm{Km}(A))$ *and such that* $|\Psi^{-1}([(\mathrm{Km}(A),\alpha)])|\geq N$.

Proof. Suppose that $\Psi([(A_1,\alpha_1)])=\Psi([(A_2,\alpha_2)])=[(\mathrm{Km}(A),\alpha)]$, i.e. suppose that there exists an isomorphism $f:\mathrm{Km}(A)\cong\mathrm{Km}(A_i)$ such that $f^*\Theta_{A_i}(\alpha_i)=\alpha$. In particular,

$$\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A_1),\Theta_{A_1}(\alpha_1))\cong\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A_2),\Theta_{A_2}(\alpha_2)).$$

Due to Theorem 4.3, the proof of (i) amounts to show that, up to isomorphisms, there are finitely many T-equivalent twisted abelian surfaces (A',β) such that $\Psi([(A',\beta)])=[(\mathrm{Km}(A),\alpha)]$. Since, up to isomorphisms, there are just finitely many abelian surfaces A' with $\mathrm{D}^{\mathrm{b}}(A')\cong\mathrm{D}^{\mathrm{b}}(A)$ (Proposition 5.3 in [3]), we can just fix a Fourier-Mukai partner A' of A and show that, up to isomorphisms, there exists a finite number of $\beta'\in\mathrm{Br}(A')$ such that (A',β) and (A',β') are T-equivalent. But this is the content of Proposition 3.4 in [12] for the case of abelian surfaces.

Applying the results in [22] and [27] to abelian surfaces, we see that, for any positive integer N, there exist N non-isomorphic abelian surfaces $A_1,\ldots,A_N$ such that $\mathrm{D}^{\mathrm{b}}(A_i)\cong\mathrm{D}^{\mathrm{b}}(A_j)$ $(i,j\in\{1,\ldots,N\})$. Due to Theorem 4.3, for any $i\in\{2,\ldots,N\}$, there is a Hodge isometry $g_i:T(A_1)\to T(A_i)$. Take $B_1\in T(A_1)\otimes\mathbb{Q}$ such that $\alpha_1:=\kappa_{A_1}(B_1)$ and $\Theta_{A_1}(\alpha_1)$ are not trivial in $\mathrm{Br}(A_1)$ and $\mathrm{Br}(\mathrm{Km}(A_1))$ respectively. We can also choose α_1 such that the order of $\Theta_{A_1}(\alpha_1)$ is n in $\mathrm{Br}(\mathrm{Km}(A_1))$. Then, for any $i\in\{2,\ldots,N\}$, define $\alpha_i:=\kappa_{A_i}(g_i(B_1))$. Obviously, (A_i,α_i) and (A_j,α_j) are T-equivalent when $i,j\in\{1,\ldots,N\}$. Theorem 4.3 implies that $(\mathrm{Km}(A_i),\Theta_{A_i}(\alpha_i))$ and $(\mathrm{Km}(A_j),\Theta_{A_j}(\alpha_j))$ are D-equivalent.

For any $i\in\{2,\ldots,N\}$, the isometry g_i induces a Hodge isometry $f_i:T(\mathrm{Km}(A_1))\to T(\mathrm{Km}(A_i))$ which (due to Theorem 1.14.4 in [21]) extends to a Hodge isometry $h_i:H^2(\mathrm{Km}(A_1),\mathbb{Z})\to H^2(\mathrm{Km}(A_i),\mathbb{Z})$. The Torelli Theorem yields an isomorphism $\varphi_i:\mathrm{Km}(A_1)\to\mathrm{Km}(A_i)$ such that $\varphi_i^*(\Theta_{A_i}(\alpha_i))=\Theta_{A_1}(\alpha_1)$ (possibly changing α_i with α_i^{-1}), for any $i\in\{2,\ldots,N\}$. This concludes the proof of (ii). $\square$

In other words, Proposition 4.6 shows that on a twisted K3 surface we can put just a finite number of non-isomorphic *twisted Kummer structures*. Nevertheless, such a number can be arbitrarily large even when the twist is non-trivial and has any possible order.

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